An Isomorphism Theorem for Holomorphic Mappings from Arithmetic Varieties of Rank ≥ 2 into Quotients of Bounded Domains of Finite Intrinsic Measure

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Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irrreducible lattice, $X := \Omega/\Gamma$. In Mok [M1,2] and To [To] Hermitian metric rigidity for the canonical Kähler-Einstein metric was established. In the locally irreducible case, it says that the latter is up to a normalizing constant the unique Hermitian metric on X of nonpositive curvature in the sense of Griffiths. This led to the rigidity result for nontrivial holomorphic mappings of X into Hermitian manifolds of nonpositive curvature in the sense of Griffiths, to the effect that up to a normalizing constant any such holomorphic mapping must be an isometric immersion totally geodesic with respect to the Hermitian connection. With an aim to studying holomorphic mappings of X into complex manifolds which are of nonpositive curvature in a more generalized sense, for instance, quotients of *arbitrary* bounded domains by torsion-free discrete groups of automorphisms, we established recently in Mok [M4] a form of metric rigidity applicable to complex Finsler metrics, including especially induced Carathéodory metrics, constructed from bounded holomorphic functions. By studying *extremal* bounded holomorphic functions in relation to Finsler metric rigidity we established rigidity theorems for nontrivial holomorphic mappings f of arithmetic varieties X of rank ≥ 2 into complex manifolds N whose universal covers admit enough nontrivial bounded holomorphic functions. A new feature of our results is that we can prove that the lifting $F : \Omega \to \widetilde{N}$ to universal covers is a holomorphic *embedding*. We called this result the Embedding Theorem.

Mok [M4] gave the first link between bounded holomorphic functions and rigidity problems. In Mok [M5] we further developed the theory by solving the Extension Problem, which is the problem of 'inverting' the holomorphic embedding $F : \Omega \to \widetilde{N}$ as a bounded holomorphic map, i.e., finding a holomorphic extension $R : \widetilde{N} \to \mathbb{C}^n$ of the inverse $i : F(\Omega) \to \Omega \Subset \mathbb{C}^n$ as a *bounded* holomorphic map. As a consequence, we proved a Fibration Theorem when f induces an isomorphism on fundamental groups. In the case when X and N are compact (N not necessarily Kähler) the Fibration Theorem says that there exists a holomorphic fibration with connected fibers $\rho : N \to X$ such that $\rho \circ f \equiv id_X$. Compactness is used to show that certain bounded plurisubharmonic functions constructed on $\widetilde{N}$ have to be constant, which allows us to show that R descends from $\widetilde{N}$ to N.

One primary objective in our research relating bounded holomorphic functions to rigidity problems is to study holomorphic mappings on X into target manifolds N which are uniformized by an *arbitrary* bounded domain D, $N := D/\Gamma'$. In this article we study the situation where $f : X \to N = D/\Gamma'$ induces an isomorphism on fundamental groups and look for necessary and sufficient conditions which would guarantee that the lifting $F : \Omega \to D$ is a biholomorphism. Our main result is to establish the latter under the assumption that N is of finite intrinsic measure with respect to the Kobayashi-Royden volume form. From the

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method of proof of the Fibration Theorem we need to show the constancy of certain bounded plurisubharmonic functions. When N is a complete Kähler manifold of finite volume, we have at our disposal the tool of integration by part on complete Kähler manifolds. We resort to such techniques, by passing first of all to the hull of holomorphy of D and making use of the canonical Kähler-Einstein metric constructed by Cheng-Yau [CY] and shown to be complete in Mok-Yau [MY]. We exploit the hypothesis that $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi-Royden volume form to prove that N can be enlarged to a complete Kähler-Einstein manifold of *finite volume*, which is enough to show that the bounded plurisubharmonic functions constructed are constant. The hypothesis that the target manifold is of finite intrinsic measure with respect to the Kobayashi-Royden volume form appears to be the most natural geometric condition, as the notion of intrinsic measure, unlike the canonical Kähler-Einstein metric, is elementary and defined for any complex manifold, and its finiteness is a necessary condition for the target manifold to be quasi-projective. The passage from a quotient of a bounded domain of finite intrinsic measure to a complete Kähler-Einstein manifold of finite volume involves an elementary *a-priori* estimate on the Kobayashi-Royden volume form of independent interest applicable to arbitrary bounded domain.

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§1 Preliminaries and statements of results.

(1.1) In Mok [M4] we proved an embedding theorem for holomorphic mappings on arithmetic varieties $X = \Omega/\Gamma$ of rank ≥ 2, $\Gamma \subset \mathrm{Aut}(\Omega)$ being torsion-free irreducible lattices, when the universal covering spaces of the target manifold admit enough bounded holomorphic functions. For the formulation we introduce a nondegeneracy condition (†) for holomorphic maps in relation to bounded holomorphic functions. Let $f : X \to N$ be a holomorphic map and denote by $F : \Omega \to \widetilde{N}$ its lifting to universal covering spaces. When Ω is irreducible we say that $(X, N; f)$ satsifies the nondegeneracy condition (†) whenever there exists a bounded holomorphic function h on $\widetilde{N}$ such that F^*h is nonconstant. In general, let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be the decomposition of Ω into irreducible factors. A subdomain $\Omega_1' = \Omega_1 \times \{(x_2, \ldots, x_m)\}$ will be called an irreducible (first) factor subdomain, etc. We say that $(X, N; f)$ satisfies the nondegeneray condition (†) if and only if for each k, $1 \leq k \leq m$, there exists a bounded holomorphic function h_k on $\widetilde{N}$ such that F^*h_k is nonconstant on some irreducible k-th factor subdomain $\Omega_k' \subset \Omega$. We proved

