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A TANDEM QUEUEING SYSTEM WITH APPLICATIONS TO PRICING STRATEGY

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Abstract. In this paper, we analyze a Markovian queueing system with multiple types of customers and two queues in tandem. All customers have to go through two stages of services. In Stage 1, the queueing system has multiple identical servers while in Stage 2, there is one single-server queue for each type of customers. The queueing discipline in the whole system is Blocked Customer Delayed (BCD). We first obtain the steady-state probability distribution of the queueing system and the expected waiting time for customers. We then apply the queueing model to solve an optimal pricing policy problem in assuming that the demand rate is dependent on the price. The objective is to minimize the number of servers in the first stage and also maximize the expected earnings by taking into account the demand and the prices. We also obtained some analytic results for the optimal pricing strategy.

1. **Introduction.** Queueing systems are useful tools for analyzing many physical systems [1, 2]. In this paper, we analyze a tandem queueing system with two stages of services. The tandem queueing systems have been studied by a number of researchers [5, 8] and have a lot of applications. In our queueing model, there are two stages of services. In Stage 1, the queueing system has only one server

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and it is Blocked Customer Delayed (BCD). In the second stage, customers go to different single-server queues depending on their types and they are also BCD queues. We then formulate the queueing system as an optimization model. The model is then applied to the design of a serving system with price analysis. The whole system can be divided into two stages. In Stage 1, the queueing system has multiple identical servers while in Stage 2, there is one single-server queue for each type of customers. We then apply the queueing model to solve an optimal pricing policy problem assuming that the demand rate is dependent on the price. Here the objective is to minimize the number of servers in the first stage and at the same time maximize the expected earnings by taking into account the demand and the prices.

The rest of the paper is structured as follows. Section 2 gives the queueing models with some performance analysis. In Section 3, we present the optimization model for finding the optimal pricing strategy. In Section 4, we further discuss the case when there is a constraint on the total sojourn time of the queueing system. Concluding remarks are then given in the last section.

2. **The Queueing System.** In this section, we present the queueing system with some analytical analysis. The queueing system being investigated is a tandem queueing system of two stages. All queues in the system are assumed to be Blocked Customer Delay (BCD) and the queueing discipline is first-come-first-served (FCFS). The service time of all servers are assumed to be independent and exponentially distributed. All arriving customers will go through the first stage and then the second stage. There are n types of customers and type i customers are assumed to arrive at a rate λ_i. The arrival of different types of customers are assumed to follow independent Poisson Processes. The total arrival rate of customers is

$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n. \tag{1}$$

In the first stage, there are s servers. The queueing system is modeled as an M/M/s/∞ queue. The service times for the servers are assumed to be independent and identically distributed exponential random variables with mean μ_1^{-1}. In the second stage, customers go on to the second stage of their service according to their types. There are n different single-server queues, one for each type of customers. The server at the queue for type i customers has exponential service time with mean $\mu_{2,i}^{-1}$, for $i = 1, 2, \ldots, n$. All the arrival rates and service rates are assumed to be known. In the following, we give a brief review on the existing results in Markovian queueing systems.

2.1. **The Queueing System in Stage One.** The first stage of the tandem queueing system is assumed to be an M/M/s/∞ queue, with service times independently and identically distributed with mean μ_1^{-1}. Therefore, we have the following results.

Lemma 2.1. *If $\lambda < s\mu_1$, the steady-state probability that there are j customers in the queueing system of stage one is given by*

$$P_1(j) = \begin{cases} \frac{a^j}{j!} P_1(0) & (j = 0, 1, 2, \ldots, s) \\ \frac{a^j}{s! s^{j-s}} P_1(0) & (j = s+1, s+2, \ldots,) \end{cases} \tag{2}$$

where $\frac{\lambda}{\mu_1} = a$ and

$$P_1(0) = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \sum_{k=s}^{\infty} \frac{a^k}{s! s^{k-s}} \right)^{-1} = \left(\sum_{k=0}^{s-1} \frac{a^k}{k!} + \frac{a^s}{(s-1)!(s-a)} \right)^{-1}.$$

Proof. See Cooper [3]. □

Lemma 2.2. *The average sojourn time of a customer in stage one is given by*

$$W_{s_1} = \frac{1}{\mu_1} + \frac{1}{\mu_1}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-1} \tag{3}$$

and the average waiting time of a customer in the queue in stage one is given by

$$W_{q_1} = \frac{1}{\mu_1}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-1}.$$

Proof. See Cooper [3]. □

Here we note that by Burke's Theorem [4], the output of the first stage of the system in its steady state is a Poisson Process with mean λ. Since the type of each customer is independent with the whole process in the first stage, the proportion of different types of customers remains unchanged, and the arrival rate for the queue of type i customers is thus λ_i in the second stage.

