A Min-Max Relation on Packing Feedback Vertex Sets

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Abstract

Let G be a graph with a nonnegative integral function w defined on $V(G)$. A collection $\mathcal{F}$ of subsets of $V(G)$ (repetition is allowed) is called a *feedback vertex set packing* in G if the removal of any member of $\mathcal{F}$ from G leaves a forest, and every vertex $v \in V(G)$ is contained in at most $w(v)$ members of $\mathcal{F}$. The *weight* of a cycle C in G is the sum of $w(v)$, over all vertices v of C. The purpose of this paper is to characterize all graphs with the property that, for any nonnegative integral function w, the maximum cardinality of a feedback vertex set packing is equal to the minimum weight of a cycle.

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1 Introduction

We begin with a brief introduction to the theory of packing and covering. More details on this subject can be found in [2, 8]. A *clutter* $\mathcal{C}$ is an ordered pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a collection of subsets of V such that $A_1 \nsubseteq A_2$, for all distinct $A_1, A_2 \in \mathcal{E}$. Members of V and $\mathcal{E}$ are called *vertices* and *edges* of $\mathcal{C}$, respectively. The *blocker* of $\mathcal{C}$ is the clutter $b(\mathcal{C}) = (V, \mathcal{E}')$, where $\mathcal{E}'$ is the collection of all minimal subsets $B \subseteq V$ such that $B \cap A \neq \emptyset$, for all $A \in \mathcal{E}$. It is well known that $b(b(\mathcal{C})) = \mathcal{C}$ holds for every clutter $\mathcal{C}$.

Let I be a set and let α be a real-valued function with domain I. Then, for any finite subset S of I, we denote by $\alpha(S)$ the sum of $\alpha(s)$, over all $s \in S$. Let $\mathbf{R}_+$ and $\mathbf{Z}_+$ denote the sets of nonnegative real numbers and nonnegative integers, respectively. Let M be the $\mathcal{E}$-V incidence matrix of a clutter $\mathcal{C} = (V, \mathcal{E})$. That is, rows and columns of the 0-1 matrix M are indexed by members of $\mathcal{E}$ and V, respectively, such that, for any $A \in \mathcal{E}$ and $v \in V$, $M_{A,v} = 1$ if and only if $v \in A$. For any vector $w \in \mathbf{Z}_+^V$, let

$$\begin{aligned}
\nu_w^*(\mathcal{C}) &= \max\{x^T \mathbf{1} : \ x \in \mathbf{R}_+^{\mathcal{E}}, \ x^T M \leq w^T\}, \\
\tau_w^*(\mathcal{C}) &= \min\{w^T y : \ y \in \mathbf{R}_+^V, \ My \geq \mathbf{1}\}, \\
\nu_w(\mathcal{C}) &= \max\{x^T \mathbf{1} : \ x \in \mathbf{Z}_+^{\mathcal{E}}, \ x^T M \leq w^T\}, \\
\tau_w(\mathcal{C}) &= \min\{w^T y : \ y \in \mathbf{Z}_+^V, \ My \geq \mathbf{1}\}.
\end{aligned}$$

Combinatorially, each vector $x \in \mathbf{Z}_+^{\mathcal{E}}$ with $x^T M \leq w^T$ can be interpreted as a collection $\mathcal{F}$ of edges (repetition allowed) of $\mathcal{C}$, for which each vertex $v \in V$ belongs to at most $w(v)$ members of $\mathcal{F}$. Such a collection is called a w-*packing* of $\mathcal{C}$. It is clear that $\nu_w(\mathcal{C})$ is the maximum size of a w-packing of $\mathcal{C}$. Similarly, $\tau_w(\mathcal{C})$ is the minimum of $w(B)$, over all edges B of $b(\mathcal{C})$. Notice that

$$\nu_w(\mathcal{C}) \leq \nu_w^*(\mathcal{C}) = \tau_w^*(\mathcal{C}) \leq \tau_w(\mathcal{C}), \tag{1.1}$$

where the equality follows from the LP Duality Theorem. One of the fundamental problems in combinatorial optimization is to identify scenarios under which either one or two of the above inequalities hold with equality. In particular, $\mathcal{C}$ is *ideal* if $\tau_w^*(\mathcal{C}) = \tau_w(\mathcal{C})$, for all $w \in \mathbf{Z}_+^V$, while $\mathcal{C}$ is *Mengerian* if $\nu_w^*(\mathcal{C}) = \nu_w(\mathcal{C})$, for all $w \in \mathbf{Z}_+^V$. It follows from a well known result of Edmonds and Giles [5] that being Mengerian is actually equivalent to $\nu_w(\mathcal{C}) = \tau_w(\mathcal{C})$, for all $w \in \mathbf{Z}_+^V$. Therefore, every Mengerian clutter is ideal.

In this paper, we consider a special class of clutters. For any simple graph $G = (V, E)$, let $\mathcal{C}_G = (V, \mathcal{E})$, where $\mathcal{E}$ consists of $V(C)$, for all induced cycles C of G. Our work is a continuation of the work done in [3, 4], by two of the authors and Xu. To clarify our motivation, we summarize the main results in [4].

We first define a few graphs. A Θ-*graph* is a subdivision of $K_{2,3}$. A *wheel* is obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle. A W-*graph* is a subdivision of a wheel. An *odd ring* (see Figure 1 below) is a graph obtained from an odd cycle by replacing each edge $e = uv$ with *either* a triangle containing e *or* two triangles uab, vcd together with two additional edges ac and bd. A subdivision of an odd ring is called an R-*graph*. Let $\mathcal{L}$ be the class of simple graphs G such that no *induced* subgraph of G is isomorphic to a Θ-graph, a W-graph, or an R-graph.

Theorem 1.1 ([4]) *The following are equivalent for every simple graph G:*

(i) $\mathcal{C}_G$ is Mengerian;

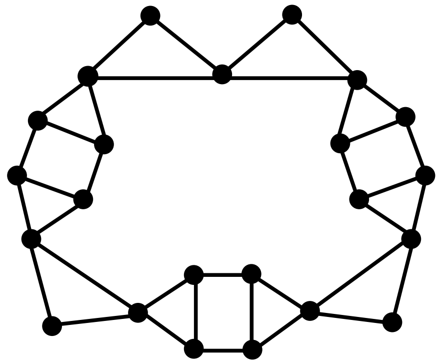

Figure 1: An odd ring obtained from a cycle of length 7.

(ii) $\mathcal{C}_G$ *is ideal;*

(iii) $G \in \mathcal{L}$.

It was proved by Lehman [7] that a clutter is ideal if and only if its blocker is ideal. Therefore, the equivalence of (ii) and (iii) in Theorem 1.1 implies the following.

Corollary 1.2 *The clutter* $b(\mathcal{C}_G)$ *is ideal if and only if* $G \in \mathcal{L}$.

A subset of $V(G)$ is a *feedback vertex set* (*FVS*) if its removal from G results in a forest. Clearly, edges of $b(\mathcal{C}_G)$ are precisely minimal feedback vertex sets of G. The idealness of $\mathcal{C}_G$ and $b(\mathcal{C}_G)$ can also be stated as follows. Let $G = (V, E) \in \mathcal{L}$. Then vertices of the polyhedron $\{x \in \mathbf{R}_+^V : \ x(V(C)) \geq 1, \forall C \in \mathcal{C}_G\}$ are precisely characteristic vectors of minimal feedback vertex sets of G. Meanwhile, vertices of the polyhedron $\{x \in \mathbf{R}_+^V : x(F) \geq 1, \forall F \in b(\mathcal{C}_G)\}$ are precisely characteristic vectors of induced cycles of G.

At this point, a natural question suggested by Guenin [6] arises: When is $b(\mathcal{C}_G)$ Mengerian? In general, the blocker of a Mengerian clutter does not have to be Mengerian (see Section 79.2 of [8]). However, the following theorem, our main result in this paper, says that $\mathcal{C}_G$ and $b(\mathcal{C}_G)$ are always Mengerian together.

Theorem 1.3 $b(\mathcal{C}_G)$ *is Mengerian if and only if* $\mathcal{C}_G$ *is.*

Using graph theoretical language, Theorem 1.3 can be restated as follows.

Theorem 1.4 *The following two statements are equivalent for every simple graph* $G = (V, E)$*:*

(i) For any $w \in \mathbf{Z}_+^V$*, the minimum of* $w(F)$*, over all feedback vertex sets* F *of* G*, is equal to the maximum number of cycles (repetition allowed) of* G *such that each* $v \in V$ *belongs to at most* $w(v)$ *of these cycles;*

(ii) For any $w \in \mathbf{Z}_+^V$*, the minimum of* $w(V(C))$*, over all cycles* C *of* G*, is equal to the maximum number of feedback vertex sets (repetition allowed) of* G *such that each* $v \in V$ *belongs to at most* $w(v)$ *of these feedback vertex sets.*

It can be seen from Theorem 1.3 and Theorem 1.1 that $b(\mathcal{C}_G)$ is Mengerian if and only if G belongs to $\mathcal{L}$. A structural characterization of these graphs is available from [4], which we will use to prove Theorem

1.3. It is worthwhile pointing out that this structural characterization yields a polynomial-time algorithm for recognizing graphs in $\mathcal{L}$. We also remark that our proof is constructive and thus can be converted into a polynomial-time algorithm to find an optimal feedback vertex set packing for graphs in $\mathcal{L}$.

