SUMS OF FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract. Let $t_\varphi(n)$ denote the nth normalized Fourier coefficient of a primitive holomorphic or Maass cusp form φ for the full modular group $SL(2,\mathbb{Z})$. In this paper we are concerned with the upper bound and omega results for the summatory function

$$\sum_{n\le x} t_\varphi(n^j).$$

Asymptotic formulae for high power moments of $t_\varphi(n)$ are (conditionally) established.

1. Introduction and Main Results

There are many hidden structures underlying the Fourier coefficients $a_f(n)$ of an automorphic form f. The famous Sato-Tate conjecture describes conjecturally their distribution. A common approach (in analytic number theory) is to look at its summatory function over a certain sequence, for instance, $\sum_{n\le x} a_f(n)$ and $\sum_{p\le x} a_f(p)$ where the former sum ranges over integers and the latter runs over primes. A number of articles in the literature are devoted to investigations in this regard. Hafner and Ivić [10] obtained, amongst other things, an O-estimate and $\Omega_\pm$-results for $\sum_{n\le x} a_f(n)$. The sum over squares $\sum_{n\le x} a_f(n^2)$ was considered in Ivić [13], Fomenko [5] and Sankaranarayanan [33], whilst the mean square (or second moment) $\sum_{n\le x} |a_f(n)|^2$ is more classical and was treated in Rankin [28] and Selberg [34]. Subsequently Rankin initiated the theme of lower and upper estimates for the power moments $\sum_{n\le x} |a_f(n)|^{2\beta}$. See, for example, [29]-[31], [25], [27], [3], [42], [43] and the references therein. In this paper we pursue the analogous problem for classical automorphic forms.

Throughout the paper, we consider the primitive holomorphic or Maass cusp forms for the full modular group. (See [14] for definitions.) Every such form has a Fourier series expansion at the cusp infinity whose coefficients are given by the eigenvalues of the Hecke operators. More precisely, for a primitive holomorphic cusp form φ, we have

$$\varphi(z) = \sum_{n\ge 1} t_\varphi(n) n^{(k-1)/2} e^{2\pi i n z}$$

where $t_\varphi(n)$ is the nth eigenvalue of the Hecke operator T_n satisfying $|t_\varphi(p)| \le 2$ (often called Deligne's bound) for all primes p. Quite similarly a primitive Maass cusp form φ admits the

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expansion

$$\varphi(z) = \rho_\varphi(1)\sqrt{y}\sum_{n\geq 1} t_\varphi(n) K_{i\kappa_\varphi}(2\pi|n|y)\mathrm{e}_\varphi(nx)$$

where K_ν is the K-Bessel function and $\mathrm{e}_\varphi(x)$ is defined as $2\cos(x)$ if φ is even, or $2i\sin(x)$ if φ is odd (i.e. according as the eigenvalues $+1$ or -1 for the reflection operator). The numbers $\rho_\varphi(1)$ and κ_φ depend on the spectral parameter (i.e. the eigenvalue of the Laplacian) for φ, and $t_\varphi(n)$ is the nth eigenvalue of the Hecke operator. However Deligne's bound is not yet available to $t_\varphi(n)$ for Maass φ.

Our main objective is to study, for a primitive form φ and $l = 3, 4, \cdots$,

$$S_l(x) = \sum_{n\leq x} t_\varphi(n^l) \qquad \text{and} \qquad H_l(x) = \sum_{n\leq x} t_\varphi(n)^l. \tag{1.1}$$

In this direction, Moreno and Shahidi [25] firstly studied the asymptotic behavior of $H_4(x)$ for the Ramanujan tau-function. More precisely they proved that

$$\sum_{n\leq x} \tau_0(n)^4 \sim cx\log x, \qquad x \to \infty$$

where $c > 0$ and $\tau_0(n) = \tau(n)/n^{\frac{11}{2}}$ is the normalized Ramanujan tau-function. When φ is a primitive holomorphic cusp form, Lü recently showed that

$$S_3(x) \ll x^{\frac{3}{4}+\varepsilon}, \qquad \text{and} \qquad S_4(x) \ll x^{\frac{7}{9}+\varepsilon}$$

in [22] and

$$H_{2j}(x) = xP_{2j}(\log x) + O\left(x^{c_{2j}+\varepsilon}\right) \qquad (j = 2, 3, 4),$$

where $\deg P_4 = 1$, $\deg P_6 = 4$, $\deg P_8 = 13$ and $c_4 = \frac{7}{8}$, $c_6 = \frac{31}{32}$, $c_8 = \frac{127}{128}$ (in [23], [24]).

In this paper we intend to extend the former results to other interesting cases (see Theorems 1 and 3); as customary, for $j \geq 5$ or for Maass form φ, some far-reaching conjectures are imperative to equip a proper environment. But even so, the general case carries the problem of factorizing the generating function, which may not be easily handled by the plain approaches in [23], [24] for small j's. Another principal result is Theorem 2 which is an unconditional omega result for $S_j(x)$ $(j = 1, 2, 3, 4)$. Below (GRC) stands for the Generalized Ramanujan Conjecture, see Section 2. The subscript $*$ in the Vinogradov symbol $\ll_*$ or the O-symbol O_* is to indicate the dependence of the implied constant on $*$.

Theorem 1. *Let φ be a primitive holomorphic or Maass cusp form. Suppose $L(\mathrm{sym}^j\varphi, s)$ is automorphic cuspidal.*

(a) *For any $\varepsilon > 0$, we have*

$$\Delta_j(x) := \sum_{n\leq x} t_{\mathrm{sym}^j\varphi}(n) \ll_{\varphi,\varepsilon} x^{\kappa_j+\varepsilon} \qquad (j \geq 2)$$

where $t_{\mathrm{sym}^j\varphi}(n)$ denotes the nth coefficient in the jth symmetric power L-function, see Section 3, and $\kappa_j = \frac{j(j+1)}{(j+1)^2+1}$.

(b) *If furthermore (GRC) holds for* φ*, we have the better estimate: both*

$$\Delta_j(x) \quad \textit{and} \quad S_j(x) \ll_\varphi x^{\frac{j}{j+2}} \qquad (j \geq 3)$$

where $S_j(x)$ *is defined as in (1.1). For the case* $j = 2$*, we have*

$$\Delta_2(x) \ll x^{1/2} \qquad \textit{but} \qquad S_2(x) \ll x^{1/2} \log x.$$

Remark 1. The results in Part (b) for holomorphic φ and $j = 2, 3, 4$ are unconditional, superseding slightly the corresponding results in [5] and [22] by a factor of $\log^2 x$ and a small power of x respectively. For $j = 1$, Hafner and Ivić [10] showed two decades ago that $S_1(x) \ll x^{1/3}$, noting $\Delta_1(x) = S_1(x)$. Indeed part (b) is proven with the method as in [10], using a result of Chandrasekharan and Narasimhan and Shiu's Brun-Titchmarsh theorem. The new (and natural) ingredient is a result in [32] stemming from the general Rankin-Selberg theory. (Some ideas are outlined below Theorem 1.3 of [22].)

Without GRC, we apply Landau's classical lemmas in place of Shiu's theorem to establish Theorem 1 (a). For a primitive Maass cusp form φ, we can derive from it that unconditionally

$$S_j(x) \ll_{\varphi,\varepsilon} x^{\kappa_j+\varepsilon} \qquad (j = 3, 4). \tag{1.2}$$

(Note that $\kappa_3 = \frac{12}{17}$ and $\kappa_4 = \frac{10}{13}$.) This will be verified in the proof of Theorem 1.

Theorem 2. *Let* φ *be a primitive holomorphic or Maass cusp form. For* $j = 1, 2, 3, 4$*, we have (unconditionally)*

$$S_j(x) = \Omega(x^{\frac{j}{2(j+1)}}).$$

Remark 2. The case $j = 1$ was treated by another method in [10], which indeed leads to a more delicate result:

$$S_1(x) = \Omega_\pm\Big(x^{1/4} \exp\Big(\frac{C(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\Big)\Big).$$

However the method is presumably intractable for $j \geq 2$. The principle of our proof is to link the (weighted) mean square of $S_j(x)$ to the mean square of the generating function along a certain vertical line L. This approach is classical, but in this case (and many other situations), we encounter the problem that the generating function has a denominator of $\zeta(2s)$ (or something similar). The possible zeros in the region $1/4 < \Re e\, s < 1/2$ cause trouble, which can of course be ruled out under RH (or GRH). Balasubramanian, Ramachandra and Subbarao [1] developed a method to get around the difficulty for low zero density cases, requiring that the number of zeros $\beta + i\gamma$ (of the denominator) near L with $|\gamma| \leq T$ is $O(T^{1-\delta})$. The method is extended to a general context in [21], which quite fits us when $j = 1, 2, 3$. (The assumption of (RC) in [21] is minor.) But for $j = 4$, the zero density result around L is $O(T^{1+\varepsilon})$, which is fatal to the application. Gratefully it is resolved due to the special form of the generating function for $S_4(x)$.

To achieve an akin omega result for $j \geq 5$ with this method, we need the automorphy of $L(\mathrm{sym}^m\varphi, s)$ $(1 \leq m \leq 2j - 4)$ and furthermore suitable zero density estimates.

Theorem 3. *Let φ be a primitive form and assume (GRC) holds for φ. Define H_l as in (1.1) and $\eta_l = 1 - 2^{1-l}$ for $l \in \mathbb{N}$.*

(a) *Let $j \geq 2$. If $L(\mathrm{sym}^r\varphi, s)$, $r = 1, 2, \cdots, j$, are all automorphic cuspidal, then for any $\varepsilon > 0$, we have*

$$H_{2j}(x) = xP_{2j}(\log x) + O_{\varphi,\varepsilon}\left(x^{\eta_{2j}+\varepsilon}\right)$$

where $P_{2j}(y)$ denotes a polynomial in y of degree $\frac{(2j)!}{j!(j+1)!} - 1$.

(b) *Suppose $L(\mathrm{sym}^r\varphi, s)$, $r = 1, 2, \cdots, j+1$, are automorphic cuspidal where $j \geq 1$. Then*

$$H_{2j+1}(x) \ll_{\varphi,\varepsilon} x^{\eta_{2j+1}+\varepsilon}.$$

Remark 3. The formula in (a) cannot cover $H_2(x)$ (since Lemma 2.4 does not apply). This case was studied long time ago by Rankin and Selberg independently, see [28] and [34]; their results read as

$$H_2(x) = Cx + O(x^{3/5})$$

whose O-term remains the sharpest to-date. Actually our proof demonstrates a sharp approach using merely the information of "convexity" for a product of L-functions of degree more than one. There is some room for improving $H_j(x)$ ($j \geq 3$), if we invoke the known subconvexity bounds (in t-aspect) of some low degree L-functions to enhance the effectiveness of Lemma 2.4. We would further discuss this in another occasion.

The degree of $P_{2j}(y)$ is consistent with the value evaluated under the assumption of the Sate-Tate conjecture (see [43, (1.10)] or [31]), but is different from the parallel (unconditional) result in [27] since the cusp forms there are not primitive forms. It is also worthwhile to remark that only the automorphy of small symmetric powers are required, e.g. $L(\mathrm{sym}^{2j}\varphi, s)$ is not necessary for $H_{2j}(x)$. The main tool for Theorem 3 is Lemma 2.4 which was nicely and extensively discussed in [6].

