Geometry of holomorphic isometries and related maps between bounded domains

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The topic of holomorphic isometries of Kähler manifolds with real-analytic Kähler metrics into complex space forms is a classical one going back to Bochner and Calabi. In the seminal work of Calabi's ([Ca], 1953) problems of existence, uniqueness and analytic continuation of germs of holomorphic isometries were established. They are applicable even to complex space forms of countably infinite dimensions.

For the purpose of studying holomorphic isometries, Calabi introduced the powerful notion of the *diastasis* on a Kähler manifold (X, g), where g is real-analytic. Fixing a base point $x \in X$ the diastasis $\delta(x, z)$ is a special potential function for the Kähler metric on some neighborhood of x. Regarding holomorphic isometries, among Calabi's results of particular relevance to us are those where the target manifold is the complex projective space $\mathbb{P}^N$, $1 \leq N \leq \infty$, equipped with the Fubini-Study metric ds^2_{FS}, normalized here to be of constant holomorphic sectional curvature equal to 2. For a simply connected complex manifold X equipped with a real-analytic Kähler metric g, it was proven in [Ca] that every germ of holomorphic isometry at any base point $x_0 \in X$ to $(\mathbb{P}^N, ds^2_{FS})$ extends holomorphically and isometrically to X.

Every irreducible Hermitian symmetric manifold (S, h) of the compact type can be holomorphically and isometrically embedded into $(\mathbb{P}^N, \lambda ds^2_{FS})$ for some $\lambda > 0$, hence the study of holomorphic isometries into an irreducible Hermitian symmetric manifold (S, h) reduces in principle to that of holomorphic isometries into finite-dimensional projective spaces equipped with the Fubini-Study metric; for instance it follows readily that any germ of holomorphic isometry $f : (S_1, h_1; x_1) \to (S_2, h_2; x_2)$ between two such manifolds is necessarily equivariant, i.e., arising from a group homomorphism $\Phi : \mathrm{Aut}_0(S_1, h_1) \to \mathrm{Aut}_0(S_2, h_2)$. From a geometric perspective, notably from the duality between Hermitian symmetric manifolds of the noncompact type and those of the compact type, it is natural to study holomorphic isometries between Hermitian symmetric manifolds of the noncompact type, which are realized as bounded symmetric domains with respect to Harish-Chandra realizations. Here the n-dimensional complex unit ball B^n, equipped with the Bergman metric, is dual to the projective space $\mathbb{P}^n$. However, by contrast to the case of compact type, B^n does not act as a universal target space. In fact, by the monotonicity on holomorphic bisectional curvatures for complex submanifolds, a bounded symmetric domain Ω of rank ≥ 2 cannot be holomorphically and isometrically embedded into the complex unit ball.

It turns out that for the study of holomorphic isometries into bounded symmetric domains the universal target space should still be the projective space. More precisely, a bounded symmetric domain Ω equipped with the Bergman metric ds^2_Ω should rather be regarded as being embedded holomorphically and isometrically into the projective space $\mathbb{P}^\infty$ of countably infinite dimensions by means of any choice of an orthonormal basis of the Hilbert space of square-integrable holomorphic functions $H^2(\Omega)$. In this

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context one can study more generally germs of holomorphic isometries up to normalizing constants between any two bounded domains equipped with the Bergman metric. Since biholomorphisms between bounded domains are holomorphic isometries with respect to the Bergman metric, the study of holomorphic isometries may be regarded as a non-equidimensional generalization of the study of biholomorphisms. For strictly pseudoconvex domains the principal interest is on boundary regularity, and Fefferman's theorem ([Fe, 1974]) on such domains is based on the asymptotic behavior of the Bergman metric. Here in the non-equidimensional situation we deal with the special case of bounded domains $D \Subset \mathbb{C}^n$ where the Bergman kernel $K_D(z,w)$ can be extended real-analytically to a neighborhood of $\overline{D} \times D$. The latter property is special, but is satisfied by bounded symmetric domains in their Harish-Chandra realizations, which have been a focus of study from the very beginning of our investigation, more generally for complete bounded circular domains D under a mild condition, and for Siegel domains in their canonical bounded realizations. The analogous problem on boundary regularity can be posed for bounded domains $D \Subset \mathbb{C}^n$ where the Bergman kernel $K_D(z,w)$ extends smoothly to $\overline{D} \times D$, as is satisfied whenever D is a strictly pseudoconvex domain with smooth boundary (Kerzman [Ker]). A study of the latter problem, which goes beyond this article, would require additional techniques yet to be developed.

The author's interest on the study of germs of holomorphic maps between bounded symmetric domains was first of all prompted by the work of Clozel-Ullmo [CU, 2003] in Arithmetic Geometry on commutators of modular correspondences on quotients $X := \Omega/\Gamma$ of an irreducible bounded symmetric domain Ω by torsion-free lattices Γ of automorphisms. Under certain conditions on the modular correspondence Z they asked the question whether an algebraic correspondence $Y \subset X \times X$ commuting with Z is necessarily modular in the sense that $Y \subset X \times X$ is a totally geodesic complex submanifold which descends from the graph of an automorphism of Ω. The problem is dynamical in nature. Write $\mathrm{pr}_i : Y \to X_i$ for the canonical projection of $Y \subset X \times X$ to the i-th factor. Iterating the commutation relation, the characterization problem on commutators is reduced in [CU] first of all to a differential-geometric problem on the characterization of germs of measure-preserving holomorphic maps. Here by taking inverse images with respect to pr_2 at a general point $x \in X$, $\mathrm{pr}_2^{-1}(x) = \{y_1, \cdots, y_{d_2}\}$, from the algebraic correspondence $Y \subset X \times X$ we have a germ of holomorphic map $h : (X; x) \to (X, y_1) \times \cdots \times (X, y_{d_2})$, which can be identified with a germ of map $f : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$ by lifting base points to $0 \in \Omega$. Denoting by $\pi_\alpha : \Omega^{d_2} \to \Omega$ the canonical projection onto the α-th direct factor; $1 \le \alpha \le d_2$; and by $d\mu_\Omega$ the volume form of Ω with respect to the Bergman metric, the algebraic correspondence $Y \subset X \times X$ is said to be measure-preserving if and only if $f^\star(\pi_1^\star d\mu_\Omega + \cdots + \pi_{d_2}^\star d\mu_\Omega) = d_1\, d\mu_\Omega$.

