A Stochastic Optimization Model for Consecutive Promotion

Ho-Yin Leung Wai-Ki Ching
Advanced Modeling and Applied Computing
Department of Mathematics
University of Hong Kong
Pokfulam Road, Hong Kong, China.
E-mail: {obliging, wching} @hkusua.hku.hk

Issic K.C. Leung
Department of MSST
The Hong Kong Institute of Education
Hong Kong, China.
E-mail: ikcleung@ied.edu.hk

2 March 2008

Abstract

Nowadays in business environment, marketing competitiveness is as demanding as ever. To survive under keen competitions, industries must keep acquiring customers and make them loyal while maximizing profit from their service subscription or product purchasing. Intensive research works have been done in answering when and what kind of promotions should be used under limited marketing communication resources to maintain a perpetual generation of revenue. In this paper, we investigate the advantages in consecutive promotion based on the framework of the model proposed in Ching et al. [1]. The customers' behavior is modelled by using a Markov chain and we aim at maximizing the expected profit using stochastic dynamic programming. We find that a multi-period promotion strategy is better then the strategy of applying several single-period promotions in our tested examples.

Key Words: Customer Behavior, Consecutive Promotion, Stochastic Dynamic Programming, Markov Process.

1 Introduction

The competitions among providers in most servicing industries are keen nowadays. Every company wants to maintain good relationship with their customers. This is usually done by providing quality service and by making prompt promotion. However, too many promotions do no good as there lay a cost burden to the company. Researchers have noticed that promotion causes a short-term growth in sales but it will diminish in the long run, see for instance [5]. Moreover, companies often have limited resources and they cannot conduct promotion all the time. In this paper, we are interested in devising an optimal promotion plan so that the promotion resources can be utilized fully and effectively. To tackle the problem, stochastic models, in particular Markovian stochastic models [7] have been proved to be useful and effective. Ching et al. [1] proposed a Markov decision approach for studying the optimal promotion policy. Esteban-Bravo [3] applied dynamic programming with the Markovian process to determine the promotion duration. They provided a practical approach for the problem. Later Lin and Lin [6] considered a Markovian stochastic for modeling the promotion duration for two competitive brands. They give both explanation of profit function and explicit algorithm for solving the optimal promotion duration.

One important step is to classify the customers by their degree of purchase (different states). Then we can find out a plan which maximizes the total expected profit, i.e., revenue minus promotional cost, using stochastic dynamic programming. In [1], Ching et al. give a comprehensive discussion under this normal setting. They propose a stochastic dynamic programming model with the Markov chain to capture the customer behaviour. The advantage of Markov chain model is that it can take into the account of the switching behaviour of the customers between the company and its competitors. Thus the customer-company relationship can be described in a stochastic setting which is more realistic, see for instance Pfeifer and Carraway [8]. Stochastic dynamic programming is then applied to solve the optimal allocation of promotion strategy with practical data in a computer services company. In [6], the authors used Markov chain, entropy and diffusion theory to study the problem of promotion duration during the transition state of customers' switching between different brands. Using Taguchi method, they are able to capture the uncertain parameters of the model to determines optimal promotion duration. We will base on the model and data in [1] and extend the model by adopting an additional consideration, the multi-period promotion proposed in [6]. It is a usual practice to decide whether to offer promotion and the type of promotion at the start of each time unit, and all promotions last for a time unit. A multi-period promotion refers to promotion that lasts for $2, 3, \ldots, R$ time units. This encourages more purchases or continuous subscriptions than a single-period promotion. We assume that customers are in the same state (not referring to transition) even if they purchase more. Hence a greater revenue is expected.

The degree of increase in revenue of each period during the r-period of consecutive promotions can be described by the following formula:

$$1 + m(1 - e^{-ni}) \quad i = 1, 2, \ldots, r \tag{1}$$

where $m \geq 0$ denotes the ceiling level, $n > 0$ determines the growth rate of the exponential curve, and $r \geq 2$, an integer denotes a r-period promotion. The increase is given to every single period and still applies if the customer changes to other consumption level (different state) under the consecutive promotion. Jain and Singh [4] suggested that the optimal acquisition and retention spending should follow the second term of the formula. We use the formula because we have reason to believe that a multi-period promotion has a significant advantage compared to a single-period one at the beginning, but then the revenue increase declines as the length of promotion increases and almost stops if the steady state of the market is reached.

