On the Zariski closure of a germ of totally geodesic complex submanifold on a subvariety of a complex hyperbolic space form of finite volume

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A fundamental object of study in projective geometry is the Gauss map. Ein [Ei, 1982] proved that the Gauss map of a linearly non-degenerate projective submanifold $W \subset \mathbb{P}^n$ is generically finite. Zak [Za, 1993] further proved that it is in fact a birational morphism. An analogous study is pursued in [Za] on submanifolds of Abelian varieties.

In differential-geometric terms, generic finiteness of the Gauss map on a linearly non-degenerate projective submanifold $W \subset \mathbb{P}^n$ is the same as the vanishing of the kernel of the projective second fundamental form at a general point. The projective second fundamental form is on the one hand by definition determined by the canonical projective connection on the projective space, on the other hand it is the same as the second fundamental form with respect to the Fubini-Study metric on the projective space. It is therefore natural to extend the study of Gauss maps on subvarieties to the context of subvarieties of complex hyperbolic space forms, i.e., subvarieties of quotients of the complex unit ball B^n, either in terms of the canonical projective connection on $B^n \subset \mathbb{P}^n$ embedded by means of the Borel embedding, or in terms of the canonical Kähler-Einstein metric, noting that the Riemannian connection of the canonical Kähler-Einstein metric is an affine connection compatible with the canonical projective connection on B^n.

In the current article the first motivation is to examine the dual analogue of Ein's Theorem on the generic finiteness of the Gauss map mentioned above in the dual situation of subvarieties $W \subset X$ of compact complex hyperbolic space forms X, which are quotients B^n/Γ of the complex unit ball by torsion-free cocompact discrete groups $\Gamma \subset \mathrm{Aut}(B^n)$. Here the Gauss map is defined on regular points of $\pi^{-1}(W)$, where $\pi : B^n \to X$ is the universal covering map, and as such the Gauss map on W is defined only up to the action of Γ, and the question of birationality is not very meaningful. An analogue to Ein's Theorem in the case of complex hyperbolic space forms is the statement that, given a projective manifold $W \subset X = B^n/\Gamma$ which is not totally geodesic, and taking $\widetilde{W} \subset \pi^{-1}(W)$ to be any irreducible component, the Gauss map on the smooth locus of $\widetilde{W}$ is of maximal rank at a general point, or, equivalently, the kernel of the second fundamental

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form vanishes at a general point. We observe in the current article that this analogue does indeed hold true, and that furthermore, in contrast to the case of projective subvarieties, it holds true more generally for any irreducible complex-analytic subvariety $W \subset X$, *without* assuming that W is nonsingular, with a proof that generalizes to the case of quasi-projective subvarieties of complex hyperbolic space forms of finite volume.

In the case of cocompact lattices $\Gamma \subset \mathrm{Aut}(B^n)$ the result already follows from the study of homomorphic foliations in Cao-Mok [CM, 1990] arising from kernels of the second fundamental form, but we give here a proof that applies to arbitrary holomorphic foliations by complex geodesic submanifolds. Using the latter proof, we are able to study the Zariski closure of a single totally geodesic complex submanifold. For any subset $E \subset W$ we denote by $\mathrm{Zar}_W(E)$ the Zariski closure of E in W. We show that, if the projective variety $W \subset X = B^n/\Gamma$ admits a germ of complex geodesic submanifold $S \subset W$, then some Zariski open subset U of $\mathrm{Zar}_W(S)$ must admit a holomorphic foliation $\mathcal{F}$ by complex geodesics such that $\mathcal{F}$ restricts to a holomorphic foliation on $S \cap U$. Our stronger result on holomorphic foliations by complex geodesic submanifolds implies that $\mathrm{Zar}_W(S)$ must itself be a totally geodesic subset. For the proof of the implication we study varieties of tangents to complex geodesics on a subvariety $W \subset X = B^n/\Gamma$ analogous to the notion of varieties of minimal rational tangents on projective subvarieties uniruled by lines, a notion extensively studied in recent years by Hwang and Mok (cf. Hwang [Hw] and Mok [Mk4]). In a certain sense, total geodesy of the Zariski closure of a germ of complex geodesic submanifold on W results from the algebraicity of varieties of tangents to complex geodesics on W and the asymptotic vanishing of second fundamental forms on locally closed complex submanifolds on B^n swept out by complex geodesics (equivalently minimal disks). This link between the study of subvarieties W of compact complex hyperbolic space forms and those of projective subvarieties is in itself of independent interest, and it points to an approach in the study of Zariski closures of totally geodesic complex submanifolds on projective manifolds uniformized by bounded symmetric domains.

For the proof of our results in the case of non-uniform lattices $\Gamma \subset \mathrm{Aut}(B^n)$ we make use of the Satake-Borel-Baily compactification ([Sa, 1960], [BB, 1966]) in the case of arithmetic lattices, and the compactification by Siu-Yau [SY, 1982] obtained by differential-geometric means, together with the description of the compactification as given in Mok [Mk3, 2009], in the case of non-arithmetic lattices.

Our result on the Zariski closure of a germ of complex geodesic submanifold in the case of complex hyperbolic space forms is a special case of a circle of problems. In general, we are interested in the characterization of the Zariski closure of a totally geodesic complex submanifold S on a quasi-projective subvariety W of a compact or finite-volume quotient X of a bounded symmetric domain Ω. The inclusion $S \subset X$ is modeled on a pair (D, Ω), where $D \subset \Omega$ is a totally geodesic complex submanifold (which is itself a bounded symmetric domain). The nature and difficulty of the problem may depend on the pair (D, Ω). A characterization of the Zariski closure relates the question of existence of germs of totally geodesic complex submanifolds $S \subset W \subset X$ to that of the global existence of certain types of complex submanifolds $\mathrm{Zar}_W(S) \subset W$. For instance, taking Ω to be biholomorphically the Siegel upper half-plane $\mathcal{H}_g$ of genus $g \geq 2$, $X = \mathcal{H}_g/\Gamma$ to be the moduli space of principally polarized Abelian varieties (where Γ has some torsion), and taking W to be the closure of the Schottky locus, S to be a totally geodesic holomorphic curve, a characterization of $\mathrm{Zar}_W(S)$ probably relates the question of local existence of totally geodesic holomorphic curves to the global existence of totally geodesic holomorphic curves and rank-1 holomorphic geodesic subspaces, and hence to a conjecture of Oort's (cf. Hain [Ha]). In such situations, granted the characterization problem can be settled, a global non-existence result may imply a non-existence result which is local with respect to the complex topology, hence completely transcendental in nature.

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§1 Statement of the main result and background materials

(1.1) *The Main Theorem on Zariski closures of germs of complex geodesic submanifolds of complex hyperbolic space forms of finite volume* Let $n \geq 3$ and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, so that $X := B^n/\Gamma$ is of finite volume with respect to the canonical Kähler-Einstein metric. Let $W \subset X$ be a complex-analytic subvariety. A simply connected open subset U of the smooth locus $\mathrm{Reg}(W)$ can be lifted to a locally closed complex submanifold $\widetilde{U}$ on B^n, and on $\widetilde{U}$ we have the Gauss map which associates each point $y \in \widetilde{U}$ to $[T_y(\widetilde{U})]$ as a point in the Grassmann of m-planes in $\mathbb{C}^n$. This way one defines a Gauss map on $\mathrm{Reg}(W)$ which is well-defined only modulo the action of the image Φ of $\pi_1(W)$ in $\pi_1(X) = \Gamma$. When the Gauss map fails to be of maximal rank we have an associated holomorphic foliation defined at general points of W whose leaves are totally geodesic complex submanifolds. Partly motivated by the study of the Gauss map on subvarieties of complex hyperbolic space forms we are led to consider the Zariski closure of a single germ of totally geodesic complex submanifold on W. We have

Main Theorem. *Let $n \geq 2$ and denote by $B^n \subset \mathbb{C}^n$ the complex unit ball equipped with the canonical Kähler-Einstein metric $ds^2_{B^n}$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice. Denote by $X := B^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric ds^2_X induced from $ds^2_{B^n}$. Denote by $\overline{X}_{\min}$ the minimal compactification of X so that $\overline{X}_{\min}$ is a projective-algebraic variety and X inherits the structure of a quasi-projective variety from $\overline{X}_{\min}$. Let $W \subset X$ be an irreducible quasi-projective subvariety, and $S \subset W$ be a locally closed complex submanifold which is totally geodesic in X with respect to ds^2_X. Then, the Zariski closure $Z \subset W$ of S in W is a totally geodesic subset.*

Here totally geodesic complex submanifolds of X are defined in terms of the canonical Kähler-Einstein. Equivalently, they are defined in terms of the canonical holomorphic projective connection on X which descends from $B^n \subset \mathbb{P}^n$ (cf. (1.3)). With the latter interpretation it is clear that the second fundamental form on a locally closed complex submanifold $Z \subset X$ is holomorphic. Total geodesy of Z means precisely the vanishing of the (holomorphic) second fundamental form.

Consider $X = B^n/\Gamma$, where $\Gamma \subset \mathrm{Aut}(B^n)$ is a non-uniform torsion-free lattice. If Γ is arithmetic, we have the Satake-Borel-Baily compactification (Satake [Sa], Borel-Baily [BB]). For the rank-1 bounded symmetric domain B^n the set of rational boundary components consists of a Γ-invariant subset Π of ∂B^n, and Γ acts on $B^n \cup \Pi$ to give $\overline{X}_{\min} := (B^n \cup \Pi)/\Gamma$, which consists of the union of X and a finite number of points, to be called cusps, such that $\overline{X}_{\min}$ can be endowed naturally the structure of a normal complex space.