The remainder of this paper is devoted to the proof of Theorem 1.3. Since every Mengerian clutter is ideal, the "only if" part of Theorem 1.3 follows immediately from Corollary 1.2 and Theorem 1.1. So it remains to prove, by Theorem 1.1, that $b(\mathcal{C}_G)$ is Mengerian if $G \in \mathcal{L}$. Our proof heavily relies on the structural characterization of graphs in $\mathcal{L}$ obtained in [4], which asserts that every graph in $\mathcal{L}$ can be constructed from some "prime" graphs by "summing" operations. This structure will be explained in detail in Section 2. Then we show in Section 3 that being Mengerian is preserved under our "summing" operations. Finally, we prove in Section 4 that all prime graphs have the required Mengerian property and thus establish our main result.

2 Structures

In this section, we summarize some results from [4] that describe how graphs in $\mathcal{L}$ can be constructed from "prime" graphs. First, we clarify our terminology.

All graphs considered in this paper are undirected, finite, and simple, unless otherwise stated. The reader is referred to [1] for undefined terminology. Let $G = (V, E)$ be a graph. For any $U \subseteq V$ or $U \subseteq E$, let $G\backslash U$ be the graph obtained from G by deleting U, and let $G[U]$ be the subgraph of G induced by U; when U is a singleton $\{u\}$, we may write $G\backslash u$ instead of $G\backslash\{u\}$.

Let G_1 and G_2 be two graphs. The 0-*sum* of G_1 and G_2 is obtained by taking the disjoint union of these two graphs; the 1-*sum* is obtained by identifying a vertex of G_1 with a vertex of G_2. A 2-*sum* of G_1 and G_2 is obtained by first choosing a triangle $x_iy_iz_i$ from G_i $(i = 1, 2)$ such that z_i has degree two in G_i, then deleting z_i from G_i $(i = 1, 2)$, and finally, identifying x_1y_1 with x_2y_2. A triangle T of a graph G is called *stable* if $G\backslash V(T)$ is connected and every vertex in T has degree at least three in G. A 3-*sum* of G_1 and G_2 is obtained by identifying a stable triangle in G_1 with a stable triangle in G_2.

A *rooted graph* consists of a graph G and a specified set R of edges such that each edge in R belongs to a triangle and each triangle in G contains at most one edge from R. By *adding pendent triangles* to the rooted graph G we mean the following operation: for each edge uv in R, we introduce a new vertex t_{uv} and two new edges ut_{uv} and vt_{uv}. The following is a reformulation of Theorem 3.1 in [4].

Lemma 2.1 *For any graph $G \in \mathcal{L}$, at least one of the following holds.*
(i) G is a k-sum of two smaller graphs, for $k = 0, 1, 2, 3$;
(ii) G is obtained from a rooted 2-connected line graph by adding pendent triangles.

An edge is *pendent* if at least one of its ends has degree one. Two distinct edges are called in *series* if they form a minimal edge cut. The following statement is contained in Lemma 4.7 of [4].

Lemma 2.2 *Two distinct edges are in series if neither is a cut edge, and every cycle that contains one must also contain the other.*

Let us also consider every edge as being in series with itself. Then being in series is an equivalence relation. We call each equivalence class a *series family.* A series family is *trivial* if it has only one edge and *nontrivial* otherwise. A graph G is *weakly even* if, for every nontrivial series family F of G with $|F|$ odd, there are two distinct edges xy and xz of F such that they are the only two edges of G that are incident with x. A graph is *subcubic* if the degree of each vertex is at most three. A graph is *chordless* if every cycle of the graph is an induced cycle. Let K_4^- be obtained from K_4 by deleting an edge, W_4^- be obtained from a wheel on five vertices by deleting a rim edge, and $K_{2,3}^+$ be obtained from $K_{2,3}$ by adding an edge between the two vertices of degree three. We shall follow convention to let $L(G)$ denote the line graph of G.

Lemma 2.3 *Suppose $G \in \mathcal{L}$ is not a 2-sum of two smaller graphs. If G is obtained from a rooted 2-connected line graph $L(Q)$ by adding pendant triangles, where Q has no isolated vertices, then*

(i) Q is connected, subcubic, and chordless;

(ii) every cut edge of Q is a pendent edge;

(iii) Q is weakly even;

(iv) if Q has a triangle, then $G \in \{K_3, K_4^-, W_4^-, K_{2,3}^+\}$.

In this lemma, statement (i) follows instantly from the assumption that G is in $\mathcal{L}$, because if Q has a vertex of degree at least four, or a cycle with a chord, then G would have one of the forbidden induced subgraphs (a subdivision of a wheel). Other statements (ii-iv) can all be found in the proof of Lemma 4.10 in [4].

The next two lemmas expose some other facts on series families, and the first one is implicit in the proof of Lemma 4.10 in [4].

Lemma 2.4 *If Q is subcubic and chordless, then every noncut edge belongs to a nontrivial series family.*

A path with end vertices u and v is called a *u-v path.* If a vertex v has degree three, then the subgraph formed by the three edges incident with v is called a *triad* with *center v*. Our next lemma follows from Lemma 4.9 in [4], where the indices are taken modulo t.

Lemma 2.5 *Suppose Q is connected and subcubic, and all its cut edges are pendent edges. If $F = \{e_1, e_2, \ldots, e_t\}$ is a nontrivial series family of Q, then $Q\backslash F$ has precisely t components $Q_1, Q_2, \ldots, Q_t$. The indices can be renamed such that each e_i is between $V(Q_i)$ and $V(Q_{i+1})$. In addition, if $|V(Q_i)| = 2$, then the only edge in $E(Q_i)$ is a pendent edge of Q and it forms a triad with e_{i-1} and e_i; if $|V(Q_i)| > 2$, and u and v are the ends of e_{i-1} and e_i in Q_i, then $u \neq v$ and Q_i has two internally vertex-disjoint u-v paths.*

The following statement is a combination of three lemmas (4.3-4.5) from [4].

Lemma 2.6 *Let $G \in \mathcal{L}$ be a k-sum of two smaller graphs. Then the following hold.*

(i) If $k \in \{0, 1, 2\}$, then G is a k-sum of two smaller graphs that belong to $\mathcal{L}$.

(ii) If G is a 3-sum of G_1 and G_2 over a triangle $x_1x_2x_3$, then all G_{ijk} ($1 \leq i \leq 2$, $1 \leq j < k \leq 3$) are in $\mathcal{L}$, where G_{ijk} is obtained from G_i by adding a new vertex x_{ijk} and two new edges $x_{ijk}x_j$ and $x_{ijk}x_k$.

Recall that the definition of 3-sum for graphs requires a stable triangle; this requirement is actually needed for the above (ii) to be true.

In the remainder of this section, we prove two lemmas. It should be pointed out that within the rest of this section, graphs may have parallel edges, but no loops.

Let $G = (V, E)$ be a graph. For any $v \in V$, let $N_G(v)$ be the set of vertices that are adjacent with v, while $\delta_G(v)$ be the set of edges between v and $N_G(v)$. The *degree* of v is defined by $d_G(v) = |\delta_G(v)|$. A 2-*edge coloring* of G is an assignment of one of two colors to every edge in E. We will say that a color is *represented* at a vertex v if at least one edge in $\delta_G(v)$ is assigned that color. It is not difficult to show that a graph G has a 2-edge coloring in which both colors are represented at each vertex of degree at least two, provided no component of G is an odd cycle (see Section 6.1 of [1]). The following is a minor modification of this fact.

Lemma 2.7 *Let $G = (V, E)$ be a graph and let $U \subseteq V$. Suppose $G[U]$ is bipartite and $d_G(u) \geq 2$, for all $u \in U$. Then G has a 2-edge coloring such that both colors are represented at every vertex in U.*

Proof. Let $v \in V - U$ be an arbitrary vertex and let $\delta_G(v) = \{e_i = vu_i : i = 1, 2, ..., d_G(v)\}$. By *disassembling* v we mean the operation of replacing v with a set $V_v = \{v_1, v_2, ..., v_{d_G(v)}\}$ of new vertices and making each e_i $(i = 1, 2, \ldots, d_G(v))$ joining from u_i to v_i, instead of v. Let us perform this operation at every $v \in V - U$. It is clear that the resulting graph G' is bipartite and $d_{G'}(u) = d_G(u) \geq 2$, for all $u \in U$. Therefore, G' has a 2-edge coloring λ such that both colors are represented at every vertex in U. Since $E(G') = E(G)$, λ is also a 2-edge coloring of G, and it is easy to see that λ has the required property. ∎

Let G' be a connected subgraph of G. Then the *contraction* of G' in G is obtained from $G \backslash E(G[V(G')])$ by identifying all vertices of $V(G')$. This is the same as the ordinary contraction except we also delete the resulting loops. Notice that the contraction of a simple graph may have parallel edges, but no loops.

Lemma 2.8 *Let $G = (V, E)$ be subcubic, chordless, and weakly even. If $G' = (V', E')$ is obtained from G by repeatedly contracting induced cycles, and $U = (V' - V) \cup \{v \in V \cap V' : d_G(v) = 3\}$, then $G'[U]$ is bipartite.*

Proof. By definition, two distinct edges are in series if and only if they form a minimal cut. Hence two edges of G' are in series if and only if they are in series in G. By definition, edges of any contracted cycle are not in series with any edge not in this cycle, so each series family of G' *is* a series family of G. Let C be a cycle of G'. Then its edges can be partitioned into series families $F_1, F_2, ..., F_t$ of G. If C is an odd cycle, it is clear that there exists an F_i with odd $|F_i|$. By Lemma 2.4, $|F_i| \neq 1$. Since G is weakly even, there exists a vertex v of G such that v is incident with two edges of F_i and $d_G(v) = 2$. Consequently, $v \in V(C)$ yet $v \notin U$, which proves that $G'[U]$ has no odd cycles and thus is bipartite. ∎

3 Sums of hypergraphs

The purpose of this section is to prove a few results, which assert that being Mengerian is preserved under some natural summing operations.