We remark that other revenue adjustment can be used. Take for example

- Linear function: $1 + mr/n$, or $1 + m \min\{1, r/n\}$ which sets an upper bound;
- S-shaped function: $m((nr + d_1)^{1/3} + d_2)$, where d_1, d_2 are shifts.

We are going to find the maximum expected profit using stochastic dynamic programming, where both finite and infinite horizon will be considered.

The rest of the paper is organized as follows. In Section 2, we review the Markov chain model discussed in [1] for customers' behavior and introduce the idea of multi-period promotion. In Section 3, we present the dynamic programming approach for the captured promotion problem. Numerical results are also given to demonstrate the method. Finally, a summary is given in Section 4 to conclude the paper.

2 A Markov Model for Customers' Behavior

In this section, customers are classified into N different states (namely $1, 2, \ldots, N$) in the Markov model according to their consumption level. The data we considered here is taken from [1]. The customer consumption data is stored in the company database and can be collected weekly. In the following, we will take the number of states to be four ($N = 4$) where States $1, 2, 3$ are low-usage, medium-usage and high-usage customers respectively, and, to make our consideration complete, State 4 is used to indicate non-customers (no usage). Thus a customer is said to be in State 4, if he or she is either a customer of the competitor company or did not subscribe any of the service during the period. It is then clear to see that at any time a customer belongs to exactly one of the states in $\{1, 2, 3, 4\}$ in our problem. With these notations, a Markov chain model is a good approach to capture the transitions of customers among the states. In fact, the Markov chain model can be characterized by an 4×4 transition matrix P. Here $P_{ij}(i, j = 1, 2, 3, 4)$ is the transition probability that a customer will make a transition to State j in the next period given that the current state is i. Thus a customer in State i will stay with the same state in the next period (retention probability) is given by P_{ii}. It can be shown that if the underlying Markov chain is irreducible then the stationary distribution $\mathbf{p}$ exists. This means that there is an unique probability vector $\mathbf{p} = (p_1, p_2, p_3, p_4)$ such that

$$\mathbf{p} = \mathbf{p}P, \quad \sum_{i=1}^{4} p_i = 1, \quad p_i \geq 0. \tag{2}$$

By making use of the stationary distribution $\mathbf{p}$, one can compute the retention probability and the customer lifetime value [1]. With the help of stochastic dynamic programming, one can also obtain the optimal promotion strategy [1]. For more details about Markov chain model, we refer readers to the book [9]. We will apply the model to a computer service company where the data is taken from Ching et al. [1]. The duration of data available was 20 weeks. The company had a promotion (e.g. special price or rate offer) for the first 8 consecutive weeks and no promotion thereafter. The customers are classified by their service time consumed as follows:

State	1	2	3	4
Minutes	$1-20$	$21-40$	>40	$=0$

Table 1: The Classification of Customers (taken from [1])

The transition probability under promotion, $P^{(1)}$, and that under no promotion, $P^{(2)}$, are respectively given by (taken from [1])

$$P^{(1)} = \begin{pmatrix} 0.4230 & 0.0992 & 0.0615 & 0.4163 \\ 0.3458 & 0.2109 & 0.2148 & 0.2285 \\ 0.2147 & 0.2034 & 0.4447 & 0.1372 \\ 0.1489 & 0.0266 & 0.0191 & 0.8054 \end{pmatrix}$$

and

$$P^{(2)} = \begin{pmatrix} 0.4146 & 0.0623 & 0.0267 & 0.4964 \\ 0.3837 & 0.1744 & 0.1158 & 0.3261 \\ 0.2742 & 0.2069 & 0.2809 & 0.2380 \\ 0.1064 & 0.0121 & 0.0053 & 0.8762 \end{pmatrix}.$$

We note that the revenue was much higher when no promotion was given due to the fact that a big discount was given to the customers in promotion period. However, it does not imply that it must be a disadvantage to carry out promotion because it can be compensated by new customers attracted. The following table gives the data.