(1.2) *Description of Satake-Baily-Borel and Mumford compactifications and for $X = B^n/\Gamma$.* We recall briefly the Satake-Baily-Borel and Mumford compactification for $X = B^n/\Gamma$ in the case of a torsion-free non-uniform arithmetic subgroup $\Gamma \subset \mathrm{Aut}\,(B^n)$ (For details cf. Mok [Mk4]). Let $E \subset \partial B^n$ be the set of boundary points b such that for the normaliser $N_b = \{\nu \in \mathrm{Aut}\,(B^n) : \nu(b) = b\}$, $\Gamma \cap N_b$ is an arithmetic subgroup. The points $b \in E$ are the rational boundary components in the sense of Satake [Sa] and Baily-Borel [BB]. Modulo the action of Γ, the set $A = E/\Gamma$ of equivalence classes is finite. Set-theoretically the Satake-Baily-Borel compactification $\overline{X}_{\min}$ of X is obtained by adjoining a finite number of points, one for each $\alpha \in A$. Fixing $b \in E$ we consider the Siegel domain presentation S_n of B^n obtained via a Cayley transform which maps b to ∞, $S_n = \big\{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im} z_n > \|z'\|^2\big\}$. Identifying B^n with S_n via the Cayley transform, we write $X = S_n/\Gamma$. Writing $z' = (z_1, \ldots, z_{n-1}); z = (z'; z_n)$, the unipotent radical of $W_b \subset N_b$ is given by

$$W_b = \Big\{\nu \in N_b : \nu(z'; z_n) = (z' + a'; z_n + 2i\,\overline{a'} \cdot z' + i\|a'\|^2 + t) : a' \in \mathbb{C}^{n-1}, t \in \mathbb{R}\Big\}\,, \tag{1}$$

where $\overline{a'} \cdot z' = \sum_{i=1}^{n-1} \overline{a}_i z_i$. W_b is nilpotent and $U_b := [W_b, W_b]$ is the real 1-parameter group of translations $\lambda_t, t \in \mathbb{R}$, where $\lambda_t(z', z) = (z', z + t)$. For a rational boundary component b, $\Gamma \cap W_b \subset W_b$ is a lattice, and $[\Gamma \cap W_b, \Gamma \cap W_b] \subset U_b$ must be nontrivial. Thus, $\Gamma \cap U_b \subset U_b \cong \mathbb{R}$ must be a nontrivial discrete subgroup, generated by some $\lambda_\tau \in \Gamma \cap U_b$. For any nonnegative integer N define

$$S_n^{(N)} = \big\{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im} z_n > \|z'\|^2 + N\big\} \subset S_n\,. \tag{2}$$

Consider the holomorphic map $\Psi : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^{n-1} \times \mathbb{C}^\star$ given by

$$\Psi(z'; z_n) = (z', e^{\frac{2\pi i z_n}{\tau}}) := (w'; w_n)\,; \quad w' = (w_1, \cdots, w_{n-1})\,; \tag{3}$$

which realizes $\mathbb{C}^{n-1} \times \mathbb{C}$ as the universal covering space of $\mathbb{C}^{n-1} \times \mathbb{C}^*$. Write $G = \Psi(S_n)$ and, for any nonnegative integer N write $G^{(N)} = \Psi(S_n^{(N)})$. We have

$$\widehat{G}^{(N)} = \big\{(w'; w_n) \in \mathbb{C} : |w_n|^2 < e^{\frac{-4\pi N}{\tau}} \cdot e^{\frac{-4\pi}{\tau}\|w'\|^2}\big\}\,, \quad \widehat{G} = \widehat{G}^{(0)}\,. \tag{4}$$

$\Gamma \cap W_b$ acts as a discrete group of automorphisms on S_n. With respect to this action, any $\gamma \in \Gamma \cap W_b$ commutes with any element of $\Gamma \cap U_b$, which is generated by the translation λ_τ. Thus, $\Gamma \cap U_b \subset \Gamma \cap W_b$ is a normal subgroup, and the action of $\Gamma \cap W_b$ descends from S_n to $S_n/(\Gamma \cap U_b) \cong \Psi(S_n) = G$. Thus, there is a group homomorphism $\pi : \Gamma \cap W_b \to \mathrm{Aut}(G)$ such that $\Psi \circ \nu = \pi(\nu) \circ \Psi$ for any $\nu \in \Gamma \cap W_b$. More precisely, for $\nu \in \Gamma \cap W_b$ of the form (1) where $t = k\tau$. $k \in \mathbb{Z}$ we have

$$\pi(\nu)(w', w_n) = \big(w' + a', e^{-\frac{4\pi}{\tau}\overline{a'} \cdot w' - \frac{2\pi}{\tau}\|a'\|^2} \cdot w_n\big)\,. \tag{5}$$

$S_n/(\Gamma \cap W_b)$ can be identified with $G/\pi(\Gamma \cap W_b)$. Since the action of W_b on S_n preserves ∂S_n, it follows readily from the definition of $\nu(z'; z_n)$ that W_b preserves the domains $S_n^{(N)}$, so that $G^{(N)} \cong S_n^{(N)}/(\Gamma \cap U_b)$ is invariant under $\pi(\Gamma \cap W_b)$. Write $\widehat{G}^{(N)} = G^{(N)} \cup (\mathbb{C}^{n-1} \times \{0\}) \subset \mathbb{C}^n, \widehat{G}^{(0)} = \widehat{G}$. $\widehat{G}^{(N)}$ is the interior of the closure of $G^{(N)}$ in $\mathbb{C}^n$. The action of $\pi(\Gamma \cap W_b)$ extends to $\widehat{G}$. Here $\pi(\Gamma \cap W_b)$ acts as a torsion-free discrete group of automorphisms of $\widehat{G}$. Moreover, the action of $\pi(\Gamma \cap W_b)$ on $\mathbb{C}^{n-1} \times \{0\}$ is given by a lattice of translations Λ_b.

Denote the compact complex torus $(\mathbb{C}^{n-1} \times \{0\})/\Lambda_b$ by T_b. The Mumford compactification $\overline{X}_M$ of X is set-theoretically given by $\overline{X}_M = X \amalg (\amalg T_b)$, the disjoint union of compact complex tori being taken over $A = E/\Gamma$. Define $\Omega_b^{(N)} = \widehat{G}^{(N)}/\pi(\Gamma \cap W_b) \supset G^{(N)}/\pi(\Gamma \cap W_b) \cong S_n^{(N)}/(\Gamma \cap W_b)$. Then the natural map $G^{(N)}/\pi(\Gamma \cap W_b) = \Omega_b^{(N)} - T_b \hookrightarrow S_n/\Gamma = X$ is an open embedding for N sufficiently large, say $N \geq N_0$. The structure of $\overline{X}_M$ as a complex manifold is defined by taking $\Omega_b^{(N)}$, $N \geq N_0$, as a fundamental system of neighborhood of T_b. From the preceding description of $\overline{X}_M$ one can equip the normal bundle $\mathcal{N}_b$ of each compactifying divisor T_b in $\Omega_b^{(N)} (N \geq N_0)$ with a Hermitian metric of strictly negative curvature, thus showing that T_b can be blown down to a normal isolated singularity by Grauert's blowing-down criterion, which gives the Satake-Baily-Borel (*alias* minimal) compactification $\overline{X}_{\min}$. T_b is an Abelian variety since the conormal bundle $\mathcal{N}_b^\star$ is ample on T_b.

In Mok [Mk4] we showed that for non-arithmetic torsion-free non-uniform lattices $\Gamma \subset \mathrm{Aut}(B^n)$, the complex hyperbolic space form $X = \Omega/\Gamma$ admits a Mumford compactification $\overline{X}_M = X \amalg (\amalg T_b)$, with finitely many Abelian varieties $T_b = \mathbb{C}^n/\Lambda_b$, such that, by collapsing each of the finitely many Abelian varieties T_b, $\overline{X}_M$ blows down to a *projective-algebraic* variety $\overline{X}_{\min}$ with finitely many isolated normal singularities. (As in the statement of the Main Theorem, identifying $X \subset \overline{X}_{\min}$ as a Zariski open subset of the projective-algebraic variety $\overline{X}_{\min}$ we will regard X as a quasi-projective manifold and speak of the latter structure as the canonical quasi-projective structure.) The picture of a fundamental system of neighborhoods $\Omega_b^{(N)}$ of T_b, $N \geq N_0$, in $\overline{X}_M$ is exactly the same as in the arithmetic case. For the topological structure of X, by the results of Siu-Yau [SY] using Busemann functions, to start with we have a decomposition of X into the union of a compact subset $K \subset X$ and finitely many disjoint open sets, called ends, which in the final analysis can be taken to be of the form $\Omega_b^{(N)}$ for some $N \geq N_0$. Geometrically, each end is *a priori* associated to an equivalence class of geodesic rays, where two geodesic rays are said to be equivalent if and only if they are at a finite distance apart from each other. From the explicit description of $\Omega_b^{(N)}$ as given in the above, the space of geodesic rays in each end can be easily determined, and we have