Remark 4. When φ is a primitive Maass cusp form, Lemma 2.4 does not apply but we may use Lemma 2.2 instead. Ultimately, we obtain that for $j = 2, 3, 4$ and any $\varepsilon > 0$,

$$H_{2j}(x) = xP_{2j}(\log x) + O_{\varphi,\varepsilon}\left(x^{\vartheta_{2j}+\varepsilon}\right) \tag{1.3}$$

where $\deg P_4 = 1$, $\deg P_6 = 4$ and $\deg P_8 = 13$ and $\vartheta_4 = \frac{15}{17}$, $\vartheta_6 = \frac{63}{65}$, $\vartheta_8 = \frac{255}{257}$. The proof will be outlined at the end of Section 7.

Remark 5. Let E be an elliptic curve over $\mathbb{Q}$ with multiplicative reduction at some prime. The associated Hasse-Weil L-function $L(s, E) = \sum_{n\geq 1} t_E(n)n^{-s}$ is shifted to center at $s = 1/2$ (so its abscissa of absolute convergence is $\Re e\, s = 1$). In light of the recent in-depth progress on Sato-Tate's conjecture, it is plausible to derive unconditionally that for $j \geq 2$,

$$H_{2j}(x) := \sum_{n\leq x} t_E(n)^{2j} \sim c_j x(\log x)^{A_j-1} \qquad (\text{as } x \to \infty) \tag{1.4}$$

where $A_j = (2j)!/(j!(j+1)!)$ and $c_j > 0$ is a constant depending on E and j. We anticipate the weaker asymptotic formula (without an explicit O-term) because the present technology reaches only the potential modularity. Let us explain the idea relying on [11].

Let $\rho_{E,\ell} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(2, \mathbb{Q}_\ell)$ denote the representation on $H^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$. The p-local factor of $L(s, E)$ is identical to $\det(I - \rho_{E,\ell}(\mathrm{Fr}_p)p^{-s-1/2})^{-1}$ for $p \nmid N := \ell N_E$, where Fr_p denotes a Frobenius element and N_E is the conductor of E. Consider the symmetric power $\rho^n_{E,\ell} = \mathrm{Sym}^{n-1}\rho_{E,\ell} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(n, \mathbb{Q}_\ell)$. The work of Harris, Shepherd-Barron and Taylor yields that for all n, the symmetric power L-function $L(s, \rho^n_E)$ has a meromorphic continuation and functional equation and is nonvanishing for $\Re\mathrm{e}\, s \geq 1$. (See [11, Theorem 5.1.1].) Now we apply Lemma 7.1 below (with a very little modification) to deduce

$$R_{2j}(s) := \sum_{\substack{n\geq 1\\ (n,N)=1}} t_E(n)^{2j} n^{-s} = F_{2j}(s)U_{2j}(s)$$

where $F_{2j}(s)$ is a product of $L(s, \rho^m_E)$ (with $0 \leq m \leq 2j$ being even) and the Dirichlet series $U_{2j}(s)$ converges absolutely on $\Re\mathrm{e}\, s = 1$. Moreover, $F_{2j}(s)$ has nonnegative coefficients and holomorphic at every s with $\Re\mathrm{e}\, s \geq 1$ except the point $s = 1$ at which $F_{2j}(s)$ has a pole of order A_j. The (generalized) Ikehara's theorem, see [41, Chapter II.7, Theorem 15], shows that

$$\sum_{n\leq x} f_{2j}(n) \sim c'_j x(\log x)^{A_j - 1}$$

where $f_{2j}(n)$ is the nth coefficient of $F_{2j}(s)$. A simple argument as in (8.2)-(8.3) gives the desired result (1.4) with the extra condition $(n, N) = 1$ in the summation. This condition can be relaxed as E is modular, $t_E(n) = t_f(n)$ for some primitive holomorphic form f and $t_f(p)^2 = 0$ or p^{-1} at every ramified prime p.

2. General L-functions

Let $L(f, s)$ be a Dirichlet series (associated to an object f) that admits an Euler product of degree $m \geq 1$, given as

$$L(f, s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_f(p, j)}{p^s}\right)^{-1},$$

where $\alpha_f(p, j), j = 1, \cdots, m$, are the local parameters of $L(f, s)$ at (finite) prime p. Suppose this series and Euler product are absolutely convergent for $\Re\mathrm{e}\, s > 1$. We denote the gamma factor by

$$L_\infty(f, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_f(j)}{2}} \Gamma\left(\frac{s + \mu_f(j)}{2}\right),$$

with the local parameters $\mu_f(j), j = 1, \cdots, m$, of $L(f, s)$ at ∞. The complete L-function $\Lambda(f, s)$ is defined as

$$\Lambda(f, s) = q(f)^{\frac{s}{2}} L_\infty(f, s) L(f, s),$$

where $q(f)$ is the conductor of $L(f, s)$. We assume that $\Lambda(f, s)$ admits an analytic continuation to the whole complex plane $\mathbb{C}$ and is holomorphic everywhere on $\mathbb{C}$ except possibly poles of finite order at $s = 0, 1$. Moreover it satisfies (hypothetically) a functional equation of Riemann type

$$\Lambda(f, s) = \epsilon_f \Lambda(\tilde{f}, 1 - s)$$

where ϵ_f is the root number with $|\epsilon_f| = 1$, and $\tilde{f}$ is the dual of f such that $\lambda_{\tilde{f}}(n) = \overline{\lambda_f(n)}$, $L_\infty(\tilde{f}, s) = L_\infty(f, s)$, and $q(\tilde{f}) = q(f)$. We say that $L(f, s) \in S_e^\#$ if it is endowed with the above conditions.

In this section we tacitly assume $L(f, s) \in S_e^\#$.

Lemma 2.1. *Assume $L(f, s)$ is entire. Then for every $\eta \geq 0$ we have*

$$\sum_{n \leq x} \lambda_f(n) \ll_f x^{\frac{1}{2} - \frac{1}{2m} + (\frac{m}{2} - \frac{1}{2})\eta} + \sum_{x < n \leq x + x^{1 - \frac{1}{m} - \eta}} |\lambda_f(n)|.$$

This is a special case of Theorem 4.1 in Chandrasekharan and Narasimhan [4] with

$$\delta = 1, \quad A = \frac{m}{2}, \quad \beta = 1, \quad u = \frac{1}{2} - \frac{1}{2m} \quad \text{and} \quad q = -\infty.$$

Lemma 2.2. *Assume the coefficients $\lambda_f(n) \geq 0$. For any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \lambda_f(n) = x P(\log x) + O_{\varepsilon, f}\left(x^{\frac{m-1}{m+1} + \varepsilon}\right)$$

where P is some polynomial of degree $\mathrm{ord}_{s=1} L(f, s) - 1$ and depends only on f.

This is a refined version of Landau's lemma, see Barthel and Ramakrishnan [2].

We say that the L-function $L(f, s)$, or simply f, satisfies the Ramanujan Conjecture if the following holds

(RC) $\lambda_f(n) \ll_\varepsilon n^\varepsilon$ ($\forall\, \varepsilon > 0$),

and satisfies the Ramanujan-Selberg Conjecture or Generalized Ramanujan Conjecture if

(GRC) $|\alpha_f(p, j)| = 1$ and $\Re\mathrm{e}\, \mu_f(j) = 0$ holds.

The Selberg Class contains those L-functions in $S_e^\#$ that fulfil (RC). The condition (RC) is quite strong, and sometimes the weak form below is sufficient for nice properties.

(RC)$^\flat$ $\displaystyle\sum_{n \leq x} |\lambda_f(n)|^2 \ll_\varepsilon x^{1+\varepsilon}$, $\forall\, \varepsilon > 0$.

The following results and proofs are now somewhat standard. We outline the necessary key points.

Lemma 2.3. *Let $\varepsilon > 0$ be any arbitrarily small number.*

(a) *We have the convexity bound:*

$$L(f, \sigma + it) \ll_\varepsilon \begin{cases} |t|^{\frac{m}{2}(1-\sigma)+\varepsilon} & (0 < \sigma < 1 + \varepsilon,\ |t| \gg 1), \\ |t|^{\frac{m}{2}(1-2\sigma)+\varepsilon} & (0 \leq -\sigma \ll 1,\ |t| \gg 1). \end{cases}$$

(b) *Suppose $L(f,s)$ satisfies (RC)$^\flat$, and let $T \ge 1$ be any number. Then*

$$\int_0^T |L(f,1/2+it)|^2\,dt \ll_\varepsilon T^{\max(1,m/2)+\varepsilon}$$

where m is the degree of $L(f,s)$.

See [23] and its references for details.

Lemma 2.4. *Suppose $L(f,s)$ is a product of two L-functions $L_1, L_2 \in S_e^\#$ with both* $\deg L_i \ge 2$*, and $L(f,s)$ satisfies (RC). For any $\varepsilon > 0$, we have*

$$\sum_{n\le x} \lambda_f(n) = M(x) + O(x^{1-\frac{2}{m}+\varepsilon})$$

where $M(x) = \mathrm{res}_{s=1} L(f,s)x^s/s$ and $m = \deg L$.

Proof. By Perron's formula (see [41, p.132]), we have

$$\sum_{n\le x}{}' \lambda_f(n) \quad = \quad \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} L(f,s)x^s\,\frac{ds}{s} + O\biggl(x^\kappa \sum_{n\ge1} \frac{|\lambda_f(n)|}{n^\kappa(1+T|\log(x/n)|)}\biggr) \tag{2.1}$$

where $T \ge 1$ and $\kappa > 1$ are some numbers at our disposal. Here as usual, $\sum'$ denotes the summation whose last summand is halved when x is an integer.

For any $\varepsilon > 0$, we have $\lambda_f(n) \ll n^\varepsilon$ under (RC). Setting $\kappa = 1+2\varepsilon$, we derive that the O-term is $O(x^\varepsilon + x^{1+\varepsilon}T^{-1})$ (by the standard argument, see for example, the proof of [41, p.133, Corollary 2.1]). By Lemma 2.3 (a), we have for $s = \sigma \pm iT$ and $\sigma \in [1/2, \kappa]$,

$$\begin{aligned} L(f,s)x^s/s \quad &\ll \quad T^{\frac{m}{2}-1+\varepsilon}\Bigl(\bigl(xT^{-\frac{m}{2}}\bigr)^{1/2} + \bigl(xT^{-\frac{m}{2}}\bigr)^{1+\varepsilon}\Bigr) \\ &\ll \quad x^{1/2}T^{\frac{m}{4}-1+\varepsilon} + x^{1+\varepsilon}T^{-1}. \end{aligned} \tag{2.2}$$

We shift the line of integration of the integral in (2.1) to $\Re e\, s = 1/2$. It gives rise to the three terms: one is the main term $M(x)$ (which is zero if L has no pole), the other is due to the horizontal line segments which is absorbed in (2.2) and the last one is

$$\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} L(f,s)x^s\,\frac{ds}{s} \quad \ll \quad x^{1/2}\biggl(\prod_{i=1,2}\int_{-T}^{T} \bigl|L_i(1/2+it)\bigr|^2\,\frac{dt}{1+|t|}\biggr)^{1/2}$$

by the Cauchy-Schwarz inequality and the decomposition $L(f,s) = L_1(s)L_2(s)$. The product inside the bracket of the last line is

$$\ll T^{(\frac{m_1}{2}-1)+(\frac{m_2}{2}-1)+\varepsilon} = T^{m/2-2+\varepsilon}$$

by Lemma 2.3 (b). The overall contribution to the error term is

$$\ll \quad x^\varepsilon + x^{1+\varepsilon}T^{-1} + +x^{1/2}T^{m/4-1+\varepsilon} \ll x^{1-\frac{2}{m}+\varepsilon}$$

by taking $T = x^{2/m}$. This completes the proof. $\square$

Define for $\sigma \ge 0$ and $T \ge 1$,

$$\mathrm{N}_L(\sigma,T) = \#\{\rho = \beta + i\gamma :\ L(f,\rho) = 0,\ \beta \ge \sigma,\ |\gamma| \le T\}.$$

