Modeling Default Data via an Interactive Hidden Markov Model

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Abstract

In this paper, we first introduce the use of an Interactive Hidden Markov Model (IHMM) for modeling and analyzing default data in a sector. Under the IHMM, the transitions of the hidden risk states of the sector depend on the observed number of bonds in the sector that default in the current time step. This models the feedback effect of the number of defaults on the transitions of the hidden risk states. This feature seems to be more realistic and does not enjoy by the traditional HMMs. We then develop a "dynamic" version of the Binomial Expansion Technique (BET) modulated by the IHMM for modeling the occurrence of defaults of bonds issued by firms in the same sector. Under the BET modulated by the IHMM, the number of bonds defaulting in each time step follows a Markov-modulated binomial distribution with the probability of defaulting of each bond depending on the states of the IHMM, which represent the hidden risk states of the sector. Efficient estimation method will be presented for estimating the model parameters in the BET modulated by the IHMM. We shall compare the hidden risk state process extracted from the IHMM with that extracted from the BET modulated by HMM in order to illustrate the significance of the feedback effect using real data. We shall also present the estimation results for the BET modulated by the IHMM and compare them with those for the BET modulated by the HMM.

Keywords: Default Data, Hidden Markov Model (HMM), Interactive Hidden Markov Model (IHMM), Binomial Expansion Technique, Feedback Effect.

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1 Introduction

Modeling portfolio credit risk is an important and practically relevant issue in credit risk analysis. It is an important research topic among academics and financial practitioners over the past few years. One important issue in modeling portfolio credit risk is to develop an appropriate model to quantify interaction effects of the credit risky entities within a portfolio. In practice, the Binomial Expansion Technique (BET) model developed by Moodys has been widely adopted in finance and insurance industries. In the BET model, it is supposed that the number of defaults within a bond portfolio in each time period is binomially distributed. The default events of the bonds in the portfolio are independent and the probability of defaulting of each bond remains the same across the bonds in the portfolio over time. In order to deal with the dependence of the defaults, the concept of diversify score has been introduced in the BET model, which groups the dependent bonds together and shrinks the number of dependent surviving bonds in each time step as the number of independent credit risky entities under which the BET model can be applied. Later Davis and Lo [4, 5] provide extensions to the BET model to model dependence of defaults and the default correlation in a bond portfolio. They focus on modeling the cross-sectional dependence of the defaults. Woo and Siu (2004) [16] model the temporal dependence of the default probabilities and introduced the use of the Bayesian filtering model for modeling the dynamics of the varying and unknown default probabilities.

In a recent paper by Giampieri et al. [8], a Hidden Markov model (HMM) is proposed for modeling the occurrence of defaults within a sector of bonds. They assume that the number of defaults within the sector / portfolio in each time step follows a binomial model with the default probability of each bond depending on the hidden states of a HMM, which are interpreted as risk states. They supposed that the risk state is common to all bonds within one particular industry, sector or region. They then employ the standard maximum likelihood estimation techniques associated with the Baum-Welch algorithm [9] developed by Baum and Welch for signal processing to estimate the model parameters. Their model can also be extended to include independent hidden risk sequences, which can then be used to incorporate the risk associated with the business cycle from the specific to the individual sector. In practice, the risk state of the bonds within the same industry or sector may depends on the financial performance of the issuing firms within that industry or sector and the number of defaults of the bonds within the industry contains important information about the financial performances of the issuing within that industry. It is reasonable to consider the impact of the number of the defaults of the bonds within the industry or sector on the common risk state of the bonds. In other words, the causality effect between the number of defaults of the bonds and the risk state of the bonds are in both direction and hence the feedback effect of the number of the defaults of the bonds on the risk state is incorporated. By incorporating this feedback effect, one can further explore the non-linear temporal behavior of the default data.

The concept of interactive hidden Markov models (IHMMs) was first introduced by Ching et al. [1, 2] in the classification of customers in a service sector. A threshold-type classifier has been obtained using the IHMM. The key idea of the IHMM is that the transitions of hidden states depend on the observable states. In a traditional HMM, the observable states are affected directly by the hidden states, but not vice versa, even though there are advanced higher-order HMMs [15]. Later Ching et al. [3] extend the IHMM to extract hidden economic conditions from observable economic data. The IHMM can be related to a discrete-time version of the class of the first-order Self-Exciting Threshold Auto-Regressive (SETAR) models first proposed by Tong [11, 12, 13, 14] for modeling non-linear time series taking numerical values. The class of SETAR models by Tong provide a piecewise linear approximation to a non-linear time series model by dividing the state-space into several regimes via the threshold principle. It provides a good first approximation to non-linear time series model beyond the class of ARMA models. The monograph by Tong [14] provides an excellent and original discussion of the SETAR models and other important non-linear time series models. The IHMM model can be considered as a discrete-state analogy of the class of the first-order SETAR models when the observable state can determine the hidden state with probability one. It can provide a first-order approximation of the non-linear behavior of categorical time series by dividing the state-space of the Markov chain process into several regimes, say two regimes.