Lemma 1. *Fix $n \geq 1$ and let $\Gamma \subset Aut(B^n)$ be a torsion-free non-uniform lattice, $X := \Omega/\Gamma$. Let $\overline{X}_M$ be the Mumford compactification and $\Omega_b^{(N)} = \widehat{G}^{(N)}/\pi(\Gamma \cap W_b)$ be a neighborhood of a divisor $T_b = \mathbb{C}^n/\Lambda_b$ at infinity. Denote by $p : G^{(N)} \to G^{(N)}/\pi(\Gamma \cap W_b)$ the canonical projection. Then, any geodesic ray $\lambda : [0, \infty) \to X$ parametrized by arc-length on the end $\Omega_b^{(N)} - T_b = G/\pi(\Gamma \cap W_b) \cong S_n^{(N)}/(\Gamma \cap W_b)$ must be of the form $\lambda(s) = p\,(\Psi(\zeta, a + Aie^{cs}))$ for some $\zeta \in \mathbb{C}^{n-1}$, $a \in \mathbb{R}$, some constant $c > 0$ determined by the choice of the canonical Kähler-Einstein metric on B^n and for some sufficiently large constant $A > 0$ (so that $\lambda(0) \in S_n^{(N)}$), where $\Psi(\zeta; \alpha) = (\zeta, e^{\frac{2\pi i \alpha}{\tau}})$, in which $\tau > 0$ and the translation $(z'; z) \to (z', z + \tau)$ is the generator of the infinite subgroup $\Gamma \cap U_b \subset U_b \cong \mathbb{R}$.*

Proof. A geodesic ray on an end $\Omega_b^{(N)} - T_b$ must lift to a geodesic ray on the Siegel domain S_n which converges to the infinity point ∞ of S_n. On the upper half-plane $\mathcal{H} = \{w > 0 : \mathrm{Im}(w) > 0\}$ a geodesic ray $\lambda : [0, \infty) \to \mathcal{H}$ parametrized by arc-length joining a point $w_0 \in \mathcal{H}$ to infinity must be of the form $\mu(s) = u_0 + iv_0 e^{cs}$ for some constant $c > 0$ determined by the Gaussian curvature of the Poincaré metric $ds^2_{\mathcal{H}}$ chosen. In what follows a totally geodesic holomorphic curve on S_n will also be referred to as a complex geodesic (cf. (1.3)). For the Siegel domain $S_n = \{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im} z_n > \|z'\|^2\}$, considered as a fibration over $\mathbb{C}^{n-1}$, the fiber $F_{z'}$ over each $z' \in \mathbb{C}^{n-1}$ is the translate of the upper half-plane by $i\|z\|^2$. $F_{z'} \subset S_n$ is a complex geodesic. It is then clear that any $\nu : [0, \infty) \to S_n$ of the form $\nu(s) = (\zeta, a + Aie^{cs})$ for $\zeta \in \mathbb{C}^{n-1}$ and for appropriate real constants a, c and A is a geodesic ray on S_n converging to ∞ (which corresponds to $b \in \partial B^n$). Any complex geodesic $S \subset S_n$ is the intersection of an affine line $L := \mathbb{C}\eta + \xi$ with S_n, and S is a disk on L unless the vector η is proportional to $e_n = (0, \cdots, 0; 1)$. Any geodesic ray λ on S_n lies on a uniquely determined complex geodesic S. Thus, $\lambda(s)$ tends to the infinity point ∞ of S_n only if S is parallel in the Euclidean sense to $\mathbb{C}e_n$, so that $\lambda([0, \infty) \subset F_{z'}$ for some $z' \in \mathbb{C}^{n-1}$. But any geodesic ray on $F_{z'}$ which converges to ∞ must be of the given form $\nu(s) = (\zeta, a + Aie^{cs})$, and the proof of Lemma 1 is complete. □

(1.3) *Holomorphic projective connections* For the discussion on holomorphic projective connections, we follow Gunning [Gu] and Mok [Mk2]. A holomorphic projective connection Π on an n-dimensional complex manifold X, $n > 1$, consists of a covering $\mathcal{U} = \{U_\alpha\}$ of coordinate open sets, with holomorphic coordinates $(z_1^{(\alpha)}, \cdots, z_n^{(\alpha)})$, together with holomorphic functions $\left({}^\alpha\Phi_{ij}^k\right)_{1 \leq i,j,k \leq n}$ on U_α symmetric in i, j satisfying the trace condition $\sum_k {}^\alpha\Phi_{ik}^k = 0$ for all i and satisfying furthermore on $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the transformation rule (†)

$$ {}^{\beta}\Phi^{\ell}_{pq} = \sum_{i,j,k} {}^{\alpha}\Phi^{k}_{ij} \frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}} \frac{\partial z_j^{(\alpha)}}{\partial z_q^{(\beta)}} \frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}} + \left[\sum_\ell \frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}} \frac{\partial^2 z_k^{(\alpha)}}{\partial z_p^{(\beta)} \partial z_q^{(\beta)}} - \delta^k_p \sigma_q^{(\alpha\beta)} - \delta^k_q \sigma_p^{(\alpha\beta)} \right], $$

where the expression inside square brackets defines the Schwarzian derivative $S(f_{\alpha\beta})$ of the holomorphic transformation given by $z^{(\alpha)} = f_{\alpha\beta}(z^{(\beta)})$, in which

$$ \sigma_p^{(\alpha\beta)} = \frac{1}{n+1} \frac{\partial}{\partial z_p^{(\beta)}} \log J(f_{\alpha\beta}) \,, $$

$J(f_{\alpha\beta}) = \det\left(\frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}}\right)$ being the Jacobian determinant of the holomorphic change of variables $f_{\alpha\beta}$. Two holomorphic projective connections Π and Π' on X are said to be equivalent if and only if there exists a common refinement $\mathcal{W} = \{W_\gamma\}$ of the respective open coverings such that for each W_γ the local expressions of Π and Π' agree with each other.

We proceed to relate holomorphic projective connections $\left({}^{\alpha}\Phi^k_{ij}\right)$ on a complex manifold to affine connections. Letting $\left({}^{\alpha}\overline{\Gamma}^k_{ij}\right)$ be any affine connection on X, we can define a torsion-free affine connection ∇ with Riemann-Christoffel symbols

$$ {}^{\alpha}\Gamma^k_{ij} = {}^{\alpha}\Phi^k_{ij} + \frac{1}{n+1} \sum_\ell \delta^k_i \; {}^{\alpha}\overline{\Gamma}^\ell_{\ell j} + \frac{1}{n+1} \sum_\ell \delta^k_j \; {}^{\alpha}\overline{\Gamma}^\ell_{i\ell} \,. \qquad (\sharp) $$

We say that ∇ is an affine connection associated to Π. Two affine connections ∇ and ∇' on a complex manifold X are said to be projectively equivalent (cf. Molzon-Mortensen [MM,§4]) if and only if there exists a smooth (1,0)-form ω such that $\nabla_\xi \zeta - \nabla'_\xi \zeta = \omega(\xi)\zeta + \omega(\zeta)\xi$ for any smooth (1,0)-vector fields ξ and ζ on an open set of X. For any complex submanifold S of X, the second fundamental form of S in X is the same for two projectively equivalent affine connections. In particular, the class of complex geodesic submanifolds S are the same. We will say that a complex submanifold $S \subset X$ is geodesic with respect to the holomorphic projective connection Π to mean that it is geodesic with respect to any (torsion-free) affine connection ∇ associated to Π. A geodesic 1-dimensional complex submanifold will simply be called a complex geodesic. Associated to a holomorphic projective connection there is a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}T_X$, called the tautological foliation, defined by the lifting of complex geodesics. We have

Lemma 2. *Let X be a complex manifold and $\pi : \mathbb{P}T_X \to X$ be its projectivized holomorphic tangent bundle. Then, there is a canonical one-to-one correspondence between the set of equivalence classes of holomorphic projective connections* *on X and the set of holomorphic foliations $\mathcal{F}$ on $\mathbb{P}T_X$ by tautological liftings of holomorphic curves.*

From now on we will not distinguish between a holomorphic projective connection and an equivalence class of holomorphic projective connections. Let M be a complex manifold equipped with a holomorphic projective connection Π, ∇ be any affine connection associated to Π by means of ($\sharp$), and $X \subset M$ be a complex submanifold. Then, the second fundamental form σ of X in M is independent of the choice of ∇. We call σ the projective second fundamental form of $X \subset M$ with respect to Π. Since locally we can always choose the flat background affine connection it follows that the projective second fundamental form is *holomorphic.*

Consider now the situation where M is a complex hyperbolic space form, a complex Euclidean space form, or the complex projective space. M is equipped with a canonical Kähler metric g of constant negative resp. zero resp. positive holomorphic sectional curvature. The universal covering space of M is the complex unit ball B^n resp. the complex Euclidean space $\mathbb{C}^n$ resp. the complex projective space $\mathbb{P}^n$ (itself), equipped with the canonical Kähler-Einstein metric resp. the Euclidean metric resp. the Fubini-Study metric. For $\mathbb{C}^n$ the family of affine lines leads to a tautological foliation $\mathcal{F}_0$ on the projective tangent bundle, and g is associated to the flat holomorphic projective connection. In the case of $\mathbb{P}^n$ the projective lines, which are closures of the affine lines in $\mathbb{C}^n \subset \mathbb{P}^n$, are totally geodesic with respect to the Fubini-Study metric g. In the case of $B^n \subset \mathbb{C}^n$, the intersections of affine lines with B^n give precisely the minimal disks which are totally geodesic with respect to the canonical Kähler-Einstein metric g. As a consequence, the tautological foliation $\mathcal{F}$ on $\mathbb{P}T_{\mathbb{P}^n}$ defined by the tautological liftings of projective lines, which is invariant under the projective linear group $\mathrm{Aut}(\mathbb{P}^n) \cong \mathbb{P}\mathrm{GL}(n+1)$, restricts to tautological foliations on $\mathbb{C}^n$ resp. B^n, and they descend to quotients Z of $\mathbb{C}^n$ resp. B^n by torsion-free discrete groups of holomorphic isometries of $\mathbb{C}^n$ resp. B^n, which are in particular projective linear transformations. The holomorphic projective connection on $\mathbb{P}^n$ corresponding to $\mathcal{F}$ will be called the canonical holomorphic projective connection. The same term will apply to holomorphic projective connections induced by the restriction of $\mathcal{F}$ to $\mathbb{C}^n$ and to B^n and to the tautological foliations induced on their quotient manifolds X as in the above. Relating the canonical holomorphic projective connections to the canonical Kähler metric g, we have the following result (cf. Mok [Mk2, (2.3), Lemma 2]) which in particular proves that the second fundamental form σ is holomorphic.