When Ω is the unit disk $\Delta \subset \mathbb{C}$, a germ of measure-preserving holomorphic map $f : (\Delta; 0) \to (\Delta; 0) \times \cdots \times (\Delta; 0)$ is equivalently a germ of holomorphic isometry $f : (\Delta, d_1\, ds^2_\Delta; 0) \to (\Delta, ds^2_\Delta; 0)^{d_2}$, where ds^2_Δ stands for the Bergman metric on Δ. In this case Clozel-Ullmo [CU] showed that $\mathrm{Graph}(f) \subset \mathbb{C} \times \mathbb{C}^{d_2}$ extends to an affine-algebraic subvariety in $\mathbb{C} \times \mathbb{C}^{d_2}$, and argued that f is totally geodesic whenever it arises from an algebraic correspondence $Y \subset X \times X$ owing to the action of the lattice Γ. When Ω is an irreducible bounded symmetric domain of dimension > 1, Clozel-Ullmo [CU] did not solve the characterization problem on germs of measure-preserving holomorphic maps, but instead reduced the problem of characterizing commutators of modular correspondences to the characterization problem on germs of holomorphic isometries $f : (\Omega, \lambda\, ds^2_\Omega; 0) \to (\Omega, ds^2_\Omega; 0) \times \cdots \times (\Omega, ds^2_\Omega; 0)$, where ds^2_Ω stands for the Bergman metric on Ω, and where *a priori* the normalizing constant λ is only known to be a positive real number. While in the case where $\mathrm{rank}(\Omega) \geq 2$ a solution to the latter problem can readily be derived from the arguments on Hermitian metric rigidity of the author's (cf. [Mk1, 1987] and [Mk2, 1989]), in the case of the complex unit ball B^n, $n \geq 2$, the problem remained unsolved in [CU].

This was the state of affairs when the author started to consider general questions on the geometry of germs of holomorphic maps between bounded symmetric domains and more generally on bounded domains and even on complex manifolds admitting non-degenerate Bergman metrics. In this article we survey on recent results on holomorphic isometries of Mok [Mk4, 6, 7], Ng [Ng1, 2] and Mok-Ng [MN1], and those on measure-preserving maps of Mok-Ng [MN2]. The starting point of Clozel-Ullmo [CU] was the use of a real-analytic functional identity on potential functions for a holomorphic isometry from the Poincaré disk into a product of Poincaré disks. The functional identity is a special case of Calabi's functional identity on diastases for holomorphic isometries between Kähler manifolds equipped with real-analytic Kähler metrics. We essentially solved the characterization problem on commutators of modular correspondences on the complex unit ball B^n in ([Mk4, 2002]) by proving first of all an extension theorem on the graph of a germ of holomorphic map $f : (B^n; 0) \to (B^n; 0) \times \cdots \times (B^n; 0)$ arising from an algebraic correspondence, which is an isometry up to a normalizing constant with respect to the Bergman metric. We polarize the real-analytic functional identity to get a family of holomorphic functional identities, and the proof in [Mk4] relies on the local rigidity of the set of common solutions to holomorphic functional identities and on Alexander's theorem on the characterization of automorphisms of B^n, $n \geq 2$ by local properties on the boundary sphere. As in the case of $\mathrm{rank}(\Omega) \geq 2$ the total geodesy of f follows without any reference to the action of a lattice. In contrast, in the case of $n = 1$, we produced in [Mk7, 2009] non-standard holomorphic isometries from the unit disk into polydisks, showing that in the case of $n = 1$, contrary to a conjecture of [CU, Conjecture 2.2], the property of being a holomorphic isometry is not enough to force total geodesy.

In [Mk7] we consider the general question of extension of a germ of holomorphic isometry up to a normalizing constant between bounded domains equipped with the Bergman metric, denoted by $f : (D; \lambda\, ds^2_D x_0) \to (\Omega, ds^2_\Omega; y_0)$, showing in general that extendibility of $\mathrm{Graph}(f)$ beyond the boundary can be established whenever the Bergman kernel $K_D(z, w)$ can be extended meromorphically in $(z, \overline{w})$ to some neighborhood of $\overline{D} \times D$ (where $\overline{D}$ denotes the topological closure of D) and the analogue holds true for Ω. The same result holds true for a canonically embeddable Bergman manifold X realized as a bounded domain on a complex manifold M. Denoting by ω_X the canonical line bundle on X and by $H^2(X, \omega_X)$ the Hilbert space of square-integrable holomorphic n-forms on X, $n := \dim(X)$, by the assumption the Bergman metric is non-degenerate on X and the canonical map $\Phi_X : X \hookrightarrow \mathbb{P}(H^2(X, \omega_X)^\star)$ is an embedding. For the extension result beyond the boundary, the key difficulty arises when the set of common solutions to the holomorphic functional identities arising from polarizing equations on diastases fails to be locally rigid. In this case we force analytic continuation beyond the boundary by imposing additional constraints. These additional constraints arise from infinitesimal variations of deformations $\zeta = f_t(z)$ of common solutions of the holomorphic functional identities, and they amount to requiring the image of f to lie in the common zero set of a family of square-integrable holomorphic n-forms. Such functions are in some sense extremal and they can be derived from the Bergman kernel form, and as such they extend meromorphically beyond the boundary if the same holds true for the Bergman kernel form.