The Embedding Theorem (Mok[M4])**.** *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice, $X := \Omega/\Gamma$. Let N be a complex manifold and denote by $\widetilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic mapping and $F : \Omega \to \widetilde{N}$ be its lifting to universal covers. Suppose $(X, N; f)$ satisfies the* *nondegeneracy condition* (†). *Then,* $F : \Omega \to \widetilde{N}$ *is a holomorphic embedding.*

In Mok [M5] we study further the holomorphic embedding $F : \Omega \to \widetilde{N}$ given by the Embedding Theorem. With applications in mind, we posed the Extension Problem, which is the problem of finding a bounded holomorpohic mapping $R : \widetilde{N} \to \mathbb{C}^n$ which serves as a left inverse of $F : \Omega \to \widetilde{N}$, i.e., $R \circ F = id_\Omega$. We have proved

Theorem (Solution the Extension Problem, (Mok[M4])**).** *Let* $\Omega \Subset \mathbb{C}^n$ *be the Harish-Chandra realization of a bounded symmetric domain of rank* ≥ 2 *and* $\Gamma \subset \mathrm{Aut}(\Omega)$ *be a torsion-free irreducible lattice,* $X := \Omega/\Gamma$. *Let* N *be a Zariski-open subset of some compact complex manifold and denote by* $\widetilde{N}$ *its universal cover. Let* $f : X \to N$ *be a nonconstant holomorphic mapping into* N*, and denote by* $F : \Omega \to \widetilde{N}$ *the lifting to universal covering spaces. Suppose* $(X, N; f)$ *satisfies the nondegeneracy condition* (†). *For the holomorphic embedding* $F : \Omega \cong F(\Omega) \subset \widetilde{N}$ *denote by* $i : F(\Omega) \to \Omega$ *the inverse mapping. Then, there exists a (not necessarily unique) bounded vector-valued holomorphic map* $R : \widetilde{N} \to \mathbb{C}^n$ *such that* $R|_{F(\Omega)} \equiv i$*, i.e.,* $R \circ F \cong id_\Omega$.

We applied the solution to the Extension Problem to the situation where $f : X \to N$ induces an isomorphism on fundamental groups. We considered the case where N can be compactified in a nice way, viz., where N is a Zariski-open subset of a compact complex manifold. In this case we prove that N can be projected onto $f(X)$. More precisely, we have

The Fibration Theorem. *Let* Ω *be a bounded symmetric domain of rank* ≥ 2 *and* $\Gamma \subset \mathrm{Aut}(\Omega)$ *be a torsion-free irreducible lattice,* $X := \Omega/\Gamma$. *Let* N *be a Zariski-open subset of some compact complex manifold and denote by* $\widetilde{N}$ *its universal cover. Let* $f : X \to N$ *be a nonconstant holomorphic mapping into* N*, and denote by* $F : \Omega \to \widetilde{N}$ *the lifting to universal covering spaces. Suppose* $(X, N; f)$ *satisfies the nondegeneracy condition* (†). *Then,* $f : X \to N$ *is a holomorphic embedding, and there exists a holomorphic fibration* $\rho : N \to X$ *with connected fibers such that* $\rho \circ f = id_X$.

(1.2) One of our primary objectives in relating bounded holomorphic functions to rigidity problems is to develop a theory applicable to holomorphic mappings from arithmetic varieties of rank ≥ 2 to complex manifolds uniformized by arbitrary bounded domains. In this case the nondegeneracy condition (†) for the Embedding Theorem are always satisfied for any nonconstant holomorphic mapping $f : X \to N$. We proved the following result as a consequence of our solution to the Extension Problem.

Theorem (Mok[M5])**.** *Let* $\Omega \Subset \mathbb{C}^n$ *be a bounded symmetric domain of rank* ≥ 2 *and* $\Gamma \subset \mathrm{Aut}(\Omega)$ *be a torsion-free irreducible lattice,* $X := \Omega/\Gamma$. *Let* D *be an* n*-dimensional bounded domain on a Stein manifold,* Γ' *be a torsion-free discrete group of automorphisms on* D*,* $N := D/\Gamma'$. *Then, any nontrivial holomorphic mapping* $f : X \to N$ *lifts to a biholomorphism* $F : \Omega \cong D$ *between covering domains.*

In the present article we further study holomorphic embeddings as given by the Embedding Theorem with target manifolds uniformized by bounded domains, assuming now as in the Fibration Theorem that the induced map on fundamental groups is an isomorphism. We look for some natural geometric condition on N which would allow us to establish an analogue of the Fibration Theorem, in which case one expects the fibers on Ω to reduce to single points, and that we would have a biholomorphism. We establish the following principal result which yields a biholomorphism under the assumption that the target manifold $N = D/\Gamma'$ is of finite measure with respect to the Kobayashi-Royden volume form, which is a necessary condition for N to admit a realization as a Zariski-open subset of some compact complex manifold. Our principal result is given by

Main Theorem (The Isomorphism Theorem). *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let D be a bounded domain on a Stein manifold, Γ' be a torsion-free discrete group of automorphisms on D, $N := D/\Gamma'$. Suppose N is of finite measure with respect to the Kobayashi-Royden volume form, and $f : X \to N$ is a holomorphic map which induces an isomorphism $f_* : \Gamma \cong \Gamma'$. Then, $f : X \to N$ is a biholomorphic map.*

Remark. We note that in the statement of the Main Theorem we do not need to assume that D is simply-connected. We will need a slight variation of the Theorem concerning the Extension Problem. There in the proof it is not important to use the universal covering space $\widetilde{N}$. We may use any regular covering $\tau : \widetilde{N} \to N$ provided that the holomorpic mapping $f : X \to N$ admits a lifting to $F : \Omega \to \widetilde{N}$.