Lemma 3. *Let (M, g) be a complex hyperbolic space form, a complex Euclidean* *space form, or the complex projective space equipped with the Fubini-Study metric. Then, the affine connection of the Kähler metric g is associated to the canonical holomorphic projective connection on M. As a consequence, given any complex submanifold $X \subset M$, the second fundamental form on X as a Kähler submanifold of (M, g) agrees with the projective second fundamental form of X in M with respect to the canonical holomorphic projective connection.*

We will make use of Lemma 3 to study locally closed submanifolds of complex hyperbolic space forms admitting a holomorphic foliation by complex geodesic submanifolds. The first examples of such submanifolds are given by level sets of the Gauss map. By Lemma 3, it is sufficient to consider the second fundamental form with respect to the flat connection in a Euclidean space, and we have the following well-known lemma. (For a proof cf. Mok [Mk1, (2.1), Lemma 2.1.3].)

Lemma 4. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$ be a closed complex submanifold. At $z \in Z$ denote by $\sigma_z : T_z(Z) \times T_z(Z) \to N_{Z|\Omega,z}$ the second fundamental form with respect to the Euclidean flat connection ∇ on Ω. Denote by $\mathrm{Ker}(\sigma_z) \subset T_z(Z)$ the complex vector subspace of all η such that $\sigma_z(\tau, \eta) = 0$ for any $\tau \in T_z(Z)$. Suppose $\mathrm{Ker}(\sigma_z)$ is of the same positive rank d on Z. Then, the distribution $z \to \mathrm{Re}(\mathrm{Ker}(\sigma_z))$ is integrable and the integral submanifolds are open subsets of d-dimensional affine-linear subspaces.*

With regard to the Gauss map, in the case of projective submanifolds we have the following result of Ein [Ei, 1982] according to which the Gauss map is generically finite on a non-linear projective submanifold. Expressed in terms of the second fundamental form, we have

Theorem (Ein [Ei]). *Let $W \subset \mathbb{P}^N$ be a k-dimensional projective submanifold other than a projective linear subspace. For $w \in W$ denote by $\sigma_w : T_w(W) \times T_w(W) \to N_{W|\mathbb{P}^n,w}$ the second fundamental form in the sense of projective geometry. Then, $\mathrm{Ker}(\sigma_w) = 0$ for a general point $w \in W$.*

In §2 we will prove a result which includes the dual analogue of Ein's result for complex submanifolds of compact complex hyperbolic space forms. As will be seen, smoothness is not essential for the validity of a dual version of the result of Ein [Ei].

For the purpose of studying asymptotic behavior of the second fundamental form on certain submanifolds of the complex unit ball B^n we will need the following standard fact about the canonical Kähler-Einstein metric $ds^2_{B^n}$. We will normalize the latter metric so that the minimal disks on B^n are of constant holomorphic sectional curvature -2. With this normalization, writing $z = (z_1, \cdots, z_n)$ for the Euclidean coordinates on $\mathbb{C}^n$, and denoting by $\|\cdot\|$ the Euclidean norm, the Kähler form ω_n of $ds^2_{B^n}$ is given by $\omega_n = i\partial\overline{\partial}(-\log(1 - \|z\|^2)$. We have

Lemma 5. *Let $n \geq 1$ and (B^n, ds_{B^n}) be the complex unit n-ball equipped with the canonical Kähler-Einstein metric of constant holomorphic sectional curvature -2. Write $\left(g_{\alpha\overline{\beta}}\right)_{1\leq\alpha,\beta\leq n}$ be the expression of $ds^2_{B^n}$ as Hermitian matrices in terms of the Euclidean coordinates $z = (z_1, \cdots, z_n)$. Let t be a real number such that $0 \leq t \leq 1$. Then, at $(t, 0, \cdots, 0) \in B^n$ we have*

$$g_{1\overline{1}}(t, 0, \cdots, 0) = \frac{1}{1 - t^2} \;;\; g_{\alpha\overline{\alpha}}(t, 0, \cdots, 0) = \frac{1}{\sqrt{1 - t^2}} \text{ for } 2 \leq \alpha \leq n\,;$$
$$g_{\beta\overline{\gamma}}(t, 0, \cdots, 0) = 0 \text{ for } \beta \neq \gamma,\, 1 \leq \beta, \gamma \leq n\,. \tag{1}$$

Proof. The automorphism group $\mathrm{Aut}(B^n)$ acts transitively on B^n. Especially, given $0 \leq t \leq 1$ we have the automorphism $\Psi_t = (\psi_t^1, \cdots, \psi_t^n)$ on B^n defined by

$$\Psi_t(z_1, z_2, \cdots, z_n) = \left(\frac{z + t}{1 + tz}, \frac{\sqrt{1 - t^2}\, z_2}{1 + tz}, \cdots, \frac{\sqrt{1 - t^2}\, z_n}{1 + tz}\right),$$

which maps 0 to $(t, 0, \cdots, 0)$. We have

$$\frac{\partial \psi_t^1}{\partial z_1}(0) = 1 - t^2 \;;\; \frac{\partial \psi_t^\alpha}{\partial z_\alpha}(0) = \sqrt{1 - t^2} \text{ for } 2 \leq \alpha \leq n\,;$$
$$\frac{\partial \psi_t^\beta}{\partial z_\gamma}(0) = 0 \text{ for } \beta \neq \gamma,\, 1 \leq \beta, \gamma \leq n\,. \tag{2}$$

Since the Kähler form ω_n of $ds^2_{B^n}$ is defined by $\omega_n = i\partial\overline{\partial}(-\log(1 - \|z\|^2)$ we have by direct computation $g_{\alpha\overline{\beta}}(0) = \delta_{\alpha\beta}$, the Kronecker delta. Lemma 5 then follows from the invariance of the canonical Kähler-Einstein metric under automorphisms, as desired. □

Any point $z \in B^n$ is equivalent modulo a unitary transformation to a point $(t, 0, \cdots, 0)$ where $0 \leq t \leq 1$. If we write $\delta(z) = 1 - \|z\|$ on B^n for the Euclidean distance to the boundary ∂B^n, then Lemma 5 says that, for a point $z \in B^n$ the canonical Kähler-Einstein metric grows in the order $\delta(z)$ in the normal direction and in the order $\sqrt{\delta(z)}$ in the complex tangential directions.

§2 **Proof of the results**

(2.1) *Total geodesy of quasi-projective complex hyperbolic space forms holomorphic foliated by complex geodesic submanifolds* As one of our first motivations we were aiming at proving a dual analogue of Ein's result on the Gauss map of projective submanifolds as stated in [(1.3), Theorem (Ein [Ei])]. By [(1.3), Lemma 4], when the Gauss map on a locally closed complex submanifold of B^n fails to be generically finite, on some neighborhood of a general point of the submanifold we obtain a holomorphic foliation whose leaves are complex geodesic submanifolds, equivalently totally geodesic complex submanifolds with respect to the canonical Kähler-Einstein metric. We consider this more general situation and prove first of all the following result which in particular implies a dual version of the result of Ein [Ei] for which the assumption of smoothness of the submanifold is no longer needed.

Proposition 1. *Let $n \geq 3$, and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, $X := B^n/\Gamma$ be the quotient manifold equipped with the canonical structure as a quasi-projective manifold. Let s, d be positive integers such that $s+d := m < n$. Write $\pi : B^n \to X$ for the universal covering map, and let $W \subset X$ be an irreducible m-dimensional quasi-projective subvariety. Denote by $\widetilde{W}$ an irreducible component of $\pi^{-1}(W)$. Let $x \in \widetilde{W}$ be a smooth point, U be a neighborhood of x on the smooth locus of $\widetilde{W}$, $Z \subset U$ be an s-dimensional complex submanifold, and $D \subset T_U$ be an integrable d-dimensional holomorphic foliation such that T_Z and D are transversal to each other on Z,* i.e., *$T_z(U) = T_z(Z) \oplus D_z$ for every $z \in Z$, and such that the leaves on U of the holomorphic foliation $\mathcal{F}$ defined by D are totally geodesic on B^n. Then, $\widetilde{W} \subset B^n$ is itself totally geodesic in B^n.*

Proof. Assume first of all that $W \subset X$ is compact. Write $z = (z_1, \cdots, z_n)$ for the Euclidean coordinates on $\mathbb{C}^n$. We choose now special holomorphic coordinates $\zeta = (\zeta_1, \cdots, \zeta_m)$ on U at $x \in Z \subset U$, as follows. Let $(\zeta_1, \cdots, \zeta_s)$ be holomorphic coordinates on a neighborhood of x in Z. We may choose $(\zeta_1, \cdots, \zeta_s)$ to be 0 at the point $x \in Z$, and, shrinking Z if necessary, assume that the holomorphic coordinates $(\zeta_1, \cdots, \zeta_s)$ are everywhere defined on Z, giving a holomorphic embedding $f : \Delta^s \xrightarrow{\cong} Z \subset U$. Again shrinking Z if necessary we may assume that there exist holomorphic D-valued vector fields $\eta_1, \cdots, \eta_d$ on Z which are linearly independent everywhere on D (hence spanning the distribution D along Z). Write $\zeta = (\zeta', \zeta'')$, where $\zeta' := (\zeta_1, \cdots, \zeta_s)$ and $\zeta'' := (\zeta_{s+1}, \cdots, \zeta_m)$. Define now $F : \Delta^s \times \mathbb{C}^d \to \mathbb{C}^n$ by

$$F(\zeta', \zeta'') = f(\zeta') + \zeta_{s+1}\eta_1(\zeta') + \cdots + \zeta_m\eta_d(\zeta') \ . \qquad (1)$$