State	1	2	3	4
Promotion	6.97	18.09	43.75	0.00
No promotion	14.03	51.72	139.20	0.00

Table 2: The Average Revenue of Customers (taken from [1])

In our revenue adjustment mechanism, we have to determine the values of m and n. We will try different values of the ceiling level of adjustment, m, in our calculations, however for n, we proposed that the increase nearly stops if it has reached the steady state. With regard to this, we calculate the sum of the absolute of the differences entry-wise between P^{100} and P^r. When it is smaller than 0.01, then we can say it reaches the "steady state" (roughly speaking) after r steps. Using this r, we want to find the value of n such that e^{-nr} is smaller than 0.05 (hence the increase is almost ceased). We remark that given an ergodic irreducible Markov chain, the process will converge to its unique stationary distribution. It is well known that the convergence rate depends on the modulus of the second largest eigenvalue $|\lambda_2| < 1$ of the transition probability matrix [2]. Roughly speaking, this means the error between P^n and P^∞ will decay at a rate proportional to $|\lambda_2|^n$ (exponentially decay). Here $P^\infty = \mathbf{1}^t\mathbf{p}$ where $\mathbf{1}$ is a row vector of all ones and $\mathbf{p}$ is the steady-state distribution. Thus this suggests an effective method for determining m and n. Notice that the matrix P we are considering is indeed $P^{(1)}$. Here we show some cases of P^r:

$$P^2 = \begin{pmatrix} 0.2884 & 0.0865 & 0.0826 & 0.5424 \\ 0.2994 & 0.1286 & 0.1665 & 0.4056 \\ 0.2771 & 0.1583 & 0.2573 & 0.3073 \\ 0.1962 & 0.0457 & 0.0387 & 0.7194 \end{pmatrix} \quad P^3 = \begin{pmatrix} 0.2504 & 0.0781 & 0.0834 & 0.5881 \\ 0.2672 & 0.1015 & 0.1278 & 0.5035 \\ 0.2730 & 0.1214 & 0.1713 & 0.4343 \\ 0.2142 & 0.0561 & 0.0528 & 0.6768 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 0.2384 & 0.0739 & 0.0805 & 0.6072 \\ 0.2505 & 0.0873 & 0.1047 & 0.5575 \\ 0.2589 & 0.0991 & 0.1273 & 0.5147 \\ 0.2221 & 0.0619 & 0.0616 & 0.6544 \end{pmatrix} \quad P^{12} = \begin{pmatrix} 0.2306 & 0.0692 & 0.0739 & 0.6263 \\ 0.2308 & 0.0694 & 0.0741 & 0.6257 \\ 0.2309 & 0.0695 & 0.0744 & 0.6253 \\ 0.2305 & 0.0691 & 0.0737 & 0.6268 \end{pmatrix}.$$

In fact, the steady-state distribution is given by $(0.2306, 0.0692, 0.0738, 0.626)$. We note that the eigenvalues of P are $1.0000, 0.0519, 0.2664$ and 0.5656. The second largest eigenvalue is 0.5656. This means that the Markov chain process in this case will converge very fast to its steady-state distribution. Therefore we find that n should be greater than 0.2496. We take $n = 0.25$. We proceed to stochastic dynamic programming in the next section.

3 Stochastic Dynamic Programming

In this section, we present the dynamic programming approach for the captured promotion problem. We adopt the notations in [1].

N	$=$	total number of states ($i = 1, \ldots, N$)
M	$=$	total number of alternatives (promotion) ($j = 1, \ldots, M$)
T	$=$	number of weeks in consideration ($t = 1, \ldots, T$)
d_j	$=$	cost needed for carrying out promotion plan j per time unit
$c_i^{(j)}$	$=$	revenue obtained from a customer in State i under alternative j per time unit
$p_{ik}^{(j)}$	$=$	transition probability for a customer to move from State i to State k under alternative j per time unit
α	$=$	discount factor.

We define $v_i(t)$ to be the maximum expected profit with t weeks remaining for a customer in State i at the beginning of the $(T-t)$th period for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. Under the normal setting, we have the following relationship:

$$v_i(t) = \max_{j=1,\ldots,M} \left\{ c_i^{(j)} - d_j + \alpha \sum_{k=1}^{N} p_{ik}^{(j)} v_k(t-1) \right\}. \quad (3)$$

A stochastic dynamic programming model is a system that can move from one distinguished state to any other possible states. In each step, the manager has to make a decision from a well-defined set of alternatives (to promote or not to promote). This action affects the transition probabilities of the next move and at the same time incurs an immediate gain or loss and subsequent gain or loss. The problem that the decision maker facing is to determine an optimal strategy (a stationary policy) of actions so that the overall gain is maximized. A stationary policy is a rule of taking actions. It describes all the decisions that should be made throughout the process. For more details about the stationary policy theorem for a stochastic dynamic programming model, we refer readers to Winston [10]. In the following subsections, we will consider both cases of infinite horizon and finite horizon under a constraint that the number of promotions is fixed.