Lemma 2.5. *Let $L(f,s)$ satisfy $(RC)^\flat$ and T be any sufficiently large number.*

(a) *Let $A>1$ be a constant. If for some $0\le\delta\le\frac14$,*

$$\int_0^T |L(f,\frac12+\delta+it)|^2\,dt \ll_\varepsilon T^{A+\varepsilon} \qquad \forall\,\varepsilon>0,$$

then, for any $\varepsilon>0$,

$$\mathrm{N}_L(\sigma,T)\ll_\varepsilon T^{\frac{2A(1-\sigma)}{1-2\delta}+\varepsilon} \qquad (\max(\tfrac23,\tfrac1A)<\sigma\le 1).$$

(b) *Let m be the degree of L and suppose $m\ge 3$. For any $\varepsilon>0$,*

$$\mathrm{N}_L(\sigma,T)\ll_\varepsilon T^{m(1-\sigma)+\varepsilon} \qquad (\max(\tfrac23,\tfrac2m)<\sigma\le 1).$$

Proof. Part (b) follows from Part (a) and Lemma 2.3 (b). The zero density result in (a) is achieved with the nowadays well-known zero-detection method, so we refer to the recent article [26] and provide only salient points of modification. To invoke our condition, we shift the line of integration to $\Re\mathrm{e}\, w = 1/2+\delta-\beta$ in the zero-detection device (see [26, p.268]). The treatment of R_1 therein remains valid (noting that $(\mathrm{RC})^\flat$ can be applied in place of (RC)). We evaluate R_2 with the same method in [26] (see p.272) but in our case, the estimate becomes

$$R_2 \ll T^{A+\varepsilon}Y^{1+2\delta-2\sigma}.$$

The choice $Y=T^{A/(1-2\delta)}$ yields our upper estimate in part (a). □

3. Symmetric Power L-functions

Let φ be a primitive holomorphic or Maass cusp form. Associated to φ is an L-function $L(\varphi,s)$, defined for $\Re\mathrm{e}\, s>1$ as

$$\begin{aligned} L(\varphi,s) &= \sum_{n=1}^\infty t_\varphi(n)n^{-s} = \prod_p (1-t_\varphi(p)p^{-s}+p^{-2s})^{-1} \\ &= \prod_p \left(1-\frac{\alpha_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p}{p^s}\right)^{-1}, \end{aligned}$$

with $\alpha_p+\beta_p=t_\varphi(p)$ and $\alpha_p\beta_p=1$. The hypothesis (GRC), i.e. $|\alpha_p|=|\beta_p|=1$, is known for holomorphic φ but for the Maass case, the current best estimate is

$$|\alpha_p|,\ |\beta_p|\le p^{\frac{7}{64}} \tag{3.1}$$

by Kim and Sarnak [17]. The jth symmetric power L-function attached to φ is defined by

$$L(\mathrm{sym}^j\varphi,s) := \prod_p\prod_{m=0}^j (1-\alpha_p^{j-m}\beta_p^m p^{-s})^{-1} \tag{3.2}$$

for $\Re\mathrm{e}\, s\gg 1$. We may express it into Dirichlet series: for $\Re\mathrm{e}\, s\gg1$,

$$L(\mathrm{sym}^j\varphi,s)=\sum_{n=1}^\infty\frac{t_{\mathrm{sym}^j\varphi}(n)}{n^s}=\prod_p\left(1+\frac{t_{\mathrm{sym}^j\varphi}(p)}{p^s}+\cdots+\frac{t_{\mathrm{sym}^j\varphi}(p^k)}{p^{ks}}+\cdots\right). \tag{3.3}$$

Apparently $t_{\mathrm{sym}^j\varphi}(n)$ is a real multiplicative function (noticing $\alpha_p \in \mathbb{R}$ or $|\alpha_p| = 1$ when $t_\varphi(p) \in \mathbb{R}$). All these symmetric power L-functions are conjecturally automorphic L-functions for $GL(j+1)$ and thus belong to the class $S_e^\#$ described in Section 2.

As is well-known, to a primitive form φ is associated an automorphic cuspidal representation π_φ of $GL_2(\mathbb{A}_\mathbb{Q})$, and hence an automorphic L-function $L(s, \pi_\varphi)$ which coincides with $L(\varphi, s)$. It is predicted that π_φ gives rise to a symmetric power lift - an automorphic representation whose L-function is the symmetric power L-function attached to φ. For the known cases the lifts are cuspidal, hence we invoke the following (stronger) hypothesis.

(SPL)$_j$ There exists an automorphic cuspidal self-dual representation, denoted by $\mathrm{sym}^j\pi_\varphi$, of $GL_{j+1}(\mathbb{A}_\mathbb{Q})$ whose L-function is the same as $L(\mathrm{sym}^j\varphi, s)$.

We say that $L(\mathrm{sym}^j\varphi, s)$ or $\mathrm{sym}^j\varphi$ is automorphic cuspidal if (SPL)$_j$ holds valid; in this case, $L(\mathrm{sym}^j\varphi, s) \in S_e^\#$ and has no poles. For $j = 1, 2, 3, 4$, (SPL)$_j$ is shown by a series of vital work. (See [8, 17, 19, 20, 36].)

The Rankin-Selberg L-function $L(\mathrm{sym}^j\varphi \times \mathrm{sym}^j\varphi, s)$ attached to $\mathrm{sym}^j\varphi$ and $\mathrm{sym}^j\varphi$ is defined as

$$\begin{aligned} L(\mathrm{sym}^j\varphi \times \mathrm{sym}^j\varphi, s) &= \prod_p \prod_{m=0}^{j} \prod_{m'=0}^{j} \left(1 - \frac{\alpha_p^{j-m}\beta_p^m \alpha_p^{j-m'}\beta_p^{m'}}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{t_{\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi}(n)}{n^s}. \end{aligned} \tag{3.4}$$

The coefficients $t_{\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi}(n)$ are nonnegative because (for $\mathrm{Re} s \gg 1$)

$$L(\mathrm{sym}^j\varphi \times \mathrm{sym}^j\varphi, s) = \prod_p \exp\left(\sum_{v=1}^{\infty} \left|\sum_{m=0}^{j} \alpha_p^{j-m}\beta_p^m\right|^2 \frac{1}{vp^{vs}}\right).$$

(Lemma 3.1 offers an alternative proof.) In addition, when $\mathrm{sym}^j\varphi$ is automorphic cuspidal (known for $j = 1, 2, 3, 4$), $L(\mathrm{sym}^j\varphi \times \mathrm{sym}^j\varphi, s)$ lies in $S_e^\#$ with simple poles at $s = 0, 1$ from the works of Jacquet and Shalika [15], [16], Shahidi [37], [38], [39], [40], and the reformulation of Rudnick and Sarnak [32]. We define analogously $L(\mathrm{sym}^i\varphi \times \mathrm{sym}^j\varphi, s)$ $(i \neq j)$, which carries the same properties except that it is now entire.

Lemma 3.1. (a) *We have*

$$|t_{\mathrm{sym}^j\varphi}(n)|^2 \le t_{\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi}(n), \quad \forall\ n \ge 1.$$

(b) *If* $\mathrm{sym}^j\varphi$ *is automorphic cuspidal, then* $L(\mathrm{sym}^j\varphi, s)$ *satisfies* (RC)$^\flat$.

Proof. Part (b) follows obviously from Part (a). The inequality is a consequence of Theorem 12.1.3 (or Proposition 7.4.20) in [9]. More concretely, let us write $\alpha_m = \alpha_p^{j-m}\beta_p^m$. Then by Cauchy's identity ([9, Proposition 7.4.20] or [7, Appendix A, (A.13)]), we have

$$\prod_{m=1}^{j}\prod_{m'=1}^{j}(1 - \alpha_m\alpha_{m'}p^{-s})^{-1} = \left(1 - p^{-js}\right)^{-1} \sum_{\substack{\underline{k}=(k_1,\cdots,k_{j-1}) \\ k_1,\cdots,k_{j-1}\ge 0}} S_{\underline{k}}(\alpha_1, \cdots, \alpha_j)^2 p^{-(k_1+2k_2+\cdots+(j-1)k_{j-1})s}$$

where $S_{\underline{k}}$ denotes a Schur polynomial (which is a symmetric function). Also by [7, Appendix A, (A.5)], we see that

$$\prod_{m=1}^{j}(1-\alpha_m p^{-s})^{-1} = \sum_{k=0}^{\infty} S_{(k,0,\cdots,0)}(\alpha_1,\cdots,\alpha_j)p^{-ks}.$$

Therefore,

$$t_{\mathrm{sym}^j\varphi}(p^k)^2 = S_{(k,0,\cdots,0)}(\alpha_1,\cdots,\alpha_j)^2 \le \sum_{\substack{\underline{k}=(k_1,\cdots,k_{j-1})\\ k_1+2k_2+\cdots+(j-1)k_{j-1}\le k}} S_{\underline{k}}(\alpha_1,\cdots,\alpha_j)^2 = t_{\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi}(p^k).$$

This proves Part (a) as its both sides are multiplicative. □

There is a handy structural formulation of the Dirichlet coefficients in (3.2)-(3.4) in the context of group representation theory. Consider the standard representation $\mathbb{C}^2$ of the group $SL_2(\mathbb{C})$, i.e. $\rho : SL_2(\mathbb{C}) \to GL_2(\mathbb{C})$, $g \mapsto \rho(g)$ and $\rho(g)v = gv$ where gv is the usual matrix multiplication. We identify, whenever no confusion arises,

$$\rho = \rho(g) \quad \text{ where } g = \mathrm{diag}(\alpha_p, \beta_p) = \begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix}$$

for any prime p, and denote $t(\rho)$ to be the trace of $\rho = \rho(g)$. Then

$$t_\varphi(p) = t(\rho), \quad t_{\mathrm{sym}^j\varphi}(p) = t(\mathrm{sym}^j\rho), \quad t_{\mathrm{sym}^i\varphi\times\mathrm{sym}^j\varphi}(p) = t(\mathrm{sym}^i\rho\otimes\mathrm{sym}^j\rho),$$

where sym^j is the jth symmetric power and $\otimes$ is the tensor product. Moreover, the p-local factors of the associated L-functions are given by the reciprocal of characteristic polynomials,

$$\begin{aligned} L(\rho, X) &= \det(I-\rho X)^{-1}, \\ L(\mathrm{sym}^j\rho, X) &= \det(I-\mathrm{sym}^j\rho X)^{-1}, \\ L(\mathrm{sym}^i\rho\otimes\mathrm{sym}^j\rho, X) &= \det(I-\mathrm{sym}^i\rho\otimes\mathrm{sym}^j\rho X)^{-1} \end{aligned}$$

with $X = p^{-s}$.