By assumption the leaves of the holomorphic foliation $\mathcal{F}$ defined by D are totally geodesic on the unit ball B^n. Now the totally geodesic complex submanifolds are precisely intersections of B^n with complex affine-linear subspaces of $\mathbb{C}^n$, so that $F(\zeta)$ lies on the smooth locus of $\widetilde{W}$ for ζ belonging to some neighborhood of $\Delta^s \times \{0\}$. Now F is a holomorphic immersion at 0 and hence everywhere on $\Delta^s \times \mathbb{C}^d$ excepting for ζ belonging to some subvariety $E \subsetneq \Delta^s \times \mathbb{C}^d$. Choose now $\xi = (\xi', \xi'') \in \Delta^s \times \mathbb{C}^d$ such that F is an immersion at ξ and such that $F(\xi) := p \in \partial B^n$, $p = F(\xi)$. Let U be a neighborhood of ξ in $\Delta^s \times \mathbb{C}^d$ and $V = B^n(p; r)$ such that $F|_U : U \to \mathbb{C}^n$ is a holomorphic embedding onto a complex submanifold $S_0 \subset V$. Let σ denote the second fundamental form of the complex submanifold $S := S_0 \cap B^n \subset B^n$ with respect to the canonical Kähler-Einstein metric $ds^2_{B^n}$ on B^n. To prove Proposition 1 it is sufficient to show that σ vanishes identically on $S := S_0 \cap B^n$. By Lemma 3 the second fundamental form $\sigma : S^2T_S \to N_{S|B^n}$ agrees with the second fundamental form defined by the canonical projective connection on S, and that in turn agrees with the second fundamental form defined by the flat Euclidean connection on $\mathbb{C}^n$. The latter is however defined not just on S but also on $S_0 \subset V$. Thus, we have actually a holomorphic tensor $\sigma_0 : S^2T_{S_0} \to N_{S_0|V}$ such that $\sigma = \sigma_0|_S$. Denote by $\| \cdot \|$ the norm on $S^2T^\star_S \otimes N_{S|V\cap B^n}$ induced by $ds^2_{B^n}$. We claim that, for any point $q \in V \cap \partial B^n$, $\|\sigma(z)\|$ tends to 0 as $z \in S$ tends to q. By means of the holomorphic embedding $F|_U : U \to S \subset V$ identify σ_0 with a holomorphic section of $S^2T^\star_U \otimes F^*T^\star_V / dF(T_U)$. For $1 \le k \le n$ write $\epsilon_k := F^* \frac{\partial}{\partial z_k}$ and $\nu_k := \epsilon \bmod dF(T_U)$. Then, writing

$$\sigma_0(\zeta) = \sum_{\alpha,\beta=1}^{n} \sigma^k_{\alpha\beta}(\zeta)\, d\zeta^\alpha \otimes d\zeta^\beta \otimes \nu_k(\zeta) \ , \tag{2}$$

for $\zeta \in U \cap F^{-1}(B^n)$ we have

$$\|\sigma(\zeta)\| \le \sum_{\alpha,\beta=1}^{n} \left|\sigma^k_{\alpha\beta}(\zeta)\right| \, \|d\zeta^\alpha\| \, \|d\zeta^\beta\| \, \|\nu_k(\zeta)\| \ , \tag{3}$$

where the norms $\| \cdot \|$ are those obtained by pulling back the norms on $T^\star_V$ and on $N_{S|V\cap B^n}$. In what follows we will replace V by $B^n(p; r_0)$ where $0 < r_0 < r$ and shrink U accordingly. Since the holomorphic functions $\sigma^k_{\alpha\beta}(\zeta)$ are defined on a neighborhood of $\overline{U}$, they are bounded on U and hence on $U \cap F^{-1}(B^n)$. From Lemma 5 we have

$$\|dz^k\| \le C_1 \sqrt{\delta(F(\zeta))} \tag{4}$$

on $B^n \subset \mathbb{C}^n$. Given $q \in V \cap \partial B^n$, by means of the embedding $F|_U : U \to V \subset \mathbb{C}^n$, the holomorphic coordinates $(\zeta_1, \cdots, \zeta_m)$ on V can be completed to holomorphic coordinates $(\zeta_1, \cdots, \zeta_m; \zeta_{m+1}, \cdots, \zeta_n)$ on a neighborhood of q in B^n. Since F is a holomorphic embedding on a neighborhood of $\overline{U}$, expressing each $d\zeta^\alpha$, $1 \le \alpha \le m$, in terms of $dz^k, 1 \le k \le n$, it follows that

$$\|d\zeta^\alpha\| \le C_2\sqrt{\delta(F(\zeta))} \tag{5}$$

for some positive constant C_2 independent of $\zeta \in U$, where $\delta(z) = 1 - \|z\|$ on B^n. (Although the expression of $d\zeta^\alpha$, $1 \le \alpha \le m$, in terms of dz^k, $1 \le k \le m$, depends on the choice of complementary coordinates $(\zeta_{m+1}, \cdots, \zeta_n)$, one can make use of any choice of the latter locally for the estimates.) On the other hand, again by Lemma 5 we have

$$\|\epsilon(\zeta)\| \le C_3\delta(F(\zeta)) \tag{6}$$

for some positive constant C_3. By definition $\|\nu_k(\zeta)\| \le \|\epsilon_k(\zeta)\|$, and the estimates (5) and (6) then yield

$$\|\sigma(\zeta)\| \le C_4 \ , \tag{7}$$

for some positive constant C_4 for $\zeta \in U \cap F^{-1}(B^n)$. For the claim we need however to show that $\|\sigma(\zeta)\|$ converges to 0 as $z = F(\zeta)$ converges to $q \in V \cap \partial B^n$. For this purpose we will use a better estimate for $\nu_k(\zeta)$. Write $q = F(\mu)$, $\mu = (\mu', \mu'')$. Assume that $F(0) = 0$, that for $1 \le i \le d$ we have $\eta_i(\mu') = (0, \cdots 0, 1, 0, \cdots 0)$ with 1 in the i-th position, and that $F(\mu) = (1, 0, \cdots, 0)$. We consider now $\zeta_t = (\mu'; t, 0 \cdots, 0)$, $0 < t < 1$, as t approaches to 1. Writing $q_t = F(\zeta_t)$, we have

$$\begin{aligned} \|\nu_k(\zeta_t)\| = \|\epsilon_k \text{ mod } dF(T_U)\| &= \left\| \frac{\partial}{\partial z_k} \text{ mod } T_{q_t}(V) \right\| \\ &\le \left\| \frac{\partial}{\partial z_k} \text{ mod } \mathbb{C}\frac{\partial}{\partial z_1} \right\| \le C\sqrt{\delta(q_t)} \ . \end{aligned} \tag{8}$$

To prove the claim we have to consider the general situation of a point $q \in V \cap \partial B^n$, $q = F(\mu)$, and consider $\zeta \in (\Delta_s \times \mathbb{C}^d) \cap F^{-1}(B^n)$ approaching q. Given such a point $\zeta = (\zeta'; \zeta'')$ there exists an automorphism φ of B^n such that $\varphi(F(\zeta'; 0)) = 0$, $\varphi(F(\zeta)) = (t, 0 \cdots, 0)$ for some $t \in (0, 1)$. Moreover, replacing $\eta_1(\zeta'), \cdots \eta_d(\zeta')$ by a basis of the d-dimensional complex vector space spanned by these d linearly independent vector fields, we may assume that $\eta_i(\zeta') = (0, \cdots, 0, 1, 0 \cdots, 0)$ with 1 in the i-th position. The estimates in (8) then holds true for $\sigma(\zeta)$ at the expense of introducing some constant which can be taken independent of $\zeta \in V \cap B^n$ (noting that U and hence V have been shrunk). As ζ converges to q, t converges to 1 too, and, replacing the estimate (6) for ϵ_k (and hence for ν_k) by the sharper estimate (8) we have shown that $\|\sigma(\zeta)\| \to 0$ as $\zeta \to q$ on $V \cap B^n$, proving the claim. (We have actually the uniform estimate $\|\sigma(\zeta)\| \le C\sqrt{(\delta(F(z))}$ for some positive constant C on $V \cap B^n$, but this estimate will not be needed in the sequel.)

We proceed now to prove Proposition 1 under the assumption that $\Gamma \subset \mathrm{Aut}(B^n)$ is cocompact. Pick any $q \in V \cap \partial B^n$. Choose a sequence of points $z_k \in V \cap B^n \subset \widetilde{W}$ converging to q, and we have $\|\sigma(z_k)\| \to 0$ as $k \to \infty$. Since $\Gamma \subset \mathrm{Aut}(B^n)$ is cocompact, X is compact, so is $W \subset X$. $W = \widetilde{W}/\Phi$ for some discrete subgroup $\Phi \subset \Gamma$ which stabilizes $\widetilde{W}$ as a set. Thus, there exists a compact subset $K \subset \widetilde{W}$ and elements $\varphi_k \in \Phi$ such that $x_k := \varphi_k^{-1}(z_k) \in K$. Passing to a subsequence if necessary we may assume that $\varphi_k^{-1}(z_k)$ converges to some point $x \in K$. Thus

$$\|\sigma(x)\| = \lim_{k\to\infty} \|\sigma(x_k)\| = \lim_{k\to\infty} \|\sigma(\varphi_k^{-1}(z_k)\| = \lim_{k\to\infty} \|\sigma(z_k)\| = 0 \ . \tag{9}$$

Now if $y \in \widetilde{W}$ is any point, for the distance function $d(\cdot,\cdot)$ on $(B^n, ds^2_{B^n})$ we have $d(\varphi_k(y), z_k) = d(\varphi_k(y), \varphi_k(x)) = d(y,x)$. Since $z_k \in V \cap \partial B^n$ converges in $\mathbb{C}^n$ to $q \in V \cap \partial B^n$, from the estimates of the metric $ds^2_{B^n}$ in Lemma 5 it follows readily that $\varphi_k(y)$ also converges to q, showing that $\|\sigma(y)\| = \lim_{k\to\infty} \|\sigma(\varphi_k(y))\| = 0$. As a consequence σ vanishes identically on $\widetilde{W}$, implying that $\widetilde{W} \subset B^n$ is totally geodesic, as desired.