A *hypergraph* H is an ordered pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a collection of subsets of V. Members of V and $\mathcal{E}$ are called *vertices* and *edges* of H. An edge is *minimal* if none of its proper subset is an edge. It is clear that clutters are special hypergraphs. Our problem is essentially a problem on clutters. We use the language of hypergraphs just to simplify our terminology.

The concept of w-packing and blocker can be extended obviously from clutters to hypergraphs. These are formalized as follows. Let $H = (V, \mathcal{E})$ be a hypergraph and let $w \in \mathbf{Z}_+^V$. A *w-packing* of H is a collection $\mathcal{F}$ of edges (repetition allowed) of H, for which each vertex $v \in V$ belongs to at most $w(v)$ members of $\mathcal{F}$. The *blocker* of H is the *clutter* $b(H) = (V, \mathcal{E}')$, where $\mathcal{E}'$ is the collection of all minimal subsets $B \subseteq V$ such that $B \cap A \neq \emptyset$, for all $A \in \mathcal{E}$. We also define $\bar{b}(H) = (V, \mathcal{E}'')$, where $\mathcal{E}''$ consists of all $B \subseteq V$ such that $B \cap A \neq \emptyset$, for all $A \in \mathcal{E}$. Finally, let $r_w(H) = \min_{A \in \mathcal{E}} w(A)$.

Lemma 3.1 *The following hold for any hypergraph $H = (V, \mathcal{E})$.*
(i) $\tau_w(b(H)) = r_w(H)$, for all $w \in \mathbf{Z}_+^V$;
(ii) The clutter $b(H)$ is Mengerian if and only if $\bar{b}(H)$ has a w-packing of size $r_w(H)$, for all $w \in \mathbf{Z}_+^V$.

Proof. In general, the equality $b(b(H)) = H$ does not hold, but it is easy to see that edges of $b(b(H))$ are precisely minimal edges of H (see Section 77.6 of [8]). Therefore,

$$\tau_w(b(H)) = \min_{A \in b(b(H))} w(A) = \min_{A \in \mathcal{E}} w(A) = r_w(H),$$

which proves (i). To prove (ii), notice that edges of $b(H)$ are precisely minimal edges of $\bar{b}(H)$. Thus $\bar{b}(H)$ has a w-packing of size $r_w(H)$ if and only if $b(H)$ has a w-packing of size $r_w(H)$, which is equivalent to $\nu_w(b(H)) \geq r_w(H) = \tau_w(b(H))$. On the other hand, since $b(H)$ is a clutter, we deduce from (1.1) that $\nu_w(b(H)) \leq \tau_w(b(H))$. Therefore, $\bar{b}(H)$ has a w-packing of size $r_w(H)$ if and only if $\nu_w(b(H)) = \tau_w(b(H))$, which proves (ii). ∎

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs. If $k := |V_1 \cap V_2| \leq 1$, then $(V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$ is called the *k-sum* of H_1 and H_2.

Lemma 3.2 *Suppose H is the k-sum $(k = 0, 1)$ of H_1 and H_2. If both $b(H_1)$ and $b(H_2)$ are Mengerian, then so is $b(H)$.*

Proof. Let $H = (V, \mathcal{E})$, $w \in \mathbf{Z}_+^V$, and $r = r_w(H)$. By Lemma 3.1, we only need to show that $\bar{b}(H)$ has a w-packing of size r. For $i = 1, 2$, let $H_i = (V_i, \mathcal{E}_i)$, and $w_i(v) = w(v)$, for all $v \in V_i$. Since $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, it follows that $r = \min\{r_{w_1}(H_1), r_{w_2}(H_2)\}$. By Lemma 3.1, as $b(H_i)$ $(i = 1, 2)$ is Mengerian, $\bar{b}(H_i)$ has a w_i-packing $\{B_1^i, B_2^i, \ldots, B_r^i\}$. In case $k = 1$, we further assume that, without loss of generality, if x is the common vertex of H_1 and H_2, then x appears only in $B_1^i, B_2^i, \ldots, B_{r_i}^i$ $(i = 1, 2)$. Now it is straightforward to verify that $\{B_1^1 \cup B_1^2, B_2^1 \cup B_2^2, \ldots, B_r^1 \cup B_r^2\}$ is a w-packing of $\bar{b}(H)$ as desired. ∎

Before proceeding, let us prove three technical lemmas which will be used in our discussions on 2- and 3-sums.

Lemma 3.3 *Suppose $\bar{b}(H)$ has no w-packing of size $r_w(H)$, for some $w \in \mathbf{Z}_+^V$. If we choose such an integral vector w so that $w(V(H))$ is minimized, then $w(v) \leq r_w(H)$, for all $v \in V(H)$.*

Proof. Let $r = r_w(H)$ and $V = V(H)$. Suppose $w(v_0) > r$ for some $v_0 \in V$. Let $w' \in \mathbf{Z}_+^V$ such that $w'(v_0) = r$ and $w'(v) = w(v)$, for all $v \in V - \{v_0\}$. It follows from $w(v_0) > r$ that v_0 does not belong to any edge of H of minimum weight. Therefore, $r_{w'}(H) = r$. By the minimality of w, $\bar{b}(H)$ has a w'-packing of size $r_{w'}(H) = r$. Notice that every w'-packing is also a w-packing, as $w' \leq w$, thus $\bar{b}(H)$ has a w-packing of size r. This contradiction proves the lemma. ∎

Lemma 3.4 *Suppose $\bar{b}(H)$ has a w-packing $\mathcal{B}$ of size r, where $w \in \mathbf{Z}_+^V$. If $\mathcal{B}$ is chosen so that $||\mathcal{B}|| = \sum_{B \in \mathcal{B}} |B|$ is maximized, subject to $|\mathcal{B}| = r$, then every vertex v of H is contained in exactly* $\min\{r, w(v)\}$ *members of $\mathcal{B}$.*

Proof. Suppose some vertex v is contained in fewer than $\min\{r, w(v)\}$ members of $\mathcal{B}$. Then, as $|\mathcal{B}| = r$, there must exist $B \in \mathcal{B}$ that does not contain v. Let $\mathcal{B}'$ be obtained from $\mathcal{B}$ by replacing B with $B \cup \{v\}$, which is another member of $\bar{b}(H)$. It is straightforward to verify that $\mathcal{B}'$ is still a w-packing of $\bar{b}(H)$ of size r, yet $\|\mathcal{B}'\| > \|\mathcal{B}\|$. This contradicts the maximality of $\|\mathcal{B}\|$ and thus every vertex must belong to at least $\min\{r, w(v)\}$ members of $\mathcal{B}$. However, since $\mathcal{B}$ is a w-packing of $\bar{b}(H)$ with $||\mathcal{B}|| = r$, no vertex is contained in more than $\min\{r, w(v)\}$ members of $\mathcal{B}$, and thus the lemma follows. ∎

Lemma 3.5 *Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be hypergraphs with $V_1 \cap V_2 = V_0$. Let $H = (V, \mathcal{E})$, where $V_0 \subseteq V \subseteq V_1 \cup V_2$ and $\mathcal{E} = \{A \in \mathcal{E}_1 \cup \mathcal{E}_2 : A \subseteq V\}$. Let $w \in \mathbf{Z}_+^V$, $w_1 \in \mathbf{Z}_+^{V_1}$, and $w_2 \in \mathbf{Z}_+^{V_2}$ such that $w(v) = w_i(v)$, for all $v \in V \cap V_i$ and $i = 1, 2$. Let $r = r_w(H)$. For $i = 1, 2$, suppose $\bar{b}(H_i)$ has a w_i-packing $\mathcal{B}_i$ of size r. Then at least one of the following holds.*

(i) $\bar{b}(H)$ has a w-packing of size r;

(ii) $\bar{b}(H)$ has no w'-packing of size $r_{w'}(H)$, for some $w' \in \mathbf{Z}_+^V$ with $w'(V) < w(V)$;

(iii) $B_1 \cap V_0 \neq B_2 \cap V_0$, for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

Proof. We prove that (i) holds if both (ii) and (iii) fail. Let $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ with $B_1 \cap V_0 = B_2 \cap V_0$. Let χ_1, χ_2, and χ be the characteristic vectors of B_1, B_2, and $B := (B_1 \cup B_2) \cap V$, which are considered as subsets of V_1, V_2, and V, respectively. We define $w_1' = w_1 - \chi_1$, $w_2' = w_2 - \chi_2$, and $w' = w - \chi$. For $i = 1, 2$, since $\bar{b}(H_i)$ has a w_i'-packing $\mathcal{B}_i - \{B_i\}$ of size $r - 1$, it follows from (1.1) and Lemma 3.1(i) that $r_{w_i'}(H_i) = \tau_{w_i'}(b(H_i)) \geq r - 1$. Therefore, $r_{w'}(H) \geq r - 1$. Notice that B is an edge of $\bar{b}(H)$. If $B = \emptyset$, then (i) holds trivially. If $B \neq \emptyset$, then $w'(V) < w(V)$. Since (ii) is false, $\bar{b}(H)$ has a w'-packing $\mathcal{B}'$ of size $r - 1$. Hence (i) holds again, as $\mathcal{B}' \cup \{B\}$ is a w-packing of $\bar{b}(H)$ of size r, which proves the lemma. ∎

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be hypergraphs with $V_1 \cap V_2 = \{x_1, x_2\}$. Suppose, for $i = 1, 2$, H_i has an edge $A_i = \{x_1, x_2, y_i\}$ which is the only edge containing y_i. Let $V_i' = V_i - \{y_i\}$ and $\mathcal{E}_i' = \mathcal{E}_i - \{A_i\}$. Then $(V_1' \cup V_2', \mathcal{E}_1' \cup \mathcal{E}_2')$ is called the 2-*sum* of H_1 and H_2.