The following are well-known identities: for $j \ge 1$,

$$\mathrm{sym}^2(\mathrm{sym}^j\rho) = \bigoplus_{0\le i\le j/2} \mathrm{sym}^{2j-4i}\rho \tag{3.5}$$

([7, p.159]) and

$$\left(\mathrm{sym}^j\rho\right)^{\otimes 2} := \mathrm{sym}^j\rho\otimes\mathrm{sym}^j\rho = \bigoplus_{0\le r\le j} \mathrm{sym}^{2r}\rho, \tag{3.6}$$

or more generally, for $a \ge b$,

$$\mathrm{sym}^a\rho\otimes\mathrm{sym}^b\rho = \bigoplus_{0\le r\le b} \mathrm{sym}^{a+b-2r}\rho, \tag{3.7}$$

([7, p.151]) where $\mathrm{sym}^0\rho$ denotes the 1-dimensional trivial representation. These identities are useful to depict the relations among the coefficients and the local-factors, for instance, we have

$$t_{\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi}(p)=\sum_{r=0}^{j} t_{\mathrm{sym}^{2r}\varphi}(p) \quad \text{and} \quad L(\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi,s)=\prod_{r=0}^{j} L(\mathrm{sym}^{2r}\varphi,s).$$

The automorphy of $\mathrm{sym}^j\varphi$ in the known cases ($j=1,2,3,4$) provides a good control on the size of $t_{\mathrm{sym}^j\varphi}(p)$. Below is an example.

Lemma 3.2. *Let $\sigma>1$ be arbitrary but fixed. Then,*

$$\sum_p |t_{\mathrm{sym}^2\varphi}(p)|^4 p^{-\sigma} \ll 1.$$

Proof. We note that

$$|t_{\mathrm{sym}^2\varphi}(p)|^4 = t_{\mathrm{sym}^2\varphi\times\mathrm{sym}^2\varphi}(p)^2 = t\big((\mathrm{sym}^2\rho\otimes\mathrm{sym}^2\rho)^{\otimes 2}\big),$$

and by (3.6),

$$\begin{aligned}(\mathrm{sym}^2\rho\otimes\mathrm{sym}^2\rho)^{\otimes 2} &= \left(\bigoplus_{r=0}^{2}\mathrm{sym}^{2r}(\rho)\right)^{\otimes 2}\\ &= (\mathrm{sym}^4\rho)^{\otimes 2}\oplus(\mathrm{sym}^2\rho)^{\otimes 2}\oplus\mathrm{sym}^0\rho\\ &\quad \oplus 2(\mathrm{sym}^4\rho\otimes\mathrm{sym}^2\rho)\oplus 2(\mathrm{sym}^4\rho)\oplus 2(\mathrm{sym}^2\rho).\end{aligned}$$

Here we write $2V$ for $V\oplus V$. The absolute convergence follows from the analytic properties of $L(\mathrm{sym}^i\varphi\times\mathrm{sym}^j\varphi,s)$ (for $i,j=0,2,4$). □

For $j\geq 1$ and $\Re\mathrm{e}\, s\gg 1$, let us define

$$L_j(s)=\sum_{n=1}^{\infty}\frac{t_\varphi(n^j)}{n^s}.$$

Apparently $L_1(s)=L(\varphi,s)$ and $L_2(s)=L(\mathrm{sym}^2\varphi,s)\zeta(2s)^{-1}$. The next lemma is for $j\geq 3$.

Lemma 3.3. *Assume φ satisfies (GRC). Then for $j\geq 3$ and $\Re\mathrm{e}\, s>1$, we have*

$$L_j(s)=L(\mathrm{sym}^j\varphi,s)\prod_{1\leq i\leq j/2} L(\mathrm{sym}^{2j-4i}\varphi,2s)^{-1}H_j(s) \tag{3.8}$$

where $H_j(s)$ converges absolutely in the half-plane $\Re\mathrm{e}\, s>\frac{1}{3}$.

When φ is a Maass form (for which (GRC) is still not known) and $j=3,4$, the identity (3.8) remains true and $H_j(s)$ has the same abscissa of absolute convergence. Moreover, $H_4(\sigma+it)^{-1}\ll_\sigma 1$ for all $\sigma\in(\frac{1}{3},\infty)\setminus\{x_1,x_2,\cdots,x_n\}$, for some real numbers $x_1,\cdots,x_n$.

Proof. We consider the local factor of $L_j(s)$, and let us write $\alpha=\alpha_p$, $\beta=\beta_p$ and $X=p^{-s}$. Recalling (see (3.2))

$$t_\varphi(p^{kj})=t_{\mathrm{sym}^{jk}}(p)=\frac{\alpha^{jk+1}-\beta^{jk+1}}{\alpha-\beta}$$

where $\rho = \rho_p = \mathrm{diag}(\alpha, \beta)$ and $\alpha\beta = 1$, we express the local factor of $L_j(s)$ as

$$\begin{aligned}
L_j(\rho, X) &= \frac{1}{\alpha - \beta}\Big(\sum_{k=1}^{\infty} \alpha^{jk+1} X^k - \sum_{k=1}^{\infty} \beta^{jk+1} X^k\Big) \\
&= \frac{1}{\alpha - \beta}\Big(\frac{\alpha}{1 - \alpha^j X} - \frac{\beta}{1 - \beta^j X}\Big) = \frac{1}{\alpha - \beta} \frac{(\alpha - \beta) + (\alpha^{j-1} - \beta^{j-1})X}{(1 - \alpha^j X)(1 - \beta^j X)} \\
&= (1 - \alpha^j X)^{-1}(1 - \beta^j X)^{-1}\big(1 + \sum_{m=0}^{j-2} \alpha^{j-2-2m} X\big) \\
&= L(\mathrm{sym}^j \rho, X)\big(1 + \sum_{m=0}^{j-2} \alpha^{j-2-2m} X\big) \prod_{0 \le m \le j-2} (1 - \alpha^{j-2-2m} X). \qquad (3.9)
\end{aligned}$$

As

$$\prod_{0 \le m \le j-2} (1 - \alpha^{j-2-2m} X) = 1 - \sum_{m=0}^{j-2} \alpha^{j-2-2m} X + \sum_{0 \le a < b \le j-2} \alpha^{2(j-2)-2(a+b)} X^2 + \cdots,$$

we see that

$$\begin{aligned}
&\big(1 + \sum_{m=0}^{j-2} \alpha^{j-2-2m} X\big) \prod_{0 \le m \le j-2} (1 - \alpha^{j-2-2m} X) \\
&= 1 - \Big(\sum_{m=0}^{j-2} \alpha^{j-2-2m}\Big)^2 X^2 + \sum_{0 \le a < b \le j-2} \alpha^{2(j-2)-2(a+b)} X^2 + \cdots \\
&= 1 - \sum_{0 \le b \le a \le j-2} \alpha^{2(j-2)-2(a+b)} X^2 + \cdots. \qquad (3.10)
\end{aligned}$$

Note that

$$L(\mathrm{sym}^2(\mathrm{sym}^r \rho), X) = \prod_{0 \le a \le b \le r} (1 - \alpha^{r-2a} \alpha^{r-2b} X)^{-1} \qquad (3.11)$$

and by (3.5),

$$L(\mathrm{sym}^2(\mathrm{sym}^r \rho), X) = \prod_{0 \le i \le r/2} L(\mathrm{sym}^{2r-4i} \rho, X). \qquad (3.12)$$

Thus we see that

$$L(\mathrm{sym}^2(\mathrm{sym}^r \rho), X)^{-1} = 1 - \sum_{0 \le a \le b \le r} \alpha^{2r-2(a+b)} X + \cdots, \qquad (3.13)$$

and in view of (3.10) and (3.13), we may write

$$\big(1 + \sum_{m=0}^{j-2} \alpha^{j-2-2m} X\big) \prod_{0 \le m \le j-2} (1 - \alpha^{j-2-2m} X) = \frac{H_j(\rho, X)}{L(\mathrm{sym}^2(\mathrm{sym}^{j-2} \rho), X^2)} \qquad (3.14)$$

where

$$
\begin{aligned}
H_j(\rho, X) &:= \big(1 + \sum_{m=0}^{j-2} \alpha^{j-2-2m} X\big) \prod_{0 \le m \le j-2} (1 - \alpha^{j-2-2m} X) L(\mathrm{sym}^2(\mathrm{sym}^{j-2}\rho), X^2) \\
&= \big(1 + \sum_{m=0}^{j-2} \alpha^{j-2-2m} X\big) \prod_{0 \le m \le j-2} (1 - \alpha^{j-2-2m} X) \prod_{1 \le i \le j/2} L(\mathrm{sym}^{2j-4i}\rho, X^2) \\
&= 1 + \sum_{r \ge 3} c_r X^r \qquad (3.15)
\end{aligned}
$$

for some $c_r \in \mathbb{C}$. We conclude (3.8) from (3.9), (3.14) and (3.12).

Under (GRC), $H_j(\rho, X)$ is given by a fraction of the form

$$
\frac{\prod(1 + O_j(X))}{\prod(1 + O_j(X^2))}.
$$

(See (3.14) and (3.11).) It follows plainly that $c_r \ll r^{O_j(1)}$ and thus $H_j(s) = \prod_p H_j(\rho_p, p^{-s})$ converges absolutely in $\Re\mathrm{e}\, s > 1/3$.

Finally we consider $j = 3, 4$ without imposing (GRC), but instead, using $1 \le |\alpha_p| \le p^\theta$ with $\theta = 7/64$. Our goal is to show that $H_j(s)$ is still absolutely convergent in $\Re\mathrm{e}\, s > 1/3$.

Apparently, we have by (3.15),

$$
\begin{aligned}
H_3(\rho, X) &= \big(1 + (\alpha + \alpha^{-1})X\big)(1 - \alpha X)(1 - \alpha^{-1} X) L(\mathrm{sym}^2\rho, X^2) \\
&= \frac{\big(1 + (\alpha + \alpha^{-1})X\big)(1 - \alpha X)(1 - \alpha^{-1} X)}{(1 - \alpha^2 X^2)(1 - X^2)(1 - \alpha^{-2} X^2)} \\
&= \frac{1 + t(\rho) X}{1 + t(\rho) X - t(\rho) X^3 - X^4} \\
&= (1 + O((|t(\rho)| + 1) X^3))^{-1}
\end{aligned}
$$

whenever $t(\rho) X \ll p^{-(\sigma - \theta)} = o(1)$. (Note $\theta = \frac{7}{64} < \frac{1}{3}$.) Taking product over p, the absolute convergence of $H_j(s)$ is equivalent to

$$
\sum_p |t_\varphi(p)| p^{-3\sigma} \ll 1,
$$

which is valid if $3\sigma > 1$.