It remains to consider the case $X = B^n/\Gamma$, where $\Gamma \subset \mathrm{Aut}(B^n)$ is a torsion-free non-uniform lattice, and $W \subset X \subset \overline{X}_{\min}$ is a quasi-projective subvariety. Recall that the minimal compactification of X is given by $\overline{X}_{\min} = X \amalg \{Q_1, \cdots, Q_N\}$ where $Q_j, 1 \le j \le N$, are cusps at infinity. Renumbering the cusps if necessary, the topological closure $\overline{W}$ of $W \subset \overline{X}_{\min}$ is given by $\overline{W} = W \cup \{Q_1, \cdots, Q_M\}$ for some nonnegative integer $M \le N$. Pick $q \in V \cap \partial B^n$ and let $(z_k)_{k=1}^{\infty}$ be a sequence of points on $V \cap B^n$ such that z_k converges to q. Recall that $\pi : B^n \to X$ is the universal covering map. Either one of the following alternatives occurs. (a) Passing to a subsequence if necessary $\pi(z_k)$ converges to some point $w \in W$. (b) There exists a cusp Q_ℓ, $1 \le \ell \le M$, such that passing to a subsequence $\pi(z_k)$ converges in $\overline{W}$ to Q_ℓ. In the case of Alternative (a), picking $x \in B^n$ such that $\pi(x) = w$, by exactly the same argument as in the case of cocompact lattices Γ it follows that the second fundamental form σ vanishes on $\widetilde{W}$ and thus $\widetilde{W} \subset B^n$ is totally geodesic. It remains to treat Alternative (b). Without loss of generality we may assume that $\pi(z_k)$ already converges to the cusp Q_ℓ.

We adopt essentially the notation in (1.2) on Mumford compactifications, and modifications on the notation will be noted. Let now $\overline{X}_M$ be the Mumford compactification of X given by $\overline{X}_M = X \amalg (T_1 \amalg \cdots \amalg T_N)$, where each $T_j = \mathbb{C}^{n-1}/\Lambda_j$ is an Abelian variety such that the canonical map $\rho : \overline{X}_M \to \overline{X}_{\min}$ collapses T_j to the cusp Q_j. (Here and henceforth we write T_j for T_{b_j}, Λ_j for Λ_{b_j}, etc. Write Ω_j for $\Omega_j^{(N)}$ for some sufficiently large integer N, so that the canonical projection $\mu_j : \Omega_j \to T_j$ realizes Ω_j as a disk bundle over T_j. Write $\Omega_j^0 := \Omega_j - T_j$. Considering each $\pi(z_k), 1 \leq k < \infty$, as a point in $X \subset \overline{X}_M$, replacing (z_k) by a subsequence if necessary we may assume that z_k converges to a point $P_\ell \in T_\ell$. Recall that the fundamental group of the bundle Ω_ℓ^0 of puncture disks is a semi-direct product $\Lambda_\ell \times \Psi_\ell$ of the lattice $\Lambda_\ell \cong \mathbb{Z}^{2(n-1)}$ with an infinite cyclic subgroup $\Psi_\ell := \Gamma \cap U_{b_\ell} \subset U_{b_\ell} = [W_{b_\ell}, W_{b_\ell}]$. Let D be a simply connected neighborhood of $P_\ell \in T_\ell$, and define $R := \mu_\ell^{-1}(D) - T_\ell \cong D \times \Delta^\star$ diffeomorphically. Then, $\pi_1(R)$ is infinite cyclic. Without loss of generality we may assume that $\pi(z_k)$ to be contained in the same irreducible component E of $R \cap W$. Consider the canonical homomorphisms $\pi_1(E) \to \pi_1(R) \to \pi_1(\Omega_\ell^0) \to \pi_1(X) = \Gamma$. By the description of the Mumford compactification the homomorphisms $\mathbb{Z} \cong \pi_1(R) \to \pi_1(\Omega_\ell^0)$ and $\pi_1(\Omega_\ell^0) \to \pi_1(X) = \Gamma$ are injective. We claim that the image of $\pi_1(E)$ in $\pi_1(R) \cong \mathbb{Z}$ must be infinite cyclic. For the justification of the claim we argue by contradiction. Supposing otherwise the image must be trivial, and E can be lifted in a univalent way to a subset $\widetilde{E} \subset \widetilde{W} \subset B^n$ by a holomorphic map $h : E \to B^n$. Let E' be the normalization of E, and $\overline{E}'_M$ be the normalization of $\overline{E}_M \subset X_M$. Composing h on the right with the normalization $\nu : E' \to E_h$ we have $h' : E' \to \widetilde{E}$. Since $\widetilde{E} \subset B^n$ is bounded, by Riemann Extension Theorem the map h' extends holomorphically to $h^\sharp : \overline{E}'_M \to \mathbb{C}^n$. Suppose $c \in \overline{E}'_M - E'$ and $h^\sharp(c) = a \in B^n$. Since $\pi(h(e)) = e$ for any $e \in E$ it follows that $\pi(h^\sharp(c)) = c \in \overline{W}_M - W$, contradicting with the definition of $\pi : B^n \to X \subset \overline{X}_M$. We have thus proven that $h^\sharp(\overline{E}'_M) \subset \overline{B^n}$ with $h^\sharp(\overline{E}'_M - E') \subset \partial B^n$, a plain contradiction to the Maximum Principle, proving by contradiction that the image of $\pi_1(E) \to \pi_1(R))$ is infinite cyclic, as claimed. As a consequence of the claim, the image of $\pi_1(E)$ in $\pi_1(X) = \Gamma$ is also infinite cyclic. Factoring through $\pi_1(E) \to \pi_1(W) \to \pi_1(X)$, and recalling that Φ is the image of $\pi_1(W)$ in $\pi_1(X)$, the image of $\pi_1(E)$ in Φ is also infinite cyclic.

Recall that $F : U \to V \subset B^n$ is a holomorphic embedding, $\zeta = (\zeta', \zeta'') \in U$, $q = F(\zeta) \in V \cap \partial B^n$, and assume that Alternative (b) occurs for any sequence of points $z_k \in V \cap B^n$ converging to q. To simplify notations assume without loss of generality that $F(\zeta', 0) = 0$, and define $\lambda_0(t) = F(\zeta', t\zeta'')$ for $t \in [0, 1]$. Then, reparametrizing λ_0 we have a geodesic ray $\lambda(s)$, $0 \leq s < \infty$ with respect to $ds^2_{B^n}$ parameterized by arc-length such that $\lambda(s)$ converges to $q \in \partial B^n$ in $\mathbb{C}^n$ as $s \to \infty$. For the proof of Proposition 1, it remains to consider the situation where Alternative (b) occurs for any choice of divergent sequence (z_k), $z_k = \lambda(s_k)$ with $s_k \to \infty$. When this occurs, without loss of generality we may assume that $\pi(\lambda([0, \infty))) \subset \Omega_\ell^0$, which is an end of X. Now by [(1.2), Lemma 1] all geodesic rays in Ω_ℓ^0 can be explicitly described, and, in terms of the unbounded realization S_n of B_n they lift as a set to $\{(z_0^1, \cdots, z_0^{n-1}, w_0) : w = u_0 + i\,\mathrm{Im}\,v, v \geq v_0\}$ for some point $(z_0^1, \cdots z_0^{n-1}; u_0 + iv_0) \in S_n$. We may choose $E \subset \pi^{-1}(R)$ in the last paragraph to contain the geodesic ray $\pi(\lambda([0,\infty))$ which converges to the point $P_\ell \in T_\ell$. Now $\Omega_\ell^0 = G_\ell^{(N)}/\pi(\Gamma' \cap W_{b_\ell})$ for the domain $G_\ell^{(N)} \subset S^n$ which is obtained as a Cayley transform of B^n mapping some $b_\ell \in \partial B^n$ to ∞. The pre-image $\pi^{-1}(\pi(\lambda([0,\infty)))$ is necessarily a countable disjoint union of geodesic rays R_i, which is the image of a parametrized geodesic ray $\rho_i : [0,\infty) \to B^n$ such that $\lim_{s\to\infty} \rho_i(s) := a_i \in \partial B^n$. For any two of such geodesic rays R_i, R_j there exists $\gamma_{ij} \in \Gamma$ such that $R_i = \gamma_{ij}(R_j)$, hence $\gamma_{ij}(a_j) = a_i$. Now both b_ℓ and q are end points of such geodesic rays on B^n and we conclude that $b_\ell = \gamma(q)$ for some $\gamma \in \Gamma$. In what follows without loss of generality we will assume that q is the same as b_ℓ.

