Graceful Tree Conjecture for Infinite Trees

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Abstract

A graceful labeling of a graph G with n edges is an injective function from the set of vertices of G to the set $\{0, 1, 2, \ldots, n\}$ such that the edge labels, the absolute difference between the two endvertex labels, are all distinct. One of the most famous open problems in graph theory is the Graceful Tree Conjecture which states that every finite tree has a graceful labeling. Despite forty years of research effort, little progress has been made towards resolving this conjecture. Today, some of the known graceful trees are caterpillars, trees with at most 4 endvertices, trees with diameter at most 5, and trees with at most 27 vertices.

In this paper, we considered the infinite version of the Graceful Tree Conjecture. First, the notions of bijective graceful $\mathbb{N}$-labeling and bijective graceful $\mathbb{N}/\mathbb{N}$-labeling of infinite graphs were introduced. Such labelings were then shown to be possible for infinite graphs built by certain types of graph amalgamations. Finally, based on the tools developed, we were able to characterize all the infinite trees that have a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling and hence solved the Graceful Tree Conjecture for infinite trees.

1 Introduction

The study of graph labeling was initiated by Rosa [9] in 1967. This involves labeling vertices or edges, or both, using integers subject to certain conditions. Ever since then, various kinds of graph labelings have been considered, and the most well-studied ones are graceful, magic and harmonious labelings. Not only interesting in its own right, graph labeling also finds a broad range of applications: the study of neofield, topological graph theory, coding theory, radio channel assignment, communication network addressing and database management. One should refer to the comprehensive survey by Gallian [6] for further details.

Rosa considered the so-called β-valuation which is commonly known as graceful labeling. A function f is called a graceful labeling of a graph G with n edges if f is an one-to-one map from the vertices of G to the set $\{0, 1, \ldots, n\}$ such that the edge labels, the absolute difference between the two endvertex labels, are all distinct. Graceful labeling was originally introduced to attack **Ringel's Conjecture** which says that a complete graph of order $2n + 1$ can be decomposed into $2n + 1$ isomorphic copies of any tree with n edges. Ringel's Conjecture is true if it could be shown that every tree has a graceful labeling. This is known as the famous **Graceful Tree Conjecture** but such seemingly simple statement defies any effort to prove it [5]. Today, some known examples of graceful trees are: caterpillars [9] (a tree such that the removal of its endvertices leaves a path), trees with at most 4 endvertices [8], trees with diameter at most 5 [7], and trees with at most 27 vertices [1].

Most of the previous works on graph labeling focused on finite graphs only. Recently, Beardon [2], and later, Combe and Nelson [3] considered magic labelings of infinite graphs over integers and infinite abelian groups. Beardon showed that infinite graphs built by certain types of graph amalgamations possess bijective edge-magic $\mathbb{Z}$-labelings. Infinite graph has the advantage that there is a much greater degree of freedom for constructing the magic labeling as both the graph and the labeling set are infinite. However, it is not known whether every countably infinite tree supports a bijective edge-magic $\mathbb{Z}$-labelings. Motivated by their ideas, we are going to study graceful labelings of infinite graphs. Along the way, we will completely solve the infinite version of the Graceful Tree Conjecture.

This paper is organized as follows. In Section 2, we give a formal definition of graceful labeling. We also consider how to construct an infinite graph by means of amalgamation, and introduce the notions of bijective graceful $\mathbb{N}$-labeling and bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. Section 3 includes two examples on graceful labelings of the semi-infinite path which illustrate the main ideas in this paper. In Section 4, our main results are presented while further generalizations are discussed in Section 5. In Section 6, we make use of the tools developed in Section 4 and characterize all infinite trees that have a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. This, in turn, settles the Graceful Tree Conjecture for infinite trees.

2 Definitions and notations

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote the set of natural numbers $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$. A graceful $\mathbb{N}$-labeling of G is an injective function f that maps $V(G)$ to $\mathbb{N}$ such that the labels $|f(x) - f(y)|$ assigned to each edge xy are all distinct. We will denote the edge label on edge xy by $f(xy)$.

Consider a graph G_n with vertex set V_n and edge set E_n. A sequence of graphs, $\{G_n\}$, is increasing if for each n, $V_n \subset V_{n+1}$ and $E_n \subset E_{n+1}$. An infinite graph, $\lim_n G_n$, is then defined to be the graph whose vertex set and edge set are $\cup_n V_n$ and $\cup_n E_n$ respectively. Throughout this thesis, we use the term infinite to mean countably infinite.

Following Beardon [2], we build an infinite graph by joining an infinite sequence of graphs through the process of amalgamation described below. Let G_1 and G_2 be two graphs with no common vertices. Select a vertex v_1 from G_1 and a vertex v_2 from G_2. The amalgamation of G_1 and G_2, $G_1 \# G_2$, is obtained by taking the disjoint union of G_1 and G_2 and identifying v_1 with v_2.

Now let $G'_1, G'_2, \ldots$ be an infinite sequence of graphs. Construct a new sequence G_n inductively by $G_1 = G'_1$ and $G_{n+1} = G_n \# G'_{n+1}$. Obviously, $\{G_n\}$ is increasing and their union is an infinite graph. Using techniques similar to those introduced by Beardon [2], we are able to show that every infinite graph generated by certain types of graph amalgamations has a graceful labeling. To be more precise, we call a graceful labeling, bijective graceful $\mathbb{N}$-labeling if there is an one-to-one correspondence between the vertex labels and $\mathbb{N}$. If both the vertex and edge labels are permutations of the natural numbers, then we call it a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.

Further definitions and notations will be introduced as our discussions proceed. The graph theory terminology used in this paper can be found in the book by Diestel [4].

3 Example: Semi-infinite Path

In this section, we will illustrate our graph labeling method and the key ideas behind by means of the semi-infinite path. Denote the semi-infinite path by P, with vertices: $v_0, v_1, v_2, \ldots$ and edges: $v_0v_1, v_1v_2, \ldots$. To simplify subsequent discussions, we choose the listing $V = \{1, 2, 3, \ldots\}$ for vertex labels and $E = \{1, 2, 3, \ldots\}$ for edge labels, and let $f(v_0) = m_0 = 1$.