3.1 The Infinite Horizon Case

We first consider the basic setting and will add back the revenue adjustment later. We can calculate the value v_i (maximum expected profit starting at State i) of a discounted, infinite-horizon Markov decision process using linear programming. The optimal values v_i satisfy the relationship [10]:

$$v_i \geq \max_{j=1,\ldots,M} \left\{ c_i^{(j)} - d_j + \alpha \sum_{k=1}^{N} p_{ik}^{(j)} v_k \right\}, \quad i = 1, \ldots, N.$$

The optimal values are the ones for which the equality holds. The optimal plan for each State i is the value of k that maximizes the right-hand-side of the inequality [10]. Therefore one can set up a linear programming as follows:

$$\begin{cases} \qquad \min \quad z = \sum_{i=1}^{N} v_i \\ \text{subject to} \\ \qquad v_i \geq c_i^{(j)} - d_j + \alpha \sum_{k=1}^{N} p_{ik}^{(j)} v_k, \quad i = 1, \ldots, N; \quad j = 1, \ldots, M. \\ \qquad v_i \geq 0, \quad i = 1, \ldots, N. \end{cases}$$

which is equivalent to the following problem in matrix form:

$$\begin{cases} \qquad \min \quad z = \sum_{i=1}^{N} v_i \\ \text{subject to} \\ \qquad \mathbf{v} \geq \mathbf{c}^{(j)} - d_j \mathbf{1} + \alpha P^{(j)} \mathbf{v}, \quad j = 1, \ldots, M. \\ \qquad \mathbf{v} \geq \mathbf{0}. \end{cases}$$

Here $\mathbf{1}$ is a matrix with all entries 1 and has the same size as $\mathbf{c}^{(j)}$.

To proceed with our revenue adjustment, we introduce one more notation as follows:

$$\mathbf{c}'^{(j)}(r) = (1 + m(1 - e^{-nr}))\mathbf{c}^{(j)} - d_j \mathbf{1}$$

where $m \geq 0, n > 0$ fixed; $j = 1, \ldots, M-1$; $r = 2, \ldots, R$ and $j \neq M$. Because we have reserved the last alternative for no action done (no promotion). Here R is the maximum duration allowed for a consecutive promotion.

Now we consider for a policy using alternative j which lasts for 2 time units. Following from one-step removal policy, its intermediate form after 1 time unit is:

$$\mathbf{v} = \mathbf{c}'^{(j)}(2) + \alpha P^{(j)} \mathbf{v}'$$

where $\mathbf{v}'$ denotes a vector of expected profit in infinite horizon given that no decision can be made in the first step.

As we have no choice to select other alternatives after 1 time unit since the promotion lasts for 2 time units, we have to use the matrix $P^{(j)}$ for transition and the same revenue $\mathbf{c}'^{(j)}(2)$. Again, from one-step removal policy, we have:

$$\mathbf{v}' = \mathbf{c}'^{(j)}(2) + \alpha P^{(j)} \mathbf{v}.$$

Combining, we have

$$\mathbf{v} = \mathbf{c}'^{(j)}(2) + \alpha P^{(j)} \left(\mathbf{c}'^{(j)}(2) + \alpha P^{(j)} \mathbf{v} \right)$$

or

$$\mathbf{v} = \left(I + \alpha P^{(j)} \right) \mathbf{c}'^{(j)}(2) + (\alpha P^{(j)})^2 \mathbf{v}.$$

Using similar argument and changing it to inequality, it can be generalized as:

$$\mathbf{v} \geq \left(I + \alpha P^{(j)} + \cdots + (\alpha P^{(j)})^{r-1}\right)\mathbf{c}'^{(j)}(r) + (\alpha P^{(j)})^r \mathbf{v}.$$

Hence the linear programming we should consider ultimately is the following:

$$\begin{cases} & \min \quad z = \sum_{i=1}^{N} v_i \\ \text{subject to} & \\ & \mathbf{v} \geq \mathbf{c}^{(j)} - d_j \mathbf{1} + \alpha P^{(j)} \mathbf{v}, \quad j = 1, \ldots, M. \\ & \mathbf{v} \geq \left(I + \alpha P^{(j)} + \cdots + (\alpha P^{(j)})^{r-1}\right)\mathbf{c}'^{(j)}(r) + (\alpha P^{(j)})^r \mathbf{v}, \\ & j = 1, \ldots, M-1; \quad r = 2, \ldots, R \\ & \mathbf{v} \geq \mathbf{0}. \end{cases}$$