For $q = b_\ell \in \partial B^n$, let $\chi : S_n \to B^n$ be the inverse Cayley transform which maps the boundary ∂S_n (in $\mathbb{C}^n$) to $\partial B^n - \{b_\ell\}$. Write $\chi\big(\widehat{G}_\ell^{(N)}\big) := H_\ell$. Denote by μ a generator of the image of $\pi_1(E)$ in $\pi_1(X) = \Gamma$. Recall that, with respect to the unbounded realization of B^n as the Siegel domain S_n, μ corresponds to an element $\mu' \in U_{b_\ell} \subset W_{b_\ell}$, where W_{b_ℓ} is the normalizer at b_ℓ (corresponding to ∞ in the unbounded realization S_n), and U_{b_ℓ} is a 1-parameter group of translations. Conjugating by the Cayley transform, W_{b_ℓ} corresponds to $W_{b_\ell}^\flat$ whose orbits are horospheres with $b_\ell \in \partial B^n$ as its only boundary point. Thus for any $z \in B^n$, $\mu^i(z)$ converges to b_ℓ as $i \to \infty$. Since F is an immersion at ζ, for $z \in \widetilde{W}$, as has been shown the norm $\|\sigma(z)\|$ of the second fundamental form σ vanishes asymptotically as z approaches q. From the invariance $\|\sigma(z)\| = \|\sigma(\mu^i(z))\|$ and the convergence of $\mu^i(z)$ to q it follows that $\|\sigma(z)\| = 0$. The same holds true for any smooth point z' on $\widetilde{W}$. In fact $\mu^i(z')$ converges to b_ℓ for any point $z' \in B^n$ as the distance $d(\mu^i(z'), \mu^i(z)) = d(z', z)$ with respect to $ds^2_{B^n}$ is fixed (while the latter metric blows up in all directions as one approaches ∂B^n). As a consequence, in any event the second fundamental form σ vanishes identically on $\widetilde{W}$, i.e., $\widetilde{W} \subset B^n$ is totally geodesic, as desired. The proof of Proposition 1 is complete. □

Remarks For the argument at the beginning of the proof showing that $V \cap B^n$ is asymptotically totally geodesic at a general boundary point $q \in V \cap \partial B^n$ with respect to the canonical Kähler-Einstein metric there is another well-known argument which consists of calculating holomorphic sectional curvature asymptotically, as for instance done in Cheng-Yau [CY]. More precisely, by direct computation it can be shown that for a strictly pseudo convex domain with smooth boundary, with a strictly plurisubharmonic function defining φ, the Kähler metric with

Kähler form $i\partial\overline{\partial}(-\log(-\varphi))$ is asymptotically of constant holomorphic sectional curvature equal to -2, which in our situation is enough to imply the asymptotic vanishing of the norm of the second fundamental form. Here we have chosen to give an argument adapted to the geometry of our special situation where $V \cap \partial B^n$ is holomorphically foliated by complex geodesic submanifolds for two reasons. First of all, it gives an interpretation of the asymptotic behavior of the second fundamental form which is not easily seen from the direct computation. Secondly, the set-up of studying holomorphic foliations by complex geodesic submanifolds in which one exploits the geometry of the Borel embedding $B^n \subset \mathbb{P}^n$ may give a hint to approach the general question of characterizing Zariski closures of totally geodesic complex submanifolds in the case of quotients of bounded symmetric domains.

(2.2) *Proof of the Main Theorem on Zariski closures of germs of complex geodesic submanifolds* In the Main Theorem we consider quasi-projective subvarieties W of complex hyperbolic space forms of finite volume. For the proof of the Main Theorem first of all we relate the existence of a germ of complex geodesic submanifold S on W with the existence of a holomorphic foliation by complex geodesics defined on some neighborhood of S in its Zariski closure, as follows.

Proposition 2. *Let $n \geq 3$, and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, $X := B^n/\Gamma$, which is endowed with the canonical quasi-projective structure. Let $W \subset X$ be an irreducible quasi-projective variety. Let $S \subset W \subset X$ be a locally closed complex geodesic submanifold of X lying on W. Then, there exists a quasi-projective submanifold $Z \subset W$ such that Z is smooth at a general point of S for which the following holds true. There is some subset $V \subset Z$ which is open with respect to the complex topology such that V is non-singular, $V \cap S \neq \emptyset$, and there is a holomorphic foliation $\mathcal{H}$ on V by complex geodesics such that for any $y \in V \cap S$, the leaf L_y of $\mathcal{H}$ passing through y must lie on S.*

Proof. Replacing W by the Zariski closure of S in W, without loss of generality we may assume that S is Zariski dense in W. In particular, a general point of S is a smooth point of W, otherwise $S \subset \mathrm{Sing}(W) \subsetneq W$, contradicting with the Zariski density of S in W. With the latter assumption we are going to prove Proposition 2 with $Z = W$. Let $x \in W$ and $\alpha \in \mathbb{P}T_x(W)$ be a non-zero tangent vector. Denote by S_α the germ of complex geodesic at x such that $T_x(S_\alpha) = \mathbb{C}\alpha$. Define a subset $A \subset \mathbb{P}T_X|_W$ as follows. By definition a point $[\alpha] \in \mathbb{P}T_x(W)$ belongs to A if and only if the germ S_α lies on W. We claim that the subset $A \subset \mathbb{P}T_X|_W$ is complex-analytic. Let $x_0 \in S \subset W$ be a smooth point and U_0

be a smooth and simply connected coordinate neighborhood of x in W which is relatively compact in W. Recall that $\pi : B^n \to X$ is the universal covering map. Let U be a connected component of $\pi^{-1}(U_0)$ lying on $\widetilde{W}$ and $x \in \widetilde{U}$ be such that $\pi(x) = x_0$. Define $S^\sharp = \pi^{-1}(S) \cap U \subset \widetilde{W}$, which is a complex geodesic submanifold of B^n. Identifying U with U_0, we use the Euclidean coordinates on U as holomorphic coordinates on U_0. Shrinking U_0 and hence U if necessary we may assume that $\pi^{-1}(W) \cap U \subset B^n$ is defined as the common zero set of a finite number of holomorphic functions $f_1, \cdots, f_m$ on U. Then $[\alpha] \in A$ if and only if, in terms of Euclidean coordinates given by $\pi|_U : U \xrightarrow{\cong} U_0$, writing $\alpha = (\alpha_1, \cdots, \alpha_n)$ we have $f_k(x_1 + t\alpha_1, \cdots, x_n + t\alpha_n) = 0$ for all k, $1 \le k \le m$, and for any sufficiently small complex number t. Consider only the subset $G \subset T_U$ consisting of non-zero tangent vectors α of length < 1 with respect to $ds^2_{B^n}$. Varying t we have a family of holomorphic functions defined on G whose common zero set descends to a subset $A \cap \mathbb{P}T_U$ in $\mathbb{P}T_U$, showing that $A \cap \mathbb{P}T_U \subset \mathbb{P}T_U$ is a complex-analytic subvariety. Since the base point $x_0 \in W$ is arbitrary, we have shown that $A \subset \mathbb{P}T_X|_W$ is a complex-analytic subvariety.

Assume first of all that X is compact. Recall that $S \subset W \subset X$ is a locally closed complex geodesic submanifold. Obviously $\mathbb{P}T_S \subset A$. Let $A_1 \subset A$ be an irreducible component of A which contains $\mathbb{P}T_S$. Denote by $\lambda : \mathbb{P}T_X \to X$ the canonical projection. Consider the subset $W_1 := \lambda(E_1) \subset W$, which contains S. By the Proper Mapping Theorem $W_1 \subset W$ is a subvariety. Since $S \subset W_1 \subset W$ and S is Zariski dense in W we must have $W_1 = W$. Thus $W \subset X = B^n/\Gamma$ is an irreducible subvariety in X filled with complex geodesics, a situation which is the dual analogue of the picture of an irreducible projective subvariety $Y \subset B$ uniruled by lines (cf. Hwang [Hw] and Mok [Mk4]). More precisely, let $\mathcal{G}$ be the Grassmannian of projective lines in $\mathbb{P}^n$, $\mathcal{G} \cong \mathrm{Gr}(2, \mathbb{C}^{n+1})$, and $\mathcal{K}_0 \subset \mathcal{G}$ be the subset of projective lines $\ell \subset \mathbb{P}^n$ such that $\ell \cap B^n$ is non-empty, and $\mathcal{K} \subset \mathcal{K}_0$ be the irreducible component which contains the set $\mathcal{S}$ of projective lines ℓ whose intersection with $S^\sharp$ contains a non-empty open subset of ℓ. Here $B^n \subset \mathbb{C}^n \subset \mathbb{P}^n$ gives at the same time the Harish-Chandra embedding $B^n \subset \mathbb{C}^n$ and the Borel embedding $B^n \subset \mathbb{P}^n$. Let $\rho : \mathcal{U} \to \mathcal{G}$ be the universal family of projective lines on $\mathbb{P}^n$, and $\rho|_{\mathcal{K}} : \rho^{-1}(\mathcal{K}) \to \mathcal{K}$ be the restriction of the universal family to $\mathcal{K}$. We will also write $\mathcal{U}|_{\mathcal{A}} := \rho^{-1}(\mathcal{A})$ for any subset $\mathcal{A} \subset \mathcal{G}$. By means of the tangent map we identify the evaluation map $\mu : \mathcal{U} \to \mathbb{P}^n$ canonically with the total space of varieties of minimal rational tangents $\mu : \mathbb{P}T_{\mathbb{P}^n} \to \mathbb{P}^n$. Thus, μ associates each $[\alpha] \in \mathbb{P}T_x(\mathbb{P}_n)$ to its base point x. Denote by $\mathcal{D} \subset \mathcal{U}|_{\mathcal{K}}$ the subset defined by $\mathcal{D} := \mathcal{U}|_{\mathcal{K}} \cap \mu^{-1}(B^n)$. Then, there exists some non-empty connected open subset

$\mathcal{E} \subset \mathcal{K}$ containing $\mathcal{S}$ such that the image of $\mu(\mathcal{D} \cap \mathcal{U}|_{\mathcal{E}})$ contains a neighborhood $U' \subset U$ of x in $\widetilde{W}$.