In Mok-Ng [MN2] we have now solved the original problem in [CU] about characterizing germs of measure-preserving maps from an irreducible bounded symmetric domain Ω to a Cartesian product $\Omega \times \cdots \times \Omega$. Key ingredients in our proof are extension results in Several Complex Variables, including the theorem on algebraic extension due to Huang [Hu] of CR-maps between strongly pseudoconvex algebraic real hypersurfaces, Alexander's theorem on the characterization of automorphisms of the complex unit ball B^n, $n \geq 2$, by its local boundary behavior at a point on the unit sphere, and Alexander-type results in the case of rank ≥ 2. On top of the well-known result of Henkin-Tumanov [TK1, 2] concerning the Shilov boundary $Sh(\Omega)$, we have also proved an Alexander-type theorem of independent interest concerning a smooth point of $\partial\Omega$.

In view of the existence of non-standard holomorphic isometries of the Poincaré disk into certain bounded symmetric domains, a natural outgrowth of our investigation is to study such holomorphic isometries. Of particular interest is the asymptotic behavior of any non-standard holomorphic isometry of the Poincaré disk into a bounded symmetric domain (equipped with the Bergman metric) which necessarily extends algebraically. It turns out that any such holomorphic isometry is necessarily asymptotically totally geodesic at a general point of the boundary circle. This is in contrast to the dual case of Hermitian symmetric spaces of the compact type, and shows that in the case of noncompact type any equivariant holomorphic map between bounded symmetric domains must be totally geodesic. Denoting by σ the second fundamental form of the holomorphic isometry, it was established in Mok [Mk6] and Mok-Ng [MN1] that $\varphi := \|\sigma\|^2$ must either vanish to the order 1 or 2 at a general boundary point, and that in the case of holomorphic isometries into the polydisk φ must satisfy an additional differential equation along the boundary. We also show that singularities must develop somewhere along the boundary circle for a non-standard holomorphic isometry of the Poincaré disk into the polydisk, and conjecture the same to hold true when the target space is a bounded symmetric domain. Finally, preliminary results on the classification problem of holomorphic isometries of the Poincaré disk into polydisks have been obtained by Ng [Ng1]. They deal with low dimensions and certain extremal cases in high dimensions. In the general case the structure of the moduli space of holomorphic isometries is unknown, and this is a source for formulating questions on the deformation of holomorphic isometries.

While the circle of problems considered arise originally from characterization problems on modular correspondences in connection with a problem on dynamics in Arithmetic Geometry, the study of germs of holomorphic isometries and measure-preserving maps on bounded symmetric domains and in more general situations has revealed a rich interplay between techniques from Kähler Geometry and Several Complex Variables. In the last section we collect and reformulate natural questions arising from our study. The general circle of problem dealt with has prompted the author to examine in a more general context the geometry of holomorphic curves in bounded symmetric domains, and hopefully this will also serve as an interface for cross-fertilization from problems and techniques belonging to a number of related areas of research.

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§1 Examples of holomorphic isometries

(1.1) *Examples of equivariant embeddings into the projective space* On the projective space $\mathbb{P}^\ell$, $\ell \geq 1$, equipped with the Fubini-Study metric $ds^2_{\mathbb{P}^\ell}$ of constant holomorphic sectional curvature equal to 2, the Kähler form $\omega_{\mathbb{P}^\ell}$ is given in terms of the homogeneous coordinates $[\zeta_0, \cdots, \zeta_\ell]$ and the canonical projection $\pi : \mathbb{C}^{\ell+1} - \{0\} \to \mathbb{P}^\ell$ by $\pi^\star\omega_{\mathbb{P}^n} = \sqrt{-1}\partial\overline{\partial}\log\sum_{k=0}^{\ell}|\zeta_k|^2$. For integers $n, m > 0$ and $N = (n+1)(m+1)-1 = nm+n+m$ consider the Segre embedding $\sigma : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ defined by

$$\sigma\big([z_0, \cdots, z_n], [w_0, \cdots w_1]\big) = \Big[\big(z_i w_j\big)_{0\leq i\leq m, 0\leq j\leq n}\Big] \ .$$

Denote by $\alpha : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $\beta : \mathbb{P}^n \to \mathbb{P}^m$ the canonical projections onto the first and the second factors. Since

$$\begin{aligned}\log\Big(\sum_{i,j}|z_i w_j|^2\Big) &= \log\Big(\sum_{i=0}^{n}|z_i|^2\Big)\Big(\sum_{j=0}^{m}|w_j|^2\Big) \\ &= \log\Big(\sum_{i=0}^{n}|z_i|^2\Big) + \log\Big(\sum_{j=0}^{m}|w_j|^2\Big) \ .\end{aligned}$$

we deduce readily that

$$\sigma^\star\omega_{\mathbb{P}^N} = \alpha^\star\omega_{\mathbb{P}^n} + \beta^\star\omega_{\mathbb{P}^m} \ ,$$