In this paper, we first introduce the use of an Interactive Hidden Markov Model (IHMM) for modeling and analyzing default data in a sector. We then compare our results with the BET modulated by Hidden Markov models (HMMs) introduced by Giampieri et al. [8]. The IHMM is different from the HMMs in Giampieri et al. [8] in the assumption of the transitions of the hidden states. Under the IHMM, the transitions of the hidden risk states of the sector depend on the observed number of bonds in the sector that default in the current time step. In other words, the IHMM can feedback the effect of the number of the defaults on the hidden risk states. Hence, the causality effect between the number of defaults and the hidden risk states are in both directions. With the IHMM model, we can further explore the temporal non-linear behavior of the default data by dividing the state-space of the Markov chain into several regimes, let us say two regimes. We then develop a "dynamic" version of the Binomial Expansion Technique (BET) modulated by the IHMM for modeling the occurrence of defaults of bonds issued by firms in the same sector. Under the BET modulated by the IHMM, the number of bonds defaulting in each time step follows a Markov-modulated binomial distribution with the probability of defaulting of each bond depending on the states of the IHMM. As in Giampieri et al. [8], the states of the IHMM here represent the hidden risk states of the sector. Efficient estimation method will be presented for estimating the model parameters in the BET modulated by the IHMM. We shall compare the hidden risk state process extracted from the the BET modulated by IHMM with that extracted from the HMM in order to illustrate the significance of the feedback effect using real data. We shall also present the estimation results for the BET modulated by the IHMM and compare them with those for the BET modulated by the HMM.

This paper is structured as follows. In Section 2, we present the main idea of the IHMM of [3]. We also describe the estimation method for the parameters of the IHMM. In Section 3, we present a general set up for the BET modulated by the IHMM. We shall also discuss the estimation issue for the BET modulated by the IHMM through a numerical example. In Section 4, we shall examine the performance of the IHMM on extracting the hidden risk state process from the observed real default data and compare its performance with that of the HMM. We also present the estimation results of the BET modulated by the IHMM and compare them with those of the BET modulated by the HMM in Giampieri et al. [8]. We shall highlight the feedback effect presented in the IHMM. Finally, concluding remarks will be given in Section 5.

2 The Interactive Hidden Markov Model

In this section, we present the main idea of the IHMM and the estimation method for the model parameters [1, 2] through the following example.

Example 2.1: Suppose we are given an observed time series of the numbers of defaults, each of which takes values in the state-space $\{0, 1, 2\}$, as below:

$$0, 1, 1, 2, 1, 0, 0, 0, 1, 2, 0, 2, 1, \ldots.$$

Here, we assume the maximum number of defaults observed is 2 in the observed sequence. Suppose further that we have a hidden state variable with two states, namely, A (Normal risk) and B (Enhanced risk). When the hidden state variable takes the value A, the probability distribution of an observable variable is assumed to be:

$$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

When the hidden state variable takes the value B, the probability distribution of an observable variable is assumed to be:

$$(0, \frac{1}{2}, \frac{1}{2}).$$

We define an augmented Markov chain with the following states: $A, B, 0, 1$ and 2. We assume that when the observable state is i, the probabilities that the hidden state is A and B in next time step are given by α_i and $1 - \alpha_i$, respectively. The transition probability matrix governing the augmented Markov chain is then given by:

$$P_2 = \left(\begin{array}{cc|ccc} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \hline \alpha_0 & 1-\alpha_0 & 0 & 0 & 0 \\ \alpha_1 & 1-\alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 1-\alpha_2 & 0 & 0 & 0 \end{array}\right).$$

Hence, to define the IHMM, one has to estimate:

$$\alpha = (\alpha_0, \alpha_1, \alpha_2)$$

from an observed data sequence. One may consider the two-step transition probability matrix:

$$P_2^2 = \left(\begin{array}{cc|ccc} \frac{\alpha_0+\alpha_1+\alpha_2}{3} & 1-\frac{\alpha_0+\alpha_1+\alpha_2}{3} & 0 & 0 & 0 \\ \frac{\alpha_1+\alpha_2}{2} & 1-\frac{\alpha_1+\alpha_2}{4} & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{\alpha_0}{3} & \frac{1}{2}-\frac{\alpha_0}{6} & \frac{1}{2}-\frac{\alpha_0}{6} \\ 0 & 0 & \frac{\alpha_1}{3} & \frac{1}{2}-\frac{\alpha_1}{6} & \frac{1}{2}-\frac{\alpha_1}{6} \\ 0 & 0 & \frac{\alpha_2}{3} & \frac{1}{2}-\frac{\alpha_2}{6} & \frac{1}{2}-\frac{\alpha_2}{6} \end{array}\right).$$

