THE PREIMAGE OF A COORDINATE

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Abstract. Let K be a field of characteristic zero. Based on the degree estimate of Makar-Limanov and J.-T.Yu, we prove the following new result: The preimage of a coordinate under an injective endomorphism of $K\langle x, y\rangle$ is also a coordinate. As by-products, we give new proofs of the following results: 1) The preimage of a coordinate under an injective endomorphism of $K[x, y]$ is also a coordinate; 2) Any automorphism of $K[x, y]$ or $K\langle x, y\rangle$ is tame.

1. Introduction and main results

In this paper, K always denotes a field of characteristic zero. Automorphisms (endomorphisms) always mean K-automorphisms (K-endomorphisms).

In Shpilrain and J.-T. Yu [12], the following problem was raised:

Problem 1. Let $p \in K\langle x_1, \ldots, x_n\rangle$ and there exists an injective endomorphism ϕ such that $\phi(p) = x$. Is then p a coordinate of $K\langle x_1, \ldots, x_n\rangle$?

In [12], a negative answer was given to Problem 1 for $n \geq 4$. But the problem remains open for $n = 2$ and $n = 3$ to the best of our knowledge.

Recently Makar-Limanov and J.-T. Yu [10] have given a sharp lower degree bound for subalgebras generated by two elements of a polynomial or free associative algebra. It has found applications for characterization of test elements by retracts for free associative algebras, see S.-J. Gong and J.-T. Yu [5].

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In this note, based on the degree estimate of Makar-Limanov and J.-T.Yu [10], we give a positive answer for Problem 1 for $n = 2$ when K has characteristic zero.

Theorem 1.1. *Let $p \in K\langle x, y\rangle$ and there exists an injective endomorphism ϕ such that $\phi(p) = x$. Then p is a coordinate of $K\langle x, y\rangle$.*

Note that the analogue of the above result in the polynomial case, which has found applications in affine algebraic geometry [9, 11, 13], is the following proposition, which was initially obtained by L.A.Campbell and J.-T.Yu [1]. But for $n \geq 3$, the analogue of Problem 1 for polynomial algebras has a negative solution by Makar-Limanov, see [4, 12].

Proposition 1.2. *Let $p \in K[x, y]$ and there exists an injective endomorphism ϕ such that $\phi(p) = x$. Then p is a coordinate of $K[x, y]$.*

The proof of the above result in [1] was somewhat complicated. Here we present a simple new proof by the degree estimate in [10].

Note that in [7], the authors considered inverse images of coordinates in some free algebras and showed that the similar results hold for all free (non-associative) algebras.

As by-products, we also give simple new proofs of the following two well-known results.

The first is due to Jung [6].

Proposition 1.3. *Any automorphism of $K[x, y]$ is tame, namely, can be decomposed as a product of linear and elementary automorphisms.*

Note that an elementary automorphism of a polynomial or associative algebra is an automorphism fixing all generators except one.

The second was given by Makar-Limanov [8] and Czerniakiewicz [3]. See also Cohn [2].

Proposition 1.4. *Any automorphism of $K\langle x, y\rangle$ is tame. Moreover, $AutK\langle x, y\rangle$ is isomorphic to $AutK[x, y]$.*

2. Proofs

The following two lemmas are Theorem 1.1 and Proposition 1.2 in [10].

Lemma 2.1. *Let $A = K\langle x_1, \cdots, x_n\rangle$ be a free associative algebra over an arbitrary field K of zero characteristic, $f, g \in A$ be algebraically* *independent, f^+ and g^+ are algebraically independent, or f^+ and g^+ are algebraically dependent and neither* $\deg(f) \mid \deg(g)$ *nor* $\deg(g) \mid \deg(f)$*, $p \in K\langle x, y\rangle$. Then*

$$\deg(p(f, g)) \geq \frac{\deg[f, g]}{\deg(fg)} w_{\deg(f),\deg(g)}(p).$$

Here deg is the homogeneous (total) degree of the corresponding element, $w_{\deg(f),\deg(g)}(p)$ is the weighted degree of p when the weight of the first variable is $\deg(f)$ and the weight of the second variable is $\deg(g)$, f^+ and g^+ are the highest homogeneous components of f and g respectively, and $[f, g] = fg - gf$ is the commutator of f and g.

Lemma 2.2. *Let $A = K[x_1, \cdots, x_n]$ be a polynomial algebra over an arbitrary field K of zero characteristic, $f, g \in A$ be algebraically independent, $p \in K[x, y]$. Then*

$$\deg(p(f, g)) \geq$$
$$w_{\deg(f),\deg(g)}(p)[1 - \frac{(\deg(f), \deg(g))(\deg(fg) - \deg(J(f, g)) - 2)}{\deg(f)\deg(g)}].$$

Here deg is the homogeneous (total) degree of the corresponding element, $w_{\deg(f),\deg(g)}(p)$ is the weighted degree of p when the weight of the first variable is $\deg(f)$ and the weight of the second variable is $\deg(g)$, $(\deg(f), \deg(g))$ is the greatest common divisor of $\deg(f)$ and $\deg(g)$, $\deg(J(f, g))$ is the largest degree of nonzero Jacobian determinants of f and g with respect to two of $x_1, \cdots, x_n$.

Lemma 2.3. *Let $A = K\langle x_1, \cdots, x_n\rangle$ be a free associative algebra over an arbitrary field K of zero characteristic, $f, g \in A$ be algebraically independent, f^+ and g^+ are algebraically dependent and neither* $\deg(f) \mid \deg(g)$ *nor* $\deg(g) \mid \deg(f)$*, $p \in K\langle x, y\rangle$. Then*

$$\deg(p(f, g)) \geq 2.$$

Proof. Applying Lemma 2.1. We may assume $2 \leq m = \deg(f) < \deg(g) = n$. Then obviously $(m + n) < 2n \leq \mathrm{lcm}(m, n)$.