The case $j = 4$ is similar. We write $t_2(\rho) = t(\mathrm{sym}^2\rho) = t_{\mathrm{sym}^2\varphi}(p)$, then

$$\begin{aligned} H_4(\rho, X) &= \big(1 + \sum_{m=0}^{2} \alpha^{2-2m} X\big) \prod_{0\le m\le 2} (1 - \alpha^{2-2m} X) \prod_{1\le i\le 2} L(\mathrm{sym}^{8-4i}\rho, X^2) \qquad (3.16) \\ &= \frac{1 + t_2(\rho)X}{(1 + t_2(\rho)X + t_2(\rho)X^2 + X^3)(1 - t_2(\rho)X^2 + t_2(\rho)X^4 - X^6)} \\ &= \frac{1 + t_2(\rho)X}{(1 + t_2(\rho)X + t_2(\rho)X^2 + X^3)(1 - t_2(\rho)X^2 + O((|t_2(\rho)| + 1)X^4))} \\ &= (1 + t_2(\rho)X)(1 + t_2(\rho)X + (1 - t_2(\rho)^2)X^3 + O((|t_2(\rho)|^2 + 1)X^4))^{-1} \\ &= (1 + O((|t_2(\rho)|^2 + 1)X^3))^{-1} \qquad (3.17) \end{aligned}$$

for $|t_2(\rho)X| \ll p^{-(\sigma-2\theta)} = o(1)$. This case boils down to

$$\sum_p |t_{\mathrm{sym}^2\varphi}(p)|^2 p^{-3\sigma} \ll 1.$$

Lemma 3.2 assures the sufficiency of $\sigma > 1/3$. Besides (3.17) shows that the p-local factor has no zero in $\Re e\, s > 1/3$ for all but finitely many primes p. The possible exception stems from the factor $1+\sum_{m=0}^{2} \alpha_p^{2-2m} p^{-s}$ (and p is small) in view of (3.16) and that $|\alpha_p|$, $|\alpha_p|^{-1} \le p^{7/64}$. If $1 + \sum_{m=0}^{2} \alpha_p^{2-2m} p^{-s} = 0$, then $|\sum_{m=0}^{2} \alpha_p^{2-2m}| = p^{-\sigma}$. So σ is determined, i.e. the possible zeros of a p-local factor lie on a vertical line in the region $\Re e\, s > 1/3$. This concludes the last assertion since only a finite number of local factors may have zeros. $\square$

4. Proof of Theorem 1

Suppose $\mathrm{sym}^j\varphi$ is automorphic cuspidal. We take $m = j + 1$ in Lemma 2.1, thus

$$\sum_{n\le x} t_{\mathrm{sym}^j\varphi}(n) \ll_\varphi x^{\frac{1}{2}-\frac{1}{2j+2}+(\frac{j+1}{2}-\frac{1}{2})\eta} + \sum_{x<n\le x+x^{1-\frac{1}{j+1}-\eta}} |t_{\mathrm{sym}^j\varphi}(n)|, \qquad (4.1)$$

holds true for every $\eta \ge 0$.

Under (GRC), we follow the approach in [10] to evaluate the sum on the right-side of (4.1). When GRC is true, we infer from (3.2) and (3.3) that

$$|t_{\mathrm{sym}^j\varphi}(p^l)| \le \frac{(j + l)!}{j!l!} = d_{j+1}(p^l) \qquad \text{and} \qquad |t_{\mathrm{sym}^j\varphi}(n)| \le d_{j+1}(n) \ll n^\varepsilon$$

where $d_{j+1}(n)$ is the divisor function whose associated Dirichlet series is $\zeta(s)^{j+1}$. Then we can apply Shiu's theorem [35, Theorem 1] to see that

$$\sum_{x<n\le x+y} |t_{\mathrm{sym}^j\varphi}(n)| \ll \frac{y}{\log x} \exp\big(\sum_{p\le x} |t_{\mathrm{sym}^j\varphi}(p)|p^{-1}\big)$$

holds uniformly for $x^{1/3} \le y \le x$. Following from [32, Proposition 2.3] and a standard Riemann-Stieltjes partial integration, we plainly have

$$\sum_{p\le x} |t_{\mathrm{sym}^j\varphi}(p)|^2 p^{-1} = \log\log x + O(1)$$

(noting that GRC implies Hypothesis H of [32]), and thus

$$\sum_{x<n\le x+y}|t_{\mathrm{sym}^j\varphi}(n)| \ll \frac{y}{\log x}\exp\left(\Big(\sum_{p\le x}|t_{\mathrm{sym}^j\varphi}(p)|^2p^{-1}\Big)^{1/2}\Big(\sum_{p\le x}p^{-1}\Big)^{1/2}\right) \ll y$$

for $x^{1/3}\le y\le x$.

Consequently we set $\eta = j/((j+1)(j+2))$ to get

$$\begin{aligned}\sum_{n\le x} t_{\mathrm{sym}^j\varphi}(n) &\ll_\varphi x^{\frac12-\frac{1}{2j+2}+(\frac{j+1}{2}-\frac12)\eta}+x^{1-\frac{1}{j+1}-\eta}\\ &\ll_\varphi x^{\frac{j}{j+2}} \qquad \text{(under (GRC)).}\end{aligned} \tag{4.2}$$

Without (GRC), we apply Lemma 2.2 to $L(\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi, s)$,

$$\sum_{n\le x} t_{\mathrm{sym}^j\varphi\times\mathrm{sym}^j\varphi}(n) = c_\varphi x + O_{\varepsilon,\varphi}\left(x^{c_j+\varepsilon}\right).$$

where $c_j = \frac{(j+1)^2-1}{(j+1)^2+1}$. Taking difference and combining with Lemma 3.1, we obtain

$$\sum_{x<n\le x+x^{1-\frac{1}{j+1}-\eta}} |t_{\mathrm{sym}^j\varphi}(n)|^2 \ll_{\varepsilon,\varphi} x^{c_j+\varepsilon}$$

if $\eta \ge 1-1/(j+1)-c_j$. This gives, by Cauchy-Schwarz's inequality, an estimate to the sum over short interval in (4.1), and consequently we obtain

$$\sum_{n\le x} t_{\mathrm{sym}^j\varphi}(n) \ll_{\varepsilon,\varphi} x^{\frac12-\frac{1}{2j+2}+(\frac{j+1}{2}-\frac12)\eta}+x^{\frac12-\frac{1}{2j+2}-\frac{\eta}{2}+c_j+\varepsilon}.$$

Set $\eta = \frac{(j+1)^2-1}{(j+1)((j+1)^2+1)} \ge 1-\frac{1}{j+1}-c_j$ and let $\kappa_j = \frac{(j+1)^2-(j+1)}{(j+1)^2+1} = \frac{j(j+1)}{(j+1)^2+1}$. We have

$$\sum_{n\le x} t_{\mathrm{sym}^j\varphi}(n) \ll_{\varepsilon,\varphi} x^{\kappa_j+\varepsilon}. \tag{4.3}$$

Now we turn to $S_j(x)$. By Lemma 3.3, we have

$$t_\varphi(n^j) = \sum_{n=ml} t_{\mathrm{sym}^j\varphi}(m)a(l)$$

where the generating function of $a(l)$ is $\prod_{1\le i\le j/2} L(\mathrm{sym}^{2j-4i}\varphi, 2s)^{-1}H_j(s)$. Suppose

$$\sum_{n\le x} t_{\mathrm{sym}^j\varphi}(n) \ll x^\kappa \quad \text{and} \quad \sum_{n=1}^\infty |a(n)|n^{-\kappa} \ll 1. \tag{4.4}$$

Then we easily see that

$$\begin{aligned}\sum_{n\le x} t_\varphi(n^j) &= \sum_{n\le x}\sum_{n=ml} t_{\mathrm{sym}^j\varphi}(m)a(l)\\ &\ll \sum_{l\le x}|a(l)|\left|\sum_{m\le x/l} t_{\mathrm{sym}^j\varphi}(m)\right| \ll \sum_{l\le x}|a(l)|(x/l)^\kappa \ll x^\kappa.\end{aligned}$$

Under (GRC), $\sum_{n=1}^{\infty} |a(n)|n^{-\sigma} \ll 1$ for $\sigma > 1/2$. If the cuspidality (and automorphy) of $\mathrm{sym}^j\varphi$ and (GRC) are fulfilled, then we get (4.2) and hence (4.4) with $\kappa = j/(j+2)$ for $j \geq 3$. It remains to consider the case $j = 2$ to complete Part (b). Recall $L_2(s) = L(\mathrm{sym}^2\varphi, s)\zeta(2s)^{-1}$. Thus $a(l)$ is supported on squares and $a(h^2) = \mu(h)$. We get by (4.2) with $j = 2$ that

$$\sum_{n\leq x} t_\varphi(n^2) \ll \sum_{h\leq\sqrt{x}} \sqrt{x/h^2} \ll x^{1/2}\log x,$$

hence complete the proof of part (b).

The proof of (1.2) is similar; by Lemma 3.3, we see that $\sum_{n=1}^{\infty} |a(n)|n^{-\sigma} \ll 1$ holds for $\sigma > 1/2$ when φ is a primitive Maass cusp form and $j = 3, 4$. As mentioned in §3, $\mathrm{sym}^j\varphi$ is automorphic cuspidal in these cases. We thus obtain (4.3) unconditionally, so now (4.4) applies with $\kappa = \kappa_j$. This proves (1.2).

5. Some Preparation for Theorem 2

We need some auxiliary results, formulated in a quite general setting. Lemma 5.1 follows from the Borel-Carathéodory inequality and a standard argument, see [21, Lemma 1] for more details. Lemma 5.2 is a core component in the proof of Theorem 2. In fact, we shall apply it to the situation that $\psi(s) = \prod_i L(f_i, 2s)$ and $L(s) = L(f, s+a)H(s)$ where the L-functions are belonged to $S_e^{\#}$, $a \in \mathbb{R}$ and $H(s)$ is a Dirichlet series convergent absolutely in the concerned region.

Lemma 5.1. *Let $0 < \alpha < 1$ and $\psi(s)$ be a Dirichlet series whose reciprocal $\psi(s)^{-1}$ also admits a Dirichlet series $\sum_{n\geq1} a_n n^{-s}$ with $a_1 = 1$. Suppose both $\psi(s)$ and $\psi(s)^{-1}$ have finite abscissas of absolute convergence, and there exists a small $\delta > 0$ such that*

- (i) *$\psi(s)$ is meromorphic in $\Re\mathrm{e}\, s > \alpha - 2\delta$ with at most a finite number of poles, and for some constants c and $t_0 > 0$, $\psi(\sigma + it) \ll |t|^c$ uniformly for $\sigma > \alpha - 2\delta$ and $|t| \gg t_0$;*
- (ii) *$\mathrm{N}_\psi(\alpha - \delta, T) \ll T^{1-c(\delta)}$ for all sufficiently large T, where $c(\delta) > 0$ is a constant depending on δ.*

Then there is a constant $C > 0$ such that for all sufficiently large T,

$$\psi(\sigma + it)^{-1} \ll T^C$$

where the implied constant is absolute, whenever $\sigma \geq \alpha$, $t \in [T, 2T]$ and

$$|t - \gamma| \geq \log^2 T, \qquad \forall\, \gamma \in \mathcal{Z}_\psi(\alpha - \delta).$$

Here $\mathcal{Z}_\psi(\sigma) := \{\gamma :\ \psi(\beta + i\gamma) = 0 \text{ for some } \beta \geq \sigma\}$.

Lemma 5.2. *Let $0 < \alpha < 1$ and $\psi(s)$ satisfy the conditions in Lemma 5.1. Suppose $L(s) = \sum_{n\geq1} b_n n^{-s}$ with $b_1 = 1$ is a Dirichlet series with finite abscissa of absolute convergence. Assume*

- (A) *$L(s)$ is meromorphic in an open set containing the region $\Re\mathrm{e}\, s \geq \alpha$ with at most a finite number of poles,*
- (B) *$L(\sigma + it) \ll |t|^C$ for all $\sigma \geq \alpha$ and $|t| \gg 1$, where $C > 0$ is some absolute constant.*

Define $G(s) = L(s)/\psi(s)$. Then for all $0 < \beta < 1$ and all sufficiently large $T \geq T_0$,

$$\int_T^{2T} \frac{|G(\lambda + it)|^2}{|\lambda + it|^{1+\beta}} \, dt \gg T^{-\beta}$$

holds uniformly for $\lambda \geq \alpha$, where the constant T_0 and the implied contant in $\gg$ are independent of λ and β. (A divergent improper integral means infinity.)