One can then extract the one-step transition probability matrix of the observable states from P_2^2 as follows:

$$\tilde{P}_2 = \begin{pmatrix} \frac{\alpha_0}{3} & \frac{1}{2}-\frac{\alpha_0}{6} & \frac{1}{2}-\frac{\alpha_0}{6} \\ \frac{\alpha_1}{3} & \frac{1}{2}-\frac{\alpha_1}{6} & \frac{1}{2}-\frac{\alpha_1}{6} \\ \frac{\alpha_2}{3} & \frac{1}{2}-\frac{\alpha_2}{6} & \frac{1}{2}-\frac{\alpha_2}{6} \end{pmatrix}.$$

The rationale of considering the matrix $\tilde{P}_2$ is that it provides us with information of the one-step transition from one observable state to another one. Even though, in this case, we do not have a closed form solution for the stationary distribution of the process. There are three parameters to be estimated. To estimate the parameter α_i, we first estimate the one-step transition probability matrix from the observed sequence, which can be done by counting the transition frequency of the states in the observed sequence as in Ching et al. [2].

Suppose the estimate of $\tilde{P}_2$ is $\hat{P}_2$. It is expected that $\tilde{P}_2 \approx \hat{P}_2$ and, hence, α_i can be obtained by solving the minimization problem:

$$\min_{\alpha_i} ||\tilde{P}_2 - \hat{P}_2||_F^2$$

subject to the constraints:

$$0 \leq \alpha_i \leq 1 \quad \text{for } i = 0, 1, 2.$$

Here, $||.||_F$ is the Frobenius norm, i.e.

$$||A||_F^2 = \sum_{i=1}^{n} \sum_{i=1}^{n} A_{ij}^2.$$

We remark that other matrix norms can also be used as the objective function. We then consider the following two cases, namely, when the matrix P is known and when it is unknown.

From Example 2.1, the one-step transition probability matrix of the observable states is:

$$\tilde{P}_2 = \alpha P$$

(P is called the emission matrix) where

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{m2} & \cdots & \alpha_{nm} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{pmatrix},$$

i.e.

$$[\tilde{P}_2]_{ij} = \sum_{k=1}^{m} \alpha_{ik} p_{kj} \quad i, j = 1, 2, \ldots, n.$$

Here α_{ij} are unknown. We first consider the case that the probabilities p_{ij} are given [2]. Suppose $[Q]_{ij}$ is the one-step transition probability matrix estimated from the observed sequence. Then, for each fixed i, $\alpha_{ij}, j = 1, 2, \ldots, m$ can be obtained by solving the constrained least squares problem as follows:

$$\min_{\alpha_{ik}} \left\{ \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \alpha_{ik} p_{kj} - [Q]_{ij} \right)^2 \right\}$$

subject to the constraints:

$$\sum_{k=1}^{m} \alpha_{ik} = 1 \quad \text{and} \quad \alpha_{ik} \geq 0 \quad \text{for } i = 1, 2, \ldots, n.$$

Ching et al. [3] also provide a comprehensive discussion for the case when P is unknown. When all of the probabilities P_{ij} are unknown, we shall use the bi-level programming technique to estimate all unknown parameters (see [3] or Appendix A). Once the model parameters are available the most likely hidden state can be obtained [3]. We shall compare the IHMM with the HMM of Giampieri et al. [8] in detecting hidden risk state in Section 4.

3 The Binomial Expansion Techniques (BET) Modulated by the IHMM

In this section, we shall present a general framework for the BET modulated by the IHMM. Here, we consider the situation that there are m hidden common risk states to all of the bonds issued by firms in the same (industrial) sector and that there are n bonds in the sector; that is, the number of surviving bonds in the portfolio at the beginning of the first period $S_0 = n$. We present the general framework in the sequel.

We consider a discrete-time economy associated with the time index set

$$\mathcal{T} := \{0, 1, 2, \ldots, T\}.$$

Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure. The probability space is rich enough to model all uncertainties in the model. Let $X := \{X_t\}_{t \in \mathcal{T}}$ denote a hidden sequence $(\Omega, \mathcal{F}, \mathcal{P})$ with state-space

$$\mathcal{S} := \{s_1, s_2, \ldots, s_m\}$$

where $s_i \in \mathcal{R}^m$, for $i = 1, 2, \ldots, m$.