The subgroup $\Phi \subset \Gamma$ acts canonically on $\mathcal{U}|_{\mathcal{K}}$ and the quotient $\mathcal{U}|_{\mathcal{K}}/\Phi$ is nothing other than $A_1 \subset \mathbb{P}T_X|_W$. Recall that $A_1 \supset \mathbb{P}T_S$. Let $A_2 \subset A_1$ be the Zariski closure of $\mathbb{P}T_S$ in A_1. Again the image $\lambda(A_2) \subset W$ equals W by the assumption that S is Zariski dense in W. Moreover, a general point $[\alpha] \in \mathbb{P}T_S$, $\alpha \in T_x(S)$, is a smooth point of A_2 and $\lambda|_{A_2} : A_2 \to W$ is a submersion at $[\alpha]$. Fix such a general point $[\alpha_0] \in \mathbb{P}T_x(S)$ and let ℓ_0 be a germ of complex geodesic passing through x such that $T_x(\ell_0) = \mathbb{C}\alpha_0$. Let $H \subset W$ be a locally closed hypersurface passing through x such that $\alpha_0 \notin T_x(H)$. Shrinking the hypersurface H if necessary there exists a holomorphic vector field $\alpha(w)$ on H transversal at every point to H such that $\alpha(x) = \alpha_0$ and $\alpha(x') \in T_{x'}(S)$ for every $x' \in H \cap S$. Then, there is an open neighborhood V of H admitting a holomorphic foliation $\mathcal{H}$ by complex geodesics such that the leaf L_w passing through $w \in H$ obeys $T_w(L_w) = \mathbb{C}\alpha(w)$, and such that, as x' runs over $S \cap H$, the family of leaves $L_{x'}$ sweeps through $V \cap S$. In particular, for $y \in V \cap S$ the leaf L_y lies on $V \cap S$. This proves Proposition 2 in the case where X is compact.

It remains to consider the case where $\Gamma \subset \mathrm{Aut}(B^n)$ is non-uniform, in which case we will make use of the minimal compactification $X \subset \overline{X}_{\min}$ of Satake [Sa] and Baily-Borel [BB]. When $W \subset X = B^n/\Gamma$ is compact the preceding arguments go through without modification. At a cusp $Q_\ell \in \overline{X}_{\min}$ the notion of a complex geodesic submanifold is undefined. In order to carry out the preceding arguments when $W \subset X$ is non-compact so that $\overline{W}$ contains some cusps $Q_\ell \in \overline{X}_{\min}$ we have to work on the non-compact manifolds X and $\mathbb{P}T_X$. In the arithmetic case by [Sa] and BB] the holomorphic tangent bundle T_X admits an extension to a holomorphic vector bundle E defined on $\overline{X}_{\min}$, and the same holds true for the non-arithmetic case by the description of the ends of X as given in (1.2). For the preceding arguments *a priori* Zariski closures on $X \subset \overline{X}_{\min}$ and on $\mathbb{P}T_X \subset \mathbb{P}E$ have to be taken in the topology with respect to which the closed subsets are complex-analytic subvarieties. We call the latter the analytic Zariski topology. Nonetheless, since the minimal compactification $\overline{X}_{\min}$ of X is obtained by adding a finite number of cusps, for any complex-analytic subvariety of $Y \subset X$ by Remmert-Stein Extension Theorem the topological closure $\overline{Y}$ in $\overline{X}_{\min}$ a subvariety. On the other hand, since $\dim(\mathbb{P}E - \mathbb{P}T_X) > 0$, the analogous statement is not *a priori* true for $\mathbb{P}T_X \subset \mathbb{P}E$. However, given any complex-analytic subvariety $Z \subset \mathbb{P}T_X$, by the Proper Mapping Theorem its image $\lambda(Z) \subset X$ under the canonical projection map $\lambda : \mathbb{P}T_X \to X$ is a subvariety. Thus $B := \lambda(Z) \subset X \subset \overline{X}_{\min}$ is quasi-projective. In the preceding arguments in which one takes Zariski closure in the compact case, for the non-compact case it remains the case that the subsets $B \subset X$ on the base manifold are quasi-projective. In the final steps of the arguments in which one obtains a subset $A_2 \subset \mathbb{P}T_X|_W$, $A_2 \supset \mathbb{P}T_S$, such that a general point $[\alpha] \in A_2 \cap \mathbb{P}T_X|_U$ over a neighborhood U of some $x \in S$ in W is non-singular and $\lambda|_{A_2}$ is a submersion at $[\alpha]$, A_2 was used only to produce a holomorphically foliated family of complex geodesics of X over some neighborhood V of $x \in S$ in W, and for that argument it is not necessary for A_2 to be quasi-projective. Thus, the arguments leading to the proof of Proposition 2 in the compact case persist in the general case of $W \subset X = B^n/\Gamma$ for any torsion-free lattice $\Gamma \subset \mathrm{Aut}(B^n)$, and the proof of Proposition 2 is complete. $\square$

Theorem 1. *Let $n \geq 2$, and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, $X := B^n/\Gamma$. Equipping $X \subset \overline{X}_{\min}$ with the structure of a quasi-projective manifold inherited from the minimal compactification $\overline{X}_{\min}$, let $W \subset X$ be a quasi-projective subvariety. Denote by $W_0 \subset W$ the smooth locus of W. Let $\widetilde{W_0} \subset B^n$ be an irreducible component (equivalently, a connected component) of $\pi^{-1}(\mathrm{Reg}(W))$. Then, the Gauss map is of maximal rank at a general point of $\widetilde{W_0} \subset B^n$ unless $W \subset X$ is a totally geodesic subset.*

Remarks When $W \subset X$ is projective, the total geodesy of $\widetilde{W} \subset B^n$ in Theorem 1 already follows from the last part of the proof of Cao-Mok [CM, Theorem 1] (inclusive of Lemma 4.1 and the arguments thereafter). We summarize the argument there, as follows. $\left(B^n, ds^2_{B^n}\right)$ is of constant Ricci curvature $-(n+1)$. An m-dimensional locally closed complex submanifold $S \subset B^n$ is totally geodesic if and only if it is of constant Ricci curvature $-(m+1)$. In general, denoting by Ric_{B^n} resp. Ric_S the Ricci curvature form of $\left(B^n, ds^2_{B^n}\right)$ resp. $\left(S, ds^2_{B^n}\big|_S\right)$, and by $\zeta = (\zeta_1, \cdots, \zeta_m)$ local holomorphic coordinates at $x \in S$, we have $\mathrm{Ric}_S = \frac{m+1}{n+1}\mathrm{Ric}_{B^n} - \rho$, where $\rho(\zeta) = \sum_{\alpha,\beta=1}^n \rho_{\alpha\overline{\beta}}(\zeta) d\zeta^\alpha \overline{d\zeta^\beta}$. In the case of Theorem 1, where $S = \widetilde{W}$ in the notations of Proposition 1, the holomorphic distribution D is given by the kernel of the second fundamental form σ, or equivalently by the kernel of the closed $(1,1)-$form ρ. Choosing the local holomorphic coordinates $\zeta = (\zeta_1, \cdots, \zeta_s; \zeta_{s+1}, \cdots, \zeta_m)$ as in the proof of Proposition 1, from the vanishing of $\rho|_L$ for the restriction of ρ to a local leaf of the holomorphic foliation $\mathcal{F}$ defined by D, it follows that $\rho_{\alpha\overline{\alpha}} = 0$ whenever $\alpha > s$. From $\rho \geq 0$ it follows that $\rho_{\alpha\overline{\beta}} = 0$ for all $\alpha > s$ and for all β $(1 \leq \beta \leq m)$. Coupling with $d\rho = 0$ one easily deduces that $\rho_{\alpha\overline{\beta}}(\zeta) = \rho_{\alpha\overline{\beta}}(\zeta_1, \cdots, \zeta_s)$, so that ρ is completely determined by its restriction $\rho|_Z$ to $Z \subset U$, and the asymptotic vanishing of ρ and hence of σ near boundary points of B^n follows from standard asymptotic estimates of the

Kähler-Einstein metric $ds^2_{B^n}$.

Finally, we are ready to deduce the Main Theorem from Proposition 2, as follows.

Proof of the Main Theorem. Recall that $X = \Omega/\Gamma$ is a complex hyperbolic space form of finite volume, $W \subset X \subset \overline{X}_{\min}$ is a quasi-projective subvariety, and $S \subset X$ is a complex geodesic submanifold lying on W. As explained in the proof of Proposition 2, the closure $Z \subset X$ of S in W with respect to the analytic Zariski topology is quasi-projective, so without loss of generality we may replace W by its Zariski closure with respect to the usual Zariski topology and proceed under that convention with proving that W is totally geodesic. By Proposition 2, there exists some $x \in S$ which is a non-singular point on W and some nonsingular open neighborhood V of x in W which admits a holomorphic foliation $\mathcal{F}$ by complex geodesics (such that the leaves of $\mathcal{F}$ passing through any $y \in V$) lies on $S \cap V$. By Proposition 2, $V \subset X$ is totally geodesic. As a consequence, $W \subset X$ is totally geodesic subset, i.e., the non-singular locus $\mathrm{Reg}(W)$ of W is totally geodesic in X, as desired. □

Remarks In the proof of Proposition 2 we remarked that any irreducible complex-analytic subvariety $Z \subset X$ of positive dimension extends by Remmert-Stein Extension Theorem to a complex-analytic subvariety in $\overline{X}_{\min}$. As a consequence, in the hypothesis of the Main Theorem, in place of assuming $W \subset X$ to be an irreducible quasi-projective subvariety we could have assumed that $W \subset X$ is simply an irreducible complex-analytic subvariety.

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