Proof. Under the assumption (ii) in Lemma 5.1, for all $T \geq T_1$ there are at least $T/(4 \log T)^2$ disjoint open intervals $I_i = (u_i, v_i) \subset [T, 2T]$ such that

(a) the width of I_i, denoted by $H_i := v_i - u_i$, is $\geq 6 \log^2 T$ and $\sum_i H_i \geq \frac{1}{2} T$,
(b) for each i, $G(s)$ has no pole in (or on the boundary of) the region R_i: $\sigma \geq \lambda$, $t \in \overline{I_i} = [u_i, v_i]$.

(T_1 is some large number independent of λ and β.)

Let $L = \log^2 T$ and $s = \lambda + it$ with $u_i + 2L \leq t \leq v_i - 2L$. Write $G(s) = \sum_{n \geq 1} g(n) n^{-s}$ and choose $1 < B \ll 1$ so that $\sum_{n \geq 1} |g(n)| n^{-B} \ll 1$. Then

$$\sum_{n \geq 1} \frac{g(n)}{n^s} (e^{-n/2} - e^{-n}) = \frac{1}{2\pi i} \int_{B - i\infty}^{B + i\infty} G(s + w) \big(2^w - 1\big) \Gamma(w) \, dw.$$

We replace the line segment $[B - iL, B + iL]$ by the contour consisting of 3 straight line segments joining $B - iL$, $-iL$, iL, $B + iL$. By (A) and Lemma 5.1, $G(\sigma + it) \ll |t|^c$ for any $\sigma + it$ lying on the contour, where $c > 0$ is some constant. Since

$$\Gamma(u + iv) \asymp |v|^{u - 1/2} e^{-\pi |v|/2} \tag{5.1}$$

for $|u| \ll 1$ and $|v| \gg 1$, the fast decay assures that

$$\int_{B - i\infty}^{B - iL} + \int_{B - iL}^{-iL} + \int_{iL}^{B + iL} + \int_{B + iL}^{i\infty} \ll T^{-1}.$$

As

$$\frac{2^{iv} - 1}{iv} \ll \min(1, \frac{1}{|v|}),$$

we have for $t \in [u_i + 2L, v_i - 2L]$,

$$\sum_{n \geq 1} \frac{g(n)}{n^s} (e^{-n/2} - e^{-n})$$
$$\ll \int_{-L}^{L} |G(s + iv)| \min(1, |v|^{-1}) \big|\Gamma(1 + iv)\big| \, dv + T^{-1}. \tag{5.2}$$

Now we square out both sides of (5.2) and multiply with $|s|^{-(1+\beta)}$. Integrating with respect to t over $[u_i+2L, v_i-2L]$, it follows from the inequality $(|a|+|b|)^2 \ll |a|^2+|b|^2$, the Cauchy-Schwarz inequality and (5.1) that

$$\int_{u_i+2L}^{v_i-2L} \left| \sum_{n\geq 1} \frac{g(n)}{n^{\lambda+it}} (e^{-n/2} - e^{-n}) \right|^2 \frac{dt}{|\lambda+it|^{1+\beta}}$$

$$\ll \int_{u_i+2L}^{v_i-2L} \int_{-L}^{L} |G(s+iv)|^2 \min(1, |v|^{-2})\, dv \frac{dt}{|\lambda+it|^{1+\beta}} + T^{-2-\beta}. \tag{5.3}$$

The double integral in the right-hand side of (5.3) is

$$\begin{aligned} &\ll \int_{u_i+L}^{v_i-L} \frac{|G(\lambda+it)|^2}{|\lambda+it|^{1+\beta}}\, dt \times \int_{-\infty}^{\infty} \min(1, |v|^{-2})\, dv \\ &\ll \int_{u_i}^{v_i} \frac{|G(\lambda+it)|^2}{|\lambda+it|^{1+\beta}}\, dt, \end{aligned} \tag{5.4}$$

while by Hilbert's inequality ([21, Lemma 2]), the left-hand side of (5.3) has a lower estimate

$$\begin{aligned} &\gg T^{-(1+\beta)} \sum_{n\geq 1} (H_i - 4L + O(n)) |g(n)(e^{-n/2} - e^{-n})|^2 n^{-2\lambda} \\ &\gg T^{-(1+\beta)} \big(H_i - O(1)\big) \\ &\gg T^{-(1+\beta)} H_i, \end{aligned} \tag{5.5}$$

for $H_i = v_i - u_i \geq 6L$, $g(1) = 1$ and $g(n)n^{-2\lambda} \ll n^B$. Consequently we deduce from (5.3)-(5.5) that for every i,

$$\int_{u_i}^{v_i} \frac{|G(\lambda+it)|^2}{|\lambda+it|^{1+\beta}}\, dt \geq c' T^{-(1+\beta)} H_i$$

for some absolute constant $c' > 0$. Summing over all i, we infer that

$$\begin{aligned} \int_{T}^{2T} \frac{|G(\lambda+it)|^2}{|\lambda+it|^{1+\beta}}\, dt &\geq \sum_i \int_{u_i}^{v_i} \frac{|G(\lambda+it)|^2}{|\lambda+it|^{1+\beta}}\, dt \\ &\geq c' T^{-(1+\beta)} \sum_i H_i \\ &\geq \frac{1}{2} c' T^{-\beta}. \end{aligned}$$

This completes the proof as c' is independent of λ and β. □

6. Proof of Theorem 2

We proceed by contradiction. Let $\epsilon > 0$ be small at our disposal and assume $|S_j(x)| \leq \epsilon x^\alpha$ for all $x \geq x_0(\epsilon)$, where $\alpha = j/(2j+2)$.

Using integration by parts, we have

$$L_j(s) = s \int_0^\infty S_j(e^y) e^{-ys}\, dy$$

for $\sigma > 1$. Under our assumption, $S_j(e^y) \ll e^{\alpha y}$ and so $L_j(s)$ is holomorphic on $\Re s > \alpha$. Rewriting into the form

$$\int_0^\infty S_j(e^y)e^{-\sigma y}e(-yt)\,dy = \frac{L_j(\sigma+it)}{\sigma+it},$$

we infer from the Plancherel theorem that

$$\int_0^\infty |S_j(e^y)|^2 e^{-2\sigma y}\,dy = \int_{-\infty}^\infty \left|\frac{L_j(\sigma+it)}{\sigma+it}\right|^2 dt. \tag{6.1}$$

As $\sigma \to \alpha+$, the left-hand side will grow up in a rate of $\epsilon^2(\sigma-\alpha)^{-1}$. Our contradiction will come up due to the inconsistency to the t-integral.

To this end, we shall apply Lemma 5.2 with the factorization of $L_j(s)$. Let us write

$$\psi_j(s) = \prod_{1\le i\le j/2} L(\mathrm{sym}^{2j-4i}\varphi, 2s).$$

In view of Lemma 3.3, we have for $\sigma > \alpha$,

$$\int_{-\infty}^\infty \left|\frac{L_j(\sigma+it)}{\sigma+it}\right|^2 dt = \int_{-\infty}^\infty \left|L(\mathrm{sym}^j\varphi, \sigma+it)H_j(\sigma+it)\psi_j(\sigma+it)^{-1}\right|^2 \frac{dt}{|\sigma+it|^2}. \tag{6.2}$$

From the functional equation of $L(\mathrm{sym}^j\varphi, s)$, we get

$$\left|L(\mathrm{sym}^j\varphi, \sigma+it)\right| \gg |t|^{\frac{j+1}{2}(1-2\sigma)}\left|L(\mathrm{sym}^j\varphi, 1-\sigma+it)\right|, \qquad \forall\ t \gg 1.$$

Note that $\overline{L(\mathrm{sym}^j\varphi, s)} = L(\mathrm{sym}^j\varphi, \overline{s})$ as $t_{\mathrm{sym}^j\varphi}(n) \in \mathbb{R}$. Inserting into (6.2), we deduce that for $\alpha < \sigma < 1/2$,

$$\int_{-\infty}^\infty \left|\frac{L_j(\sigma+it)}{\sigma+it}\right|^2 dt \quad \gg \quad \int_{-\infty}^\infty \frac{\left|G(\sigma+it)\right|^2}{\left|\sigma+it\right|^{1+\beta}}\,dt \tag{6.3}$$

where $G(\sigma+it) = L(\mathrm{sym}^j\varphi, \sigma+it+1-2\sigma)H_j(\sigma+it)\psi_j(\sigma+it)^{-1}$ and $\beta = 2(j+1)(\sigma-\alpha)$. We are ready to invoke Lemma 5.2 to derive that for all sufficiently large T,

$$\int_T^{2T} \frac{\left|G(\sigma+it)\right|^2}{\left|\sigma+it\right|^{1+\beta}}\,dt \gg T^{-2(j+1)(\sigma-\alpha)}, \tag{6.4}$$

once the condition of zero density for $\psi(s)$ is verified. (The implied constant in $\gg$ is independent of σ.)

Denote by $N_j(\sigma,T)$ the number of zeros of $\psi_j(s)$ in the rectangle with corners at $\sigma \pm iT$ and $2 \pm iT$. The cases $j = 1, 2$ are plain, more concretely, $\psi_1(s) = 1$, $\psi_2(s) = \zeta(2s)$ and $H_1(s) = H_2(s) = 1$. We have $N_1(\sigma,T) = 0$, and by [12, Theorem 11.1],

$$N_2(\sigma,T) \ll T^{12(1-2\sigma)/5+\varepsilon} \quad (\sigma \in [\tfrac{1}{4}, \tfrac{1}{2}]).$$

Note that $N_2(\sigma,T) \ll T^{9/10+\varepsilon}$ for $\sigma = \alpha_2 - \frac{1}{48} = \frac{15}{48}$ in this case. (We write α_j for α when we want to emphasize the value of j and recall $\alpha_j = j/(2j+2)$.) Therefore, (6.4) holds in these two cases by virtue of Lemma 5.2

We turn to the cases $j = 3, 4$, where

$$\psi_3(s) = L(\mathrm{sym}^2\varphi, 2s) \quad\text{and}\quad \psi_4(s) = L(\mathrm{sym}^4\varphi, 2s)\zeta(2s).$$

For $j = 3$, we apply Lemma 2.5 (b) and so

$$N_3(\sigma, T) \ll T^{3(1-2\sigma)+\varepsilon} \qquad (\sigma \in (\tfrac{1}{3}, \tfrac{1}{2}]),$$

which is $O(T^{1-c})$ when $\sigma = \alpha_3 - \delta$ for some constants $c, \delta > 0$. This case is straightforward.