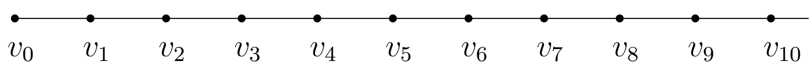


Figure 1

Bijective graceful ℕ-labeling of the semi-infinite path

Our goal is to label the vertices of P using ℕ such that the vertex labels correspond one-to-one to the set of the natural numbers and the edge labels are all distinct. We will proceed in a manner similar to that in [1].

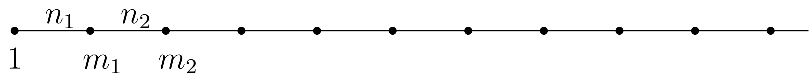


Figure 2

Let $f(v_1) = m_1$, $f(v_2) = m_2$, $f(v_0v_1) = n_1 = |m_0 - m_1|$ and $f(v_1v_2) = n_2 = |m_1 - m_2|$. Take m_2 to be the smallest integer in V not yet used for vertex labeling which is 2. Now, we can choose m_1 to be sufficiently large so that n_1 and n_2 are distinct and have not appeared in the edge labels. $m_1 = 3$ will do, and we have $n_1 = 2$ and $n_2 = 1$.

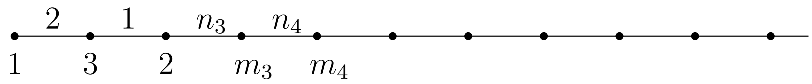


Figure 3

Now let $f(v_3) = m_3$, $f(v_4) = m_4$, $f(v_2v_3) = n_3 = |m_2 - m_3|$ and $f(v_3v_4) = n_4 = |m_3 - m_4|$. Again, take m_4 to be the smallest integer in V not yet appeared which is 4. Choose m_3 to be sufficiently large so that n_3 and n_4 are distinct and have not appeared in the edge labels. Pick $m_3 = 7$, and we have $n_3 = 5$ and $n_4 = 3$.

The above process can be repeated indefinitely. Since for each $n \in \mathbb{N}$, we can choose $f(v_{2n})$ to be the smallest unused integer in V, f is surjective. By construction, f is also injective and all edge labels are distinct. Hence, we have constructed a bijective graceful ℕ-labeling of the semi-infinite path.

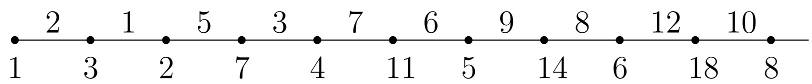


Figure 4

Bijective graceful ℕ/ℕ-labeling of the semi-infinite path

In the previous example, we require that all natural numbers appear in the vertex labels. A natural question arises: can we also require that all natural numbers appear in the edge labels? As will be shown below, this is possible for the semi-infinite path. Recall that we choose the listing $V = \{1, 2, 3, \ldots\}$ for vertex labels and $E = \{1, 2, 3, \ldots\}$ for edge labels, and let $f(v_0) = m_0 = 1$.

Let $f(v_1) = m_1$, $f(v_2) = m_2$, $f(v_0v_1) = n_1$ and $f(v_1v_2) = n_2$. We choose n_2 to be the smallest integer in E not used in the edge labels. Hence, $n_2 = 1$.

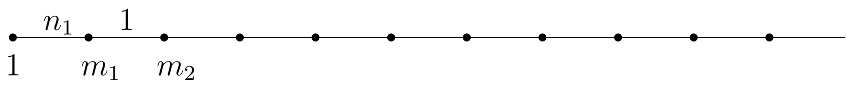


Figure 5

Now we would like to choose m_1 and m_2 that satisfy the following conditions:

1. m_1 and m_2 are distinct, different from 1 (vertex labels already used) and $|m_1 - m_2| = 1$, and
2. $|1 - m_1|$ is different from 1 (edge labels already used).

This is always possible if we choose m_1 and m_2 to be sufficiently large so that n_1 has not appeared before. (This key idea will be used in the proof of Lemma 2.) But in this particular example, $m_1 = 3$ and $m_2 = 2$ will do.

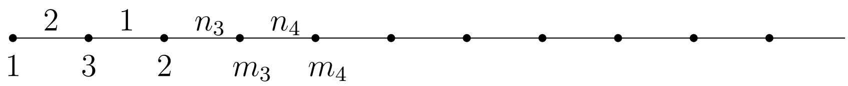


Figure 6

Let $f(v_3) = m_3$, $f(v_4) = m_4$, $f(v_2v_3) = n_3$ and $f(v_3v_4) = n_4$. This time we choose m_4 to be the smallest integer in V not yet appeared in the vertex labels. So $m_4 = 4$. Now choose m_3 sufficiently large so that n_3 and n_4 have not appeared in the edge labels. Pick $m_3 = 7$, and we have $n_3 = 5$ and $n_4 = 3$.

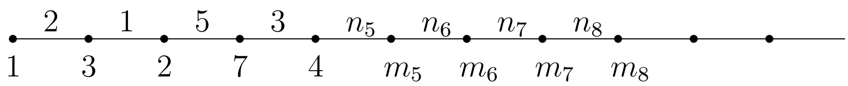


Figure 7

Repeat the above two procedures. Let $f(v_5) = m_5$, $f(v_6) = m_6$, $f(v_7) = m_7$, $f(v_8) = m_8$ and $f(v_4v_5) = n_5$, $f(v_5v_6) = n_6$, $f(v_6v_7) = n_7$, $f(v_7v_8) = n_8$. Choose n_6 to be smallest unused edge label which is 4. Pick m_5 and m_6 sufficiently large so that $|m_5 - m_6| = 4$ and n_5 has not appeared. We have $m_5 = 10$, $m_6 = 6$ and $n_5 = 6$.

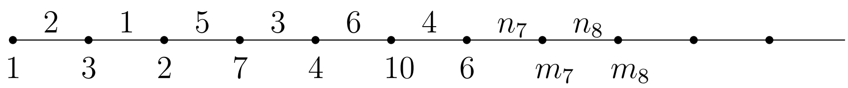


Figure 8

Next choose m_8 to be the smallest unused vertex label which is 5. Pick m_7 sufficiently large so that n_7 and n_8 have not appeared. Therefore, $m_7 = 13$, $n_7 = 7$ and $n_8 = 8$.