2.2. **The Queueing System in Stage Two.** Since the queues in stage two are independent M/M/1/∞ queues, the probability that there are j customers in the i^{th} queue in the steady state is given by [6]

$$p_{i,j} = \rho_i^j(1-\rho_i) \quad \text{for } j = 0, 1, 2, 3, \dots \tag{4}$$

where $\rho_i = \dfrac{\lambda_i}{\mu_{2,i}}$ given $\rho_i < 1$. We have the following two results for queues in tandem [3].

Proposition 1. *If $\rho_i < 1$ for $i = 1, 2, \dots, n$, the joint distribution of the number of customers in the n queues in stage two in steady state is given by*

$$P_2(j_1, j_2, \dots, j_n) = \prod_{i=1}^{n} \rho_i^{j_i}(1-\rho_i) \quad \text{for } j_1, j_2, \dots, j_n = 0, 1, 2, 3, \dots. \tag{5}$$

2.3. **The Tandem Queueing System.**

Proposition 2. *If $\lambda < s\mu_1$ and $\lambda_i < \mu_{2,i}$ for $i = 1, 2, \dots, n$, the steady-state probablility that there are i customers in the first stage and $j_1, j_2, \dots, j_n$ in each of the queues in the second stage is given by*

$$P(i, j_1, j_2, \dots, j_n) = \begin{cases} C\dfrac{a^i}{i!}\displaystyle\prod_{k=1}^{n}\rho_k^{j_k} & (i = 0, 1, 2, \dots, s) \\ C\dfrac{a^i}{s!s^{i-s}}\displaystyle\prod_{k=1}^{n}\rho_k^{j_k} & (i = s+1, s+2, \dots) \end{cases} \tag{6}$$

where

$$a = \frac{\lambda}{\mu_1}, \quad \rho_k = \frac{\lambda_k}{\mu_{2,k}}$$

and

$$C = \left(\prod_{i=1}^{n}(1-\rho_i)\right)\left(\sum_{k=0}^{s-1}\frac{a^k}{k!} + \frac{a^s}{(s-1)!(s-a)}\right)^{-1}$$

for $j_1, j_2, \dots, j_n = 0, 1, 2, \dots$.

3. **The Optimization Problem.** In the queueing system, one may want to set upper bounds on the average length of the queue and the average waiting time of customers. These constraints are practical in many situations, e.g. making sure that customers do not wait too long in the queue and that the queueing place does not get too crowded. Mathematically, one may impose the conditions as follow:

$$L_q \le l \quad \text{and} \quad W_q \le w \tag{7}$$

where l and w are constants to be determined by the practical situations in the queueing model.

3.1. **Minimizing the Number of Servers in Stage One.** Given the arrival rate for each type of customers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the service rates μ_1 for the first stage and $\mu_{2,1}, \mu_{2,2}, \ldots, \mu_{2,n}$ for the second stage, minimize s, the number of servers required in the first stage subject to the constraints on the average length of the queue L_{q_1} and the average waiting time of customers W_{q_1}. That is, we would like to find

$$s_{min} = min\{s \in \mathbb{N} | L_{q_1} \le l \text{ and } W_{q_1} \le w\} \tag{8}$$

where l and w are given constants.

We note that by Little's Formula, $L_{q_1} = \lambda W_{q_1}$. Since λ is constant in this problem, the constraints $L_{q_1} \le l$ and $W_{q_1} \le w$ can be re-written as

$$\lambda W_{q_1} \le l \text{ and } W_{q_1} \le w$$

or equivalently

$$W_{q_1} \le \frac{l}{\lambda} \text{ and } W_{q_1} \le w,$$

i.e. $W_{q_1} \le K$ where $K = min(\frac{l}{\lambda}, w)$.