As the zeros of $\psi_4(s)$ (in $\Re\mathrm{e}\, s > 1/3$) come from $L(\mathrm{sym}^4\varphi, 2s)$ or $\zeta(2s)$ and the zero density of $\zeta(2s)$ is under control (see $N_2(\sigma, T)$). It remains to count the zeros of $L(\mathrm{sym}^4\varphi, 2s)$. The useful tool Lemma 5.2 based on the existing mean square estimate

$$\int_0^T \left|L(\mathrm{sym}^4\varphi, 1/2 + it)\right|^2 dt \ll T^{5/2+\varepsilon}$$

is, however, not enough for our purpose. Fortunately we have a bypass in this hypothetical situation, based on the observation from a rearrangement of (3.8),

$$L(\mathrm{sym}^4\varphi, s) = L_4(s)L(\mathrm{sym}^4\varphi, 2s)\zeta(2s)H_4(s)^{-1}.$$

Using Lemma 3.3, we may choose a constant $0 < \delta < 36^{-1}$ to avoid any possible zeros of $H_4(s)$ so that

$$L(\mathrm{sym}^4\varphi, 1 + 2\delta + it)\zeta(1 + 2\delta + it)H_4(\frac{1}{2} + \delta + it)^{-1} \ll_\delta 1,$$

hence

$$\int_1^T \left|L(\mathrm{sym}^4\varphi, 1/2 + \delta + it)\right|^2 dt \;\ll\; \int_1^T \left|L_4(1/2 + \delta + it)\right|^2 dt.$$

Under the assumption on the size of $S_4(x)$, we infer that

$$\begin{aligned}\int_0^T \left|L_4(1/2 + it)\right|^2 dt \;&\ll\; T^2 \int_{-\infty}^{\infty} \left|\frac{L_4(1/2 + it)}{1/2 + it}\right|^2 dt \\ &\ll\; T^2 \int_0^{\infty} e^{2(\alpha - 1/2 - \delta)y}\, dy \ll T^2,\end{aligned}$$

by (6.1). (Here $\alpha = \alpha_4 = 2/5$.) Consequently we obtain the *conditional* estimate

$$\int_0^T \left|L(\mathrm{sym}^4\varphi, 1/2 + \delta + it)\right|^2 dt \ll T^2.$$

We thus apply Lemma 2.5 (a), and hence obtain

$$N_4(\sigma, T) \ll T^{\frac{4(1-2\sigma)}{1-2\delta}+\varepsilon} \qquad (\sigma \in (\tfrac{1}{3}, \tfrac{1}{2}]),$$

which is $\ll T^{16/17+\varepsilon}$ when $\sigma = \alpha - \frac{1}{90} = \frac{35}{90}$. So now the conditions for Lemma 5.2 are fulfilled, and (6.4) also holds when $j = 4$.

From (6.4) and (6.3), we choose a sufficiently large but fixed T_0 so that

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{L_j(\sigma+it)}{\sigma+it}\right|^2 dt &\gg \sum_{n\geq 0}\int_{2^nT_0}^{2^{n+1}T_0}\frac{|G(\sigma+it)|^2}{|\sigma+it|^{1+\beta}}\,dt\\ &\gg T_0^{-\beta}\sum_{n\geq 0}2^{-n\beta}\\ &\gg \frac{T_0^{-\beta}}{1-2^{-\beta}}\\ &\gg \beta^{-1}T_0^{-\beta}.\end{aligned}$$

Hence when $|S(x)|\leq \epsilon x^{\alpha}$ ($\forall\ x\geq x_0(\epsilon)$), we obtain from (6.1) the inequality

$$\epsilon^2(\sigma-\alpha)^{-1}+O_\epsilon(1)\gg \beta^{-1}T_0^{-\beta}\gg(\sigma-\alpha)^{-1}-O(1)$$

for $\sigma\in(\alpha,1/2)$, where the implied constants are independent of σ. Taking $\sigma\to\alpha+$, it follows

$$\epsilon^2\gg 1,$$

which leads to a contradiction for suitable small ϵ. This completes the proof.

7. Some Preparation for Theorem 3

Let $l\geq 2$ and define

$$R_l(s)=\sum_{n=1}^{\infty}\frac{t_\varphi(n)^l}{n^s}.$$

We begin with a decomposition of $R_l(s)$.

Lemma 7.1. *For $\Re\mathrm{e}\, s\gg 1$,*

$$R_l(s)=F_l(s)U_l(s) \tag{7.1}$$

where

$$\begin{aligned}F_{2j}(s)&=\zeta(s)^{A_j}L(\mathrm{sym}^{2j}\varphi,s)\prod_{1\leq r\leq j-1}L(\mathrm{sym}^{2r}\varphi,s)^{C_j(r)} &&(l=2j),\\ F_{2j+1}(s)&=L(\varphi,s)^{B_j}L(\mathrm{sym}^{2j+1}\varphi,s)\prod_{1\leq r\leq j-1}L(\mathrm{sym}^{2r+1}\varphi,s)^{D_j(r)} &&(l=2j+1)\end{aligned}$$

where the constants A_j, B_j, $C_j(r)$, $D_j(r)$ $(1\leq r\leq j-1)$ are given by

$$A_j=\frac{(2j)!}{j!(j+1)!},\qquad B_j=2\frac{(2j+1)!}{j!(j+2)!},$$

$$C_j(r)=\frac{(2j)!(2r+1)}{(j-r)!(j+r+1)!},\qquad D_j(r)=\frac{(2j+1)!(2r+2)}{(j-r)!(j+r+2)!}.$$

The L-function $F_l(s)$ is of degree 2^l, and for even $l=2j$ all coefficients of $F_{2j}(s)$ are nonnegative. Under (GRC) for φ, $U_j(s)$ is a Dirichlet series absolutely convergent in $\Re\mathrm{e}\, s>1/2$.

Proof. Again we only need to consider the local factors of all large primes. Observing that

$$\alpha^m + \beta^m = t(\mathrm{sym}^m \rho) - t(\mathrm{sym}^{m-2}\rho)$$

if $\rho = \mathrm{diag}(\alpha, \beta)$, we carry out binomial expansion to get

$$\begin{aligned} t(\rho)^{2j} &= \sum_{r=0}^{j-1} \binom{2j}{r} \left(t(\mathrm{sym}^{2j-2r}\rho) - t(\mathrm{sym}^{2j-2r-2}\rho) \right) + \binom{2j}{j} \\ &= t(\mathrm{sym}^{2j}\rho) + \sum_{r=1}^{j-1} \left(\binom{2j}{r} - \binom{2j}{r-1} \right) t(\mathrm{sym}^{2j-2r}\rho) + \binom{2j}{j} - \binom{2j}{j-1} \\ &= A_j + t(\mathrm{sym}^{2j}\rho) + \sum_{r=1}^{j-1} C_j(r) t(\mathrm{sym}^{2r}\rho) \qquad (7.2) \end{aligned}$$

after replacing r by $j - r$, because

$$A_j = \frac{(2j)!}{j!(j+1)!} = \left(\binom{2j}{j} - \binom{2j}{j-1} \right)$$

and for $1 \le r \le j-1$,

$$C_j(r) = \frac{(2j)!(2r+1)}{(j-r)!(j+r+1)!} = \binom{2j}{j-r} - \binom{2j}{j-r-1}.$$

Similarly, we calculate that

$$\begin{aligned} t(\rho)^{2j+1} &= \sum_{r=0}^{j-1} \binom{2j+1}{r} \left(t(\mathrm{sym}^{2j-2r+1}\rho) - t(\mathrm{sym}^{2j-2r-1}\rho) \right) + \binom{2j+1}{j} t(\rho) \\ &= B_j t(\rho) + t(\mathrm{sym}^{2j+1}\rho) + \sum_{r=1}^{j-1} D_j(r) t(\mathrm{sym}^{2r+1}\rho) \qquad (7.3) \end{aligned}$$

where

$$B_j = \left(\binom{2j+1}{j} - \binom{2j+1}{j-1} \right) = \frac{2(2j+1)!}{j!(j+2)!}$$

and for $1 \le r \le j-1$,

$$D_j(r) = \binom{2j+1}{j-r} - \binom{2j+1}{j-r-1} = \frac{(2j+1)!(2r+2)}{(j-r)!(j+r+2)!}.$$

Formally, we can express

$$1 + \sum_{\ell \ge 1} t(p^\ell) p^{-\ell s} = \exp\left(t(p)p^{-s}\right) \left(1 + \sum_{\ell \ge 2} p^{-\ell s} \sum_{h+k=\ell} c_{h,k} t(p^h) t(p)^k \right).$$

Under (GRC), the sum $\sum_{\ell \ge 2}$ is $\ll p^{-2\sigma}$. We thus define $U_l(s) = R_l(s)/F_l(s)$, and its p-local factor is of the form $1 + O(p^{-2\sigma})$ by (7.2) or (7.3). So the Euler product (hence the Dirichlet series) of $U_l(s)$ converges absolutely in $\Re e\, s > 1/2$ if (GRC) holds.

The degree of $F_{2j}(s)$ equals

$$A_j + (2j+1) + \sum_{r=1}^{j-1} C_j(r)(2r+1) = 2^{2j},$$

which can be obtained by taking $\alpha = \beta = 1$ in (7.2). Similarly the degree of $F_{2j+1}(s)$ is 2^{2j+1} with (7.3). To see that $F_{2j}(s)$ has nonnegative coefficients, it suffices to check

$$\log F_{2j}(s) = \sum_n b_n n^{-s} \qquad \text{with all } b_n \geq 0,$$

since $F_{2j}(s) = \exp(\log F_{2j}(s))$. Taking logarithm of the p-local factor leads to

$$-A_j \log(1-X) - \log \prod_{m=0}^{2j} (1-\alpha^{2j-m}X) - \sum_{r=1}^{j-1} C_j(r) \log \prod_{m=0}^{2r} (1-\alpha^{2r-m}X)$$

where $\alpha = \alpha_p$ and $X = p^{-s}$. Expanding out, we get that

$$\begin{aligned} & A_j \sum_{\ell \geq 1} \frac{X^\ell}{\ell} + \sum_{m=0}^{2j} \sum_{\ell \geq 1} \alpha^{(2j-m)\ell} \frac{X^\ell}{\ell} + \sum_{r=1}^{j-1} C_j(r) \sum_{m=0}^{2r} \sum_{\ell \geq 1} \alpha^{(2j-r)\ell} \frac{X^\ell}{\ell} \\ = & \sum_{\ell \geq 1} f_\ell \frac{X^\ell}{\ell}, \quad \text{say,} \end{aligned}$$

and

$$\begin{aligned} f_\ell &= A_j + \sum_{m=0}^{2j} \alpha^{(2j-m)\ell} + \sum_{r=1}^{j-1} C_j(r) \sum_{m=0}^{2r} \alpha^{(2j-r)\ell} \\ &= A_j + t(\mathrm{sym}^{2j} \rho^\ell) + \sum_{r=1}^{j-1} C_j(r) t(\mathrm{sym}^{2r} \rho^\ell) \end{aligned}$$

with $\rho^\ell = \mathrm{diag}(\alpha^\ell, \beta^\ell)$. Comparing with (7.2), we conclude

$$f_\ell = t(\rho^\ell)^{2j} = (\alpha^\ell + \beta^\ell)^{2j} \geq 0$$

(recalling $\beta = \overline{\alpha}$ or $\alpha, \beta \in \mathbb{R}$) and hence $b_n \geq 0$ (as $b_n = \ell^{-1} f_\ell$ if $n = p^\ell$). □

Remark 6. The formulae in (7.2) and (7.3) can also be derived from the decomposition:

$$V^{\otimes d} \cong \bigoplus_{\lambda \in \mathfrak{S}_d} \mathbb{S}_\lambda V^{\otimes m_\lambda}$$

where $\mathfrak{S}_d$ is the permutation group for d objects, $\mathbb{S}_\lambda V$ is the image of the Young symmetrizer c_λ and m_λ is the dimension of the irreducible representation V_λ of $\mathcal{C}_d$ corresponding to λ. (See [7, Theorem 6.3] for details.) The decomposition gives

$$t(\rho)^d = \sum_{\lambda \in \mathfrak{S}_d} S_\lambda(\alpha, \beta)^{m_\lambda}$$

where $S_\lambda(x_1, x_2)$ is a Schur polynomial. Note that $\dim V = 2$ in our case. The sum runs over $\lambda = (d-r, r)$ where $0 \le 2r \le d$, for otherwise, $\mathbb{S}_\lambda V = 0$. Our formulas (7.2) and (7.3) will follow from a little calculation with [7, (4.11) and (A.5)].