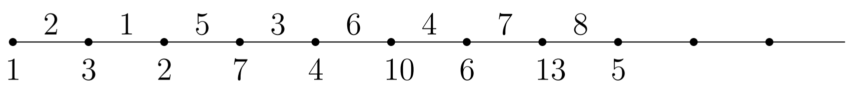


Figure 9

The above labeling process can go on indefinitely. Since for each $n \in \mathbb{N}$, we are able to choose $f(v_{4n-3}v_{4n-2})$ and $f(v_{4n})$ to be the first unused integer in E and V respectively, $f : E(P) \to \mathbb{N}$ and $f : V(P) \to \mathbb{N}$ are surjective. By construction, f is also injective. Therefore, we have successfully obtained a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling of the semi-infinite path.

Summing up, the crucial element that makes bijective graceful $\mathbb{N}$-labeling of the semi-infinite path possible is that during the labeling process, one can find a vertex that is not adjacent to all the previously labelled vertices. Such vertex can then be labelled using the smallest unused vertex label. Likewise, one can find an edge that is not incident to all the previously labelled vertices. Such edge can be labelled using the smallest unused edge label allowing one to obtain a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling of the semi-infinite path.

4 Main Results

Here we put the ideas developed in the previous chapter into Lemma 1 and 3 which are the key to our main results on graceful labelings of infinite graphs. Along the way, type-1 and type-2 graph amalgamations will be introduced.

Type-1 graph amalgamation

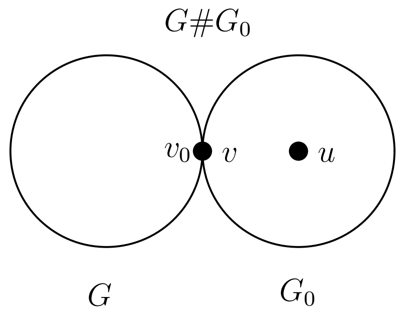


Figure 10

Lemma 1. *Let f_0 be an injective graceful $\mathbb{N}$-labeling of a finite graph G_0. Let V_0 be the set of integers taken by f_0 on $V(G_0)$ and E_0 be the resulting edge labels on $E(G_0)$. Suppose that $m \in \mathbb{N} \setminus V_0$. Let G be any finite graph and form an amalgamated graph $G_0\#G$ by identifying a vertex v_0 of G_0 with a vertex v of G. If G has a vertex u not adjacent to v, then $G_0\#G$ is called a type-1 amalgamation (Figure 10) and f_0 can be extended to an injective graceful $\mathbb{N}$-labeling f of $G_0\#G$ such that $f(u) = m$.*

Proof. First define f to be f_0 on G_0 and $f(u) = m$. Let $v_1, \ldots, v_k$ be the vertices in G other than u and v. Define $f(v_i) = m_i$ for $i = 1, \ldots, k$ where m_i's are parameters to be determined. Having identified v with v_0, we write $m_v = f_0(v_0)$. Now, each edge in G is of one of the forms: vv_i, uv_i or v_iv_j for $1 \leq i \neq j \leq k$ with edge labels $|m_v - m_i|$, $|m - m_i|$, and $|m_i - m_j|$ respectively. Notice that the edge label for edge e is the absolute value of a non-constant linear polynomial $p_e(m_1, \ldots, m_k)$. To make f injective, we want to choose m_i, for $i = 1, \ldots, k$, so that:

1. $m_i \neq m_j$ for $i \neq j$,
2. $m_1, \ldots, m_k \notin V_0 \cup \{m\}$,
3. $p_{e_i}(m_1, \ldots, m_k) \notin E_0$,
4. $p_{e_i}(m_1, \ldots, m_k) \neq p_{e_j}(m_1, \ldots, m_k)$ for $i \neq j$, and
5. $p_{e_i}(m_1, \ldots, m_k) \neq -p_{e_j}(m_1, \ldots, m_k)$ for $i \neq j$.

This is always possible by the following lemma. □

Lemma 2. *Let N_0 be a finite subset of $\mathbb{N}$. Consider, for $m_1, \ldots, m_k$ in $\mathbb{N}$, the $5^k - 1$ non-trivial expressions of the form $L_j(m_1, \ldots, m_k) = a_{j1}m_1 + \ldots + a_{jk}m_k$ where each a_{ij} is $-2, -1, 0, 1, 2$. Then there exists a choice of $m_1, \ldots, m_k$ in $\mathbb{N}$ such that no $L_j(m_1, \ldots, m_k)$ is in N_0.*

Proof. We prove by induction. For $k = 1$, we can choose m_1 so that $-2m_1, -m_1, m_1, 2m_1$ are all outside N_0. Suppose the statement holds for every finite subset N_0 and for $k = 1, \ldots, n$. Now consider the variables, $m_1, \ldots, m_n, m_{n+1}$ and any finite subset N_0 of $\mathbb{N}$. Choose m_{n+1} so that $-2m_{n+1}, -m_{n+1}, m_{n+1}, 2m_{n+1}$ are all outside N_0. By induction hypothesis, we can choose $m_1, \ldots, m_n$ so that for each j, $L_j(m_1, \ldots, m_n) \notin B$ where $B = (-2m_{n+1} + N_0) \cup (-m_{n+1} + N_0) \cup N_0 \cup (m_{n+1} + N_0) \cup (2m_{n+1} + N_0)$. Now any linear form in the variables $m_1, \ldots, m_{n+1}$ is of the form $L_j(m_1, \ldots, m_n) + am_{n+1}$ where $a = -2, -1, 0, 1, 2$. Obviously, in each case, $L_j(m_1, \ldots, m_n) + am_{n+1} \notin N_0$. Hence, the statement is true for $k = n + 1$ and the proof is complete. □

Type-2 graph amalgamation

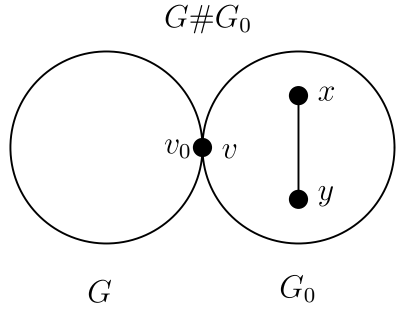


Figure 11

Lemma 3. *Let f_0 be an injective graceful $\mathbb{N}$-labeling of a finite graph G_0. Let V_0 be the set of integers taken by f_0 on $V(G_0)$ and E_0 be the resulting edge labels on $E(G_0)$. Suppose that $n \in \mathbb{N} \setminus E_0$. Let G be any finite graph and form an amalgamated graph $G_0\#G$ by identifying a vertex v_0 of G_0 with a vertex v of G. If G has an edge xy such that x and y are different from v, then $G_0\#G$ is called a type-2 amalgamation (Figure 11) and f_0 can be extended to an injective graceful $\mathbb{N}$-labeling f of $G_0\#G$ such that $f(xy) = n$.*