§2 Complete Kähler-Einsten metrics and estimates on the Kobayashi- Royeden volume form

(2.1) For the Isomorphism Theorem we are interested in the case where the target manifold N is uniformized by a bounded domain D on a Stein manifold. In our study of such manifolds we will need to resort to the use of canonical complete Kähler metrics. When D is assumed furthermore to be a domain of holomorphy, we have the canonical Kähler-Einstein metric. The existence of the metric was established by Cheng-Yau [CY], and its completeness by Mok-Yau [MY]. More precisely, we have

Existence Theorem on Kähler-Einstein Metrics. *Let M be a Stein manifold and $D \Subset M$ be a bounded domain of holomorphy on M. Then, there exists on D a unique complete Kähler-Einstein metric ds_{KE} of Ricci curvature $-(n+1)$. The metric is furthermore invariant under $\mathrm{Aut}(D)$.*

Remark. We note that invariance of ds^2_{KE} under $\mathrm{Aut}(D)$ follows from uniqueness and the Ahlfors-Schwarz Lemma for volume forms. Furthermore, for the existence of ds^2_{KE} the bounded domain $D \Subset M$ has to be assumed a domain of holomorphy. It was in fact proven in [MY] that any bounded domain on M admitting a complete Kähler-Einstein metric of negative Ricci curvature satisfies the Kontinuitätssatz of Oka's, and must therefore be a domain of holomorphy.

In the formulation of the Isomorphism Theorem we assume that the target manifold $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi-Royden volume form.

This notion of intrinsic measure (cf. (2.2)) is defined for any complex manifold. For the proof of the Isomorphism Theorem we need nonetheless to work with the complete Kähler-Einstein metric. This is done by first passing to the hull of holomorphy $\widehat{D}$ of D. For the passage and for estimates in the proof it is necessary to compare various canonical metrics and volume forms, as given in the following Comparison Lemma which results from the Ahlfors-Schwarz Lemma for Kähler metrics and for volume forms (cf. Mok [M2] and the references given there).

The Comparison Lemma. *Let D be a bounded domain on some n-dimensional Stein manifold, ds^2_{KE} be the canonical complete Kähler-Einstein metric of constant Ricci curvaute $-(n+1)$, and denote by dV_{KE} its volume form. Then, for the Carathéodory metric κ and the Kobayashi-Royden volume form dV_{KR} on D, we have*

$$ds^2_{KE} \geq \frac{2\kappa}{n+1} \ , \quad dV_{KE} \leq dV_{KR} \ .$$

(2.2) Let $ds^2_{B^n}$ be the Poincaré metric on the unit ball $B^n \subset \mathbb{C}^n$ normalized to have constant Ricci curvature $-(n+1)$, with volume form $dV_{Poin.}$. On a complex manifold M let $\mathcal{K}_M$ be the space of all holomorphic maps $f : B^n \to M$. For a holomorphic n-vector η its norm with respect to the Kobayashi-Royden volume form dV_{KR} is given by $\|\eta\|_{dV_{KR}} = \inf\{\|\xi\|_{dV_{Poin.}} : f_*\xi = \eta \text{ for some } f \in \mathcal{K}_M\}$. We will need the following estimate for the Kobayashi-Royden volume form on a bounded domain in $\mathbb{C}^n$ in terms of distances to the boundary.

Proposition 1. *Let $U \Subset \mathbb{C}^n$ be a bounded domain, and denote by $\rho = \rho_U$ the Kobyashi-Royden volume form on U. For $z \in U$ denote by $\delta(z)$ the Euclidean distance of z from the boundary ∂U. Write dV for the Euclidean volume form on $\mathbb{C}^n$. Then, there exists a positive constant c depending only on n and the diameter of U such that*

$$\rho(z) > \frac{c}{\delta(z)} \ .$$

Proof. We will first deal with the case where $n = 1$. In this case, the Kobayashi-Royden form is the same as the infinitesimal Kobayashi-Royden metric, which agrees with the Poincaré metric, and we have the stronger estimate where $\frac{c}{\delta(z)}$ is replaced by $\frac{c}{\delta^2(z)(log\delta)^2}$ (Mok-Yau [MY]). The latter estimate relies on the Uniformization Theorem and does not carry over to the case of general n. We will instead give the weaker estimate as stated in Proposition 1 for $n = 1$ using Cauchy estimates and give the necessary modification for general n.

Let $z \in U$ and $f : \Delta \to U$ be a holomorphic function such that $f(0) = z$. Denote by w the Euclidean coordinate on Δ. We will show that for some absolute constant C to be determined, we have $|f'(0)| \leq C\sqrt{\delta(z)}$, which gives the estimate $\|\frac{\partial}{\partial z}\|^2 \geq \frac{c}{\delta(z)}$ for $C = \frac{1}{c}$. Write $|f'(0)| = A$. Let $b \in \partial U$ be such that $|z - b| = \delta(z)$. To get an upper estimate for A we are going to show that if A were too large, then b would lie in the image f, leading to a contradiction. To this end consider the function $h(w) = f(w) - b$. The linear part of h at 0 is given by $L(w) = f'(0)w + z - b$. Write $h(w) = L(w) + E(w)$. Assume that $A \geq C\sqrt{\delta}$. We assert that there is a constant $a>0$ for which the following holds whenever $\delta(z)$ is sufficiently small:

$$\begin{aligned}&\text{(a)}\quad |L(w)|>2\delta(z)\quad\text{whenever}\quad |w|=a\sqrt{\delta(z)}\ ;\\&\text{(b)}\quad |E(w)|<\delta(z)\quad\text{whenever}\quad |w|=a\sqrt{\delta(z)}\ .\end{aligned}$$

From (a) and (b) it follows that $|h(w)|>\delta(z)$ whenever $|w|=a\sqrt{\delta(z)}$.