So the problem can be re-written as follows:

$$s_{min} = min\{s \in \mathbb{N} | W_{q_1} \le K\}$$

where K is a a constant. We remark that from Lemma 2.2, we have

$$W_{q_1} = \frac{1}{\mu_1} \left(\frac{(s-1)!(s-a)^2}{a^s} (\sum_{k=0}^{s-1} \frac{a^k}{k!}) + (s-a) \right)^{-1}. \tag{9}$$

Proposition 3. *The average waiting time in the stage one, $W_{q_1}(s)$ is strictly decreasing in s, i.e. for $s_1, s_2 \in \boldsymbol{N}$ such that $s_1 < s_2$, $W_{q_1}(s_1) > W_{q_1}(s_2)$.*

Proof. For $0 < s_1 < s_2$, we have

$$(s_1 - 1)! < (s_2 - 1)! \quad \text{and} \quad (s_1 - a)^2 < (s_2 - a)^2$$

and $a^{s_1} > a^{s_2}$ for $0 < a < 1$ which implies

$$\frac{1}{a^{s_1}} < \frac{1}{a^{s_2}}.$$

Moreover,

$$\sum_{k=0}^{s_1-1} \frac{a^k}{k!} < \sum_{k=0}^{s_2-1} \frac{a^k}{k!}$$

so

$$\left(\frac{(s_1-1)!(s_1-a)^2}{a^{s_1}} (\sum_{k=0}^{s_1-1} \frac{a^k}{k!}) \right) < \left(\frac{(s_2-1)!(s_2-a)^2}{a^{s_2}} (\sum_{k=0}^{s_2-1} \frac{a^k}{k!}) \right).$$

Since $(s_1 - a) < (s_2 - a)$, we have

$$\left(\frac{(s_1-1)!(s_1-a)}{a^{s_1}}^2 (\sum_{k=0}^{s_1-1} \frac{a^k}{k!}) + (s_1 - a)\right) < \left(\frac{(s_2-1)!(s_2-a)}{a^{s_2}}^2 (\sum_{k=0}^{s_2-1} \frac{a^k}{k!}) + (s_2 - a)\right)$$

and therefore

$$\frac{1}{\mu_1}\left(\frac{(s_1-1)!(s_1-a)}{a^{s_1}}^2 (\sum_{k=0}^{s_1-1} \frac{a^k}{k!}) + (s_1 - a)\right)^{-1} > \frac{1}{\mu_1}\left(\frac{(s_2-1)!(s_2-a)}{a^{s_2}}^2 (\sum_{k=0}^{s_2-1} \frac{a^k}{k!}) + (s_2 - a)\right)^{-1}.$$

and therefore $W_{q_1}(s_1) > W_{q_1}(s_2)$. ☐

¿From Proposition 3, we know that s_{min} is the unique positive integer which satisfies the inequalities

$$\begin{cases} \dfrac{1}{\mu_1}\left(\dfrac{(s_{min}-1)!(s_{min}-a)}{a^{s_{min}}}^2 (\sum_{k=0}^{s_{min}-1} \dfrac{a^k}{k!}) + (s_{min} - a)\right)^{-1} \le K \\ \dfrac{1}{\mu_1}\left(\dfrac{(s_{min}-2)!(s_{min}-1-a)}{a^{s_{min}-1}}^2 (\sum_{k=0}^{s_{min}-2} \dfrac{a^k}{k!}) + (s_{min} - 1 - a)\right)^{-1} \ge K. \end{cases} \tag{10}$$

If

$$\frac{a}{\mu_1(1-a)} \ge K,$$

i.e., having only one server cannot fulfill the constraints on the average waiting time.

If the condition on K does not hold, $s_{min} = 1$. Assuming that the condition on K holds. Although we do not have an explicit formula for s_{min}, given s_{min} as the unique positive integer which satisfies the above two inequalities, we can use a searching algorithm to find s_{min} as follows:

```
Step 1:  s = 1, p = a, b = 1, f = 1
Step 2:  d = f*(s-a)^2*b/p+(s-a), w = 1/(μ_1*d)
Step 3:  if w ≤ K, return s and stop the program.
Step 4:  f = f*s, b = b+p/f, p = p*a, s = s+1, then go to Step 2.
```

This algorithm takes s_{min} iterations to find s_{min}. We note that, the given arrival rates and service rates are fixed, the departure process of the first stage in the equilibrium is a Poisson Process with rate λ [4], which is unaffected by the number of servers in the first stage and the given equilibrium conditions are all satisfied.