so that the Segre embedding $\sigma : \big(\mathbb{P}^m, ds^2_{\mathbb{P}^m}\big) \times \big(\mathbb{P}^n, ds^2_{\mathbb{P}^n}\big) \hookrightarrow \big(\mathbb{P}^N, ds^2_{\mathbb{P}^N}\big)$ is a holomorphic isometry. When $m = n$, restricting to the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$ we obtain the Veronese embedding $\nu : \big(\mathbb{P}^n, ds^2_{\mathbb{P}^n}\big) \hookrightarrow \big(\mathbb{P}^\ell, ds^2_{\mathbb{P}^\ell}\big)$, where $\ell = \frac{n(n+3)}{2}$. The latter is a special case of the the Veronese embeddings $\nu_k : \mathbb{P}^n \hookrightarrow \mathbb{P}^{\ell_k}$, $k \geq 2$, which are the equivariant embedding defined by $\nu_k([\eta]) = [\eta \otimes \cdots \otimes \eta]$, noting that the image of ν_k lies in $\mathbb{P}(S^k\mathbb{C}^{n+1})$. Another standard example of an equivariant isometric embedding is given by the Plücker embedding. Given a finite-dimensional complex vector space W and $1 \leq p \leq \dim(W) - 1$ the Plücker embedding on the Grassmann manifold $\mathrm{Gr}(p, W)$, denoted by $\tau : \mathrm{Gr}(p, W) \hookrightarrow \mathbb{P}(\Lambda^p W)$, is defined by setting $\tau([E]) = [e_1 \wedge \cdots \wedge e_p]$, where $(e_1, \cdots, e_p)$ is a basis of the p-dimensional vector subspace $E \subset W$. The examples discussed are special cases of equivariant holomorphic isometric embeddings into the projective space. Let (X,g) and (X',g') be Kähler manifolds. Suppose $G := \mathrm{Aut}_0(X,g)$ acts transitively on X and likewise $G' := \mathrm{Aut}_0(X',g')$ acts transitively on X', a mapping $F : X \to X'$ will be said to be equivariant if and only if there exists a group homomorphism $\Phi : G \to G'$ such that $F(\gamma(z)) = \Phi(\gamma)(z)$ whenever $\gamma \in G$. It follows readily from Calabi [Ca] that for any Hermitian symmetric manifold (X,g) of the compact type, any germ of holomorphic isometry $f : (X,g;x_0) \to \left(\mathbb{P}^N, ds^2_{\mathbb{P}^N}; y_0\right)$ necessarily extends to an equivariant holomorphic isometric embedding, cf. (2.1).

In the verification that the Segre embedding is a holomorphic isometry we equate two potential functions for the Kähler forms, after lifting to affine spaces. The same verification could have been done using inhomogeneous coordinates by checking that local potential functions for $\sigma^\star\omega_{\mathbb{P}^N}$ and for $\alpha^\star\omega_{\mathbb{P}^m} + \beta^\star\omega_{\mathbb{P}^n}$ agree. On $\mathbb{P}^\ell$ with homogeneous coordinates $[\zeta_0, \cdots, \zeta_\ell]$ we will write $(\zeta_1', \cdots, \zeta_\ell')$ for inhomogeneous coordinates at the point $[1, 0, \cdots, 0]$ (corresponding to 0 in inhomogeneous coordinates), and the same applies to $[z_0, \cdots, z_m]$ and $[w_0, \cdots, w_n]$. We have $\omega_{\mathbb{P}^\ell}\big|_{\mathbb{C}^\ell} = \sqrt{-1}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^\ell |\zeta_k'|^2\right)$. Then, the equality on potentials can be reformulated as

$$\log\left(1 + \sum_{i=1}^m |z_i'|^2 + \sum_{j=1}^n |w_j'|^2 + \sum_{i=1}^m\sum_{j=1}^n |z_i'w_j'|^2\right)$$
$$= \log\left(1 + \sum_{i=1}^n |z_i'|^2\right) + \log\left(1 + \sum_{j=0}^m |w_j'|^2\right) .$$

As will be seen in (2.1), this verification on potential functions is the same as verifying a functional identity for holomorphic isometries on the *diastasis*, a notion introduced by Bochner [Bo, 1947] and Calabi [Ca, 1953], which are in some sense potential functions in normal form.

Consider irreducible Hermitian symmetric manifolds (X,g) and (X',g') of the compact type. (X',g') can be holomorphically and isometrically embedded into a finite-dimensional projective space by some $\sigma : (X',g') \to \left(\mathbb{P}^N, \mu\, ds^2_{\mathbb{P}^N}\right)$ for some $\mu > 0$. Hence, given any $\lambda > 0$, the study of germs of holomorphic isometries $f : (X, \lambda g; x_0) \to (X', g'; x_0')$ can in principle be reduced to the study of $\sigma\circ f : (X,\lambda g; x_0) \to \left(\mathbb{P}^N, \mu\, ds^2_{\mathbb{P}^N}; \sigma(x_0')\right)$. In place of Hermitian symmetric manifolds of the compact type, one can pose in the case of dual manifolds the problem of classifying holomorphic isometries between Hermitian symmetric manifolds of the noncompact type. Here the picture is not really dual to the case of compact type. While the projective space acts as a universal target space for irreducible Hermitian symmetric spaces of the compact type, the analogue is not true for the complex unit ball B^ℓ. In fact, by the monotonicity on holomorphic bisectional curvatures, any bounded symmetric domain D cannot be holomorphically and isometrically immersed in B^ℓ unless D is itself of rank 1. Furthermore, any germ of holomorphic isometry from $(B^k;0)$ into $(B^\ell;0)$ is necessarily totally geodesic, by Umehara [Um]. As it turns out, germs of holomorphic isometries between bounded symmetric domains should still be studied through holomorphic isometries into the projective space $\mathbb{P}^\infty$ of countably infinite dimension, as will be explained in (2.3).