1) If $w_{\deg(f),\deg(g)}(p) < \deg(fg) = (m + n) < \mathrm{lcm}(m, n)$, then in $p(f, g)$, f^+ and g^+ cannot cancel out, hence $\deg(p(f, g)) \geq \deg(f) \geq 2$;

2) Otherwise $w_{\deg(f),\deg(g)}(p) \geq \deg(fg)$, it follows that $\deg(p(f, g)) \geq \deg[f, g] \geq 2$. □

Note that in the above proof, we use the well-known fact: Two elements $f, g \in K\langle x_1, \cdots, x_n\rangle$ are algebraically independent over K if and only if $[f,g] \neq 0$ if and only if $\deg[f,g] \geq 2$. See, for instance, Cohn [2].

Lemma 2.4. *Let $A = K[x_1, \cdots, x_n]$ be a polynomial algebra over an arbitrary field K of zero characteristic, $f, g \in A$ be algebraically independent, f^+ and g^+ are algebraically dependent and neither* $\deg(f) \nmid \deg(g)$ *nor* $\deg(g) \nmid \deg(f)$*, $p \in K[x,y]$. Then*

$$\deg(p(f,g)) \geq 2.$$

Proof. Applying Lemma 2.2. We may assume $2 \leq m = \deg(f) < \deg(g) = n$.

1) If $w_{\deg(f),\deg(g)}(p) < \mathrm{lcm}(m,n)$, then in $p(f,g)$, f^+ and g^+ cannot cancel out, hence $\deg(p(f,g)) \geq \deg(f) \geq 2$;

2) Otherwise $w_{\deg(f),\deg(g)}(p) \geq \mathrm{lcm}(m,n) = mn/(m,n)$. We also have $mn = (m,n)\mathrm{lcm}(m,n) \geq (m,n)(m+n)$. Hence $\deg(p(f,g)) \geq \deg(J(f,g)) + 2 \geq 2$. □

Note that in the above proof, we use the well-known fact: Two elements $f, g \in K[x_1, \ldots, x_n]$ are algebraically independent over K if and only if $J(f,g) \neq 0$ if and only if $\deg(J(f,g)) \geq 0$. See, for instance, J.-T. Yu [14].

Proof of Theorem 1.1.

Let $\phi(x) = f, \phi(y) = g$). Then f and g are algebraically independent. Set $\deg(f) = m,\ \deg(g) = n$. Let $h(x,y)$ be the hightest (m,n) homogeneous component of $p(x,y)$

1) If f^+ and g^+ are algebraically independent, by $p(f,g) = x$, we get $h(f^+, g^+) = x$. Then h must be linear. So is p. Hence p is a coordinate.

2) If f^+ and g^+ are algebraically dependent, by Lemma 2.3, $m \mid n$ or $n \mid m$. Suppose $m \mid n$, $n = km$. Replace g by $g_1 = g - f^k$ and $p(x,y)$ by $p_1(x,y) = p(x, y + x^k)$. We get $p_1(f, g_1) = x$. Note that $\deg(g_1) < \deg(g)$ and p_1 is a coordinate if and only if so is p.

Repeating the process in 2) inductively, after a finite number of steps we would return to the case 1). Therefore p is a coordinate.□

Proof of Proposition 1.2.

Similar to the above proof. Just replace Lemma 2.3 by Lemma 2.4 in the proof. □

Proof of Proposition 1.3.

Let $\phi = (f, g)$ be an automorphism of $K[x, y]$. Then there exist p, q such that $p(f, g) = x,\ q(f, g) = y$. Set $\deg(f) = m, \deg(g) = n$.

1) If f^+ and g^+ are algebraically independent. Since $p(f, g) = x$ and $q(f, g) = y$, both f and g are linear, since f^+ and g^+ cannot cancel out in $p(f, g)$ and $q(f, g)$.

2) If f^+ and g^+ are algebraically dependent. Then by Lemma 2.4, $m \mid n$ or $n \mid m$. Suppose $m \mid n,\ n = km$. Replace g by $g_1 = g - f^k$ Note $\deg(g_1) < \deg(g)$ and (f, g) is composition of the automorphism (f, g_1) with the elementary automorphism $(x, y + x^k)$.

Repeating the process in 2) inductively, after a finite number of steps we would return to the case 1). Therefore ϕ is a composition of linear and elementary automorphisms, hence tame.□

Proof of Proposition 1.4.

First, similar to the Proof of Proposition 1.3, we can prove that any automorphism of $K\langle x, y\rangle$ is tame (just replace Lemma 2.4 by Lemma 2.3 in the proof).

An automorphism $\phi = (f, g) \in \mathrm{Aut}K\langle x, y\rangle$ is a product of linear and elementary automorphisms: $(f, g) = (f_1, g_1) \dots (f_s, g_s)$. Take the map $\mathrm{Aut}K\langle x, y\rangle \to \mathrm{Aut}K[x, y]$ induced by the abelianization from $K\langle x, y\rangle$ onto $K[x, y]$, we get the automorphism $\bar{\phi} = (\bar{f}, \bar{g})$ of $K[x, y]$ as a product of corresponding linear and elementary automorphisms of $K[x, y]$: $(\bar{f}, \bar{g}) = (\bar{f}_1, \bar{g}_1) \dots (\bar{f}_s, \bar{g}_s)$. Note that the linear and elementary automorphisms of $K\langle x, y\rangle$ and $K[x, y]$ are 'identical'. Therefore, the map $\mathrm{Aut}K\langle x, y\rangle \to \mathrm{Aut}K[x, y]$ is bijective, hence it is an isomorphism between the two groups.□

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