Example 1. Suppose $\lambda = 3.68, \mu_1 = 2.71$, so $a = \dfrac{\lambda}{\mu_1} = 1.36$. Set $l = 30$ and $w = 10$ in the optimization problem. The constraints

$$W_{q_1} < 0.5 \quad \text{and} \quad L_{q_1} < 10$$

can be re-written as

$$W_{q_1} < 0.5 \quad \text{and} \quad W_{q_1} < \frac{10}{3.68},$$

i.e.,$W_{q_1} < 0.5$. For statistical equilibrium to hold, we must have $\dfrac{\lambda}{s\mu_1} < 1$, so $s > a$, i.e. $s > 1$. When $s = 2$,

$$W_{q_1} = \frac{1}{2.71}\left(\frac{(2-1)!(2-1.36)^2}{1.36^2}(1 + \frac{1.36}{1!}) + (2 - 1.36)\right)^{-1} = 0.32 < 0.5.$$

Therefore, the minimum number of servers in the first stage is 2 under the given constraints.

As the queues in the second stage are M/M/1/∞ queue, the average waiting time of the i^{th} queue is given by

$$W_{q_2,i} = \frac{\rho_i}{\mu_{2,i}(1-\rho_i)} \quad \text{where } \rho_i = \frac{\lambda_i}{\mu_{2,i}}. \tag{11}$$

Therefore, if we want to minimize the number of servers in the first stage while keeping the average total waiting time for each type of customers in the two stages below a constant value, the problem can be written as finding

$$s_{min} = min\{s \in \mathbb{N} | W_{q_1} + W_{q_2,i} \leq K \text{ for } i = 1, 2, \ldots, n\} \tag{12}$$

where K is a constant. Since

$$W_{q_1} + W_{q_2,i} \leq K \text{ for } i = 1, 2, \ldots, n \tag{13}$$

iff

$$W_{q_1} + max(W_{q_2,i}) \leq K,$$

the problem can be re-written as finding

$$s_{min} = min\{s \in \mathbb{N} | W_{q_1} \leq K'\} \tag{14}$$

where $K' = K - max(W_{q_2,i})$ is a constant. Proposition 3 can then be applied to the problem and it can be solved by the same method above.

3.2. **Maximizing the Earnings by Assuming a Linear Relationship between Price and Demand.** First we denote price of the product (food) for customer type i by P_i, $i = 1, 2, \ldots, n$. Assuming that the arrival rate of each type of customer $\lambda_1, \lambda_2, \ldots, \lambda_n$ has a negative linear relationship with price for customer of that type, i.e.,

$$\lambda_i = c_i - k_i P_i \quad \text{for } i = 1, 2, \ldots, n. \tag{15}$$

Here c_i and k_i are positive constants, and $0 \leq P_i \leq \frac{c_i}{k_i}$. This assumption is a simplification of the Law of Demand in Economics [7]. The total profit rate V is defined as

$$V = \sum_{i=1}^{n} \lambda_i P_i = \sum_{i=1}^{n} P_i(c_i - k_i P_i). \tag{16}$$

Given the service rates $\mu_{2,1}, \mu_{2,2}, \ldots, \mu_{2,n}$ for the second stage, we would like to determine P_i^*'s, the price for each type of customers, so that the total earning rate V is maximized subject to a common constraint on the average sojourn time $W_{s_2,i}$. Mathematically, we want to maximize

$$V = \sum_{i=1}^{n} P_i(c_i - k_i P_i) \tag{17}$$

subject to the constraints

$$W_{s_2,i} \leq w \text{ for } i = 1, 2, \ldots, n,$$

where w is a given constant.

Since the queues in the second stage are M/M/1/∞ queues, the average sojourn time of type i customers in stage two is given by [6]

$$W_{s_2,i} = \frac{1}{\mu_{2,i}(1-\rho_i)},$$

where

$$\rho_i = \frac{\lambda_i}{\mu_{2,i}} = \frac{c_i - k_i P_i}{\mu_{2,i}}.$$

We then have

$$\frac{\partial W_{s_{2,i}}}{\partial P_i} = \frac{\partial}{\partial P_i}\left(\frac{1}{\mu_{2,i} - c_i + k_i P_i}\right) = -\frac{k_i}{(\mu_{2,i} - c_i + k_i P_i)^2} < 0. \quad (18)$$