Proof. The proof is almost identical to that of Lemma 1 except for some minor modifications. Let $m_v = f_0(v_0)$. By choosing m_x and m_y sufficiently large, we can ensure that (i) $m_x, m_y \in \mathbb{N} \setminus V_0$, (ii) $|m_x - m_y| = n$, (iii) $|m_x - m_v| \notin E_0 \cup \{n\}$ if x is adjacent to v, and (iv) $|m_y - m_v| \notin E_0 \cup \{n\}$ if y is adjacent to v. Define f to be f_0 on G_0, $f(x) = m_x$ and $f(y) = m_y$. Let $v_1, \ldots, v_k$ be the vertices in G other than v, x and y. Define $f(v_i) = m_i$ for $i = 1, \ldots, k$ where m_i's are parameters to be determined. Now, each edge e in G except xy (and possibly vx and vy) is of one of the forms: vv_i, xv_i, yv_i or v_iv_j for $1 \leq i \neq j \leq k$ with edge labels $|m_v - m_i|$, $|m_x - m_i|$, $|m_y - m_i|$ and $|m_i - m_j|$ respectively. Notice that every such edge label is the absolute value of a non-constant linear polynomial $p_e(m_1, \ldots, m_k)$ in the variables $m_1, \ldots, m_k$. To make f injective, we want to choose m_i, for $i = 1, \ldots, k$, so that:

1. $m_i \neq m_j$ for $i \neq j$,
2. $m_1, \ldots, m_k \notin V_0 \cup \{m_x\} \cup \{m_y\}$,
3. $p_{e_i}(m_1, \ldots, m_k) \notin E_0 \cup \{n\}$,
4. $p_{e_i}(m_1, \ldots, m_k) \neq m_x - m_v$ if x is adjacent to v,
5. $p_{e_i}(m_1, \ldots, m_k) \neq m_v - m_x$ if x is adjacent to v,
6. $p_{e_i}(m_1, \ldots, m_k) \neq m_y - m_v$ if y is adjacent to v,
7. $p_{e_i}(m_1, \ldots, m_k) \neq m_v - m_y$ if y is adjacent to v,
8. $p_{e_i}(m_1, \ldots, m_k) \neq p_{e_j}(m_1, \ldots, m_k)$ for $i \neq j$, and
9. $p_{e_i}(m_1, \ldots, m_k) \neq -p_{e_j}(m_1, \ldots, m_k)$ for $i \neq j$.

The remaining part of the proof is identical to that of Lemma 1. □

Here comes the two theorems that tell us what particular types of infinite graphs can have a bijective graceful $\mathbb{N}$-labeling or a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.

Theorem 1. *Let $\{G'_n\}$ be an infinite sequence of finite graphs. Let $G_1 = G'_1$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n\#G'_{n+1}$. If there are infinitely many type-1 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{N}$-labeling.*

Proof.

Since there are infinitely many type-1 amalgamations, without loss of generality, we can assume that every amalgamation is a type-1 amalgamation.

Let f_1 be an injective graceful $\mathbb{N}$-labeling of G_1. This is always possible by using labels such as $\{1, 2, 2^2, \ldots\}$. Let V_1 and E_1 be the set of vertex and edge labels of G_1 respectively. Assume $1 \in V_1$. Let $m_2 = \min\{\mathbb{N} \setminus V_1\}$. We form a type-1 amalgamated graph $G_2 = G_1 \# G'_2$ by identifying a vertex v_1 of G_1 with a vertex v'_2 of G'_2. Suppose u_2 is a vertex in G'_2 not adjacent to v'_2. By Lemma 1, we can extend f_1 to an injective graceful $\mathbb{N}$-labeling f_2 of G_2 such that $f_2(u_2) = m_2$.

Let V_2 and E_2 be the set of vertex and edge labels of G_2 respectively. Let $m_3 = \min\{\mathbb{N} \setminus V_2\}$. Proceed as above, we can extend f_2 to an injective graceful $\mathbb{N}$-labeling f_3 of $G_3 = G_2 \# G'_3$ such that $m_3 \in f_3(V(G_3))$. By repeating the above process indefinitely, we obtain an injective graceful $\mathbb{N}$-labeling of $\lim_n G_n$.

Now denote the set of vertex labels of G_i by V_i. Let $m_1 = 1$ and $m_{i+1} = \min\{\mathbb{N} \setminus V_i\}$. By the construction above, $m_i \in V_i$. To prove that f is surjective, it suffices to show that $\{1, 2, \ldots, n\} \subset V_n$. Notice that $V_1 \subset V_2 \subset V_3 \subset \ldots$. Obviously, $\{1\} \subset V_1$. Now suppose $\{1, 2, \ldots, k\} \subset V_k$. If $k + 1 \in V_k$, then $k + 1 \in V_{k+1}$ and we are done. If $k + 1 \notin V_k$, then $m_{k+1} = \min\{\mathbb{N} \setminus V_k\} = k + 1$ and we have $k + 1 = m_{k+1} \in V_{k+1}$. By induction, $\{1, 2, \ldots, n\} \subset V_n$. Hence, f is surjective and the proof is complete. □

Theorem 2. *Let $\{G'_n\}$ be an infinite sequence of finite graphs. Let $G_1 = G'_1$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 and type-2 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.*

Proof. Since there are infinitely many type-1 and type-2 amalgamations, without loss of generality, we can assume that the amalgamation process alternates between type-2 and type-1 amalgamations indefinitely.

Let f_1 be an injective graceful $\mathbb{N}$-labeling of G_1 (e.g. choose labels from $\{1, 2, 2^2, \ldots\}$), and V_1 and E_1 be the set of vertex and edge labels of G_1 respectively. Without loss of generality, assume $1 \in V_1$. Let $n_2 = \min\{\mathbb{N} \setminus E_1\}$. We form a type-2 amalgamated graph $G_2 = G_1 \# G'_2$ by identifying a vertex of v_1 of G_1 with a vertex v'_2 of G'_2. Suppose x_2y_2 is an edge in G'_2 such that x_2 and y_2 are different from v'_2. By Lemma 3, we can extend f_1 to an injective graceful $\mathbb{N}$-labeling f_2 of G_2 so that $f_2(x_2y_2) = n_2$. Let V_2 and E_2 be the set of vertex and edge labels of G_2 respectively. If $1 \in E_1$, then $1 \in E_2$. Otherwise if $1 \notin E_1$, we have $n_2 = 1 \in E_2$.