(1.2) *Non-standard holomorphic isometries of the Poincaré disk into polydisks* The unit disk is conformally equivalent to the upper half-plane, the unbounded realization of the unit disk by means of the inverse Cayley transform. For our construction it is essential to make use of the Euclidean coordinate of the upper half-plane $\mathcal{H}$. For $\tau \in \mathcal{H}$, $\tau = re^{i\theta}$, where $r > 0$, $0 < \theta < \pi$, and for $p \geq 2$ a positive integer, we write $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$. Then, we have

Proposition 1.2.1 (Mok [Mk7, Prop.3.2.1]). *Let $p \geq 2$ be a positive integer. Equip the upper half-plane $\mathcal{H}$ with the Poincaré metric $ds^2_{\mathcal{H}} = 2\mathrm{Re}\frac{d\tau \otimes d\overline{\tau}}{2(\mathrm{Im}\tau)^2}$ of constant Gaussian curvature -1 and $\mathcal{H}^p$ with the product metric. Then, writing $\gamma = e^{\frac{\pi i}{p}}$, the proper holomorphic mapping $\rho_p : \big(\mathcal{H}, ds^2_{\mathcal{H}}\big) \to (\mathcal{H}, ds^2_{\mathcal{H}})^p$ defined by*

$$\rho_p(\tau) = \big(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \dots, \gamma^{p-1}\tau^{\frac{1}{p}}\big),$$

called the p-th root map, is a holomorphic isometric embedding.

Write $\tau = s + it$; where s and t are real variables. Then, $\sqrt{-1}\partial\overline{\partial}(-2\log t) = \frac{\sqrt{-1}d\tau \wedge d\overline{\tau}}{2t^2}$ is the Kähler form $\omega_{\mathcal{H}}$ of $ds^2_{\mathcal{H}}$. Write $\omega_{\mathcal{H}^p}$ for the Kähler form on $\mathcal{H}^p$ with respect to the product metric $ds^2_{\mathcal{H}^p}$. In the simplest case of $p = 2$, the fact that the square root map ρ_2 is an isometry follows readily from

$$\begin{aligned}\rho_2^\star(\omega_{\mathcal{H}^2}) &= -2\sqrt{-1}\partial\overline{\partial}\Big(\log\big(\mathrm{Im}(\sqrt{\tau})\big) + \log\big(\mathrm{Im}(i\sqrt{\tau})\big)\Big) \\ &= -2\sqrt{-1}\partial\overline{\partial}\log\Big(\big(\mathrm{Im}(\sqrt{\tau})\big)\big(\mathrm{Im}(i\sqrt{\tau})\big)\Big),\end{aligned}$$

and from the identity $\mathrm{Im}(\sqrt{\tau})\mathrm{Im}(i\sqrt{\tau}) = \frac{1}{2}\mathrm{Im}(\tau)$. The general case of Proposition 1.2.1 follows from the trigonometric identity

$$\sin\theta \sin\Big(\frac{\pi}{p} + \theta\Big) \cdots \sin\Big(\frac{(p-1)\pi}{p} + \theta\Big) = c_p \sin(p\theta)$$

for some positive constant c_p.

A holomorphic map of the form $\Phi \circ \rho_p \circ \varphi$, where $\varphi \in \mathrm{Aut}(\Delta)$ and $\Phi \in \mathrm{Aut}(\Delta^p)$ will be referred to as being congruent to ρ_p. Composing a map congruent to the p-th root map with a map congruent to the q-th root acting on an individual factor of Δ^p and iterating the process one obtains families of holomorphic isometries of the Poincaré disk into polydisks. It is not known whether all holomorphic isometries of the Poincaré disk into polydisks are generated by the set of p-th root maps in the way described. Some results on classification in low dimensions and in some extreme cases are obtained by Ng [Ng1], cf. (3.4).

(1.3) *A non-standard holomorphic isometry of the Poincaré disk into a Siegel upper half-plane* In Mok [Mk7] we also constructed an example of a holomorphic isometry of the Poincaré disk into the Siegel upper half-plane $\mathcal{H}_3$ of genus 3, given by

Proposition 1.3.1 (Mok [Mk7, Prop.3.3.1]. *For $\zeta \in \mathcal{H}$, $\zeta = \rho e^{i\varphi}$, $\rho > 0$, $0 < \varphi < \pi$, n a positive integer, we write $\zeta^{\frac{1}{n}} := \rho^{\frac{1}{n}} e^{\frac{i\varphi}{n}}$. Then, the holomorphic mapping $G : \mathcal{H} \to M_s(3, \mathbb{C})$ defined by*

$$G(\tau) = \begin{bmatrix} e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}} & \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & 0 \\ \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}} \end{bmatrix}$$

maps $\mathcal{H}$ into $\mathcal{H}_3$, and $G : \big(\mathcal{H}, 2ds^2_{\mathcal{H}}\big) \to \big(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\big)$ is a holomorphic isometry.

By considering the boundary behavior of holomorphic isometries of the Poincaré disk, it can be shown that the image of the holomorphic isometric embedding $G : \mathcal{H} \to \mathcal{H}_3$ is not contained in any maximal polydisk, cf. (3.3), especially Proposition 3.3.1.

(1.4) *Examples of holomorphic isometries with arbitrary normalizing constants $\lambda > 1$* Examples of holomorphic isometric embeddings $F : D \to \Omega$ up to normalizing constants with respect to the Bergman metric are given by holomorphic totally geodesic isometric embeddings between bounded symmetric domains, where D is assumed to be irreducible. For such holomorphic embeddings, which have been classified by Satake [Sa1] and Ihara [Ih], the normalizing constants are necessarily rational numbers $\lambda \geq 1$.

Let $D \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. For any real number $\alpha > 0$ we have constructed in Mok [Mk7] a totally geodesic holomorphic embedding $F : D \to \Omega$ from D into a bounded domain $\Omega_\alpha \Subset \mathbb{C}^{n+1}$ such that $F : (D, (1+\alpha)ds^2_D) \to (\Omega_\alpha, ds^2_{\Omega_\alpha})$ is a holomorphic isometry. $\Omega_\alpha \subset D \times \Delta \Subset \mathbb{C}^N$ are bounded domains on which the Bergman metric is complete and they are realized as disk bundles on D homogeneous under a natural action of $\mathrm{Aut}(D)$.