Thus $W_{s_{2,i}}$ is strictly decreasing with respect to P_i. Let Q_i be the solution of P_i in the equation $W_{2s,i} = w$. Then we have

$$\frac{1}{\mu_{2,i} - c_i + k_i Q_i} = w, \quad (19)$$

that is

$$Q_i = \frac{1}{k_i}(\frac{1}{w} - \mu_{2,i} + c_i). \quad (20)$$

Since $W_{s_{2,i}}$ is strictly decreasing with P_i, $W_{s_{2,i}} \leq w$ if and only if $P_i \geq Q_i$. Thus, we can reformulate the optimization problem as

$$\text{maximize } V = \sum_{i=1}^{n} P_i(c_i - k_i P_i) \quad (21)$$

subject to the constraints

$$P_i \geq Q_i \text{ for } i = 1, 2, \ldots, n,$$

where Q_i is a given constant.

Proposition 4. *The optimal set of prices are*

$$P_i^* = max(\frac{c_i}{2k_i}, Q_i), \quad \text{for } i = 1, 2, \ldots, n, \quad (22)$$

where

$$Q_i = \frac{1}{k_i}(\frac{1}{w} - \mu_{2,i} + c_i).$$

Proof. Let

$$\mathbf{P}^* = (P_1^*, P_2^*, \ldots, P_n^*)^T$$

where

$$P_i^* = max(\frac{c_i}{2k_i}, Q_i).$$

Suppose the price vector $\mathbf{P}^*$ is not optimal. Denote the optimal price vector by $\mathbf{P'} = (P_1', P_2', \ldots, P_n')^T$ which satisfies:

$$\begin{cases} \mathbf{P'} \neq \mathbf{P}^* \\ V(\mathbf{P'}) > V(\mathbf{P}^*) \\ P_i \geq Q_i \text{ for } i = 1, 2, \ldots, n. \end{cases}$$

Since $\mathbf{P'} \neq \mathbf{P}^*$, there exists m such that

$$P_m' \neq P_m^* = max(\frac{c_i}{2k_i}, Q_i).$$

We then consider

$$\mathbf{P}^{**} = (P_1', P_2', \ldots, P_{m-1}', P_m^*, P_{m+1}', \ldots, P_n')^T.$$

Now

$$\begin{aligned} V(\mathbf{P}^{**}) &= \sum_{i=1,i\neq m}^{n} P'_i(c_i - k_i P'_i) + P^*_m(c_m - k_m P^*_m) \\ &= V(\mathbf{P}') + P^*_m(c_m - k_m P^*_m) - P'_m(c_m - k_m P'_m) \\ &= V(\mathbf{P}') + \left[-k_m(P^*_m - \frac{c_m}{2k_m})^2 + \frac{c_m^2}{4k_m} \right] - \left[-k_m(P'_m - \frac{c_m}{2k_m})^2 + \frac{c_m^2}{4k_m} \right] \\ &= V(\mathbf{P}') - k_m(P^*_m - \frac{c_m}{2k_m})^2 + k_m(P'_m - \frac{c_m}{2k_m})^2. \end{aligned}$$

If

$$Q_m \leq \frac{c_m}{2k_m}, \quad P^*_m = max(Q_m, \frac{c_m}{2k_m}) = \frac{c_m}{2k_m}$$

then

$$\begin{aligned} V(\mathbf{P}^{**}) &= V(\mathbf{P}') - k_m(\frac{c_m}{2k_m} - \frac{c_m}{2k_m})^2 + k_m(P'_m - \frac{c_m}{2k_m})^2 \\ &= V(\mathbf{P}') + k_m(P'_m - \frac{c_m}{2k_m})^2 > V(\mathbf{P}'). \end{aligned}$$

If

$$Q_m > \frac{c_m}{2k_m}, \quad P^*_m = max(Q_M, \frac{c_m}{2k_m}) = Q_m.$$

Note that $\frac{c_m}{2k_m} < Q_m < P'_m$. So

$$V(\mathbf{P}^{**}) = V(\mathbf{P}') - k_m(Q_m - \frac{c_m}{2k_m})^2 + k_m(P'_m - \frac{c_m}{2k_m})^2 > V(\mathbf{P}').$$

Since

$$k_m(P'_m - \frac{c_m}{2k_m})^2 > k_m(Q_m - \frac{c_m}{2k_m})^2$$

as $P'_m > Q_m$. In both cases we find that $V(\mathbf{P}^{**}) > V(\mathbf{P}')$. Thus it means that $\mathbf{P}'$ cannot be optimal. There is a contradiction and thus, the optimal price vector $\mathbf{P}$ must equal $\mathbf{P}^*$. □

We end this section by the following numerical example.