We can start with (b), which is valid for a sufficiently small $a>0$ depending on the diameter of U but independent of $\delta(z)$, by means of Cauchy estimates, since the "error" term $E(w)$ satisfies $E(0)=E'(0)=0$. Choose now the constant C such that $C>\frac{3}{a}$. Then, for $|w|=a\sqrt{\delta(z)}$,

$$\big|L(w)\big|\geq A|w|-\delta(z)>\frac{3}{a}\sqrt{\delta(z)}\big(a\sqrt{\delta(z)}\big)-\delta(z)>2\delta(z)\ , \tag{1}$$

so that (1) holds true. Given (1) and (2) we apply Rouché's Theorem. In view of the generalization to several variables, we give the proof here. Consider $h_t(w)=L(w)+tE(w)$ for t real, $0\leq t\leq 1$. For $t=0$ the affine-linear function L admits a zero at $w=0$. The number of zeros of h_t on the disk $\Delta\big(c\sqrt{\delta(z)}\big)$ is counted, by Stokes' Theorem, by the boundary integral

$$\frac{1}{2\pi}\int_{\partial\Delta\left(a\sqrt{\delta(z)}\right)}\sqrt{-1}\ \overline{\partial}\mathrm{log}|h_t|^2=\int_{\Delta\left(a\sqrt{\delta(z)}\right)}\sqrt{-1}\partial\overline{\partial}\mathrm{log}|h_t|^2\ . \tag{2}$$

The boundary integral is well-defined, takes integral values by the Argument Principle, and varies continuously with t, so that it is independent of t, implying that there exists a zero of h_t on the disk $\Delta\big(\sqrt{\delta(z)}\big)$; $0\leq t\leq 1$. In particular $f(w)=b$ has a solution, contradicting with the assumption that $b\in\partial U$.

We now generalize the argument to several variables. Let $f:B^n\to U$ be such that $f(0)=z$. Let again $b\in\partial U$ be a point such that $\|z-b\|=\delta(z)$. Consider the linear map $df(0)$. Since U is bounded, eigenvalues of $df(0)$ are bounded by a fixed constant. By Cramer's rule it follows that, $df(0)\big(B^n(r)\big)\supset B\big(e|\det(df(0)|\big)$ for some constant e depending on U but independent of $f\in\mathcal{K}_U$. In terms of the Euclidean coordinates $w=(w_1,...,w_n)$ of the domain manifold define as for $n=1$ the holomorphic map $h(w)=f(w)+z-b$. Decomposing $h(w)=L(w)+E(w)$ as in the case of $n=1$, $L(w)=df(0)(w)+z-b$, and using exactly the same argument there we have a real one-parameter family of holomorphic maps $h_t(w)=L(w)+tE(w)$, $0\leq t\leq 1$, such that $h_t(w)\neq 0$ for any $w\in\partial B^n\big(a\sqrt{\delta(z)}\big)$. For the analogue of Rouché's Theorem we note that $L(w)=0$ has a unique solution on $B^n\big(a\sqrt{\delta(z)}\big)$. Suppose for some t, $0<t\leq 1$, $h_t(w)=0$ is not solvable on $B^n\big(c\sqrt{\delta(z)}\big)$. Writing $h_t(w)=(h_{t,1}(w),\cdots,h_{t,n}(w))$, $1\leq k\leq n$, it follows that the components $h_{t,k}(w)$, $1\leq k\leq n$, cannot be simultaneously zero, so that $[h_t]:B^n\to\mathbb{P}^{n-1}$ is well-defined, and $\big(\sqrt{-1}\partial\,\overline{\partial}\mathrm{log}|h_t|^2\big)^n\equiv 0$, since the (1,1)-form inside the parenthesis is nothing other than the pull-back of the Kähler form of the Fubini-Study metric on $\mathbb{P}^{n-1}$. If that happens, by Stokes' Theorem we have

$$I(t):=\frac{1}{(2\pi)^n}\int_{\partial B^n\left(a\sqrt{\delta(z)}\right)}\sqrt{-1}\overline{\partial}\mathrm{log}|h_t|^2\wedge\big(\sqrt{-1}\partial\overline{\partial}\mathrm{log}|h_t|^2\big)^{n-1}=0\ . \tag{3}$$

The boundary integral is well-defined for $0 \le t \le 1$, with $I(0) = 1$. Obviously $I(t)$ varies continuously with t, but it is less clear that $I(t)$ is an integer for each t. To reach a contradiction to the assumption $b \in \partial U$ (as in the use of Rouché's Theorem for $n = 1$), we proceed as follows. $h_t = L + tE$ makes sense for any real t, and, for ϵ sufficiently small, in the interval $-\epsilon \le t \le 1+\epsilon$, h_t is not equal to 0 on $\partial B^n\big(a\sqrt{\delta(t)}\big)$. Hence, the boundary integral $I(t)$ remains well-defined. $I(t)$ then varies as a real-analytic function in t. For t sufficiently small, h_t is a biholomorphism of $B^n\big(a\sqrt{\delta(z)}\big)$ onto its image. The current $\big(\sqrt{-1}\partial\overline{\partial}\log|h_t|)^2\big)^n$ over $B^n\big(a\sqrt{\delta(z)}\big)$ is given by $(2\pi)^n\delta_{x(t)}$, where $x(t)$ is the unique zero of h_t, and δ_x denotes the delta measure at x. Hence $I(t) = 1$ for t sufficiently small. It follows that $I(t) = 1$ for $0 \le t \le 1$ by real-analyticity, and we have a contradiction at $t = 1$. The proof of Proposition 1 is complete. $\square$

The Kobayashi-Royden volume form on a complex manifold M arises from the space $\mathcal{K}_M$ of holomorphic maps $f : B^n \to M$. As is well-known, in the event where M is a bounded domain U in a Stein manifold Z, estimates for the Kobayashi-Royden form can be localized using Cauchy estimates. More precisely, if b lies on the boundary ∂U on Z, and $B \subset Z$ is a small Euclidean coordinate ball centred at b, any holomorphic map $f : B^n \to U$ must map $B^n(r)$ into $B \cap U$ for the Euclidean ball $B^n(r)$ centred at o of radius r, for some $r > 0$ independent of $f \in \mathcal{K}_U$. This leads to an upper bound on the Kobayashi-Royden volume form of $B \cap U$ in terms of that of U. We formulate it in a more general form as follows, noting the monotonicity property of the Kobayashi-Royden volume form.