Next form a type-1 amalgamated graph $G_3 = G_2 \# G'_3$ by identifying a vertex v_2 of G_2 with a vertex v'_3 of G'_3. Let $m_3 = \min\{\mathbb{N} \setminus V_2\}$. Suppose u_3 is a vertex in G'_3 not adjacent to v'_3. By Lemma 1, we can extend f_2 to an injective graceful $\mathbb{N}$-labeling f_3 of G_3 such that $f_3(u_3) = m_3$.

By repeating the above process indefinitely, we obtain an injective graceful $\mathbb{N}$-labeling

f of $\lim_n G_n$. Now it remains to show that f is surjective. First denote the set of vertex and edge labels of G_i by V_i and E_i respectively. Let $m_{i+1} = \min\{\mathbb{N} \setminus V_i\}$ and $n_{i+1} = \min\{\mathbb{N} \setminus E_i\}$. From the above construction, we have $n_{2k} \in E_{2k}$ and $m_{2k+1} \in V_{2k+1}$.

To prove the surjectivity of f, we will first show that $\{1, 2, \ldots, n\} \subset V_{2n-1}$. Note that $V_1 \subset V_2 \subset V_3 \subset \ldots$. From above, we have $\{1\} \subset V_1$. Now suppose that $\{1, 2, \ldots, k\} \subset V_{2k-1}$. If $k+1 \in V_{2k}$, then $k+1 \in V_{2k+1}$. Otherwise, $k+1 \notin V_{2k}$. But since $\{1, 2, \ldots, k\} \subset V_{2k-1} \subset V_{2k}$, we have $m_{2k+1} = \min\{\mathbb{N} \setminus V_{2k}\} = k+1 \in V_{2k+1}$ and $\{1, 2, \ldots, k+1\} \subset V_{2k+1}$. By induction, $\{1, 2, \ldots, n\} \subset V_{2n-1}$.

To show that $\{1, 2, \ldots, n\} \subset E_{2n}$, again notice that $E_1 \subset E_2 \subset E_3 \subset \ldots$. From above, we have $\{1\} \subset E_2$. Now suppose $\{1, 2, \ldots, k\} \subset E_{2k}$. If $k+1 \in E_{2k+1}$, then $k+1 \in E_{2k+2}$. Otherwise, $k+1 \notin E_{2k+1}$. But since $\{1, 2, \ldots, k\} \subset E_{2k} \subset E_{2k+1}$, we have $n_{2k+2} = \min\{\mathbb{N} \setminus E_{2k+1}\} = k+1 \in E_{2k+2}$. In either case, we have $\{1, 2, \ldots, k+1\} \subset E_{2k+2}$. Therefore, $\{1, 2, \ldots, n\} \subset E_{2n}$ by induction. We have thus shown that f is surjective and the proof is complete. □

5 Generalizations

As mentioned in [2], the amalgamation process described above can be generalized to one that identifies a finite set of vertices in one graph with a finite set of vertices in another graph. Based on this more general amalgamation, we can derive the more general versions of Theorem 1 and 2. As a result, we are able to prove the following two theorems which are important for the characterizations of graphs that have a bijective graceful $\mathbb{N}$-labeling and graphs that have a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.

Proposition 1. *Let G be an infinite graph. If every vertex of G has a finite degree, then G has a bijective graceful $\mathbb{N}$-labeling.*

Proof. We will show that such a graph can be constructed inductively by type-1 amalgamation. For $W \subset V(G)$, denote the neighbor of W (i.e. all vertices other than W that are adjacent to some vertex in W) by $N(W)$ and the subgraph of G induced by W by $G[W]$. Choose a vertex v_1 in G and let $G_1 = G'_1 = \{v_1\}$. Since the degree of v_1 is finite, $|N(G_1)|$ is finite. Therefore, there exists $v_2 \in G$ such that $v_2 \notin G_1 \cup N(G_1)$. Let $G'_2 = G[G_1 \cup N(G_1) \cup \{v_2\}]$. Form a type-1 amalgamated graph $G_2 = G_1 \# G'_2$ by identifying G_1. Interestingly, we have $G_2 = G'_2$. Now there exists $v_3 \notin G_2 \cup N(G_2)$. Let $G'_3 = G[G_2 \cup N(G_2) \cup \{v_3\}]$. Form a type-1 amalgamated graph $G_3 = G_2 \# G'_3$ by identifying G_2. By repeating the above process, we see that G_n is increasing and $G = \lim_n G_n$. Hence, by Theorem 1, G has a bijective graceful $\mathbb{N}$-labeling. □

Proposition 2. *Let G be an infinite graph with infinite number of edges. If every vertex of G has a finite degree, then G has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.*

Proof. The proof is similar to that of Proposition 1. Here we form both type-1 and type-2 amalgamations instead and apply Theorem 2. $\square$

Although our discussions so far only make use of $\mathbb{N}$ for graph labeling, all the above results still hold for any infinite torsion-free abelian group $\mathbb{A}$ (written additively). An abelian group $\mathbb{A}$ is torsion-free if for all $n \in \mathbb{N}$ and for all $a \in \mathbb{A}$, $na \neq 0$. Here, $na = a + \ldots + a$ (n times). In such general settings, the absolute difference will no longer be meaningful and we have to consider directed graphs instead. Denote the directed edge from x to y by xy. Let $f(x)$ and $f(y)$ be the vertex labels of x and y respectively. We will define the edge label for xy to be $f(y) - f(x)$. Now we are ready for the more general versions of Theorem 1 and 2 but first we need the following three lemmas.