Lemma 3.6 *Let H be a 2-sum of H_1 and H_2. If both $b(H_1)$ and $b(H_2)$ are Mengerian, then so is $b(H)$.*

Proof. Let $H = (V, \mathcal{E})$. Like in the proof of Lemma 3.2, we will show that, for all $w \in \mathbf{Z}_+^V$,

(*) $\bar{b}(H)$ has a w-packing of size $r := r_w(H)$.

Let us use the terminology in the definition of a 2-sum. Suppose (*) is false for some $w \in \mathbf{Z}_+^V$. Then we choose such a w with $w(V)$ as small as possible. By Lemma 3.3,

(3.6.1) $w(v) \le r$ for all $v \in V$.

Let $i \in \{1,2\}$. We define $w_i \in \mathbf{Z}_+^{V_i}$ such that $w_i(y_i) = \max\{0, r - w(x_1) - w(x_2)\}$ and $w_i(v) = w(v)$, for all $v \in V_i - \{y_i\}$. Notice that other than A_i every edge of H_i is an edge of H, so $r_{w_i}(H_i) \ge r$. Since $b(H_i)$ is Mengerian, we conclude from Lemma 3.1 that $\bar{b}(H_i)$ has a w_i-packing $\mathcal{B}_i$ of size r. We choose such a $\mathcal{B}_i$ with $\|\mathcal{B}_i\| = \sum_{B \in \mathcal{B}_i} |B|$ as *large* as possible. Then, by (3.6.1) and Lemma 3.4,

(3.6.2) for any $i, j \in \{1,2\}$, x_j is contained in exactly $w(x_j)$ members of $\mathcal{B}_i$.

Furthermore, by Lemma 3.5,

(3.6.3) $B_1 \cap \{x_1, x_2\} \ne B_2 \cap \{x_1, x_2\}$, for all $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

If $w(x_1) + w(x_2) > r$, we deduce from (3.6.2) that, for $i = 1,2$, there exists $B_i \in \mathcal{B}_i$ that contains both x_1 and x_2, which contradicts (3.6.3). So we must have $w(x_1) + w(x_2) \le r$. By the definition of w_i $(i = 1,2)$, we have $w_i(A_i) = r$. Therefore, $|B_i \cap A_i| = 1$, for all $B_i \in \mathcal{B}_i$ $(i = 1,2)$, as every edge of H_i should meet every edge of $\bar{b}(H_i)$. Since (*) does not hold, $r > 0$ and thus $\mathcal{B}_1 \ne \emptyset$. Take any $B_1 \in \mathcal{B}_1$. There must exist $a_1 \in A_1$ such that $B_1 \cap A_1 = \{a_1\}$. It follows that $w_1(a_1) \ne 0$. Let $a_2 \in A_2$ be the vertex corresponding to a_1. Then $w_2(a_2) = w_1(a_1) \ne 0$. By (3.6.2), some $B_2 \in \mathcal{B}_2$ contains a_2. Since $|B_2 \cap A_2| = 1$, we conclude that $B_1 \cap \{x_1, x_2\} = B_2 \cap \{x_1, x_2\}$, contradicting (3.6.3), which completes our proof of the lemma. ∎

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be hypergraphs with $V_1 \cap V_2 = \{x_1, x_2, x_3\}$. Suppose $A = \{x_1, x_2, x_3\}$ is an edge of H_1 and H_2. Then $(V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$ is called the 3-*sum* of H_1 and H_2 over A.

Lemma 3.7 *Let H be the 3-sum of H_1 and H_2 over $A = \{x_1, x_2, x_3\}$. For $i = 1,2$ and $1 \le j < k \le 3$, let H_{ijk} be obtained from H_i by adding a new vertex x_{ijk} and a new edge $A_{ijk} = \{x_{ijk}, x_j, x_k\}$. If all $b(H_{ijk})$ are Mengerian, then so is $b(H)$.*

Proof. Let $H = (V, \mathcal{E})$. Again, we prove that, for all $w \in \mathbf{Z}_+^V$,

(*) $\bar{b}(H)$ has a w-packing of size $r := r_w(H)$.

We use the terminology in the definition of a 3-sum. Suppose (*) is false for some $w \in \mathbf{Z}_+^V$. Then we choose such a w with $w(V)$ as small as possible. Our proof is very similar to the proof of the last lemma. First, by Lemma 3.3,

(3.7.1) $w(v) \le r$ for all $v \in V$.

Let $1 \le i \le 2$ and $1 \le j < k \le 3$. Let $V_{ijk} = V_i \cup \{x_{ijk}\}$, which is the vertex set of H_{ijk}. We define $w_{ijk} \in \mathbf{Z}_+^{V_{ijk}}$ with $w_{ijk}(x_{ijk}) = \max\{0, r - w(x_j) - w(x_k)\}$ and $w_{ijk}(v) = w(v)$ for all $v \in V_i$. Since other than A_{ijk} every edge of H_{ijk} is an edge of H, it follows that $r_{w_{ijk}}(H_{ijk}) \ge r$. Then, as $b(H_{ijk})$ is Mengerian, we conclude from Lemma 3.1 that $\bar{b}(H_{ijk})$ has a w_{ijk}-packing $\mathcal{B}_{ijk}$ of size r. We choose such a $\mathcal{B}_{ijk}$ with $\|\mathcal{B}_{ijk}\| = \sum_{B \in \mathcal{B}_{ijk}} |B|$ as *large* as possible. Then, by (3.7.1) and Lemma 3.4, (3.7.2) for any $1 \le i \le 2$, $1 \le j < k \le 3$, and $1 \le h \le 3$, x_h belongs to exactly $w(x_h)$ members of $\mathcal{B}_{ijk}$.

Furthermore, by Lemma 3.5,

(3.7.3) $B \cap A \neq B' \cap A$, for all $B \in \mathcal{B}_{1jk}$ and $B' \in \mathcal{B}_{2j'k'}$ with $1 \le j < k \le 3$, and $1 \le j' < k' \le 3$.

Since $r_w(H) = r$, it follows that $w(A) \ge r$. We prove that

(3.7.4) $w(A) > r$.

If $w(A) = r$, then A is an edge of H_{i12} ($i = 1, 2$) of minimum weight. It follows that $|B \cap A| = 1$, for all $B \in \mathcal{B}_{i12}$ ($i = 1, 2$), as $|\mathcal{B}_{i12}| = r$ and every member of $\mathcal{B}_{i12}$ should meet every edge of H_{i12}. Since (*) does not hold, we have $r > 0$ and so, $w(x_h) \neq 0$, for some $h = 1, 2, 3$. Therefore, by (3.7.2), there exists $B_{i12} \in \mathcal{B}_{i12}$ ($i = 1, 2$) with $B_{112} \cap A = \{x_h\} = B_{212} \cap A$. This contradicts (3.7.3) and thus (3.7.4) is proved.

(3.7.5) $w(x_j) + w(x_k) > r$, for all $1 \le j < k \le 3$.

Suppose otherwise. By symmetry, we assume that $w(x_1)+w(x_2) \le r$. Then we deduce from the definition of w_{i12} that $w_{i12}(A_{i12}) = r$ ($i = 1, 2$). It follows that A_{i12} ($i = 1, 2$) is an edge of H_{i12} of minimum weight, and thus no member of $\mathcal{B}_{i12}$ can contain both x_1 and x_2.

If $w(x_1) + w(x_3) > r$, by (3.7.2) there exists $B_{i12} \in \mathcal{B}_{i12}$ ($i = 1, 2$) that contains both x_1 and x_3. Therefore, $B_{112} \cap A = \{x_1, x_3\} = B_{212} \cap A$, contradicting (3.7.3). Hence $w(x_1) + w(x_3) \le r$. By symmetry, we must also have $w(x_2) + w(x_3) \le r$. Therefore, the conclusion we made in the previous paragraph holds not only for $\mathcal{B}_{i12}$, but for all $\mathcal{B}_{ijk}$. That is, $|B \cap \{x_j, x_k\}| \le 1$, for all edges $B \in \mathcal{B}_{ijk}$. On the other hand, by (3.7.4) and (3.7.2), each $\mathcal{B}_{ijk}$ has an edge B_{ijk} with $|B_{ijk} \cap A| > 1$. Thus, by (3.7.3), $\{B_{1jk} \cap A,\, B_{2jk} \cap A\} = \{\{x_j, x_\ell\}, \{x_k, x_\ell\}\}$ ($1 \le j < k \le 3$), where $\ell \in \{1, 2, 3\} - \{j, k\}$.

Without loss of generality, let $B_{112} \cap A = \{x_1, x_3\}$, and $B_{212} \cap A = \{x_2, x_3\}$. We deduce from (3.7.3) that $B_{113} \cap A \neq \{x_2, x_3\}$. Thus $B_{113} \cap A = \{x_1, x_2\}$, and $B_{213} \cap A = \{x_2, x_3\}$. Now $B_{223} \cap A$ is $\{x_1, x_2\}$ or $\{x_1, x_3\}$. But (3.7.3) is violated in either case, which completes the proof of (3.7.5).

(3.7.6) $|B \cap A| \le 2$, for all $B \in \mathcal{B}_{ijk}$, where $1 \le i \le 2$ and $1 \le j < k \le 3$.

Suppose the claim is false. Without loss of generality, we assume that some $\mathcal{B}_{1jk}$ has a member $B_0 \supseteq A$. It follows that $|B \cap A| \le 2$, for all $B \in \mathcal{B}_{212} \cup \mathcal{B}_{213} \cup \mathcal{B}_{223}$. Let $1 \le j < k \le 3$. Since $w(x_j) + w(x_k) > r$, $\mathcal{B}_{2jk}$ has a member B_{2jk} that contains both x_j and x_k, which implies $B_{2jk} \cap A = \{x_j, x_k\}$. Therefore, by (3.7.3), $|B \cap A| \neq 2$, for all $B \in \mathcal{B}_{112} \cup \mathcal{B}_{113} \cup \mathcal{B}_{123}$.