We interpret the states of X as the common risk states of the bonds in the sector. In particular, "1" represents the lowest risk state and "m" represents the highest risk state. Without loss of generality, we can take the state-space $\mathcal{S}$ for X to be the set $\{e_1, e_2, \ldots, e_m\}$ of unit vectors in $\mathcal{R}^m$. This is called the canonical representation of a Markov chain process introduced by Elliott [6] (see also Elliott et al. [7]). We suppose that X is modeled by an IHMM model which is described in the following.

Let $\{S_t\}_{t \in \mathcal{T}}$ denote a stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$, where S_t represents the number of surviving bonds at the end of the t^{th} period, for each $t \in \mathcal{T}$. Then, for each $t \in \mathcal{T} \backslash \{0, 1\}$ and each $k = 0, 1, 2, \ldots, S_{t-2}$, the transition probabilities of the hidden risk state process X are specified as below:

$$a_{ij}(k) = \mathcal{P}(X_t = e_j | X_{t-1} = e_i, M_{t-1} = k) \ , \tag{3.1}$$

and

$$A(k) = [a_{ij}(k)]_{1 \le i,j \le m}.$$

Here $\{M_t\}_{t \in \mathcal{T}}$ is a stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$ and M_t represents the number of defaults in the t^{th} period. Now, we describe the BET modulated by the IHMM for modeling the number of defaults of the bonds over time. Let $\mathcal{F}_t^S$ denote the $\mathcal{P}$-augmentation of the σ-algebra $\sigma\{S_0, S_1, \ldots, S_t\}$ generated by the process $\{S_t\}_{t \in \mathcal{T}}$ up to and including time $t \in \mathcal{T}$. $\mathcal{F}_t^S$ represents observable information up to and including time $t \in \mathcal{T}$. For each $t \in \mathcal{T}$, conditional on $\mathcal{F}_t^S$, M_{t+1} takes values in the set $\{0, 1, 2, \ldots, S_t\}$.

We define the probability of the default of each bond in the sector at time t as:

$$\Theta_t := \Theta(t, X_t) := \langle \boldsymbol{\Theta}, X_t \rangle = \sum_{i=1}^{m} \Theta_i \langle X_t, e_i \rangle \ , \tag{3.2}$$

where

$$\boldsymbol{\Theta} := (\Theta_1, \Theta_2, \ldots, \Theta_m) \in \mathcal{R}^m, \quad \Theta_i \in (0,1) \quad \text{for } i = 1, 2, \ldots, m$$

and

$$\Theta_1 < \Theta_2 < \cdots < \Theta_m.$$

For each $t \in \mathcal{T}$, given $\mathcal{F}_t^S$, the conditional probability distribution of M_{t+1} under $\mathcal{P}$ is given by the following regime-switching binomial distribution:

$$\begin{aligned} P(M_{t+1} = j | X_t, \mathcal{F}_t^S) &= P(M_{t+1} = j | X_t, S_t) & (3.3) \\ &= \binom{S_t}{j} (< \boldsymbol{\Theta}, X_t >)^j (1 - < \boldsymbol{\Theta}, X_t >)^{S_t - j} \\ &= \sum_{i=1}^{m} \binom{S_t}{j} (\Theta_i)^j (1 - \Theta_i)^{S_t - j} < X_t, e_i > \ , & (3.4) \end{aligned}$$

for $j = 1, 2, \ldots, S_t$.

The following example provides an illustration of the model. The version of the BET model presents in the example resembles that of Giampieri et al. [8], except that the IHMM is used to incorporate the feedback effect.

Example 3.1: Consider the situation that the hidden risk state takes two possible values, namely, "N" (normal risk) and "E" (enhanced risk). In the normal risk state, the number of observed defaults in each time step is modeled by a BET model with the default probability of a bond P_N. In the enhanced risk state, let P_E denote the default probability of a bond. Here $P_E > P_N$, which means that the probability of the default of a bond in the enhanced risk state is higher than that in the normal risk state.