More precisely, denote by $\pi : L \to D$ the anti-canonical line bundle on D, which is equipped with the Hermitian metric h induced by the Bergman kernel, and identify L with $D \times \mathbb{C}$ in the obvious way using the Harish-Chandra coordinates on D. The action of the automorphism group $\mathrm{Aut}(D)$ on D induces an action on L. Thus, given any $z \in D$ and $\gamma \in \mathrm{Aut}(D)$ we have $\gamma_*\big(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\big)(z) = \det(d\gamma(z)) \cdot \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}$, and the action of $\mathrm{Aut}(D)$ on L is given by $\Phi(\gamma)(z,t) = (\gamma(z), \det(d\gamma(z)) \cdot t)$. We write e for the holomorphic section of $\pi : L \to D$ corresponding to $(z,1)$, and define $h_0 := h(e,e)$. Let $\Omega \subset L$ be the unit disk bundle, i.e., the open subset of all n-vectors η of length < 1 with respect to h. Let now $\alpha > 0$ be a real number, and define $L^\alpha := D \times \mathbb{C}$ set-theoretically to be the same as L, but regard $\pi_\alpha : L^\alpha \to D$ as being equipped with the Hermitian metric h^α, such that, writing e^α for the holomorphic section of $\pi_\alpha : L^\alpha \to D$ corresponding to $(z,1)$, we have $h^\alpha(e^\alpha, e^\alpha) = (h_0)^\alpha$. Let $\Omega_\alpha \subset L^\alpha$ be the unit disk bundle with respect to h^α, $\Omega_\alpha \Subset \mathbb{C}^{n+1}$, noting that $L = L^1$ and $\Omega_1 = \Omega$. With this set-up we have

Proposition 1.4.1 (Mok [Mk7, Proposition 3.1.2]. *Let α be a positive real number and $F : D \to \Omega_\alpha$ be the embedding given by $F(z) = (z, 0)$. Then, $F : (D, \lambda ds^2_D) \to (\Omega_\alpha, ds^2_{\Omega_\alpha})$ is a totally geodesic holomorphic isometric embedding for $\lambda = 1 + \alpha$. Furthermore, (D, ds^2_D) and $(\Omega_\alpha, ds^2_{\Omega_\alpha})$ are complete Kähler manifolds.*

We will explain here only the verification that $F : (D, (1+\alpha)ds^2_D) \to (\Omega_\alpha, ds^2_{\Omega_\alpha})$ is a holomorphic isometry. Since D is simply connected, a holomorphic logarithm $\log\big(\det(d\gamma(z))\big)$ can be defined, and $\varphi(z,\eta) := \big(\gamma(z), \exp\left(\alpha \log\left(\det(d\gamma(z))\right) \eta\right)\big)$ defines an automorphism of $\pi_\alpha : L^\alpha \to D$ as a holomorphic line bundle which preserves the Hermitian metric h^α. Identifying D as the zero section of $\pi_\alpha : L^\alpha \to D$, $D \subset L^\alpha$ is homogeneous under the action of the automorphism group $\mathrm{Aut}(\Omega_\alpha)$. In particular, the restriction of the Bergman kernel K_{Ω_α} to D can be computed from a single point. For

$z \in D$, let $\gamma \in \mathrm{Aut}(D)$ be such that $\gamma(0) = z$. Then,

$$K_{\Omega_\alpha}\left((z,0),(z,0)\right) = \left|\det(d\gamma(0))\right|^{-2(1+\alpha)} K_{\Omega_\alpha}(0,0)\ ;$$
$$K_D(z,z) = \left|\det(d\gamma(0))\right|^{-2} K_D(0,0)\,.$$

Hence, for some positive constant c_α we have

$$K_{\Omega_\alpha}\left((z,0),(z,0)\right) = c_\alpha \cdot K_D(z,z)^{1+\alpha}\,.$$

Taking logarithms and applying $\partial\overline{\partial}$ we conclude that $F^\star\omega_{\Omega_\alpha} = (1+\alpha)\omega_D$ for the Kähler form ω_D of (D, ds_D^2) and the Kähler form ω_{Ω_α} of $(\Omega, ds^2_{\Omega_\alpha})$, hence $F^\star ds^2_{\Omega_\alpha} = (1+\alpha)ds_D^2$.

As an example of Ω_α, the $(n+1)$-ball $B^{n+1} \Subset \mathbb{C}^{n+1}$ can be regarded as the unit disk bundle of $L^{\frac{1}{n+1}}$ over $B^n \Subset \mathbb{C}^n$.

§2 Analytic continuation of germs of holomorphic isometries

(2.1) *Analytic continuation of holomorphic isometries into the projective space equipped with the Fubini-Study metric* Let $G \subset \mathbb{C}^n$ be a Euclidean domain equipped with a Kähler metric g with Kähler form ω_g. If G is Stein and $H^2(G,\mathbb{R}) = 0$, there exists a smooth function φ on G such that $\sqrt{-1}\partial\overline{\partial}\varphi = \omega_g$. In what follows we assume that g is real-analytic, in which case φ is necessarily real-analytic. In general for any real-analytic function φ on a Euclidean domain G there exists a function $\Phi(z,w)$ defined on some neighborhood of the diagonal of $G \times G$ which is holomorphic in z and anti-holomorphic in w such that $\varphi(z) = \Phi(z,z)$. If $0 \in G$ and $\varphi(z) = \sum a_{I\overline{J}} z^I \overline{z}^J$ in a neighborhood of 0, then $\Phi(z,w) = \sum a_{I\overline{J}} z^I \overline{w}^J$. Here the summations are performed over multi-indexes $I = (i_1, \cdots, i_n)$ and $J = (j_1, \cdots, j_n)$ of nonnegative integers.