We limit R to be 4 in our numerical examples. The result is computed using *Scilab* and the program code is available at "http://hkumath.hku.hk/∼ wkc/MDP2.zip". We denote D_i to be the optimal plan taken when a customer is in State i. Here 0 means no promotion should be carried out and 1 means we should carry out a single-period promotion, and so on. We have set $M = 2$ (promotion and no promotion), $N = 4$ (four states of customers). We remark that when $m = 0$, the situation reduces to the case studied in [1]. Tables 3 to 8 present the optimal solutions.

d	α	m	v_1	v_2	v_3	v_4	D_1	D_2	D_3	D_4
0	0.9	0	119	179	297	92	1	0	0	1
		0.5	119	179	297	92	1	0	0	1
		1	121	180	298	93	2	0	0	1
		1.5	134	189	307	101	4	0	0	1
		2	150	209	320	111	4	4	0	1
	0.95	0	234	295	415	205	1	0	0	1
		0.5	234	295	415	205	1	0	0	1
		1	237	297	418	207	2	0	0	1
		1.5	261	318	437	225	4	0	0	1
		2	292	354	466	250	4	4	0	1
	0.99	0	1144	1206	1329	1113	1	0	0	1
		0.5	1144	1206	1329	1113	1	0	0	1
		1	1158	1220	1342	1126	2	0	0	1
		1.5	1269	1327	1448	1231	4	4	0	1
		2	1414	1481	1592	1370	4	4	0	1

Table 3: $d = 0$

d	α	m	v_1	v_2	v_3	v_4	D_1	D_2	D_3	D_4
2	0.9	0	102	164	282	74	1	0	0	1
		0.5	102	164	282	74	1	0	0	1
		1	102	164	282	74	2	0	0	1
		1.5	115	173	291	82	4	0	0	1
		2	131	189	303	92	4	4	0	1
	0.95	0	199	262	383	169	1	0	0	1
		0.5	199	262	383	169	1	0	0	1
		1	200	263	384	170	2	0	0	1
		1.5	224	283	403	188	4	0	0	1
		2	253	315	429	211	4	4	0	1
	0.99	0	966	1030	1153	934	1	0	0	1
		0.5	966	1030	1153	934	1	0	0	1
		1	976	1040	1163	943	2	0	0	1
		1.5	1080	1140	1263	1042	4	0	0	1
		2	1220	1286	1400	1175	4	4	0	1

Table 4: $d = 2$

d	α	m	v_1	v_2	v_3	v_4	D_1	D_2	D_3	D_4
4	0.9	0	88	151	269	58	0	0	0	1
		0.5	88	151	269	58	0	0	0	1
		1	88	151	269	58	0	0	0	1
		1.5	96	157	275	63	4	0	0	1
		2	111	170	286	72	4	4	0	1
	0.95	0	164	230	351	134	0	0	0	1
		0.5	164	230	351	134	0	0	0	1
		1	164	230	351	134	0	0	0	1
		1.5	186	247	369	150	4	0	0	1
		2	214	276	393	172	4	4	0	1
	0.99	0	788	854	978	755	1	0	0	1
		0.5	788	854	978	755	1	0	0	1
		1	793	860	983	761	2	0	0	1
		1.5	893	955	1079	855	4	0	0	1
		2	1025	1091	1208	981	4	4	0	1

Table 5: $d = 4$

3.2 The Finite Horizon Case

We consider optimization in finite horizon. We limit the number of promotion available for each customer in a period of planning to make it closer to reality. We will use the values obtained in the last subsection as boundary conditions. We introduce some more notations:

$$\begin{aligned} w &= \text{number of weeks remaining} \\ p &= \text{number of promotions remaining} \\ q &= \text{number of weeks remaining until the next decision echo} \\ r &= \text{currently under a } r\text{-period promotion} \end{aligned}$$