Example 2. Suppose we have

$$s = 3, \lambda = 3.68, \mu_1 = 2.71, \lambda_1 = 3.04, \lambda_2 = 0.64, \mu_{2,1} = 4.07, \mu_{2,2} = 1.53,$$

with the parameters of the supply and demand model

$$c_1 = 6.55, k_1 = 0.23, c_2 = 2.39 \quad \text{and} \quad k_2 = 0.080.$$

If we would like to keep the waiting time of customers in the second stage below 3 minutes, while maximizing the rate of earnings, then our optimization problem would be, mathematically,

$$\text{maximize } V = \sum_{i=1}^{n} P_i(c_i - k_i P_i)$$

subject to the constraints

$$W_{s_{2,i}} \leq 3 \text{ for } i = 1, 2.$$

According to our previous discussion, this can be re-written as

$$\text{maximize } V = \sum_{i=1}^{n} P_i(c_i - k_i P_i)$$

subject to the constraints

$$P_i \geq Q_i \text{ for } i = 1, 2$$

where

$$Q_i = \frac{1}{k_i}(\frac{1}{3} - \mu_{2,i} + c_i).$$

Here

$$Q_1 = \frac{1}{0.23}(\frac{1}{3} - 3.04 + 6.55) = 16.7$$

and

$$Q_2 = \frac{1}{0.08}(\frac{1}{3} - 0.64 + 2.39) = 26.0.$$

And

$$\frac{c_1}{2k_1} = 14.2 \quad \text{and} \quad \frac{c_2}{2k_2} = 14.9.$$

¿From Proposition 4, the optimal prices are

$$P_1^* = max(\frac{c_1}{2k_1}, Q_1) = 16.7 \quad \text{and} \quad P_2^* = max(\frac{c_2}{2k_2}, Q_2) = 26.0.$$

4. **Maximizing the Earnings Based on Constraints on the Total Sojourn Time.** In this section, we look at a similar problem of maximizing the earnings based on constraints on the total sojourn time of the tandem queues. As in the previous optimization problem, we assume the same negative linear relationship between price and demand.

However, instead of putting limit the average sojourn time of the second stage, we impose constraints on the total sojourn time of the tandem queue, $W_{s1} + W_{s_{2,i}}$. Mathematically, we would like to maximize

$$V = \sum_{i=1}^{n} P_i(c_i - k_i P_i) \tag{23}$$

subject to the constraints

$$W_{s_1} + W_{s_{2,i}} \leq w$$

and

$$0 \leq P_i \leq \frac{c_i}{k_i}$$

for $i = 1, 2, \ldots, n$, where w is a given constant.

¿From the earlier results, we have

$$W_{s_1} = \frac{1}{\mu_1} + \frac{1}{\mu_1}\left(\frac{(s-1)!(s-a)^2}{a^s}(\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-1},$$

where

$$a = \frac{\lambda}{\mu_1} = \frac{1}{\mu_1}\sum_{i=1}^{n}(c_i - k_i P_i)$$

assuming $a < s$ and

$$W_{s_{2,i}} = \frac{1}{\mu_{2,i}(1 - \rho_i)}$$

where

$$\rho_i = \frac{\lambda_i}{\mu_{2,i}} = \frac{c_i - k_i P_i}{\mu_{2,i}}$$

assuming $\rho_i < 1$. We note that W_{s_1} and $W_{s_{2,i}}$ are continuous function of P_i on $[0, \frac{c_i}{k_i}]$ for $i = 1, 2, \ldots, n$. We have

$$\frac{\partial W_{s_1}}{\partial P_i} = \frac{\partial}{\partial P_i}\left[\frac{1}{\mu_1} + \frac{1}{\mu_1}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-1}\right].$$