Let $1 \le j < k \le 3$ and $\ell \in \{1, 2, 3\} - \{j, k\}$. Let $\mathcal{B}'_{1jk}$ consist of members of $\mathcal{B}_{1jk}$ that contains x_ℓ. By (3.7.2), $|\mathcal{B}'_{1jk}| = w(x_\ell)$. Notice that (3.7.5) implies $w_{1jk}(x_{1jk}) = 0$. Since every member of $\mathcal{B}_{1jk}$ must meet A_{1jk}, which is an edge of H_{1jk}, we deduce that every member of $\mathcal{B}_{1jk}$ contains at least one of x_j and x_k. In particular, every member of $\mathcal{B}'_{1jk}$ contains at least one of x_j and x_k. Therefore, $B \supseteq A$, for all $B \in \mathcal{B}'_{1jk}$, as $|B \cap A| \neq 2$. Consequently, $w(x_j) \ge w(x_\ell)$. Since j, k, ℓ were chosen arbitrarily, it follows that $w(x_1) = w(x_2) = w(x_3)$. It also follows that $\mathcal{B}'_{1jk} = \mathcal{B}_{1jk}$, and thus $w(x_1) = w(x_2) = w(x_3) = r$. On the other hand, since $|B \cap A| \leq 2$, for all $B \in \mathcal{B}_{212}$, we deduce from (3.7.2) that $r = |\mathcal{B}_{212}| \geq w(A)/2 = 3r/2$, a contradiction, which completes the proof of (3.7.6).

Finally, let $i \in \{1, 2\}$. By (3.7.5), $w(x_1)+w(x_2) > r$, which implies $\mathcal{B}_{i12}$ has a member B_{i12} that contains both x_1 and x_2. Then, by (3.7.6), we must have $B_{i12} \cap A = \{x_1, x_2\}$, which contradicts (3.7.3). The lemma is proved. ∎

We finally point out that performing k-sums on graphs agrees with performing k-sums on hypergraphs. We omit the proof of the next lemma since it follows from the corresponding definitions immediately.

Lemma 3.8 *Let G be a k-sum ($k = 0, 1, 2, 3$) of G_1 and G_2. Then $\mathcal{C}_G$ is the k-sum of $\mathcal{C}_{G_1}$ and $\mathcal{C}_{G_2}$. Moreover, if $H = \mathcal{C}_G$ and $H_i = \mathcal{C}_{G_i}$ ($i = 1, 2$), then each hypergraph H_{ijk} defined in Lemma 3.7 is precisely $\mathcal{C}_{G_{ijk}}$, where G_{ijk} is the graph defined in Lemma 2.6.*

4 Packing feedback vertex sets

We have shown in the preceding section that the Mengerian property is preserved under summing operations. To prove Theorem 1.3, it remains to verify that every prime graph G enjoys the desired Mengerian property. Recall Lemma 2.1, G is obtained from a rooted 2-connected line graph $L(Q)$ by adding pendent triangles. Since G belongs to a finite set of sporadic graphs if Q has a triangle (see Lemma 2.3), it is natural to divide our proof into two parts, depending on the presence or absence of a triangle in Q. Our next two lemmas are concerning the case when Q is triangle-free, while the opposite case is established by the third lemma.

When Q is triangle-free, we start with an arbitrary collection $\mathcal{F}$ of subsets of $V(G)$ such that $|\mathcal{F}| = r_w(\mathcal{C}_G)$ and every vertex v belongs to exactly $w(v)$ members of $\mathcal{F}$. We then adjust the collection to make it more efficient, meaning to make sets in $\mathcal{F}$ to meet as many induced cycles as possible. We prove that if $\mathcal{F}$ is optimal then it is a w-packing of feedback vertex sets. When we make the adjustments, we distinguish triangles and other cycles. They behave quite differently because triangles in G correspond to triads in Q, as Q is triangle-free, and other induced cycles of G correspond to cycles of Q. We first make sets in $\mathcal{F}$ meet all triangles and then other cycles. One of the key steps for making these adjustments is to find a better partition of the union of two members of $\mathcal{F}$. This is our first lemma.

Lemma 4.1 *Let G be obtained from a rooted line graph $L(Q)$ by adding pendent triangles, where Q is triangle-free and satisfies (i-iii) in Lemma 2.3. Let $\mathcal{C}$ be a collection of induced cycles in G, which includes all triangles of G. Suppose $S \subseteq V(G)$ with $|S \cap V(C)| \geq 2$, for every $C \in \mathcal{C}$. Then S can be partitioned into R and B such that $R \cap V(C) \neq \emptyset \neq B \cap V(C)$, for every $C \in \mathcal{C}$.*

Proof. Suppose the lemma is false. Then we can choose a counterexample $\Omega = (G, \mathcal{C}, S)$ such that

(a) $|\mathcal{C}|$ is minimized;

(b) subject to (a), $t_\Omega = |\{C \in \mathcal{C} : |V(C)| = 3 \text{ and } V(C) \subseteq S\}|$ is minimized;

(c) subject to (a) and (b), $d_\Omega = |\{v \in V(G) : d_G(v) \geq 4\}|$ is minimized.

A pair (R, B) that satisfies the conclusion of the lemma is called a *certificate* for $(G, \mathcal{C}, S)$. Since $L(Q)$ is an induced subgraph of G, we will view elements of $V(L(Q))$ indifferently as vertices of G or edges of Q. We proceed by proving a sequence of claims.

(1) $|\mathcal{C}| \geq 2$.

This is clear since Ω is a counterexample.

(2) If $x \in V(G)$ belongs to a triangle T of G and $d_G(x) = 2$, then $x \notin S$.

Let x, y, z be the three vertices of T. If T is a pendent triangle added to $L(Q)$, then $y, z \in E(Q)$. If T is not such a pendent triangle, then x is a pendent edge of Q, as Q has no triangles. In both cases, it is clear that $\Omega' = (G \backslash x, \mathcal{C} - \{T\}, S - \{x\})$ satisfies the assumptions of the lemma. By the minimality of $|\mathcal{C}|$, Ω' has a certificate (R, B). Since S contains at least two vertices of T, at least one of R and B, say R, contains either y or z. Therefore, the partition $(R, B \cup \{x\})$ of S is a certificate for Ω, a contradiction, which proves (2).

It follows immediately from (2) that

(3) $S \subseteq E(Q)$.

Let $x = uv$ be an edge of Q. Let $\delta'_Q(u) = \delta_Q(u) - \{x\}$ and $\delta'_Q(v) = \delta_Q(v) - \{x\}$. Since Q is simple, $\delta'_Q(u)$ and $\delta'_Q(v)$ are disjoint. On the other hand, $N_{L(Q)}(x) = \delta'_Q(u) \cup \delta'_Q(v)$. Let Q_x be obtained from Q by subdividing x with a new vertex w. Observe that every series family of Q_x that does not contain uw or wv is also a series family of Q, while the series family containing uw must also contain wv, and $d_Q(w) = 2$, thus Q_x is weakly even as well. Moreover, $L(Q_x)$ can be obtained from $L(Q)$ by replacing x with two adjacent vertices uw and vw, such that uw is adjacent to all vertices in $\delta'_Q(u)$, and vw is adjacent to all vertices in $\delta'_Q(v)$. Since Q has no triangles, $L(Q)$ has no edges between $\delta'_Q(u)$ and $\delta'_Q(v)$. Therefore, this "splitting" operation performed on $L(Q)$ (to get $L(Q_x)$) does not destroy any of its triangles, which means that there is a natural correspondence between triangles in $L(Q)$ and triangles in $L(Q_x)$. It follows that $L(Q_x)$ can be rooted the same ways as $L(Q)$ was rooted. Let G_x be obtained from the rooted $L(Q_x)$ by adding pendent triangles. Then the following is clear.

(4) If x is not a pendent edge of Q, then G_x satisfies the assumption of the lemma.

For any induced cycle C of G, we define an induced cycle C_x of G_x as follows. If C is a triangle, let C_x be the triangle in G_x that naturally corresponds to C. If C has length at least four, then C can be expressed as $L(D)$ for a cycle D of Q. If $x \notin E(D)$, then we define $C_x = C$. If $x \in E(D)$, let D_x be obtained from D by subdividing x with w and let $C_x = L(D_x)$. Finally, let $\mathcal{C}_x = \{C_x : C \in \mathcal{C}\}$.

(5) $t_\Omega = 0$. That is, $|S \cap V(T)| = 2$, for all triangles T of G.

Suppose (5) is false. Then G has a triangle T such that all its three vertices, say, x, y, z, belong to S. By (3), T is not a pendent triangle. Let us name these three vertices such that, if T contains a root edge at which a pendent triangle is attached, then this root edge is yz. Suppose, as an edge of Q, that x has ends u

and v, where u is the common end of x, y, z. Then x is not a pendent edge of Q, for otherwise $d_G(x) = 2$, so $x \notin S$ by (2), a contradiction. Then it is routine to verify, using (4), that $\Omega_x = (G_x, \mathcal{C}_x, (S - \{x\}) \cup \{wv\})$ satisfies the assumptions of the lemma. Moreover, $|\mathcal{C}_x| = |\mathcal{C}|$, but $t_{\Omega_x} = t_\Omega - 1$. By the minimality of t_Ω, we conclude that Ω_x has a certificate (R, B). Now it is easy to see that replacing wv with x in (R, B) results in a partition of S, which is a certificate of Ω. This is a contradiction and thus (5) is proved.

(6) $G = L(Q)$.