Then, in the normal state N, given the number of surviving bonds at the current time is k, the conditional probability distribution of the number of defaults in the next period is:

$$P(m|Nk) = \binom{k}{m} (P_N)^m (1 - P_N)^{k-m} \ , \quad m = 1, 2, \ldots, k \ . \tag{3.5}$$

In the enhanced state E, the corresponding probability distribution is

$$P(m|Ek) = \binom{k}{m} (P_E)^m (1 - P_E)^{k-m} \ . \tag{3.6}$$

In the following example, we shall define an augmented Markov chain associated with the BET modulated by an IHMM with two hidden risk states. The state-space of the augmented Markov chain is formed by both hidden risk states and observable number of defaults. We obtain the transition probability matrix of the augmented Markov chain, which provides complete description of the probabilistic behaviors of the transitions of both observable states and hidden states. The transition probability matrix also plays a key role for the estimation of the unknown parameters of the IHMM.

Example 3.2: Consider the BET in Example 3.1 again. Suppose there are two surviving bonds at the beginning of the first period, i.e., $S_0 = 2$. In this case, there are three observable states $0, 1$ and 2 while there are six unobservable states (or more precisely, partially unobservable), namely, $N2, N1, N0, E2, E1$ and $E0$. Here, $N2$ represents that the credit risk is normal and there are two surviving bonds and $E0$ means that the credit risk is enhanced and there is no surviving bond. We shall describe the probabilistic behavior of the transitions of both observable states and unobservable states by an augmented Markov chain with the following 9 states:

$$\{N2, N1, N0, E2, E1, E0, 2, 1, 0\}.$$

We further suppose that when the observable state is k $(k = 0, 1, 2)$, the probabilities that the hidden state is N and E in next time step are given by α_k and $1 - \alpha_k$, respectively.

Then, the transition probability matrix governing the augmented Markov chain is:

$$P_2 = \left(\begin{array}{cccccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & P(0|N2) & P(1|N2) & P(2|N,2) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P(0|N1) & P(1|N1) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P(0|N0) \\
0 & 0 & 0 & 0 & 0 & 0 & P(0|E2) & P(1|E2) & P(2|E2) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P(0|E1) & P(1|E1) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P(0|E0) \\
\hline
\alpha_2 & 0 & 0 & 1-\alpha_2 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 1-\alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_0 & 0 & 0 & 1-\alpha_0 & 0 & 0 & 0
\end{array}\right).$$

Suppose

$$P_N = 0.15 \quad \text{and} \quad P_E = 0.3.$$

Then, the transition probability matrix is:

$$P_2 = \left(\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0.7225 & 0.2550 & 0.0225 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8500 & 0.1500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4900 & 0.4200 & 0.0900 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7000 & 0.3000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \\ \hline \alpha_2 & 0 & 0 & 1-\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 1-\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_0 & 0 & 0 & 1-\alpha_0 & 0 & 0 & 0 \end{array}\right).$$

3.1 The Estimation of the BET Modulated by the IHMM

To illustrate how to use the above estimation method to estimate the BET modulated by the IHMM, we first consider the case when P_N and P_E are known. To define the IHMM for the hidden risk states, one has to estimate $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ from the observed sequence of the default data. We consider the 3×3 sub-matrix related to the observable states of the two-step transition probability matrix P_2^2 as below:

$$\tilde{P}_2 = \begin{pmatrix} \alpha_2 P(0|N,2) + (1-\alpha_2)P(0|N,2) & \alpha_2 P(1|N,2) + (1-\alpha_2)P(1|N,2) & \alpha_2 P(2|N,2) + (1-\alpha_2)P(2|N,2) \\ 0 & \alpha_1 P(0|N,1) + (1-\alpha_2)P(0|N,1) & \alpha_1 P(1|N,1) + (1-\alpha_2)P(1|N,1) \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose $P_N = 0.15$ and $P_E = 0.3$. Then,

$$\tilde{P}_2 = \begin{pmatrix} 0.2325\alpha_2 + 0.4900 & -0.1650\alpha_2 + 0.4200 & -0.0675\alpha_2 + 0.0900 \\ 0.0000 & 0.1500\alpha_1 + 0.7000 & -0.1500\alpha_1 + 0.3000 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix}.$$

Note that α_0 is not important in the model and can be assumed to be 0. We also notice that $P(0|N0) = P(0|E0) = 1$.