As before, we consider the case $M = 2$ and $N = 4$. The recursive relation is given by:

$$v_i(w,p,q,r) = \begin{cases} \max\limits_{t=0,\ldots,\min\{w,p\}} \left\{ \widetilde{c}_i(t) + \alpha \sum\limits_{k=1}^{N} p_{ik}^{(m_1)} v_i(w-1, m_2, m_3, m_4) \right\} & \text{if } q = 0 \\ \widetilde{c}_i(r) + \alpha \sum\limits_{k=1}^{N} p_{ik}^{(1)} v_i(w-1, p-1, q-1, r) & \text{if } q \geq 2 \\ \widetilde{c}_i(r) + \alpha \sum\limits_{k=1}^{N} p_{ik}^{(1)} v_i(w-1, p-1, 0, 0) & \text{if } q = 1 \end{cases}$$

where

$$m_1 = \begin{cases} 2 & \text{if } t = 0 \\ 1 & \text{otherwise} \end{cases} \qquad m_2 = \begin{cases} p & \text{if } t = 0 \\ p-1 & \text{otherwise} \end{cases}$$

$$m_3 = \begin{cases} 0 & \text{if } t = 0 \\ t-1 & \text{otherwise} \end{cases} \qquad m_4 = \begin{cases} 0 & \text{if } t = 0, 1 \\ t & \text{otherwise} \end{cases}$$

and

$$\widetilde{c}_i(t) = \begin{cases} (1 + m(1 - e^{-nr}))c_i^{(1)} - d_1 & \text{if } t \geq 2 \\ c_i^{(1)} - d_1 & \text{if } t = 1 \\ c_i^{(2)} - d_2 & \text{if } t = 0 \end{cases}$$

where $m \geq 0, n > 0$ are fixed.

In our promotion planning, we set $w_{\max} = 52$, $p_{\max} = 4$. The solution is presented as a list like

$$t_1, t_2, t_3, t_4, v^*$$

where v^* is the maximum expected profit, t_i is the week to employ a promotion, "-" refers as no promotion and being enclosed by square brackets means consecutive promotion. For example, $[1, 2]$, $51, 52, 83$ means we should have a consecutive promotion at week 1 and 2, i.e. a two-period promotion in week 1, and two single-period promotion in weeks 51 and 52, and the maximum expected profit is 83. Tables 9 to

d	α	m	State 1	State 2	State 3	State 4
0	0.9	0	1,47,50,52,95	-,-,-,-,158	-,-,-,-,276	1,2,3,4,67
		0.5	1,47,50,52,95	-,-,-,-,158	-,-,-,-,276	1,2,3,4,67
		1	[1,2],[50,51],95	-,-,-,-,158	-,-,-,-,276	1,2,3,4,67
		1.5	[1,2,3,4],101	-,-,-,-,161	-,-,-,-,279	1,2,3,4,68
		2	[1,2,3,4],109	[1,2,3,4],168	-,-,-,-,284	[1,2,3,4],70
	0.95	0	45,48,50,52,169	-,-,-,-,234	-,-,-,-,355	1,2,3,4,138
		0.5	45,48,50,52,169	-,-,-,-,234	-,-,-,-,355	1,2,3,4,138
		1	[45,46],[50,51],169	-,-,-,-,234	-,-,-,-,356	1,2,3,4,138
		1.5	[1,2,3,4],177	-,-,-,-,239	-,-,-,-,361	1,2,3,4,140
		2	[1,2,3,4],187	[1,2,3,4],249	-,-,-,-,368	[1,2,3,4],145
	0.99	0	47,49,50,51,963	-,-,-,-,1031	-,-,-,-,1155	1,2,3,4,929
		0.5	47,49,50,51,963	-,-,-,-,1031	-,-,-,-,1155	1,2,3,4,929
		1	[47,48],[50,51],971	-,-,-,-,1039	-,-,-,-,1162	1,2,3,4,937
		1.5	[1,2,3,4],1039	[49,50,51,52],1106	-,-,-,-,1230	[48,49,50,51],1003
		2	[1,2,3,4],1132	[20,21,22,23],1198	-,-,-,-,1322	[49,50,51,52],1094