Localization Lemma for the Kobayashi-Royden volume form. *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, and $W \subset Z$ be any open subset. Let $K \subset W$ be a compact subset. Then, there exists a positive constant C depending on U, W and K such that for any $z \in K$ we have*

$$\mu_U(z) \le \mu_{U\cap W}(z) \le C\mu_U(z) \ .$$

Proposition 2. *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, and $W \subset U$ be an open subset. Let $x \in U - W$ and $B \subset U$ be an open coordinate neighborhood of x in U, which we will identify as a Euclidean open set, endowed with the Lebesgue measure λ. Suppose* Volume$(B \cap W, \mu_B) < \infty$. *Then, the closed subset $B - W \subset B$ is of zero Lesbesgue measure.*

Proof. The problem being local, we may consider the following special situation. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Let I denote the unit interval $[0,1]$. Let $E \subset I^{2n}$ be a closed subset contained in $I^{2n-1} \times [\epsilon, 1-\epsilon]$ for some $\epsilon > 0$ such that $(I^{2n-1} \times \{0\}) \cap E = \emptyset$. On $\mathbb{C}^n - E$ denote by δ the Euclidean distance to E, i.e. $\delta(x) = \sup\big\{r : B^n(x;r) \cap E = \emptyset\big\}$ for $x \notin E$. By Proposition 1 in (2.2), we have

$$\int_{I^{2n}-E} \frac{dV}{\delta} < \infty \ , \tag{1}$$

where dV denotes the Euclidean volume form on $\mathbb{R}^{2n}$. Then, we need to prove that $\lambda(E) = 0$ for the Lebesgue measure λ. Let $S \subset I^{2n-1}$ be the closed subset consisting of those s such that $(\{s\} \times I) \cap E \neq \emptyset$. Denote by t the Euclidean variable for the last direct factor of I^{2n}. For each $s \in S$ observe that

$$\int_{(\{s\}\times I)-E} \frac{dt}{\delta} = \infty \ . \tag{2}$$

In fact, for each $s \in S$, $\delta(s,t) \leq |t - t_0|$ for any t_0 such that $(s, t_0) \in E$. Taking t_0 to be the mimimal possible value, then the integral above dominates the integral $\int_0^{t_0} \frac{dt}{t} = \infty$, as observed. As a consequence, by Fubini's Theorem, the closed subset $S \subset I^{2n-1}$ is of zero Lebesgue measure, so that $E \subset S \times I$ is of zero Lebesgue measure, as desired. □

We note that, given any unramified covering map $\nu : M' \to M$, the Kobayashi-Royden volume form μ_M on M agrees with that on M' by lifting, since B^n is simply-connected. Using Proposition 2 we deduce the following result crucial to the proof of the Main Theorem. It relates the covering domain D to its hull of holomorpy $\widehat{D}$, and allows us to enlarge N to a manifold admitting a complete Kähler-Einstein metric of finite volume.

Proposition 3. *Let $D \subset Z$ be a bounded domain on a Stein manifold Z, $\Gamma' \subset \mathrm{Aut}(D)$ be a torsion-free discrete group of automorphisms of D such that $N = D/\Gamma'$ is of finite measure with respect to μ_D. Let $\pi : \widehat{D} \to Z$ be the hull of holomorphy of D. Then, Γ' extends to a torsion-free discrete group of automorphisms $\widehat{\Gamma}'$ of $\widehat{D}$ such that, writing $\widehat{N} := \widehat{D}/\widehat{\Gamma}'$, $\widehat{N}$ is of finite volume with respect to $\mu_{\widehat{N}}$.*

Proof. Since $\mu_{\widehat{N}} \leq \mu_N$ on N, $\mathrm{Volume}(N, \mu_{\widehat{N}}) \leq \mathrm{Volume}(N, \mu_N) < \infty$. On the other hand, $\mathrm{Volume}(\widehat{N} - N, \mu_{\widehat{N}})$ is obtained by integrating $\mu_{\widehat{N}}$ over $\widehat{N} - N$. In terms of local holomorphic coordinates which give local Lebesgue measures, we can write $\mu_{\widehat{N}} = \varphi \cdot \lambda$, where φ is a continuous function. Since $\widehat{N}$ can be covered by a countable number of open Euclidean open sets B_α such that $B_\alpha \cap (\widehat{N} - N)$ is of zero Lebesgue measure of each index α, we conclude that $\mathrm{Volume}(\widehat{N}, \mu_{\widehat{N}}) = \mathrm{Volume}(N, \mu_{\widehat{N}}) < \infty$, as desired. □

From Proposition 3 and the Existence Theorem on Kähler-Einstein metrics in (1.1) on bounded domains of holomorphy, we have immediately

Corollary 1. *Let $\widehat{N} \supset N$ be the complex manifold as in Proposition 3. Then $\widehat{N}$ admits a unique complete Kähler-Einstein metric g_{KE} of finite volume and of constant Ricci curvature $-(n+1)$, $n = \dim N$.*