Suppose x is a vertex of G that does not belong to $E(Q)$. Then there is a unique pendent triangle T that contains x. From the minimality of $|\mathcal{C}|$ we deduce that $\Omega' = (G, \mathcal{C} - \{T\}, S)$ has a certificate (R, B). Let y, z be the other two vertices of T. By the rule of adding pendent triangles, $L(Q)$ must have a triangle, say T', that contains both y and z. It follows from (2) and (5) that $S \cap V(T') = \{y, z\}$, and thus each of R and B contains precisely one of y and z. Therefore, (R, B) is a certificate for Ω, a contradiction, which proves (6).

It follows instantly from (6) and (2) that

(7) If $x = uv$ is a pendent edge of Q, then $x \notin S$.

From (6) we also deduce that members of $\mathcal{C}$ are line graphs of triads and cycles of Q. This fact suggests that all properties of Ω can be verified via Q. An edge $uv \in E(Q)$ is *maximum* if $d_Q(u) = d_Q(v) = 3$.

(8) Every maximum edge of Q belongs to S

Suppose there exists a maximum edge $x \notin S$. Then x is not a pendent edge in Q. It is straightforward to verify that $\Omega_x = (G_x, \mathcal{C}_x, S)$ satisfies the assumptions of the lemma. Moreover, $|\mathcal{C}_x| = |\mathcal{C}|$, $t_{\Omega_x} = t_\Omega$, but $d_{\Omega_x} = d_\Omega - 1$. By the minimality of d_Ω, Ω_x has a certificate (R, B). Then the definition of G_x implies that (R, B) is a certificate for Ω, a contradiction, which proves (8).

By (5), every triad T of Q contains precisely two edges in S. Let S_T be the set of these two edges. Let $\mathcal{D}$ be the collection of cycles D of Q such that $L(D) \in \mathcal{C}$.

(9) $|S_T \cap E(D)| < 2$, for all triads T of Q and all cycles $D \in \mathcal{D}$.

Suppose $S_T \subseteq E(D)$, for a triad T of Q and a cycle $D \in \mathcal{D}$. By the minimality of $|\mathcal{C}|$, there exists a certificate (R, B) for $\Omega' = (G, \mathcal{C} - \{L(D)\}, S)$. Since $L(T) \in \mathcal{C} - \{L(D)\}$ and $|S_T| = 2$, it follows that $R \cap S_T \neq \emptyset \neq B \cap S_T$. Therefore, $R \cap E(D) \neq \emptyset \neq B \cap E(D)$, which implies that (R, B) is a certificate for Ω, a contradiction, which proves (9).

(10) No cycle in $\mathcal{D}$ contains a maximum edge.

Suppose x is a maximum edge contained in $D \in \mathcal{D}$. By Lemma 2.2, two distinct edges are in series iff neither is a cut edge, and every cycle that contains one must also contain the other, so the series family F that contains x is a subset of $E(D)$. Since x is a maximum edge, we deduce from Lemma 2.4 and Lemma 2.3(ii) that $t := |F| \geq 2$. Let edges $x_1, x_2, ..., x_t$ of F and components $Q_1, Q_2, ..., Q_t$ of $Q \backslash F$ be indexed as in Lemma 2.5. If $|V(Q_i)| = 2$, for some i, by Lemma 2.5, the only edge y of $E(Q_i)$ is a pendent edge of Q and it forms a triad with x_{i-1}, x_i. Now (7) implies $y \notin S$ but (9) implies $y \in S$. This contradiction proves that $|V(Q_i)| \neq 2$ for all i. Let $I = \{i : 1 \leq i \leq t \text{ and } |V(Q_i)| > 2\}$. It follows from the choice of x that $|I| \geq 2$. For each $i = 1, 2, ..., t$, let $x_i = u_i v_i$ such that $u_i \in V(Q_i)$ and $v_i \in V(Q_{i+1})$, where $Q_{t+1} = Q_1$.

Claim 1. For each $i \in I$, vertices v_{i-1} and u_i are nonadjacent, where $v_0 = v_t$.

Assume the contrary: v_{i-1} and u_i are adjacent. Since F is a series family, Lemma 2.5 guarantees the existence of two edge-disjoint paths linking v_{i-1} and u_i in Q_i; let P denote one of them that is different from edge $v_{i-1}u_i$. Let $D' = D$ if D does not contain edge $v_{i-1}u_i$, and let D' be the cycle obtained from D by replacing edge $v_{i-1}u_i$ with path P otherwise. Then $v_{i-1}u_i$ is a chord of cycle D', contradicting the hypothesis that Q is chordless. So claim 1 is justified.

Let us proceed by distinguishing between two cases.

We first assume that $I = \{1, 2, ..., t\}$. It follows from Lemma 2.5 that all edges in F are maximum, and thus they are in S, by (8). Let $Z_2 = Q_2$ and $Z_1 = Q \backslash V(Q_2)$. For $i = 1, 2$, let Q'_i be obtained from Q by contracting Z_j ($j \in \{1, 2\} - \{i\}$) into a vertex w_i, and then adding a pendent edge y_i at w_i.

Claim 2. Q'_i is triangle-free and satisfies (i-iii) in Lemma 2.3, for $i = 1$ and 2.

From Claim 1 we see that Q'_i is triangle-free and chordless. Since Q is connected, subcubic, and every cut edge of Q is a pendent edge, from the construction of Q'_i we deduce that all these properties also hold on Q'_i. Observe that each series family of Q'_i is either a series family of Q, or it contains both x_1 and x_2. In the latter case, the series family is just $\{x_1, x_2\}$ if $i = 2$, while it is F if $i = 1$, but $|F|$ must be even since all the edges in F are maximum in Q. So Q'_i is weakly even, and thus Claim 2 is established.

For $i = 1$ and 2, let $G_i = L(Q'_i)$, $\mathcal{C}_i = \{C \in \mathcal{C} : V(C) \subseteq V(G_i)\} \cup \{x_1 x_2 y_i\}$, and $S_i = (S \cap E(Z_i)) \cup \{x_1, x_2\}$. It follows from $\{x_1, x_2\} \subseteq S$ that S_i contains at least two vertices from every cycle in $\mathcal{C}_i$. Notice that Z_j ($j \in \{1, 2\} - \{i\}$) contains at least two vertices v with $d_Q(v) = 3$, so $|\mathcal{C}_i| < |\mathcal{C}|$. By Claim 2 and the minimality of $\mathcal{C}$, $(G_i, \mathcal{C}_i, S_i)$ has a certificate (R_i, B_i). Since x_1 and x_2 are the only members of S_i that are contained in the triangle $x_1 x_2 y_i$, we may assume that $x_1 \in R_i$ and $x_2 \in B_i$. Now it is routine to verify that $(R_1 \cup R_2, B_1 \cup B_2)$ is a certificate for Ω, which is a contradiction.

Next, we assume that $|I| < t$. Without loss of generality, let $1 \notin I$. Take any $i \in I$. Let $Q'_i = Q[E(Q_i) \cup E(D)]$ and $Z_i = E(Q_i) \cup \{x_{i-1}, x_i\}$. Let $G_i = L(Q'_i)$, $S_i = S \cap Z_i$, and $\mathcal{C}_i = \{C : C \in \mathcal{C}$ and $V(C) \subseteq Z_i\} \cup \{L(D)\}$. It follows from the definition of I, Lemma 2.5, and the definition of Q'_i that $d_{Q'_i}(v_{i-1}) = d_{Q'_i}(u_i) = 3$. Let $T_{v_{i-1}}$ and T_{u_i} be the two triads centered as these two vertices. Since $|E(D) \cap T_{v_{i-1}}| = |E(D) \cap T_{u_i}| = 2$, by Claim 1, v_{i-1} and u_i are nonadjacent. Therefore, we deduce from (5) that $|S_i \cap E(D)| \geq 2$, and thus $|S_i \cap V(C)| \geq 2$, for every $C \in \mathcal{C}_i$. Clearly $\Omega_i = (G_i, \mathcal{C}_i, S_i)$ satisfies the hypothesis of the lemma. Notice that, for any $j \in I - \{i\}$, Q_j contains at least two vertices v with $d_Q(v) = 3$, so $|\mathcal{C}_i| < |\mathcal{C}|$. By the minimality of $\mathcal{C}$, Ω_i has a certificate (R_i, B_i). In addition, since $1 \notin I$, renaming R's and B's if necessary, we may assume that, for each i with $\{i, i+1\} \subseteq I$, if $S_i \cap S_{i+1} \neq \emptyset$ (which implies that x_i is their common edge), then x_i belongs to either both R_i and R_{i+1}, or both B_i and B_{i+1}. It follows that $(R, S - R)$, where $R = \cup_{i \in I} R_i$, is a certificate for Ω. This contradiction completes the proof of (10).

For each cycle $D \in \mathcal{D}$, edges of Q that have precisely one end in $V(D)$ are called *connectors* of D.

(11) Every $D \in \mathcal{D}$ has at least two connectors.

If D has no connectors, then, as Q is connected, $Q = D$, which contradicts (1). If D has only one connector x, then x is a cut edge of Q, which means x is a pendent edge. Now (7) implies $x \notin S$ but (9) implies $x \in S$. This contradiction proves (11).

(12) Cycles in $\mathcal{D}$ are pairwise vertex-disjoint.

Suppose some $D \in \mathcal{D}$ shares a common vertex with another cycle in $\mathcal{D} - \{D\}$. By the minimality of $\mathcal{C}$, $\Omega' = (G, \mathcal{C} - \{L(D)\}, S)$ has a certificate (R, B). Since (R, B) is not a certificate for Ω, we may assume that $S \cap E(D) \subseteq R$. Since $E(T) \cap B \neq \emptyset$ for each triad T and $E(D) \cap B = \emptyset$, all connectors of D belong to B.