To estimate the parameters α_i $(i = 1, 2)$ given P_N and P_E, we first estimate the one-step transition probability matrix from the observed sequence of default data. This can be done by counting the transition frequencies among the states in the observed sequence of default data and then follow by a normalization, see for instance [2, 3, 9]. Suppose that the estimated transition probability matrix is given by

$$\hat{P}_2 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

It is expected that $\tilde{P}_2 \approx \hat{P}_2$ and, hence, α_i can be obtained by solving the following minimization problem:

$$\min_{\alpha_i} ||\tilde{P}_2 - \hat{P}_2||_F^2$$

subject to:

$$0 \leq \alpha_i \leq 1 \ , \quad i = 1, 2 \ .$$

Here, $||.||_F$ represents the Frobenius norm. Solving the above optimization problem,

$$\alpha_1^* = 0.3333 \quad \text{and} \quad \alpha_2^* = 0.9602.$$

Hence,

$$P_2 = \left(\begin{array}{cccccc|ccc} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.7225 & 0.2550 & 0.0225 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.8500 & 0.1500 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4900 & 0.4200 & 0.0900 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.7000 & 0.3000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ \hline 0.9602 & 0.0000 & 0.0000 & 0.0398 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.3333 & 0.0000 & 0.0000 & 0.6667 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \end{array} \right) .$$

Now, we consider the situation that both P_N and P_E are unknown. In this case, we shall use the bi-level programming technique presented in Appendix A to estimate the model parameters for the BET modulated by the IHMM with two risk states. We shall present the estimation results of the IHMMs using the observed default data in the next section.

4 Numerical Examples and Comparison

In this section, we present the estimation results of the IHMMs using the observed default data in [8]. In [8], Giampieri et al. apply the HMM to the quarterly bond defaults data of four sectors (consumer, energy, media and transportation) in United States taken from Standard & Poors' ProCredit6.2 database. The data set covers the period from January 1981 to December 2002. The total number of bonds in January 1981 was 281 while the total number of bonds in December 2002 was 222. For the convenience of comparison, we extracted the credit default data and also the most likely hidden risk state directly from the figures in [8]. We then apply our IHMM to the extracted data and comparisons are made and the results are reported in Appendix B. The details of the computational procedures are given below. All computations were done on a Pentium 4HT PC with MATLAB.

Now, we need to set the initial probability for each sector and for each model. We assume that the initial probability matrices of the consumer sector for both the IHMM and the HMM are the same and that the common probability matrix is:

$$P^{(0)} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \end{pmatrix} \quad (4.1)$$

We recall that the number of observable states is equal to the maximum number of defaults observed plus one in the IHMM. The initial probability matrices of the energy sector and the transportation sector for both the IHMM and the HMM are supposed to be the same and the common probability matrix is:

$$P^{(0)} = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix} \quad (4.2)$$

The initial probability matrices of the media sector for both the IHMM and the HMM are assumed to be the same and the common probability matrix is:

$$P^{(0)} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix} \quad (4.3)$$

Figures 1-4 (see Appendix B) display the results of the most likely hidden risk state process extracted from the IHMM (Figures 1(b)-4(b)) and also those extracted from the figures in [8] (Figures 1(a)-4(a)) using the observed default data in the consumer/service sector, the energy and natural resources sector, the leisure time/media sector and the transportation sector, respectively. There are 13, 6, 8 and 6 observable states in the consumer/service sector, the energy and natural resources sector, the leisure time/media sector and the transportation sector, respectively.

[Figures 1-4 about here]

From Figures 1-4, we can see that our IHMM is more sensitive in detecting the changes in the hidden risk states than the HMM in [8] for the consumer/service sector, the energy and natural resources sector, the leisure time/media sector and the transportation sector. This reveals that the incorporation of the feedback effect by the IHMM can improve the ability in detecting the changes in the hidden risk states. As expected, the IHMM gives a threshold-type classification of the hidden risk states. For example, the IHMM classifies those periods having six or more defaults as enhanced risk in the consumer/service sector. The threshold values for the remaining three sectors are $3, 4$ and 3 defaults respectively. For the HMM, generally speaking, it is very unlikely to have two transitions of hidden risk states in three consecutive transitions. Therefore, the HMM might not be adaptive to the rapid changes of hidden risk states. This is consistent with the numerical examples in Figures 1(a)-4(a) in Appendix B.

Table 4.1.

			IHMM			HMM in [8]			
Sectors	Total	Default	α	P_N	P_E	q	p	P_N	P_E
Consumer	1041	251	0.63	0.0022	0.0075	0.95	0.81	0.0026	0.0159
Energy	420	71	0.68	0.0015	0.0085	0.95	0.88	0.0014	0.0099
Media	650	133	0.50	0.0015	0.0085	0.96	0.83	0.0027	0.0194
Transport	281	59	0.63	0.0017	0.0153	0.97	0.78	0.0025	0.0223

We then apply the BET modulated by the IHMM discussed in Section 3 to the default data again. In this case, since the number of model parameters α_i is much more than the data available, we assume that $\alpha_i = \alpha$, for all i, in the estimation. Based on the observed default data and the hidden risk state process for each sector extracted by our IHMM [3] the likelihood function or the joint probability distribution for the hidden risk states with the observed default data can be obtained in the following form:

$$\alpha^P(1-\alpha)^Q P_N^R(1-P_N)^S P_E^T(1-P_E)^U. \tag{4.4}$$

Here P, Q, R, T and U can be obtained from the observed default data. The estimates of all model parameters are then obtained by maximizing the above likelihood function (4.4).