Lemma 4. *Let $\mathbb{A}$ be an infinite torsion-free abelian group and $\mathbb{A}_0$ be a finite subset of $\mathbb{A}$. Then there exists $m \in \mathbb{A}$ such that for all $k \in \mathbb{Z}\backslash\{0\}$, $km \notin \mathbb{A}_0$.*

Proof. Let $B = \mathbb{A}_0 \cup -\mathbb{A}_0$. Since B is finite, there exists $a \in \mathbb{A}$ such that $a \notin B$. Consider $C = \{a, 2a, 3a, \ldots\}$ in which all elements are distinct as $\mathbb{A}$ is torsion-free. Now, only finitely many elements of C lie in B. Otherwise there exist $p < q$ such that $pa = qa \in B$ which is impossible as $\mathbb{A}$ is torsion-free. Similarly, only finitely many elements of $-C$ lie in B. Therefore, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $ka \notin B$ and $-ka \notin B$. Take $m = Na$. We have for all $k \in \mathbb{Z}\backslash\{0\}$, $km \notin B$ and hence $km \notin \mathbb{A}_0$. $\square$

Lemma 5. *Let $\mathbb{A}$ be an infinite torsion-free abelian group and $\mathbb{A}_0$ be a finite subset of $\mathbb{A}$. Consider, for $m_1, \ldots, m_k$ in $\mathbb{A}$, the $5^k - 1$ non-trivial expressions of the form $L_j(m_1, \ldots, m_k) = a_{j1}m_1 + \ldots + a_{jk}m_k$ where each a_{ij} is $-2, -1, 0, 1, 2$. Then there exists a choice of $m_1, \ldots, m_k$ in $\mathbb{A}$ such that no $L_j(m_1, \ldots, m_k)$ is in $\mathbb{A}_0$.*

Proof. The proof is identical to that of Lemma 2. Here we use Lemma 4 to make sure that we can choose m so that $-2m, -m, m, 2m$ are all outside $\mathbb{A}_0$. $\square$

Lemma 6. *Let $\mathbb{A}$ be an infinite abelian group and suppose $m \in \mathbb{A}$. Then there exists infinitely many pairs $x, y \in \mathbb{A}$ such that $x - y = m$.*

Proof. Obvious. For each $y \in \mathbb{A}$, choose $x = y + m$. $\square$

Using Lemma 5 and 6, we can obtain results similar to Lemma 1 and 3 for any infinite torsion-free abelian group. The reason is that the polynomials we are dealing with are of the form described in Lemma 5. Lemma 6 ensures that we can choose m_x and m_y as desired for Lemma 3. As a result, we have the following generalizations of Theorem 1 and 2.

Theorem 3. *Suppose $\mathbb{A}$ is an infinite torsion-free abelian group. Let $\{G'_n\}$ be an infinite sequence of finite graphs. Let $G_1 = G'_1$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{A}$-labeling.* □

Theorem 4. *Suppose $\mathbb{A}$ is an infinite torsion-free abelian group. Let $\{G'_n\}$ be an infinite sequence of finite graphs. Let $G_1 = G'_1$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 and type-2 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{A}/\mathbb{A} \setminus \{0\}$-labeling.* □

We can generalize even further by examining the so-called bijective graceful V or V/E-labeling where V and E are infinite subsets of an infinite abelian group. To illustrate this idea, let us consider an infinite graph with a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. Now multiply each vertex label by q and then add r to it where $0 \leq r < q$. The result is a bijective graceful $q\mathbb{N}+r/q\mathbb{N}$-labeling of the original graph. The reverse process can also be performed. This shows that bijective graceful $\mathbb{N}/\mathbb{N}$-labeling and $q\mathbb{N}+r/q\mathbb{N}$-labeling are equivalent. We will demonstrate the usefulness of such general notion of graceful labeling in the next section.

6 Graceful Tree Theorem for Infinite Trees

In this section, we make use of the tools developed earlier and characterize all infinite trees that have a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. This in turn solves the Graceful Tree Conjecture for infinite trees. Let us start off with the following two propositions.

Proposition 3. *Let T be an infinite tree with a semi-infinite path. Then T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.*

Proof. The infinite tree T can be constructed inductively by the following procedure.

0. Order the vertices of T so that $V(T) = \{v_1, v_2, v_3, \ldots\}$ and v_1 is the starting vertex of the semi-infinite path. Let $T_1 = T[\{v_1\}]$. Set $i = 1$.

1. Consider the neighbor of T_i in T, $N(T_i)$, where the order of vertices is induced by $V(T)$. Choose the first vertex in $N(T_i)$ say u and let x be the unique neighbor of u in T_i.

2. If u is on the semi-infinite path, then x is the vertex immediately before u on the semi-infinite path. Let y be the vertex immediately after u on the semi-infinite path. Now amalgamate the path xuy to T_i by identifying x. This is a type-1 and type-2 amalgamation and we have $T_{i+1} = T_i \# xuy$.

3. If u is not on the semi-infinite path, then amalgamate the edge xu to T_i by identifying x. We have $T_{i+1} = T_i \# xu$.

4. $i = i + 1$. Goto step 1.

We will show that the above amalgamation process includes every vertex and edge of T eventually. Consider a vertex w in T. Since T is connected, there is a finite path connecting v_1 and w namely $v_1 = v_{a_1} v_{a_2} \dots v_{a_k} = w$. It is easy to see that after at most $\max\{a_1, \dots, a_k\} - 1$ iterations, w will appear in the amalgamated tree.

Consider an edge xy in T. Denote the unique path from p to q on T by pTq. Consider v_1Tx and v_1Ty and let z be the last vertex on v_1Tx that lies on v_1Ty. If $z \neq x$ and $z \neq y$, then $zTx + xy + yTz$ is a cycle in T which contradicts T is a tree. Therefore, $x \in v_1Ty$ or $y \in v_1Tx$. Without loss of generality, suppose $x \in v_1Ty$. Since x will appear in the amalgamated tree eventually, so will y and xy as T is a tree.

Now the presence of the semi-infinite path guarantees that there are infinitely many type-1 and type-2 amalgamations. Therefore, by Theorem 2, $T = \lim_n T_n$ has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. □

Proposition 4. *Let T be an infinite tree with at least 1 vertex of infinite degree denoted by v. If v has infinitely many neighbors of degree > 1, then T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.*

Proof. The infinite tree T can be constructed inductively by the following procedure.

0. Denote the set of neighbors of v with degree > 1 by N. Notice that $|N|$ is infinite. Order the vertices of T so that $V(T) = \{v_1, v_2, v_3, \dots\}$ and suppose that $v_1 = v$. Let $T_1 = T[\{v_1\}]$. Set $i = 1$.

1. Consider the neighbor of T_i in T, $N(T_i)$, where the order of vertices is induced by $V(T)$. Choose the first vertex in $N(T_i)$ say u and let x be the unique neighbor of u in T_i.

2. If $u \in N$, then $x = v$. Let y be a neighbor of u in T other than x. Amalgamate the path xuy to T_i by identifying x. This is a type-1 and type-2 amalgamation and we have $T_{i+1} = T_i \# xuy$.