§3 Proof of the Isomorphism Theorem

(3.1) We recall briefly the argument for proving the Fibration Theorem in the case when X and N are compact, after the Extension Problem had been solved. Denote by $\tau : \widetilde{N} \to N$ the universal covering map. For any bounded holomorphic function θ on $\widetilde{N}$ consider the continuous plurisubharmonic function $\psi_\theta : N \to \mathbb{R}$ defined by $\psi_\theta(q) = \sup\{|\theta(p)| : \tau(p) = q\}$. ($\theta$ is continuous, actually Lipschitz, by Cauchy estimates.) From the compactness of N it follows that ψ_θ must be constant. We applied this argument on linear projections of the bounded vector-valued holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$ to show that $R(\widetilde{N}) \subset \Omega$. Using $f_* : \Gamma \cong \Gamma'$ we identify Γ with Γ'. Consider now the vector-valued holomorphic map $T_\gamma : \widetilde{N} \to \mathbb{C}^n$

given by $T_\gamma = R(\gamma(p)) - \gamma(R(p))$, which vanishes identically on $F(\Omega)$. Applying the above argument on θ again to components of T_γ we conclude that the latter vanishes identically on $\widetilde{N}$, i.e., we have the identity $R \circ \gamma \equiv \gamma \circ R$ on all of $\widetilde{N}$. It follows that the holomorphic mapping $R : \widetilde{N} \to \Omega$ descends to $\rho : N \to X$, which gives the Fibration Theorem in the compact case.

For the proof of the Isomorphism Theorem we proceed now to justify the same line of argument by first proving the constancy of analogous functions ψ_θ. This will be demonstrated by integrating by part on complete Kähler manifolds, for which purpose we will pass to the hull of holomorphy $\widehat{D}$ of D and make use of complete Kähler-Einstein metrics as explained in §2. A further argument, again related to the vanishing of certain bounded plurisubharmonic functions, will be needed to show that the holomorphic fibration obtained is trivial.

For the proof of the Main Theorem along this line of thoughts we will need

Lemma 1. *Let (Z, ω) be an s-dimensional complete Kähler manifold of finite volume, and u be a uniformly Lipschitz bounded plurisubharmonic function on Z. Then, u is a constant function.*

Proof. Fix a base point $z_0 \in Z$. For $R > 0$ denote by B_R the geodesic ball on (Z, ω) of radius R centred at z_0. There exists a smooth nonnegative function ρ_R on Z, $0 \leq \rho_R \leq 1$, such that $\rho \equiv 1$ on B_R, $\rho \equiv 0$ outside B_{R+1}, and such that $\|d\rho_R| \leq \frac{2}{R}$. By Stokes' Theorem, we have

$$0 = \int_Z d\big(\sqrt{-1}\rho_R u \overline{\partial} u \wedge \omega^{n-1}\big) = \int_Z \sqrt{-1} d(\rho_R u) \wedge \overline{\partial} u \wedge \omega^{s-1} + \int_Z \rho_R \sqrt{-1} u \partial\overline{\partial} u \wedge \omega^{s-1} \ . \quad (1)$$

so that

$$\begin{aligned} \int_{B_R} \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega^{s-1} &\leq \int_Z \rho_R \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega^{s-1} \\ &= -\int_Z \sqrt{-1} u \partial\rho_R \wedge \overline{\partial} u \wedge \omega^{s-1} - \int_Z \rho_R \sqrt{-1} u \partial\overline{\partial} u \wedge \omega^{s-1} \ . \end{aligned} \quad (2)$$

In terms of norms on (Z, ω), $\|du\|$ is by assumption uniformly bounded. Furthermore, $\|d\rho_R\| \leq \frac{2}{R}$, and its support is contained in $Z - B_R$, so that the second last term of (2), up to a fixed constant, is bounded by Volume$(Z - B_R, \omega)$, which decreases to 0 as $R \to \infty$ since Volume$(Z, \omega) < \infty$ by assumption. On the other hand the last term of (2) (without the minus sign) is nonnegative since $u \geq 0$ and u is plurisubharmonic. Fix any $R_0 > 0$. It follows readily that for any $R > R_o$,

$$\int_{B_{R_0}} \|\partial u\|^2 \leq \int_{B_R} \|\partial u\|^2 \to 0 \qquad \text{as} \qquad R \to \infty \ . \quad (3)$$

As a consequence $\partial u \equiv 0$, so that $u \equiv C$ for some constant C, as desired. $\square$

We are now ready to prove the Main Theorem, as follows.