The estimates of the parameters of the BET modulated by the IHMM and the BET modulated by the HMM are presented for the four sectors are presented in Table 4.1. Under the BET modulated by the HMM, the hidden risk state is assumed to follow a first-order Markov chain having the transition probability matrix:

$$\begin{pmatrix} q & 1-q \\ 1-p & p \end{pmatrix}.$$

Here q is the probability of remaining in the normal risk state while p represents the probability of remaining in the enhanced risk state. We observe that the default probabilities under the enhanced risk state estimated using the BET modulated by the HMM are always significantly greater (can be over 200% in the Media sector) than those estimated by BET modulated by IHMM. While the default probabilities under the normal risk state obtained by both of the models are more or less consistent with each other.

5 Concluding Remarks

We considered the IHMM and a BET modulated by an IHMM for modeling the occurrence of defaults of bonds issued by firms in the same sector. The main idea of the two models is to assume that the transitions of the hidden risk states of the sector depend on the current observed number of defaulting within the sector. We presented an efficient estimation method for the model parameters and an efficient method for extracting the most likely hidden risk state process. We conducted empirical studies on the models and compared the hidden risk state process extracted from the IHMM model with that extracted from the HMM using the real default data from Giampieri et al. [8]. We found that the incorporation of the interactive or feedback effect can provide a more sensitive way to detect the transitions in the hidden risk states.

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6 Appendix A

The bi-level optimization algorithm (Taken from [3]).

Initialize $p_{ij}^{(0)}$; $e = 1$; $h = 1$;

Solve $\alpha_{ik}^{(h)}$

$$\min_{\alpha_{ik}^{(h)}} \left\{ \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \alpha_{ik}^{(h)} p_{kj}^{(h-1)} - [Q]_{ij} \right)^2 \right\}$$

subject to:

$$\sum_{k=1}^{m} \alpha_{ik}^{(h)} = 1 \quad \text{and} \quad \alpha_{ik}^{(h)} \geq 0;$$

Solve $p_{ik}^{(h)}$

$$\min_{p_{ik}^{(h)}} \left\{ \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \alpha_{ik}^{(h)} p_{kj}^{(h)} - [Q]_{ij} \right)^2 \right\}$$

subject to:

$$\sum_{k=1}^{n} p_{ik}^{(h)} = 1 \quad \text{and} \quad p_{ik}^{(h)} \geq 0$$

While $e <$ tolerance,

$h := h + 1$;
Solve $\alpha_{ik}^{(h)}$

$$\min_{\alpha_{ik}^{(h)}} \left\{ \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \alpha_{ik}^{(h)} p_{kj}^{(h-1)} - [Q]_{ij} \right)^2 \right\}$$

subject to:

$$\sum_{k=1}^{m} \alpha_{ik}^{(h)} = 1 \quad \text{and} \quad \alpha_{ik}^{(h)} \geq 0$$

Solve $p_{ik}^{(h)}$

$$\min_{p_{ik}^{(h)}} \left\{ \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \alpha_{ik}^{(h)} p_{kj}^{(h)} - [Q]_{ij} \right)^2 \right\}$$

subject to:

$$\sum_{k=1}^{n} p_{ik}^{(h)} = 1 \quad \text{and} \quad p_{ik}^{(h)} \geq 0$$

$e := ||\alpha^{(h)} - \alpha^{(h-1)}||_2^2 - ||P^{(h)} - P^{(h-1)}||_2^2$;
end;

7 Appendix B

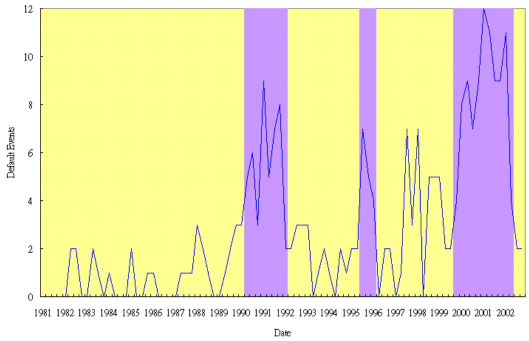

Figure 1(a): Consumer/service sector. (HMM in [8])