Returning to our situation of real-analytic potential functions φ for real-analytic Kähler metrics g, the functions φ are uniquely determined only up to a pluriharmonic function. Assume in what follows also $H^1(G,\mathbb{R}) = 0$. If ψ also solves $\sqrt{-1}\partial\overline{\partial}\psi = \omega_g$ on G, then $\psi = \varphi + h + \overline{h}$ for some holomorphic function h on G. In Calabi [Ca, 1953], following Bochner [Bo, 1947], the *diastasis* $\delta_G(x,y)$ on (G, ds_G^2) is defined by $\delta_G(x,y) = \Phi(x,x) - \Phi(x,y) - \Phi(y,x) + \Phi(y,y)$. If we replace φ by $\psi = \varphi + 2\mathrm{Re}(h)$, then Φ is replaced by $\Psi = \Phi + H$ where $H(x,y) = h(x) + \overline{h(y)}$. If we tentatively denote by $\delta'_G(x,y)$ the function analogous to $\delta_G(x,y)$ defined using ψ in place of φ, then, substituting $H(x,y) = h(x) + \overline{h(y)}$, we have obviously $\delta'_G(x,y) - \delta_G(x,y) = H(x,x) - H(x,y) - H(y,x) + H(y,y) = 0$, showing that $\delta'_G(x,y) = \delta_G(x,y)$ and hence that the diastasis is well-defined independent of the choice of potential function φ. It follows that given any Kähler manifold (X,g) with a real-analytic metric (X,g), we have a well-defined diastasis $\delta_X(x,y)$ defined on a neighborhood $\mathcal{U}$ of the diagonal of $X \times X$. Actually, δ_X resembles the square of the distance function $d_X(x,y)$ on (X,g). For an open subset $V \Subset X$ and for a neighborhood $\mathcal{V}$ of $\mathrm{diag}(\overline{V} \times \overline{V})$ such that $\mathcal{V} \Subset \mathcal{U}$ we have $\delta_X(x,y) = d_X(x,y)^2 + O\left(d_X(x,y)^4\right)$ for $(x,y) \in \mathcal{V}$, cf. Calabi [Ca, Proposition 4].

The diastasis is especially suited for the study of holomorphic isometries between complex manifolds endowed with real-analytic Kähler metrics. Suppose (X,g) and (Y,h) are two such manifolds, $x_0 \in X$, $y_0 \in Y$ and $f : (X,g;x_0) \to (Y,h;y_0)$ is a germ of holomorphic isometry. Then, by pulling back potential functions it follows from the definition of the diastasis that we have $\delta_Y(f(x_1), f(x_2)) = \delta_X(x_1, x_2)$ for x_1 and x_2 sufficiently close to the base point x_0.

Given $x_0 \in X$, and a local potential φ of the real-analytic Kähler metric g on a neighborhood of x_0, let $\varphi(z) = \sum a_{I\overline{J}} z^I \overline{z}^J$ be the Taylor expansion of φ at x_0 in terms of local holomorphic coordinates $(z_1, \cdots, z_n)$, with respect to which x_0 corresponds to the origin, on a neighborhood of x_0. Then, $\delta_X(z, x_0)$ is the potential function $\varphi'(z) = \sum_{I,J\neq 0} a_{I\overline{J}} z^I \overline{z}^J$. We may call φ' the normal form of the potential function of (X, g) at x_0. In the Taylor expansion in $(z, \overline{z})$, a term is called a pure term if it is a constant, or if it is attached to either z^I or $\overline{z^J}$, otherwise it is called a mixed term. From the definition of the diastasis it follows that $\delta_X(z, x_0)$ is the unique germ of potential function at x_0 whose Taylor expansion at x_0 consists solely of mixed terms, i.e., it is devoid of pure terms. As an example, the potential function $\varphi(z) = \log\left(1 + \|z\|^2\right)$ at $0 \in \mathbb{C}^N$ for the Fubini-Study metric ds^2_{FS} of constant holomorphic sectional curvature equal to 2 on $\mathbb{P}^N$ is such a function, so that it equals to $\delta_{\mathbb{P}^N}(z, 0)$ with respect to ds^2_{FS}, and the verification in (1.1) that the Segre embedding $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ is a holomorphic isometry up to a normalizing constant is in fact the verification that $\delta_Y(f(z), 0) = \delta_X(z, 0)$, a condition which is, as we have seen, not only sufficient but also necessary for f to be a holomorphic isometry. A sufficient condition for a potential function φ to be in the normal form at $x_0 \in X$ is to have $\varphi = \sum_k \pm|f_k|^2$ as a convergent sum, where each f_k is a holomorphic function vanishing at x_0. In particular, by Taylor expansion this is the case for $\varphi = \log\left(1 + \sum_k |h_k|^2\right)$ where each h_k is a holomorphic function vanishing at $x_0 \in X$ and the infinite sum is convergent. As an example, this is the case at 0 for the Bergman metric ds^2_D on a bounded symmetric domain $D \Subset \mathbb{C}^n$ in its Harish-Chandra realization, and more generally when D is a complete bounded circular domain, cf. (2.3).