Proof of Main Theorem (The Isomorphism Theorem). Here and in what follows by the intrinsic measure we will always mean the measure given by the Kobayashi-Royden volume form. By (2.1), Proposition 1, we can 'complete' D to a bounded domain of holomorphy $\widehat{D}$ and extend Γ' to a torsion-free discrete group of automorphisms $\widehat{\Gamma}'$, such that $\widehat{N} = \widehat{D}/\widehat{\Gamma}'$ is of finite intrinsic measure. By Corollary 1 in (2.2), $\widehat{N}$ carries a unique Kähler-Einstein metric g_{KE} of constant Ricci curvature $-(n+1)$, $n = \dim(N)$. Denote by $\omega_{_{KE}}$ the Kähler form of $g_{_{KE}}$. By invariance, $g_{_{KE}}$ and $\omega_{_{KE}}$ descend to $\widehat{N}$, and we use the same notations on $\widehat{N}$. Again by the Comparison Lemma in (2.1), the Kähler-Einstein volume form on $\widehat{N}$ is bounded by a constant multiple of the Kobayashi-Royden volume form, so that $(\widehat{N}, \omega_{_{KE}})$ is also of finite volume. We may consider the holomorphic map $f : X \to N$ to have image in $\widehat{N}$. Applying our solution to the Extension Problem as given in (1.2), we can extend the inverse map $i : F(\Omega) \cong \Omega$ to $\widehat{R} : \widehat{D} \to \mathbb{C}^n$ as a bounded holomorphic map. We claim, in analogy to the proof of the Fibration Theorem, that $\widehat{R}(\widehat{D}) \subset \Omega$. The proof there relies on showing that the bounded plurisubharmonic function ψ_θ is a constant. From the construction, $d\psi_\theta$ is uniformly bounded with respect to the induced Carathéodory metric κ on $\widehat{N}$. By the Comparison Lemma in (2.1), $g_{_{KE}}$ dominates a constant multiple of κ, so that $\|d\psi_\theta\|_{g_{_{KE}}}$ is uniformly bounded on $\widehat{N}$ (cf. Eqn. (1) below for details in an analogous situation). By Lemma 1 it follows that ψ_θ is a constant, so that $\widehat{R}(\widehat{D}) \subset \Omega$. The same argument applied to the bounded vector-valued holomorphic functions $T_\gamma = \widehat{R} \circ \gamma - \gamma \circ \widehat{R}$ yields the equivariance of $\widehat{R}$ under Γ. As a consequence, the analogue of the Fibration Theorem remains valid, i.e., there exists a holomorphic map $\rho : \widehat{N} \to X$ such that $f \circ \rho \equiv id_X$. To complete the proof of the Isomorphism Theorem it remains to show that $f : X \to N$ is an open embedding. Knowing this, we will have $\rho \circ f \equiv id_{\widehat{N}}$ by the identity theorem, so that f maps X biholomorphically onto $\widehat{N}$. But, by hypothesis $f(X) \subset N$, so that $\widehat{N} = N$ and we will have established that $f : X \to N$ is a biholomorphism.

We proceed to prove that $f : X \to N \subset \widehat{N}$ is an open embedding. Suppose otherwise. Then, $n = \dim(N) > \dim(X) := m$ and the fibers $\rho^{-1}(x)$ of $\widehat{\rho} : \widehat{N} \to X$ are positive-dimensional. Let $x_0 \in X$ be a regular value of $\widehat{\rho} : \widehat{N} \to X$, and $L \subset \widehat{\rho}^{-1}(x_0)$ be a connected component, $\dim(L) = n-m > 0$. We claim that L lifts in a univalent way to $\widehat{D}$. To this end let $\widetilde{x}_0 \in \Omega$ be such that $\pi(\widetilde{x}_0) = x_0$, and $\widetilde{L} \subset \widehat{D}$ be a connected component of $\widehat{R}^{-1}(\widetilde{x}_0)$, such that $\tau(\widetilde{L}) = L$ for the covering map $\tau : \widehat{D} \to \widehat{N}$. Suppose $\gamma \in \Gamma$ acts as a covering transformation on $\widehat{D}$ such that $\gamma(\widetilde{L}) = \widetilde{L}$. By the Γ-equivariance of $\widehat{R}$ we have $\widehat{R}(\gamma(p)) = \gamma(\widehat{R}(p))$. Applying this to $p \in \widetilde{L}$, $\widehat{R}(\gamma(p)) = \widehat{R}(p)$, so that $\gamma(\widehat{R}(p)) = \widehat{R}(p)$, implying that γ acts as the identity map on Ω since $\Gamma \subset \mathrm{Aut}(\Omega)$ is torsion-free. This means precisely that $\tau|_{\widetilde{L}}$ maps $\widetilde{L}$ bijectively onto L, as claimed.

Recall that g_{KE} is the complete Kähler-Einstein metric on $\widehat{N}$ of constant Ricci curvature $-(n+1)$, ω_{KE} is its Kähler form. From the liftings $\widetilde{L}$ we are going to derive a contradiction. Let σ be a bounded holomorphic function on the bounded domain $\widehat{D}$ such that $\sigma|_{\widetilde{L}}$ is not identically a constant. Then, $u := |\sigma|^2$ gives a nonnegative plurisubharmonic function on $\widetilde{L} \cong L$. If we know that $(L, \omega_{_{KE}}|_L)$ is of finite volume, then Lemma 1 applies to yield a contradiction. We only know that $(\widehat{N}, \omega_{_{KE}})$ is of finite volume. Let again $x_0 \in X$ be a regular value of $\rho : \widehat{N} \to X$, $q_0 := f(x_0)$. Let V be a simply connected open neighborhood of x_0 in X. For $x \in V$ denote by $L_q \subset \widehat{\rho}^{-1}(x) \subset \widehat{N}$ the connected component of $\widehat{\rho}^{-1}(x)$ containing $q := f(x)$. Since V is simply connected there is an open subset $\widetilde{V} \subset \Omega$ such that