Table 6: $d = 0$

d	α	m	State 1	State 2	State 3	State 4
2	0.9	0	49,50,51,52,89	-,-,-,-,152	-,-,-,-,271	1,2,3,4,60
		0.5	49,50,51,52,89	-,-,-,-,152	-,-,-,-,271	1,2,3,4,60
		1	[49,50],[51,52],89	-,-,-,-,152	-,-,-,-,271	1,2,3,4,60
		1.5	[1,2,3,4],95	-,-,-,-,155	-,-,-,-,274	1,2,3,4,61
		2	[1,2,3,4],102	[1,2,3,4],161	-,-,-,-,278	[1,2,3,4],63
	0.95	0	48,50,51,52,160	-,-,-,-,225	-,-,-,-,347	1,2,3,4,128
		0.5	48,50,51,52,160	-,-,-,-,225	-,-,-,-,347	1,2,3,4,128
		1	[48,49],[51,52],160	-,-,-,-,225	-,-,-,-,347	1,2,3,4,128
		1.5	[1,2,3,4],167	-,-,-,-,230	-,-,-,-,351	1,2,3,4,130
		2	[1,2,3,4],177	[1,2,3,4],239	-,-,-,-,359	[49,50,51,52],136
	0.99	0	48,49,51,52,849	-,-,-,-,917	-,-,-,-,1041	1,2,3,4,815
		0.5	48,49,51,52,849	-,-,-,-,917	-,-,-,-,1041	1,2,3,4,815
		1	[48,49],[51,52],855	-,-,-,-,923	-,-,-,-,1047	1,2,3,4,821
		1.5	[1,2,3,4],920	-,-,-,-,987	-,-,-,-,1110	[49,50,51,52],883
		2	[1,2,3,4],1008	[34,35,36,37],1075	-,-,-,-,1199	[49,50,51,52],971

Table 7: $d = 2$

d	α	m	State 1	State 2	State 3	State 4
4	0.9	0	-,-,-,-,84	-,-,-,-,147	-,-,-,-,266	1,2,3,4,54
		0.5	-,-,-,-,84	-,-,-,-,147	-,-,-,-,266	1,2,3,4,54
		1	-,-,-,-,84	-,-,-,-,147	-,-,-,-,266	1,2,3,4,54
		1.5	[1,2,3,4],88	-,-,-,-,149	-,-,-,-,268	1,2,3,4,54
		2	[1,2,3,4],95	[41,42,43,44],154	-,-,-,-,273	[49,50,51,52],57
	0.95	0	-,-,-,-,151	-,-,-,-,217	-,-,-,-,338	1,2,3,4,119
		0.5	-,-,-,-,151	-,-,-,-,217	-,-,-,-,338	1,2,3,4,119
		1	-,-,-,-,151	-,-,-,-,217	-,-,-,-,338	1,2,3,4,119
		1.5	[1,2,3,4],157	-,-,-,-,221	-,-,-,-,342	48,49,50,51,121
		2	[1,2,3,4],167	[1,2,3,4],229	-,-,-,-,350	[49,50,51,52],128
	0.99	0	49,50,51,52,736	-,-,-,-,804	-,-,-,-,928	1,2,3,4,701
		0.5	49,50,51,52,736	-,-,-,-,804	-,-,-,-,928	1,2,3,4,701
		1	[49,50],[51,52],739	-,-,-,-,807	-,-,-,-,931	1,2,3,4,705
		1.5	[1,2,3,4],801	-,-,-,-,868	-,-,-,-,992	[49,50,51,52],765
		2	[1,2,3,4],885	[39,40,41,42],952	-,-,-,-,1076	[49,50,51,52],849

Table 8: $d = 4$

14 present the optimal strategy. Again we remark that when $m = 0$, it reduces to the situation in [1] and the results are re-produced.

From the numerical results, we observe that there is no need to conduct promotion to customers in State 3. For customers in State 2, promotion is useful only when m is large. For customers in State 4, generally speaking, promotion should be conducted in the very beginning in order to keep the customer. If we let the value of m to increase, when it is greater than a certain threshold, the promotion pattern differs from that with $m = 0$ (the situation that consecutive promotions have no beneficial effect). From that level onwards, we can say consecutive promotion has a significant benefit over single-period promotions. Also, when m is large, either no promotion is used or 4-period promotions are employed. In other words, although consecutive promotion gives up the opportunity to observe the states after one period transition and set strategy accordingly, the benefit derived exceeds significantly the cost of opportunity we gave up.

Finally, we remark that the new model actually gives better objective values when compare to the previous model.

4 Summary

This paper is an extension of the work of Ching et al. [1]. In [1] they have proposed a stochastic dynamic model for infinite and finite horizon with budget constraint. Adopting the idea from Lin [6] that consecutive promotion is more revenue-enhanced, we used an exponential decay function to increase the revenue according to the duration of promotion. Both infinite and finite horizon are discussed. We have used *Scilab* to compute the numerical examples. The software is very similar to the powerful mathematical tool *Matlab*. *Scilab* can be downloaded at "http://www.scilab.org". The model is then applied to the data in [1] of a computer service company. The numerical results suggest that consecutive promotion is recommended. Besides the intensity of promotion, it cannot be overlooked in an optimal promotion planning.