Since

$$\begin{aligned}
& \frac{\partial}{\partial P_i}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right) \\
= & \frac{\partial}{\partial a}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)\cdot\frac{\partial a}{\partial P_i} \\
= & \left(\sum_{k=0}^{s-1}\frac{(s-1)!}{k!}\frac{\partial}{\partial a}\frac{(s-a)^2}{a^{s-k}} - 1\right)\cdot\left(-\frac{k_i}{\mu_1}\right) \\
= & -\frac{k_i}{\mu_1}\left(\sum_{k=0}^{s-1}\frac{(s-1)!}{k!}\cdot\frac{-2(s-a)a^{s-k} - (s-k)(s-a)^2 a^{s-k-1}}{a^{2(s-k)}} - 1\right) \\
= & \frac{k_i}{\mu_1}\left(\sum_{k=0}^{s-1}\frac{(s-1)!}{k!}\cdot\frac{2(s-a)a^{s-k} + (s-k)(s-a)^2 a^{s-k-1}}{a^{2(s-k)}} + 1\right) \\
> & \ 0 \text{ when } 0 < a < s.
\end{aligned}$$

We then have

$$\begin{aligned}
\frac{\partial W_{s_1}}{\partial P_i} & = \frac{\partial}{\partial P_i}\left[\frac{1}{\mu_1} + \frac{1}{\mu_1}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-1}\right] \\
& = -\frac{1}{\mu}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-2}\cdot \\
& \quad \frac{\partial}{\partial P_i}\left(\frac{(s-1)!(s-a)}{a^s}^2 (\sum_{k=0}^{s-1}\frac{a^k}{k!}) + (s-a)\right)^{-1} \\
& < \ 0 \text{ when } 0 < a < s. \text{ i.e., when } P_i \le \frac{c_i}{k_i}
\end{aligned}$$

and

$$\frac{\partial W_{s_{2,i}}}{\partial P_i} = \frac{\partial}{\partial P_i}\left(\frac{1}{\mu_{2,i} - c_i + k_i P_i}\right) = -\frac{k_i}{(\mu_{2,i} - c_i + k_i P_i)^2} < 0 \ \text{ for } P_i \le \frac{c_i}{k_i}.$$

Thus the sojourn time of both stages, W_{s_1} and $W_{s_{2,i}}$ are decreasing functions of P_i for $i = 1, 2, \ldots, n$, for $P_i \in [0, \frac{c_i}{k_i}]$.

We present analytical results when $n = 1$, i.e, there is only one queue in the second stage of the tandem queueing system.

Proposition 5. *For the optimization problem (23), if $n = 1$ and*

$$w \ge \frac{1}{\mu_1} + \frac{1}{\mu_{2,1}},$$

the optimal price is

$$P_1^* = \begin{cases} \dfrac{c_1}{2k_1} & \text{if } (W_{s_1} + W_{s_{2,1}})|_{P_1 = \frac{c_1}{2k_1}} \le w \\ Q_{1,w} & \text{otherwise} \end{cases}$$

where

$$(W_{s_1} + W_{s_{2,1}})\big|_{P_1=Q_{1,w}} = w.$$

If $w < \frac{1}{\mu_1} + \frac{1}{\mu_{2,1}}$, *the problem is infeasible.*

Proof. Suppose

$$w < \frac{1}{\mu_1} + \frac{1}{\mu_{2,1}}.$$

In this case, since $W_{s_1} + W_{s_{2,1}}$ is decreasing with P_1 on $[0, \frac{c_1}{k_1}]$, for any

$$0 \le P_1 \le \frac{c_i}{k_i}, W_{s_1} + W_{s_{2,1}} \ge (W_{s_1} + W_{s_{2,1}})\big|_{P_1=\frac{c_1}{k_1}} = \frac{1}{\mu_1} + \frac{1}{\mu_{2,1}} > w,$$

the problem is infeasible.

Now let us suppose that

$$w \ge \frac{1}{\mu_1} + \frac{1}{\mu_{2,1}}.$$

We note

$$\frac{\partial V}{\partial P_1} = c_1 - 2k_1P_1 \quad \text{and} \quad \frac{\partial^2 V}{\partial P_1^2} = -2k_1 < 0.$$

Thus when $P_1 = c_1/2k_1$ the unconstrained global maximum of V can be achieved.

If

$$(W_{s_1} + W_{s_{2,1}})\big|_{P_1=\frac{c_1}{2k_1}} \le w$$

then $P_1^* = \frac{c_1}{2k_1}$ is feasible and thus the optimal solution.

