A CANCELLATION CONJECTURE FOR FREE ASSOCIATIVE ALGEBRAS

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Abstract. We develop a new method to deal with the Cancellation Conjecture of Zariski in different environments. We prove the conjecture for free associative algebras of rank two. We also produce a new proof of the conjecture for polynomial algebras of rank two over fields of zero characteristic.

1. **Introduction and main results**

There is a famous

Conjecture 1.1. (**Concellation Conjecture of Zariski**) *Let R be an algebra over a field K. If $R[z]$ is K-isomorphic to $K[x_1,\dots,x_n]$, then R is isomorphic to $K[x_1,\dots,x_{n-1}]$.*

Conjecture 1.1 was proved for $n=2$ by Abhyankar, Eakin and Heizer [1], and Miyanishi [10]. For $n=3$, the Conjecture was proved by Fujita [5], and Miyanishi and Sugie [11] for zero characteristic, and by Russell [12] for arbitrary fields K. For $n\geq 4$, the Conjecture remains open to the best of our knowledge. See [4, 6, 7, 8, 9, 14] for Zariski's conjecture and related topics.

Denote by $A*B$ the free product of two K-algebras A and B. In view of Conjecture 1.1, it is natural and interesting to raise

Conjecture 1.2. (**Concellation Conjecture for Free Associative Algebras**) *Let R be an algebra over a field K. If $R*K[z]$ is K-isomorphic to $K\langle x_1,\dots,x_n\rangle$, then R is K-isomorphic to $K\langle x_1,\dots,x_{n-1}\rangle$.*

2000 *Mathematics Subject Classification.* Primary 16S10. Secondary 13B10, 13F20, 14R10, 16W20.

Key words and phrases. Cancellation Conjecture of Zariski, algebras of rank two, polynomial algebras, free associative algebras, centralizers, Jacobians, algebraic dependence.

The research of V.Drensky was partially supported by the Grant MI-1503/2005 of the Bulgarian National Science Fund.

The research of Jie-Tai Yu was partially supported by an RGC-CERG Grant.

In this paper we develop a new method based on the conditions of algebraic dependence, which can be used in different environments. In particular, by this method we prove Conjecture 1.2 for $n = 2$:

Theorem 1.3. *Let R be an algebra over an arbitrary field K. If $R * K[z]$ is K-isomorphic to $K\langle x, y\rangle$, then R is K-isomorphic to $K[x]$.*

We also produce a new and simple proof for Conjecture 1.1 for $n = 2$ in the zero characteristic case [1]:

Proposition 1.4. *Let R be an algebra over a field K of zero characteristic. If $R[t]$ is K-isomorphic to $K[x, y]$, then R is isomorphic to $K[x]$.*

2. **Preliminaries**

Call a set of elements of an associative K-algebra *algebraically dependent* over K if the K-subalgebra generated by the elements is not free on that generating set. To prove the main results, we need the well-known necessary and sufficient conditions for algebraic dependence.

Lemma 2.1. *Let K be an arbitrary field, $f, g \in K\langle x_1, \dots, x_n\rangle$. Then f and g are algebraically dependent over K if and only if $[f, g] = 0$, where $[f, g] = fg - gf$ is the commutator of f and g.*

See, Corollary 6.7.4, p.338, Cohn [3].

Lemma 2.2. *Let K be a field of zero characteristic, $f,\ g \in K[x_1, \dots, x_n]$. Then f and g are algebraically dependent over K if and only if $J_{x_i,x_j}(f, g) = 0$ for all $1 \le i < j \le n$, where $J_{x_i,x_j}(f, g)$ is the Jacobian determinant of f and g with respect to x_i and x_j.*

See, for instance, Jie-Tai Yu [15], for a proof.

We also need a description of the subset of all elements of a polynomial or free associative algebra which are algebraically dependent on a fixed element. The following result is due to Bergman [2]. See also Cohn [3].

Lemma 2.3. *Let K be an arbitrary field, $f \in K\langle x_1, \dots, x_n\rangle - K$, $\mathcal{C}(f)$ the subset of $K\langle x_1, \dots, x_n\rangle$ such that for all $g \in \mathcal{C}(f)$, $[f, g] = 0$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K\langle x_1, \dots, x_n\rangle$.*

For polynomial algebras, the analogue of the above result has been obtained by Shestakov and Umirbaev [13]:

Lemma 2.4. *Let K be a field of zero characteristic, $f \in K[x_1, \ldots, x_n] - K, \mathcal{C}(f)$ the subset of $K[x_1, \ldots, x_n]$ such that for all $g \in \mathcal{C}(f)$, $J_{x_i,x_j}(f, g) = 0$ for all $1 \le i < j \le n$. Then $\mathcal{C}(f) = K[u]$ for some $u \in K[x_1, \ldots, x_n]$.*

3. Proofs of the main result

Proof of Theorem 1.3. Let $R * K[z] \cong K\langle x, y\rangle$ and let (z) be the ideal of $R * K[z]$ generated by z. Clearly, $(R * K[z])/(z) \cong R$. Since the algebra $R * K[z]$ is isomorphic to the free algebra of rank 2, it is two-generated and the same holds for its homomorphic image $(R*K[z])/(z) \cong R$. Hence R is generated by $v, w \in R$. Now we use that R is a subalgebra of the free associative algebra $R * K[z] \cong K\langle x, y\rangle$. If v and w are algebraically independent over K, then R is isomorphic to the free algebra $K\langle t_1, t_2\rangle$ and $R*K[z] \cong K\langle t_1, t_2, z\rangle$ is the free algebra of rank 3, which is impossible. Hence v and w are algebraically dependent. By Lemma 2.1, it's easy to deduce that any element $f = f(v, w) \in R$ and v are algebraically dependent over K. By Lemma 2.1 and Lemma 2.3, $R \subset K[u]$ for some $u \in R*K[z]$. Write $u = u_0 + u_1$, where $u_0 \in R$, u_1 contains only monomials occurring in u with z-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, h is a polynomial over K in one variable. Substituting $z = 0$, $f = h(u_0)$. Therefore $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. This forces $R = K[u_0]$. Therefore R is K-isomorphic to $K[x]$. □

Proof of Proposition 1.4. As $R[z]$ is K-isomorphic to $K[x, y]$, it is easy to deduce that R has transcendence degree 1 over K. Therefore there exists a $g \in R - K$ such that for all $f \in R$, f and g are algebraically dependent over K. By Lemma 2.2 and Lemma 2.4, $R \subset K[u]$ for some $u \in R[t]$. Write $u = u_0 + u_1$, where $u_0 \in R$, u_1 contains only monomials occurring in u with z-degree at least 1. For any $f \in R$, $f = h(u) = h(u_0 + u_1)$, h is a polynomial over K in one variable. Substituting $z = 0$, $f = h(u_0)$. Therefore $R \subset K[u_0]$. Now $K[u_0] \subset R \subset K[u_0]$. This forces $R = K[u_0]$. Therefore R is K-isomorphic to $K[x]$. □

4. Acknowledgements

The authors are grateful to the Beijing International Center for Mathematical Research for warm hospitality during their visit when this work was carried out. They also would like to thank L.Makar-Limanov and V.Shpilrain for stimulating discussions, and an anonymous referee for very helpful comments and suggestions.

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