The next lemma explains the sufficiency of the automorphy of the first $l/2$ symmetric powers rather than all l powers in Theorem 3.

Lemma 7.2. *Let $j \ge 2$ and and $h = [j/2]$. Then we have*

$$\begin{aligned} F_{2j}(s) &= \zeta(s)^{A_j - C_j(h+1)} \prod_{1 \le r \le h} L(\mathrm{sym}^{2r}\varphi, s)^{C_j(r) - C_j(h+1)} \\ &\quad \times L(\mathrm{sym}^j\varphi \times \mathrm{sym}^j\varphi, s) \prod_{h+1 \le r \le j-1} L(\mathrm{sym}^r\varphi \times \mathrm{sym}^r\varphi, s)^{C_j(r) - C_j(r+1)}, \end{aligned}$$

and

$$\begin{aligned} F_{2j+1}(s) &= L(\varphi, s)^{B_j - D_j(h+1)} \prod_{1 \le r \le h} L(\mathrm{sym}^{2r+1}\varphi, s)^{D_j(r) - D_j(h+1)} \\ &\quad \times L(\mathrm{sym}^{j+1}\varphi \times \mathrm{sym}^j\varphi, s) \prod_{h+1 \le r \le j} L(\mathrm{sym}^{r+1}\varphi \times \mathrm{sym}^r\varphi, s)^{D_j(r) - D_j(r+1)} \end{aligned}$$

where all the exponents are nonnegative.

In particular, we have

$$\begin{aligned} F_4(s) &= \zeta(s) L(\mathrm{sym}^2\varphi, s)^2 L(\mathrm{sym}^2\varphi \times \mathrm{sym}^2\varphi, s) \\ F_6(s) &= L(\mathrm{sym}^2\varphi, s)^4 L(\mathrm{sym}^2\varphi \times \mathrm{sym}^2\varphi, s)^4 L(\mathrm{sym}^3\varphi \times \mathrm{sym}^3\varphi, s) \\ F_8(s) &= \zeta(s)^7 L(\mathrm{sym}^2\varphi, s)^{21} L(\mathrm{sym}^4\varphi, s)^{13} \\ &\quad \times L(\mathrm{sym}^3\varphi \times \mathrm{sym}^3\varphi, s)^6 L(\mathrm{sym}^4\varphi \times \mathrm{sym}^4\varphi, s). \end{aligned}$$

and

$$\begin{aligned} F_3(s) &= L(\varphi, s) L(\mathrm{sym}^2\varphi \times \varphi, s), \\ F_5(s) &= L(\varphi, s)^4 L(\mathrm{sym}^3\varphi, s)^3 L(\mathrm{sym}^3\varphi \times \mathrm{sym}^2\varphi, s) \\ F_7(s) &= L(\varphi, s)^8 L(\mathrm{sym}^3\varphi, s)^8 L(\mathrm{sym}^3\varphi \times \mathrm{sym}^2\varphi, s)^5 L(\mathrm{sym}^4\varphi \times \mathrm{sym}^3\varphi, s). \end{aligned}$$

Proof. This is shown with (3.6) and (3.7) respectively. The exhausting part is the nonnegativity of the exponents. We first note that $A_j = C_j(0)$ and $C_j(j) = 1$ and similar for the odd case. Consider $C_j(r) - C_j(r+1)$, which equals by Lemma 7.1,

$$\begin{aligned} &\frac{(2j)!}{(j-r)!(j+r+2)!} \{(2r+1)(j+r+2) - (2r+3)(j-r)\} \\ = \; &\frac{2(2j)!}{(j-r)!(j+r+2)!} \big(2r^2 + 4r + 1 - j\big). \end{aligned}$$

As $h = [j/2] \ge (j-1)/2$, we see that $C_j(r) - C_j(r+1) > 0$ for $r \ge h$ and for all j.

Next we evaluate

$$\begin{aligned} & C_j(r) - C_j(h+1) \\ = \quad & \frac{(2j)!}{(j-r)!(j+h+2)!}\Big\{(2r+1)\underbrace{(j+h+2)\cdots(j+r+2)}_{h-r+1} - (2h+3)\underbrace{(j-r)\cdots(j-h)}_{h-r+1}\Big\}. \end{aligned}$$

Observe that $\frac{j+h+2-a}{j-r-a} \ge \frac{3}{2}$ for $a \ge 0$, we have

$$\frac{(j+h+2)\cdots(j+r+2)}{(j-r)\cdots(j-h)} \ge \big(\frac{3}{2}\big)^{h-r+1}$$

while

$$\frac{2h+3}{2r+1} = \frac{2(h-r)+2}{2r+1} + 1 \le 2(h-r)+3.$$

Note that the inequality $\big(\frac{3}{2}\big)^{x+1} \ge 2x+3$ holds when $x \ge 6$. Write $r = h-s$, then we are left to $1 \le s \le 6$ and

$$\begin{aligned} 2r^2+4r+1-j &= 2(h-s)^2+4(h-s)+1-j \ge 2(h-s)^2+2(h-s)-2s \\ &\ge 2(h-s)^2+2(h-s)-12 \end{aligned} \tag{7.4}$$

as $h \ge (j-1)/2$. When $r = h-s \ge 3$, $C_j(r) - C_j(r+1) > 0$ by (7.4) and therefore $C_j(r) - C_j(h+1) > 0$. When $j \ge 18$, the value of h is ≥ 9, so $h-s \ge 3$ for $s \le 6$. It remains to handle the cases $j \le 17$, which can be verified directly.

Now,

$$\begin{aligned} & D_j(r) - D_j(r+1) \\ = \quad & \frac{(2j+1)!}{(j-r)!(j+r+3)!}\{(2r+2)(j+r+3)-(2r+4)(j-r)\} \\ = \quad & \frac{2(2j+1)!}{(j-r)!(j+r+3)!}\big(2r^2+6r+3-j\big). \end{aligned}$$

So $D_j(r) - D_j(r+1) > 0$ for $r \ge h$. Also, we have

$$\begin{aligned} & D_j(r) - D_j(h+1) \\ = \quad & \frac{(2j+1)!}{(j-r)!(j+h+3)!}\Big\{(2r+2)\underbrace{(j+h+3)\cdots(j+r+3)}_{h-r+1} - (2h+4)\underbrace{(j-r)\cdots(j-h)}_{h-r+1}\Big\}. \end{aligned}$$

This leads to consider

$$\big(\frac{3}{2}\big)^{h-r+1} \ge \frac{h+2}{r+1} = \frac{h-r+1}{r+1} + 1 \ge (h-r)+2.$$

Note $\big(\frac{3}{2}\big)^{x} \ge x+2$ holds for $x \ge 4$. It reduces to $j \le 7$, which is again a routine checking.

For small j we list the following for $F_l(s)$ $(3 \le l \le 8)$:

l	j	A_j	$C_j(1)$	$C_j(2)$	$C_j(3)$	$C_j(4)$
4	2	2	3	***1***		
6	3	5	9	***5***	1	
8	4	14	28	20	***7***	1

l	j	B_j	$D_j(1)$	$D_j(2)$	$D_j(4)$
3	1	2	***1***		
5	2	5	4	***1***	
7	3	14	14	***6***	1

The entry in bold italic is $C_j(h+1)$ or $D_j(h+1)$ in the respective case. $\square$

8. Proof of Theorem 3

Let $F_l(s) = \sum_{n\geq 1} c_l(n)n^{-s}$ be defined as in Lemma 7.1. Note that $\deg F_l = 2^l$ and (GRC) is assumed until we turn to the proof of Remark 4.

By Lemma 2.4, it follows immediately that for $l \geq 3$,

$$\sum_{n\leq x} c_l(n) = xQ_l(\log x) + O(x^{\eta_l+\varepsilon}) \tag{8.1}$$

where $Q_{2j+1} \equiv 0$ if $l = 2j+1$ is odd while $\deg Q_{2j} = (2j)!/(j!(j+1)!) - 1$ for even $l = 2j$, and $\eta_l = 1 - 2^{1-l}$. We have, by Lemma 7.1, the convolution

$$t_\varphi(n)^l = \sum_{n=uv} c_l(u)b(v). \tag{8.2}$$

and

$$\sum_{v\geq 1} |b(v)|v^{-\sigma} \ll_\sigma 1 \qquad (\forall\ \sigma > 1/2).$$

With (8.1), we infer that

$$\begin{aligned}
\sum_{n\leq x} t_\varphi(n)^l &= \sum_{v\leq x} b(v) \sum_{u\leq x/v} c_l(u) \\
&= x\sum_{v\geq 1} \frac{b(v)}{v} Q_l(\log \frac{x}{v}) + O(x^{1+\varepsilon}\sum_{v\geq x} |b(v)|v^{-1}) + O(x^{\eta_l+\varepsilon}) \\
&= xP_l(\log x) + O(x^{\eta_l+\varepsilon})
\end{aligned} \tag{8.3}$$

as $\sum_{v\geq x} |b(v)|v^{-1} \ll x^{-1/2}$. This completes our proof.

Lastly we justify Remark 4. Without (GRC), the factorization of $R_{2j}(s)$ and $F_{2j}(s)$ in Lemmas 7.1 and 7.2 remains true, except possibly for a different abscissa of absolute convergence for $U_{2j}(s)$. Hence we have (8.2). Noting $c_{2j}(n) \geq 0$ (see Lemma 7.1), we infer from Lemma 2.2 that (unconditionally)

$$\sum_{n\leq x} c_{2j}(n) = xP_{2j}(\log x) + O(x^{\vartheta_{2j}+\varepsilon}) \qquad (j = 2, 3, 4)$$

where $\vartheta_{2j} = \frac{2^{2j}-1}{2^{2j}+1}$. The final task is to verify

$$\sum_{v\geq 1} |b(v)|v^{-\vartheta_{2j}} \ll 1,$$

i.e. to check $U_{2j}(s)$ is absolutely convergent at $\Re e\, s = \vartheta_{2j}$. For these cases ($j = 2, 3, 4$), we may expand, by brute-force, the local factors of $R_{2j}(s)F_{2j}(s)^{-1}$. After some calculation, it is seen that $U_{2j}(s)$ is absolutely convergent for $\sigma > \sigma_*(2j)$ where $\sigma_*(4) = \frac{53}{96}$, $\sigma_*(6) = \frac{53}{64}$, $\sigma_*(8) = \frac{95}{96}$. These values are not optimal but give $\vartheta_{2j} > \sigma_*(2j)$ ($j = 2, 3, 4$) for our purpose, whence (1.3) is done.

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