Let $D_1 \in \mathcal{D} - \{D\}$ with $V(D_1) \cap V(D) \neq \emptyset$. Since Q is subcubic, D_1 must contain two (or more) connectors of D; let x_1 denote one of them and let y_1, z_1 be the two edges of D that are incident with x_1 such that $y_1 \in R$. By (10), $\{x_1, y_1, z_1\}$ is the only triad in Q that contains x_1 or y_1. So there must exist a cycle D_2 in $\mathcal{D} - \{D, D_1\}$ such that $R \cap E(D_2) = \{y_1\}$ and $x_1 \notin E(D_2)$, for otherwise (R', B'), with $R' = (R - \{y_1\}) \cup \{x_1\}$ and $B' = (B - \{x_1\}) \cup \{y_1\}$, would be a certificate for Ω, a contradiction. Let P be a maximal common segment of D and D_2 with $S \cap E(P) = \{y_1\}$, and let $\{x_i, y_i, z_i\}$, $i = 2, 3$, denote the triads whose centers are the two ends of P, such that x_2, x_3 are connectors of D contained in D_2. From the choice of D_2, we see that $\{y_2, y_3\} \subseteq S \cap (E(D) - E(P))$ and that x_1, x_2, x_3 are distinct. Let v_i be the center of the triad $\{x_i, y_i, z_i\}$ for $i = 1, 2, 3$. Symmetry allows us to assume that y_1 is on the subpath of P from v_1 to v_2. Now define $R'' := (R - \{y_2\}) \cup \{x_2\}$ and $B'' := (B - \{x_2\}) \cup \{y_2\}$. Let us show that (R'', B'') is a certificate for Ω.

For any triad T of Q, since both x_2 and y_2 are contained in cycles of $\mathcal{D}$, it follows from (10) that either T has center v_2 or T contains neither x_2 nor y_2. Therefore $R'' \cap E(T) \neq \emptyset \neq B'' \cap E(T)$ in either case. Next, suppose $D' \in \mathcal{D}$. If $v_2 \notin V(D')$, it is clear that $R'' \cap E(D') = R \cap E(D') \neq \emptyset \neq B \cap E(D') = B'' \cap E(D')$. If $D' = D$, we have $y_1 \in R'' \cap E(D)$ and $y_2 \in B'' \cap E(D)$. Finally, suppose $D' \neq D$ yet $v_2 \in V(D')$.

If one of $R'' \cap E(D')$ and $B'' \cap E(D')$ is empty then, as D' contains at least two indicators of D that are in B, we have $R \cap E(D') = \{y_2\}$ and $x_2 \notin E(D')$. Since $S \cap E(P) = \{y_1\}$ and since, by (10), D contains no maximum edge, x_1, x_2, x_3 are the only connectors of D incident with vertices on P. Thus from $y_2 \in E(D')$ and $x_2 \notin E(D')$, we deduce that $y_1 \in E(D')$. Recall that $R \cap E(D') = \{y_2\}$, so $y_1 = y_2$, contradicting the fact that $y_1 \in E(P)$ while $y_2 \notin E(P)$. Hence (12) holds.

Let $E_{\mathcal{D}} = \cup_{D \in \mathcal{D}} E(D)$ and $V_{\mathcal{D}} = \cup_{D \in \mathcal{D}} V(D)$. Let $Q_{\mathcal{D}}$ be obtained from Q by contracting D, for every cycle $D \in \mathcal{D}$, into a vertex v_D. Let $U = \{v_D : D \in \mathcal{D}\} \cup \{v \in V(Q) - V_{\mathcal{D}} : d_Q(v) = 3\}$. Then, by Lemma 2.8, $Q_{\mathcal{D}}[U]$ is a bipartite graph. Let $S' \subseteq E(Q_{\mathcal{D}})$ be the set of edges corresponding to those in $S - E_{\mathcal{D}}$, and let $Q' = Q_{\mathcal{D}}[S']$. Then $Q'[U]$ is bipartite. By (5), (11) and (9), $d_{Q'}(u) \geq 2$, for all $u \in U$. Thus, by Lemma 2.7, Q' has a 2-edge coloring λ' such that both colors are represented at every vertex in U. Let us view S' as a subset of S. Then λ' can be extended into a 2-edge coloring on S as follows. For each $x \in E_{\mathcal{D}}$, if x is not incident with a connector of D then we assign a color to x arbitrarily; if x is incident with a connector x' of D, then x' is unique, by (10), and thus we assign x a color different from that of x'. Let R, B be the color classes. Then it is routine to verify that (R, B) is a certificate for Ω, a contradiction, which completes the proof of the lemma. ∎

Lemma 4.2 *Let $G = (V, E)$ be obtained from a rooted line graph $L(Q)$ by adding pendent triangles, where Q is triangle-free and satisfies (i-iii) in Lemma 2.3. Then $b(\mathcal{C}_G)$ is Mengerian.*

Proof. Let $w \in \mathbf{Z}_+^V$. By Lemma 3.1, we need to show that $\bar{b}(\mathcal{C}_G)$, which consists of all feedback vertex sets of G, has a w-packing of size $r := r_w(\mathcal{C}_G)$, which is the minimum of $w(V(C))$, over all cycles C of G. By decreasing the value of $w(v)$ to r, if necessary, we may assume that $w(v) \le r$, for all $v \in V$.

Let $\mathcal{C}'$ consist of all triangles in G and $\mathcal{C}''$ consist of all other cycles in G. For any $F \subseteq V$, let $\alpha(F)$ and $\beta(F)$ be the number of cycles in $\mathcal{C}'$ and $\mathcal{C}''$, respectively, that F meets. Clearly, there is a collection $\mathcal{F}$ of subsets of V such that

(a) $|\mathcal{F}| = r$; and

(b) every $v \in V$ is contained in exactly $w(v)$ members of $\mathcal{F}$.

We choose such an $\mathcal{F}$ such that

(c) $\alpha(\mathcal{F}) = \sum_{F \in \mathcal{F}} \alpha(F)$ is maximum, and

(d) subject to (c), $\beta(\mathcal{F}) = \sum_{F \in \mathcal{F}} \beta(F)$ is maximum.

We prove that every member of $\mathcal{F}$ is an FVS of G. This implies that $\mathcal{F}$ is a w-packing of $\bar{b}(\mathcal{C}_G)$ of size r, which would prove the lemma.

(1) $F \cap V(C) \neq \emptyset$, for all $F \in \mathcal{F}$ and $C \in \mathcal{C}'$.

Suppose, by contradiction, that there exist $F_0 \in \mathcal{F}$ and $C_0 \in \mathcal{C}'$ with $F_0 \cap V(C_0) = \emptyset$. From (a), (b), and the fact $w(V(C_0)) \ge r$ we deduce that there exists $F_1 \in \mathcal{F}$ with $|F_1 \cap V(C_0)| \ge 2$. As usual, let $F_0 \Delta F_1 = (F_0 - F_1) \cup (F_1 - F_0)$. Let $F_{01}^Q = (F_0 \Delta F_1) \cap E(Q)$ and $F_{01}^G = (F_0 \Delta F_1) - E(Q)$. Let $\mathcal{C}_0'$ be the collection of all cycles $C \in \mathcal{C}'$ with $V(C) \cap (F_0 \cap F_1) = \emptyset$ and $|V(C) \cap F_{01}^Q| \ge 2$. Clearly, for each $C \in \mathcal{C}_0'$, there exists a triad in Q that contains all members of $V(C) \cap F_{01}^Q$. Let U be the set of centers of all these triads.

For each pendent triangle $C \in \mathcal{C}_0'$, we perform the following operation on Q. Let x, y be the two edges in $V(C) \cap F_{01}^Q$. Let u be their common end and let $z = uv$ be the other edge incident with u. We replace z with $u'v$, where u' is a new vertex. In other words, edges in $\delta_Q(u)$ are split into two groups, x, y are still incident with u, but z is moved from u to the new vertex u'. Let Q' be the resulting graph, after performing this operation over all pendent triangles $C \in \mathcal{C}_0'$. Let $Q'' = Q'[F_{01}^Q]$. Then the definitions of U and Q' imply that $d_{Q''}(u) \ge 2$, for all $u \in U$. On the other hand, by Lemma 2.8, $Q[U]$ is bipartite, so $Q'[U]$ is bipartite, which in turn implies that $Q''[U]$ is bipartite. By Lemma 2.7, Q'' has a 2-edge coloring so that both colors are represented at each vertex of U. Let R_0 and R_1 denote the two color classes.

For each $z \in V(G) - E(Q)$, let T_z denote the unique (pendent) triangle of G that contains z. Let $S_0 = \{z \in F_{01}^G : |V(T_z) \cap R_0| < |V(T_z) \cap R_1|\}$ and $S_1 = F_{01}^G - S_0$. For $i = 0, 1$, let $F_i' = (F_0 \cap F_1) \cup R_i \cup S_i$. Let $\mathcal{F}' = (\mathcal{F} - \{F_0, F_1\}) \cup \{F_0', F_1'\}$. Notice that $(R_0 \cup S_0, R_1 \cup S_1)$ is a partition of $F_0 \Delta F_1$, thus $\mathcal{F}'$ satisfies (a) and (b). Let $\mathcal{D} = \{C \in \mathcal{C}' : V(C) \cap (F_0 \cup F_1) \neq \emptyset\}$. Let $\mathcal{D}_0 = \{C \in \mathcal{D} : V(C) \cap (F_0 \cap F_1) \neq \emptyset\}$, $\mathcal{D}_1 = \{C \in \mathcal{D} - \mathcal{D}_0 : |V(C) \cap (F_0 \Delta F_1)| = 1\}$, and $\mathcal{D}_2 = \mathcal{D} - \mathcal{D}_0 - \mathcal{D}_1$. Then it is clear that $C_0 \in \mathcal{D}_2$ and $\alpha(F_0) + \alpha(F_1) \le 2|\mathcal{D}_0| + |\mathcal{D}_1| + 2|\mathcal{D}_2 - \{C_0\}| + |\{C_0\}|$. On the other hand, the definition of (R_0, R_1) implies that each cycle in $\mathcal{C}_0'$ meets both R_0 and R_1. Similarly, the definition of (S_0, S_1) implies that every cycle in $\mathcal{D}_2 - \mathcal{C}_0'$ meets both $R_0 \cup S_0$ and $R_1 \cup S_1$. Therefore, every cycle in $\mathcal{D}_2$ meets both F_0' and F_1', which means

$\alpha(F_0') + \alpha(F_1') = 2|\mathcal{D}_0| + |\mathcal{D}_1| + 2|\mathcal{D}_2| > \alpha(F_0) + \alpha(F_1)$. It follows that $\alpha(\mathcal{F}') > \alpha(\mathcal{F})$, contradicting (c), and thus (1) is proved.