3. If $u \notin N$, then amalgamate the edge xu to T_i by identifying x. We have $T_{i+1} = T_i \# xu$.

4. $i = i + 1$. Goto step 1.

As in the proof of Proposition 3, every vertex and edge of T will be included by the amalgamation process eventually. Now $|N| = \infty$ implies that there are infinitely many type-1 and type-2 amalgamations. Therefore, by Theorem 2, $T = \lim_n T_n$ has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. □

We are now ready for the **Graceful Tree Theorem for Infinite Trees**.

Theorem 5. *Every infinite tree has a bijective graceful* $\mathbb{N}/\mathbb{N}$*-labeling except when the infinite tree does not contain any semi-infinite path, has more than one but finitely many vertices of infinite degree, and every infinite degree vertex has finitely many neighbors of degree greater than one.*

The proof will be divided into four cases: (i) Infinite tree with no infinite degree vertices, (ii) Infinite tree with exactly one infinite degree vertices, (iii) Infinite tree with more than one but finitely many infinite degree vertices, and (iv) Infinite tree with infinitely many infinite degree vertices.

(i) Infinite tree with no infinite degree vertices

Proposition 5. *Every infinite tree with all vertices of finite degree has a bijective graceful* $\mathbb{N}/\mathbb{N}$*-labeling.*

Proof. By Proposition 2. Another proof is by Proposition 3 and the following lemma.

Lemma 7. *Let T be an infinite tree in which every vertex has finite degree. Then T has a semi-infinite path.*

Proof. Choose any vertex v_0 of T. Since T is infinite, there exist infinitely many paths starting from v_0 namely one to each vertex. As there are only finitely many edges leaving v_0, there is a vertex v_1 such that infinitely many paths start with the edge v_0v_1. Now the same reasoning shows that there is a vertex v_2 such that infinitely many paths start with the path $v_0v_1v_2$. We can define a sequence $\{v_n\}$ inductively in this way and this sequence defines a semi-infinite path from v_0. ☐

(ii) Infinite tree with exactly one infinite degree vertices

Lemma 8. *Every finite tree T has a V/E-labeling. Here $V = \{n_1, n_2, \ldots, n_k\}$ and $E = \{n_1, n_2, \ldots, n_{k-1}\}$ where k is the number of vertices of T and $n_1 < n_2 < \ldots < n_{k-1} < n_k$ are to be determined.*

Proof. Pick any vertex $v \in V(T)$ to be the root of T. Let $S(l)$ be the set of vertices in T that are of distance l from v. Let $T(l)$ be the subtree induced by the vertices of distance $\leq l$ from v.

Label v by 1. We want to obtain a labeling of $T(1)$ that satisfies the condition described by the lemma. To this end, we multiply the label of v by $2p$ where p is a sufficiently large odd number. Now the vertices of $S(1)$ can be labelled using $\{1, 3, \ldots, 2p-1\}$ with the rule that if x is used, then so is $2p - x$. Also if $|S(1)|$ is odd, then p is used. Notice that the label of $T(0)$ is even while the labels of $S(1)$ are all odd.

Now suppose we have obtained a labeling for $T(l)$ such that the labels of $T(l-1)$ are all even and the labels of $S(l)$ are all odd. We would like to extend it to $T(l+1)$. Again, the idea is to multiply the labels of $T(l)$ by $2q$ where q is a sufficiently large odd number and choose the labels for $S(l+1)$ using appropriate odd numbers.

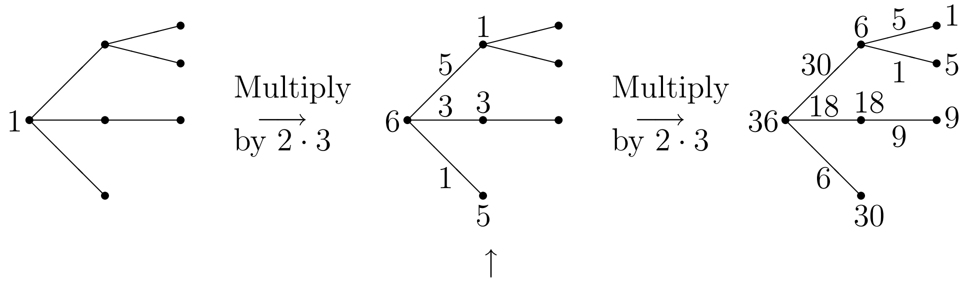
Let $v_1, v_2, \ldots, v_s$ be the vertices of $S(l)$ and their respective labels be $x_1, x_2, \ldots, x_s$ which are all odd. For each v_i, let $v_{i1}, v_{i2}, \ldots, v_{it_i}$ be its neighbors in $S(l+1)$. Multiply the labels of $T(l)$ by $2q$ where q is an odd number to be determined. The labels of $T(l)$ now become all even and still satisfy the condition stated in the lemma. In particular, the labels for $v_1, v_2, \ldots, v_s$ now become $2qx_1, 2qx_2, \ldots, 2qx_s$.

Observe that the set of $2t_i + 1$ consecutive integers $\{qx_i - t_i, \ldots, qx_i - 1, qx_i, qx_i + 1, \ldots, qx_i + t_i\}$ contains at least t_i odd numbers. The labels of $v_{i1}, v_{i2}, \ldots, v_{it_i}$ can then be chosen from these odd numbers according to the rule: If x is used, so is $2qx_i - x$. If t_i is odd, then qx_i is used.

Finally, to ensure the feasibility of the labeling, we require that: $0 < qx_1 - t_1, qx_1 + t_1 < qx_2 - t_2, \ldots, qx_{s-1} + t_{s-1} < qx_s - t_s$ or equivalently $q > \frac{t_1}{x_1}, q > \frac{t_2+t_1}{x_2-x_1}, \ldots, q > \frac{t_s+t_{s-1}}{x_s-x_{s-1}}$ which is always possible by choosing a sufficiently large odd number q. Hence we obtain a labeling of $T(l+1)$ satisfying the condition of the lemma.

By repeating the above procedure, we obtain a V/E-labeling of T with the desired properties. Note that the root of T, v, is labelled by n_k. $\square$

We illustrate the above labeling procedure by the following example.



$$x_1 = 1,\ t_1 = 2,\ q > \tfrac{t_1}{x_1} = \tfrac{2}{1} = 2$$
$$x_2 = 3,\ t_2 = 1,\ q > \tfrac{t_2+t_1}{x_2-x_1} = \tfrac{2+1}{3-1} = \tfrac{3}{2}$$
$$x_3 = 5,\ t_3 = 0,\ q > \tfrac{t_3+t_2}{x_3-x_2} = \tfrac{0+1}{5-3} = \tfrac{1}{2}$$

Therefore, we can choose $q = 3$.