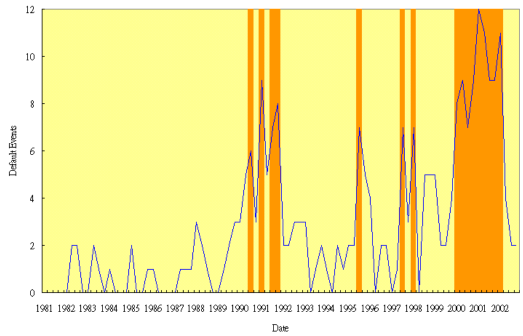

Figure 1(b): Consumer/service sector. (IHMM)

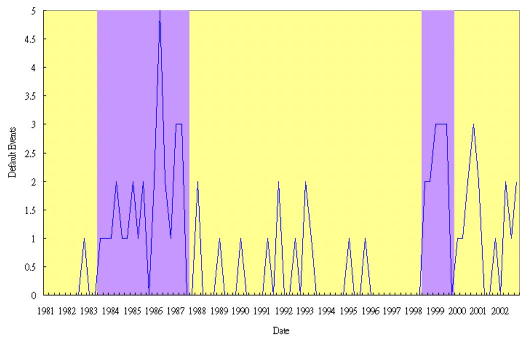

Figure 2(a): Energy and natural resources sector. (HMM in [8])

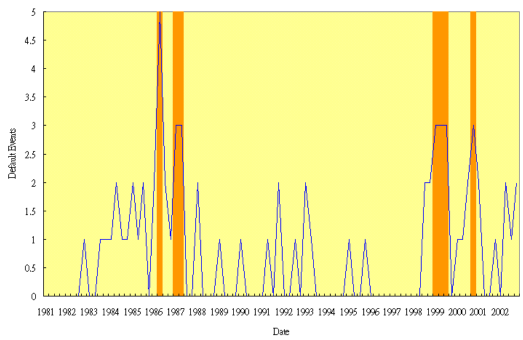

Figure 2(b): Energy and natural resources sector. (IHMM)

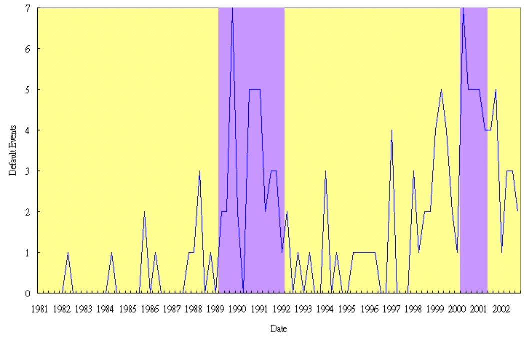

Figure 3(a): Leisure time/media sector. (HMM in [8])

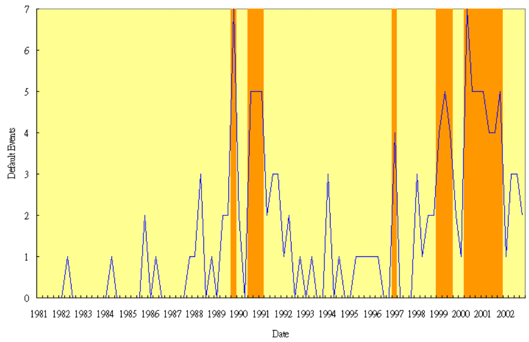

Figure 3(b): Leisure time/media sector. (IHMM)

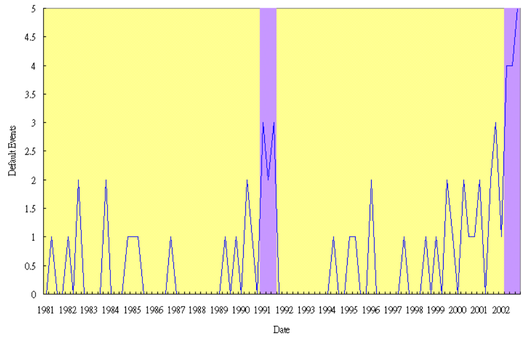

Figure 4(a): Transportation sector. (HMM [8])

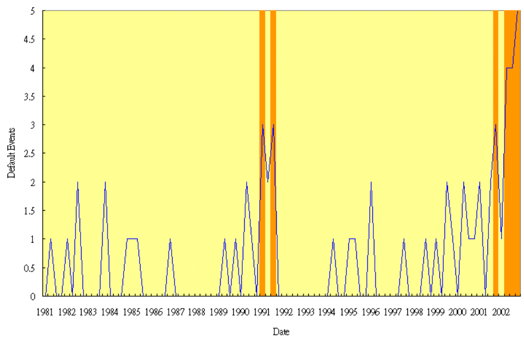

Figure 4(b): Transportation sector. (IHMM)