In a real application, the parameters of the revenue-adjusting function as well as the function itself can be estimated using practical data from manager. Consecutive promotion can be realized not only by revenue enhancement but also adjustment of the transition matrix. It is believed that if the consecutive promotion period is longer, it is likely to have more loyal users.

Acknowledgment: The authors would like to thank the anonymous referee for the helpful comments and constructive suggestions in revising the paper.

References

[1] Ching, W.K., Ng, M.K., Wong, K.K., and Altman, E. (2004). Customer Lifetime Value: Stochastic Optimization Approach. *Journal of the Operational Research Society*, 55, 860–868.

[2] Ching, W.K. and Ng, M.K. (2006). Markov Chains: Models, Algorithms and Applications. Springer's international Series in Operations Research and Management Science, Springer, N.Y., U.S.

[3] Esteban-Bravo, M., Mugica, J.M. and Vidal-Sanz, J.M. (2005) Optimal Duration of Magazine Promotions *Marketing Letters*, 16, 99114.

[4] Jain, D., and Singh, S.S. (2002). Customer Lifetime Value Research in Marketing: A Review and Future Directions. *Journal of Interactive Marketing*, 16(2), 34–46.

[5] Kahn, B.E., and Louie, T.A. (1990). Effects of retraction of Price Promotions on Brand Choice Behavior for Variety-seeking and Last-purchase-loyal Consumers. *Journal of Marketing Research*, 27, 279–289.

[6] Lin, C. and Lin, Y.T. (2007). Robust Analysis on Promotion Duration for Two Competitive Brands *Journal of the Operational Research*, 1–8.

[7] Massy, W.F., Montgomery, D.B. and Morrison, D.G. (1970) *Stochastic Models of Buying Behavior*. MIT Press: Cambridge, MA.

[8] Pfeifer, P. and Carraway R. (2000) Modeling Customer Relationships as Markov Chain. *Journal of Interactive Marketing*, 14, 43–55.

[9] Ross, S. (2003). *Introduction to Probability Models*, 8th Edition, Academic Press.

[10] Winston, W.L. (1994). *Operations Research: Applications and Algorithms*, Third Edition, Belmont Calif.: Duxbury Press.

Ho-Yin LEUNG is a M. Phil. student at the Department of Mathematics, the University of Hong Kong. He got his B. Sc. in Mathematics, the University of Hong Kong with first class honour. He was in the Dean's honour list for three years (2005-2007) and he got the following scholarships: CV Starr Scholarships (2007), HSBC Hong Kong Scholarship (2006) and the Koo Shui Ting Memorial Scholarship (2005-2007). He also obtained the following academic prizes: Alan John Ellis Prizes in Mathematics (2005), B. Sc. Class of 1971 Prize (2006), Ho Sin Hang Price in Science (2007) and Wong Yung Chow Prize in Mathematics (2006). His research interests are stochastic process and modelling with applications.

Wai-Ki CHING is a lecturer at the department of Mathematics, the University of Hong Kong. He obtained his B. Sc. (1991) and M. Phil. (1994) degrees in Mathematics from the University of Hong Kong. He then obtained his Ph.D. degree in Systems Engineering and Engineering Management (1998) from the Chinese University of Hong Kong and was a visiting post-doc fellow at the Judge Business School of the Cambridge University (1999-2000). He has previously taught at the Hong Kong Polytechnic University and the University of Science and Technology. Before joining his Alma Mater, he was a lecturer at the University of Southampton (2000-2001). His research interests include data modelling, optimization algorithms, systems engineering and bioinformatics.

Issic K.C. Leung was a lecturer at the University of Macau for two years (1990-1992) after he had obtained his Bachelor and Master degrees from the University of Manitoba, Canada (1990). He then studied further and graduated with his Ph.D. in Mathematics from the Flinders University of South Australia (1996) at there he was awarded the Flinders University Research Scholarship (1993-1996). He was a senior market analyst in PCCW, one of the giants in the telecom industry in Hong Kong, for many years before he became an assistant professor of mathematics education at the Hong Kong Institute of Education (2006-2008). His research interest is in mathematics in business modelling, quantitative methods in marketing analyst, mathematics education and teacher competence in mathematics.