(2) For any $x \in V$, if G' is a block of $G\backslash x$, then there exists a triangle-free graph Q', which satisfies (i-iii) in Lemma 2.3, and such that G' is obtained from $L(Q')$ by adding pendent triangles.

Clearly, we may assume that $|V(G')| \geq 3$. Let $Q_1 = Q[V(G') \cap E(Q)]$. Since G' is 2-connected and has three or more vertices, for each $z \in V(G') - E(Q)$, the unique (pendent) triangle T_z of G that contains z is contained in G'. If $T_z \backslash z$ is contained in a triangle of $L(Q_1)$, for all $z \in V(G') - E(Q)$, then it is routine to verify that $Q' = Q_1$ has the required properties. Therefore, we may assume that some $T_z \backslash z$ is not contained in any triangle of $L(Q_1)$. Let Z be the set of all such vertices z. Observe that $G' \backslash (V(G') - E(Q) - Z)$ is isomorphic to $L(Q')$, where Q' is obtained from Q_1 by adding $|Z|$ pendent edges. It follows that G' can be obtained from $L(Q')$ by adding pendent triangles. Furthermore, it is routine to verify that Q' is triangle-free and satisfies (i-ii) in Lemma 2.3. To verify that Q' is weakly even, notice that Q' and Q_1 have the same nontrivial series families, since the two graphs only differ by some pendent edges. In addition, every nontrivial series family of Q_1 can be partitioned into series families of Q, as Q_1 is a subgraph of Q. Therefore, each odd series family S_1 of Q_1 contains an odd series family S of Q. By Lemma 2.4, $|S| > 1$. It follows that Q has a degree-two vertex u, which is incident with two edges of S. From the way we add pendent edges we deduce that $d_{Q'}(u) = 2$, which proves that Q' is weakly even, and thus (2) is proved.

Now we are ready to prove that each $F \in \mathcal{F}$ is an FVS. Suppose otherwise. By (1), there exist $F_0 \in \mathcal{F}$ and $C_0 \in \mathcal{C}''$ with $F_0 \cap V(C_0) = \emptyset$. Again, there must exist $F_1 \in \mathcal{F}$ with $|F_1 \cap V(C_0)| \geq 2$. Suppose that $G_1, G_2, \ldots, G_k$ are all blocks of $G \backslash (F_0 \cap F_1)$. Let $i \in \{1, 2, ..., k\}$. By (2), G_i is obtained from $L(Q_i)$ by adding pendent triangles, where Q_i is triangle-free and satisfies (i-iii) in Lemma 2.3. Let $S_i = (F_0 \Delta F_1) \cap V(G_i)$ and let $\mathcal{D}_i$ be the collection of cycles C of G_i with $|V(C) \cap S_i| \geq 2$. By (1), $\mathcal{D}_i$ contains all triangles of G_i. Then, by Lemma 4.1, S_i can be partitioned into (B_i, R_i) such that each cycle in $\mathcal{D}_i$ meets both B_i and R_i. We further assume, by interchanging B_i with R_i when necessary, that if any distinct S_i and S_j have a common vertex v then either $v \in R_i \cap R_j$ or $v \in B_i \cap B_j$. Let $B = B_1 \cup B_2 \cup ... \cup B_k$ and $R = R_1 \cup R_2 \cup ... \cup R_k$. Let $F_0' = (F_0 \cap F_1) \cup B$ and $F_1' = (F_0 \cap F_1) \cup R$. Let $\mathcal{F}' = (\mathcal{F} - \{F_0, F_1\}) \cup \{F_0', F_1'\}$. Since (B, R) is a partition of $F_0 \Delta F_1$, $\mathcal{F}'$ satisfies (a) and (b). By (1) and by the constructions of (B, R), we see that $\alpha(\mathcal{F}') \geq \alpha(\mathcal{F})$. However, $\beta(\mathcal{F}') - \beta(\mathcal{F}) = \beta(F_0') + \beta(F_1') - (\beta(F_0) + \beta(F_1)) \geq 2|\mathcal{D}''| - (2|\mathcal{D}'' - \{C_0\}| + |\{C_0\}|) = 1$, where $\mathcal{D}''$ consists of all cycles C in $\mathcal{C}''$ that are contained in $G \backslash (F_0 \cap F_1)$ with $|V(C) \cap (F_0 \Delta F_1)| \geq 2$. This contradicts the maximality of $\beta(\mathcal{F})$, which completes the proof of the Lemma. ∎

Lemma 4.3 *If $K \in \{K_3, K_4^-, W_4^-, K_{2,3}^+\}$, then $b(\mathcal{C}_K)$ is Mengerian.*

Proof. Let $K = (V, E)$ and $w \in \mathbf{Z}_+^V$. Suppose $v_1, v_2, \ldots, v_{|V|}$ are all the vertices in K such that $v_1v_2v_3$ is a triangle in K of minimum weight and that $d_K(v_1) \geq d_K(v_2) \geq d_K(v_3)$. We only need to exhibit an FVS w-packing $\mathcal{F}$ of K of size $w(v_1) + w(v_2) + w(v_3)$. Let $V' = V - \{v_1, v_2\}$. It is routine to verify that, if $K \in \{K_3, K_4^-, K_{2,3}^+\}$, then $\mathcal{F} = \{w(v_1)\{v_1\},\ w(v_2)\{v_2\},\ w(v_3)V'\}$ is as desired, where the number $w(x)$ in front of each set indicates the number of times the set appears in $\mathcal{F}$.

Next, suppose $K = W_4^-$. Then $d_K(v_1) = 4$. When $v_4v_5 \notin E$, we may assume that vertices on the path $K \backslash v_1$ are ordered as $v_5v_2v_3v_4$. It follows that $w(v_2) \leq w(v_4)$ and $w(v_3) \leq w(v_5)$. Again, it is straightforward to verify that $\mathcal{F} = \{w(v_1)\{v_1\},\ w(v_2)\{v_2, v_4\},\ w(v_3)\{v_3, v_5\}\}$ has the required property. It remains to consider the case where $v_4v_5 \in E$. Let us assume vertices on the path $K\backslash v_1$ are ordered as $v_3v_2v_4v_5$. Then $w(v_3) \leq w(v_4)$. In this case, we have to consider two subcases. If $w(v_2) \leq w(v_5)$, then $\mathcal{F} = \{w(v_1)\{v_1\},\ w(v_3)\{v_3, v_4\},\ w(v_2)\{v_2, v_5\}\}$ has the required property; if $w(v_2) > w(v_5)$ then $\mathcal{F} = \{w(v_1)\{v_1\},\ w(v_3)\{v_3, v_4\},\ w(v_5)\{v_2, v_5\},\ (w(v_2) - w(v_5))\{v_2, v_4\}\}$ is as desired for $w(v_2) + w(v_3) \leq w(v_4) + w(v_5)$. ■

Remark. For each $K \in \{K_3, K_4^-, W_4^-, K_{2,3}^+\}$, let $v_1v_2v_3$ be a triangle in K with $d_K(v_1) \geq d_K(v_2) \geq d_K(v_3)$, and let the vertices on the path $W_4^-\backslash v_1$ be ordered as $v_5v_2v_3v_4$. It is easy to see that the edge sets of $b(\mathcal{C}_K)$ ($K = K_3,\ K_4^-,\ K_{2,3}^+,\ W_4^-$) are $\{\{v_1\}, \{v_2\}, \{v_3\}\}$, $\{\{v_1\}, \{v_2\}, \{v_3, v_4\}\}$, $\{\{v_1\}, \{v_2\}, \{v_3, v_4, v_5\}\}$, $\{\{v_1\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}\}$, respectively. In every case, the incidence matrix for $b(\mathcal{C}_K)$ is totally unimodular, which also implies that $b(\mathcal{C}_K)$ is Mengerian (See Section 83.3 of [8]).

Proof of Theorem 1.3. As observed at the end of Section 1, we only need to show that, if G is in $\mathcal{L}$ then $b(\mathcal{C}_G)$ is Mengerian. We apply induction on $|V(G)|$. The case $|V(G)| = 1$ is trivial, so we proceed to the induction step. By Lemma 2.6, Lemma 3.2, and Lemmas 3.6-3.8, we may assume that G cannot be represented as a k-sum ($k = 0, 1, 2, 3$) of two smaller graphs, for otherwise we are done by induction. Then we conclude from Lemma 2.1 that G is obtained from a rooted 2-connected line graph $L(Q)$ by adding pendant triangles. Clearly we may assume that Q has no isolated vertices. If Q has a triangle, then we are done by Lemma 2.3(iv) and Lemma 4.3. Thus we may assume that Q is triangle-free and satisfies (i-iii) in Lemma 2.3. Now the result follows from Lemma 4.2. ■

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