If

$$(W_{s_1} + W_{s_{2,1}})\big|_{P_1=\frac{c_1}{2k_1}} > w$$

together with

$$(W_{s_1} + W_{s_{2,1}})\big|_{P_1=\frac{c_1}{k_1}} = \frac{1}{\mu_1} + \frac{1}{\mu_{2,1}} \le w.$$

Since $W_{s_1} + W_{s_{2,1}}$ is continuous and decreasing on $[\frac{c_1}{2k_1}, \frac{c_1}{k_1}]$, there exists $Q_{1,w}$ such that

$$(W_{s_1} + W_{s_{2,1}})\big|_{P_1=Q_{1,w}} = w.$$

We note that the feasible region

$$\begin{aligned}
S &= \{P_1 : (W_{s_1} + W_{s_{2,1}}) \le w \text{ and } 0 \le P_1 \le \frac{c_1}{k_1}\} \\
&= \{P_1 : (W_{s_1} + W_{s_{2,1}}) \le w\} \cap [0, \frac{c_1}{k_1}] \\
&= \{P_1 : P_1 \ge Q_{1,w}\} \cap [0, \frac{c_1}{k_1}] \ \ (\text{ since } W_{s_1} + W_{s_{2,1}} \text{ is decreasing on } [0, \frac{c_1}{k_1}]) \\
&= [Q_{1,w}, \frac{c_1}{k_1}]
\end{aligned}$$

Since

$$\frac{c_1}{2k_1} < Q_{1,w}, \quad \frac{\partial V}{\partial P_1} < 0$$

for all $P_1 \ge Q_{1,w}$. Thus V achieves its maximum on $[Q_{1,w}, \frac{c_1}{k_1}]$ when $P_1 = Q_{1,w}$. Therefore $P_1^* = Q_{1,w}$ is optimal.

Finally combining the two cases, we have

$$P_1^* = \begin{cases} \dfrac{c_1}{2k_1} & \text{if } (W_{s_1} + W_{s_{2,1}})\big|_{P_1=\frac{c_1}{2k_1}} \le w \\ Q_{1,w} & \text{otherwise} \end{cases}$$

is the optimal solution of the optimization problem. □

5. **Concluding Remarks.** In this paper, we study the steady-state distribution of a tandem queueing system with n types of customers, which consists of a M/M/s/∞ queue in the first stage and n M/M/$1/\infty$ queues in the second stage, one for each type of customers. We also study the average waiting time and sojourn time for the customers in the first stage. We then introduce two optimization problems, one to minimize the number of servers in the first stage, another to maximize the earnings rate by adjusting the price, both under constraints on the average waiting time of customers.

All queues in the second stage of the tandem queueing system have been assumed to be single-server queues. The case where these queues can have more than one servers may be further explored in the future. Also, the linear model between the prices and the arrival rates relationship may be replaced by more realistic and sophisticated models. For example, we could have a constant elasticity relationship instead of a linear relationship. Results can be obtained in a similar, but more complicated way.

Our model as a whole does not consider those customers who will leave the queues even during waiting at the first stage. In a marketing context, we call this "churn", a measure of customer attrition and the possibility is called the churn rate. It is in the sense that customers have decided to purchase a service at the very beginning but shift to the competitors during or after the actual service is provided. Finally we will extend the results in Section 4 to the case of $n > 1$.

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REFERENCES

[1] W. Ching (2001) *Iterative Methods for Queuing and Manufacturing Systems*. Springer Monographs in Mathematics, Springer, London.

[2] W. Ching and M. Ng (2006) *Markov Chains : Models, Algorithms and Applications*. International Series on Operations Research and Management Science, Springer, New York.

[3] R. B. Cooper (1972) *Introduction to Queueing Theory*. New York: Macmillan. pp 71 - 72

[4] P. J. Burke, *The Output of a Queueing System*, Operations Research 4 (1956) pp 699 - 704

[5] Y. Li, X. Cai, F. Tu and X. Shao, *Optimization of Tandem Queue Systems with Finite Buffers*, Comput. Oper. Res. 31 (2004) 963-984.

[6] S. Ross *Introduction to Probability Models* (8th Edition). San Diego, Calif. : Academic Press. (2003) Ch. 8

[7] J. B. Taylor *Principles of Microeconomics* (4th Edition). New York: Houghton Mifflin. (2004) 104-123.

[8] J. van Leeuwaarden and J. Resing, *A Tandem Queue with Coupled Processors: Computational Issues*, Queueing Syst. 51 (2005), 29-52.

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