Figure 12

Proposition 6. *An infinite star with its center attached to a finite number of finite trees each by an edge has a bijective graceful* $\mathbb{N}/\mathbb{N}$*-labeling.*

Proof. Denote the infinite star by S, its center by v, and the finite trees by $T_1, T_2, \ldots, T_m$. Each T_i is joined to S by an edge vu_i where $u_i \in V(T_i)$. The resulting infinite tree T can be described as $S \cup (\cup_{i=1}^m T_i) + \sum_{i=1}^m vu_i$. By taking u_i to be the root of T_i, we can obtain a V_i/E_i-labeling of T_i for $i = 1, \ldots, m$ as described in Lemma 8. Hence we have a $(c_iV_i + 1)/c_iE_i$-labeling of T_i. Choose c_i's so that the sets $\{c_iV_i + 1\}_{i=1}^m$ are pairwise disjoint. This is always possible by choosing c_i sequentially with each one sufficiently larger than the previous one. Now label the center of the star by 1 and the leaves of the star by $\mathbb{N} \setminus \cup_{i=1}^m (c_iV_i + 1) \cup \{1\}$. The result is a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling of T. $\square$

Proposition 7. *Every infinite tree with exactly one vertex of infinite degree has a bijective graceful* $\mathbb{N}/\mathbb{N}$*-labeling.*

Proof. Let T stand for the infinite tree and v be the vertex of infinite degree. Denote the set of neighbors of v with degree > 1 by N.

If T has a semi-infinite path, then by Proposition 3, T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.

Suppose T does not have a semi-infinite path. If $|N|$ is finite, then T is an infinite star with its center attached to a finite number of finite trees each by an edge. By Proposition

6, T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. If $|N|$ is infinite, then T is an infinite tree with one infinite degree vertex v where v has infinitely many neighbors of degree > 1. By Proposition 4, T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. □

(iii) Infinite tree with more than one but finitely many infinite degree vertices

Lemma 9. *Let G be an amalgamation of a finite graph $G_1 = (V_1, E_1)$ and k infinite stars by identifying k distinct vertices of G_1 with the k centers of the stars. Suppose that $|E_1| \geq |V_1| - 1$ and G has a bijective graceful $\mathbb{N}$-labeling. Then $k = 1$, the center of the infinite star is labelled 1, and $|E_1| = |V_1| - 1$.*

Proof. Denote the k centers by $v_1, v_2, \ldots, v_k$. Consider v_1 and take a vertex v adjacent to v_1 such that the label of v is greater than that of any vertices of G_1. Let the label of v be n. Consider the subgraph H of G induced by the vertices labelled $\{1, 2, \ldots, n\}$. Let n_i be the number of common edges between H and the infinite star centered at v_i. We have $|V(H)| = n = |V_1| + n_1 + \ldots + n_k$ and $|E(H)| = |E_1| + n_1 + \ldots + n_k$. Hence $|E(H)| \geq |V(H)| - 1$ as $|E_1| \geq |V_1| - 1$. Since the edge labels of H are all distinct, $|E(H)|$ must be less than $|V(H)|$ implying that $|E_1| < |V_1|$. So $|E_1| = |V_1| - 1$ and $|E(H)| = |V(H)| - 1$. Now H has $n - 1$ edges which must be labelled by $\{1, 2, \ldots, n-1\}$. The edge labelled $n - 1$ must be incident with the two vertices labelled 1 and n. Since the vertex labelled n is v, v_1 is labelled 1. However, the same argument also implies that v_2 is labelled 1 if $k > 1$ which is a contradiction. So $k = 1$. □

Proposition 8. *Every infinite tree with more than one but finitely many vertices of infinite degree has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling except for the case when the tree does not contain any semi-infinite path and every infinite degree vertex has finitely many neighbors of degree greater than one.*

Proof. Let T be the infinite tree and U be the set of vertices of infinite degree. If T has a semi-infinite path, then by Proposition 3, T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling.

Suppose T does not have a semi-infinite path. If T has a vertex v of infinite degree such that v has infinitely many neighbors of degree > 1, then by Proposition 4, T has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. Now suppose every vertex of infinite degree has finitely many neighbors of degree > 1. Remove from T all degree 1 neighbors of v for all $v \in U$. The resulting graph T' is a finite tree. This means that T is an amalgamation of T' and $|U|$ infinite stars by identifying $|U|$ vertices of T' with the $|U|$ centers of the stars. By Lemma 9, T does not have a bijective graceful $\mathbb{N}$-labeling. □

(iv) Infinite tree with infinitely many vertices of infinite degree

Proposition 9. *Every infinite tree with infinitely many vertices of infinite degree has a bijective graceful* $\mathbb{N}/\mathbb{N}$*-labeling.*

Proof. The infinite tree T can be constructed by the following procedure.

0. Order the vertices of T so that $V(T) = \{v_1, v_2, v_3, \ldots\}$. Denote the set of vertices of infinite degree by U. Let $T_1 = T[\{v_1\}]$. Set $i = 1$.

1. Consider the neighbors of T_i, $N(T_i)$, where the order of the vertices is induced by $V(T)$. Choose the first vertex in $N(T_i)$ say u and let x be the unique neighbor in T_i adjacent to u.

2. If $u \in U$, then there exists a vertex y other than x that is adjacent to u. Amalgamate the path xuy to T_i by identifying x. This is a type-1 and type-2 amalgamation and we have $T_{i+1} = T_i \# xuy$.

3. If $u \notin U$, then amalgamate the edge xu to T_i by identifying x. We have $T_{i+1} = T_i \# xu$.

4. $i = i + 1$. Goto step 1.

As in the proof of Proposition 3, every vertex and edge of T will be included by the amalgamation process eventually. Since T has infinitely many vertices of infinite degree, this guarantees that infinitely many type-1 and type-2 amalgamations will occur. By Theorem 2, $T = \lim_n T_n$ has a bijective graceful $\mathbb{N}/\mathbb{N}$-labeling. □

The proof of the Graceful Tree Theorem for Infinite Trees is therefore complete.

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