$\pi|_{\widetilde{V}} : \widetilde{V} \to X$ maps $\widetilde{V}$ bijectively onto V for the universal covering map $\pi : \Omega \to X$. For $x \in V$ denote by $\widetilde{x} \in \widetilde{V}$ the unique point such that $\pi(\widetilde{x}) = x$ and write $\widetilde{L}_q \subset \Omega$ for the irreducible component of $\pi^{-1}(L_q)$ containing $\widetilde{x}$. For almost all $x \in V$, x is a regular value of $\rho : \widetilde{N} \to X$, and $L_q \subset \widetilde{N}$, $q = f(x)$, is a complex submanifold of $\widetilde{N}$ of dimension equal to $\dim(N) - \dim(X) = n - m$. For a singular value x, it remains the case that $\dim(L_q) = n - m$, but L_q may have singularities. The arguments in the preceding paragraph remain valid to show that $\tau|_{\widetilde{L}_q}$ maps $\widetilde{L}_q$ bijectively onto L_q. Let $W \subset \widehat{N}$ be the union of $\widetilde{L}_q$. Let σ now be a bounded holomorphic function on the bounded domain $\widehat{D}$ such that $\sigma|_{\widetilde{L}_{q_0}}$ is not identically a constant. Since $\tau|_{\widetilde{W}} : \widetilde{W} \to \widehat{N}$ maps $\widetilde{W}$ bijectively onto W we may regard σ as a bounded holomorphic function on W. Write $u := |\sigma|^2$. Then, u is a nonnegative bounded plurisubharmonic function on W. Recall that κ is the induced Carathéodory metric on $\widehat{N} = \widehat{D}/\Gamma'$. By the Comparison Lemma of (2.1), $g_{KE} \geq \text{Const.} \times \kappa$. Since $\partial u = \overline{\sigma}\partial\sigma$ and σ is bounded, we have

$$\begin{aligned}\|\partial u(y)\|_{g_{KE}} &\leq \text{Const.} \times \|\partial\sigma(y)\|_{g_{KE}} \\ &= \text{Const.} \times \sup\left\{|\partial\sigma(\eta)| : \eta \in T_y(\widehat{D}), \|\eta\|_{g_{KE}} \leq 1\right\} \\ &\leq \text{Const.}' \times \sup\left\{|\partial\sigma(\eta)| : \eta \in T_y(\widehat{D}), \|\eta\|_{\kappa} \leq 1\right\} \\ &< \infty\ ,\end{aligned} \tag{1}$$

where the last inequality follows from the definition of the Carathéodory metric. Denote by $R_\rho \subset f(V)$ the subset of all $q = f(x)$, where $x \in V$ is a regular value of ρ. Consider the fibration $\widehat{R} : \widehat{D} \to \Omega$. Then, the Carathéodory metric $\kappa_{\widehat{D}}$ on $\widehat{D}$ dominates the pull-back of the Carathéodory metric κ_Ω on Ω. By the Comparison Lemma of (2.1), the Kähler-Einstein metric g_{KE} on $\widehat{D}$ dominates a constant multiple of the Carathéodory metric $\kappa_{\widehat{D}}$ on $\widehat{D}$, so that

$$g_{KE} \geq \text{Const.} \times \widehat{R}^*\kappa_\Omega\ . \tag{2}$$

Descend to $\widehat{N}$ and consider the fibration $\rho|_W : W \to V$. In what follows we impose the condition that $V \Subset X$ and denote by $d\lambda$ the restriction of a smooth volume form on X to V. From (1) it follows

$$\omega_{KE}^n \geq (\text{Const.} \times \rho^* d\lambda) \wedge \omega_{KE}^{n-m}\ . \tag{3}$$

By Fubini's Theorem we conclude from the estimates that

$$\begin{aligned}\int_{q \in R_\rho} \text{Volume}(L_q, \omega_{KE}|_{L_q}) d\lambda(q) &\leq \text{Const.} \times \text{Volume}(W, \omega_{KE}) \\ &\leq \text{Const.} \times \text{Volume}(\widehat{N}, \omega_{KE}) < \infty\ ,\end{aligned} \tag{4}$$

so that $q \in R_\rho$ and $\text{Volume}(L_q, \omega_{KE}|_{L_q}) < \infty$ for almost all $q \in V$. Applying Lemma 1 to a regular fiber L_q with q sufficiently to q_0, $\text{Volume}(L_q, \omega_{KE}|_{L_q}) < \infty$ and to the plurisubharmonic function $u = |\sigma|^2$ we obtain a contradiction to Lemma 1, proving by contradiction that $f : X \to N$ is an open embedding, with which we have completed the proof of the Main Theorem. $\square$

We have the following variation of the Main Theorem when the fundamental groups of X and N are only assumed to be isomorphic as abstract groups.

Variation of the Main Theorem. *Suppose in the statement of the Main Theorem in place of assuming that $f_* : \Gamma \cong \Gamma'$ we assume instead that $\Gamma \cong \Gamma'$ as abstract groups and that $f : X \to N$ is nontrivial. Then, $f : X \to N$ is a biholomorphism.*

Proof. Fix an isomorphism between Γ and Γ' as abstract groups and hence identify Γ' with Γ. f_* is thus regarded as a group endomorphism of Γ. Let $G = \mathrm{Aut}_0(\Omega)$ be the identity component of the automorphism group of Ω. Replacing Γ (and hence Γ') by a subgroup of finite index we may assume that $\Gamma \subset G$. Since G is semisimple, connected and of real rank ≥ 2, and $\Gamma \subset G$ is an irreducible lattice, by Margulis' Superrigidity Theorem [Ma], either $f_*(\Gamma)$ is finite, or else $f_* : \Gamma \to \Gamma$ extends to a group automorphism $\varphi : G \to G$. In the former case we would have a lifting X to the covering domain D of N, which would force f to be constant by the Maximum Principle, since the Satake compactification of X is obtained by adding a variety of dimension $\leq \dim(X) - 2$. In other words, the nontriviality of f forces $f_* : \Gamma \to \Gamma$ to extend to a group automorphism $\varphi : G \to G$. In particular, f_* is injective. With respect to a fixed Haar measure on the semsimiple Lie group G, which is invariant under the automorphism φ, Volume(G/Γ) must agree with Volume$(G/f_*(\Gamma))$. Since $f_*(\Gamma) \subset \Gamma$, it follows that $f_*(\Gamma) = \Gamma$, so that $f_* : \Gamma \cong \Gamma \cong \Gamma'$, and we are back to the original formulation of the Main Theorem. $\square$

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