Let $b \in \mathbb{R}$, $1 \leq N \leq \infty$, and $F(N, b)$ be the Fubini-Study space of constant holomorphic sectional curvature $2b$ according to Calabi [Ca, Chapter 4]. Given a Kähler manifold (X, g) with a real-analytic metric g, the local and global existence problem of holomorphic isometries from (X, g) into $F(N, b)$ is solved in [Ca, Theorems 8-11] in terms of the notion of b-resolvability. In what follows we consider only the case $b > 0$. In this case $F(N, b)$ is the projective space $\mathbb{P}^N$ equipped with the Fubini-Study metric of constant holomorphic sectional curvature $2b$, where by $\mathbb{P}^\infty$ we mean the projectivization of the Hilbert space of countably infinite dimension. In a nutshell, given $x_0 \in X$ it is proven in [Ca, Theorem 8] that the existence of a germ of holomorphic isometry $f : (X, g; x_0) \to \left(\mathbb{P}^N, \frac{1}{b} ds^2_{FS}; y_0\right)$, where $y_0 \in \mathbb{P}^N$ is arbitrary, is equivalent to the statement that the real-analytic Kähler metric g is b-resolvable of rank N, a condition which is defined in terms of coefficients of the Taylor expansions of the diastasis. This condition is then proved to be global in nature, i.e., the condition is valid at every base point $x \in X$ if and only if it is valid at one base point $x_0 \in X$, by [Ca, Theorem 10]. Thus, the existence of a germ of holomorphic isometry $f : (X, g; x_0) \to \left(\mathbb{P}^N, \frac{1}{b} ds^2_{FS}; y_0\right)$ implies the same when x_0 is replaced by any base point $x \in X$. Furthermore, in [Ca, Theorem 9] a local rigidity theorem for holomorphic isometric embeddings is proved to the effect that, assuming that the image of the germ of map f above in $\mathbb{P}^N$ is not contained in any proper closed linear subspace, then f is uniquely determined up to a unitary transformation. The latter uniqueness result modulo unitary transformations can then be used to find a global holomorphic isometry by means of the developing map, assuming that there is no topological obstruction, e.g., when X is simply connected, yielding

Theorem 2.1.1 (Calabi [Ca, Theorem 11]). *Let (X, g) be a simply connected complex manifold equipped with a real-analytic Kähler metric. Let $b > 0$, $1 \le N \le \infty$, and $f : (X, g; x_0) \to \left(\mathbb{P}^N, \frac{1}{b} ds^2_{FS}; y_0\right)$ be a germ of holomorphic isometry. Then, f admits an extension to a holomorphic isometry $F : (X, g) \to \left(\mathbb{P}^N, \frac{1}{b} ds^2_{FS}\right)$.*

In the proof we note that, replacing $\mathbb{P}^N$ by the topological closure of the projective-linear span of $f(U)$ for an open neighborhood U of x_0 in X on which the germ of map f is defined, without loss of generality we may assume that the image of f is not contained in any proper closed linear subspace, so that results on local rigidity can be applied.

As we will be discussing in this article extension results on germs of holomorphic isometries up to normalizing constants between bounded domains equipped with the Bergman metric, Theorem 2.1.1 becomes relevant. In fact, for any bounded Euclidean domain $G \Subset \mathbb{C}^n$, there is a canonical embedding $\Phi_G : G \hookrightarrow \mathbb{P}(H^2(G)^\star)$ which can be defined in terms of any choice of orthonormal basis $(h_i)_{i=0}^{\infty}$ by $\Phi_G(z) = [h_0(z), \cdots, h_k(z), \cdots]$, and the Bergman metric ds^2_G on G is the pull-back of ds^2_{FS} by Φ_G. Thus, in the case where we have a germ $f : (D, \lambda\, ds^2_D; x_0) \to (\Omega, ds^2_\Omega; y_0)$ of holomorphic isometry between bounded domains equipped with Bergman metric up to the normalizing constant $\lambda > 0$, we have by composition with the canonical embedding a germ of holomorphic isometry $\Phi_\Omega \circ f : (D, ds^2_D; x_0) \to \left(\mathbb{P}(H^2(\Omega)^\star), \frac{1}{\lambda} ds^2_{FS}; \Phi_\Omega(y_0)\right)$. Results about holomorphic isometries into $\mathbb{P}^\infty$ then becomes relevant for the study of germs of holomorphic isometries up to a normalizing constant with respect to the Bergman metric. Actually, we can apply the following precise result.

Theorem 2.1.2 (Calabi [Ca, Theorem 12]). *Let (X, g) be a complex manifold equipped with a real-analytic Kähler metric. Let $b > 0$, $1 \le N \le \infty$, and $\varphi : (X, g; x_0) \to \left(\mathbb{P}^N, \frac{1}{b} ds^2_{FS}; y_0\right)$ be a germ of holomorphic isometry. Suppose for each $x \in X$, the maximal analytic extension of the diastasis $\delta_X(x, y)$ is single-valued. Then, f admits an extension to a holomorphic isometry $\Phi : (X, g) \to \left(\mathbb{P}^N, \frac{1}{b} ds^2_{FS}\right)$. Furthermore, assuming that $\delta_X(x, y) = 0$ if and only if $x = y$, then F is injective. In other words, F is a holomorphic isometric embedding.*

The application of Theorem 2.1.2 to the study of extension results concerning the Bergman metric will be discussed in the first paragraphs of (2.3).

(2.2) *An extension and rigidity problem arising from commutators of modular correspondences* We will now be dealing with bounded symmetric domains in their Harish-Chandra realizations and more generally bounded domains for which the boundary behavior of Bergman kernels satisfies a certain regularity property. Here and in what follows for a bounded domain $G \Subset \mathbb{C}^n$ we denote by $K_G(z, w)$ its Bergman kernel, and by ds^2_G the Bergman metric on G.

For the author the original motivation for studying germs of holomorphic isometries and also measure-preserving maps between bounded symmetric domains stems from the questions raised in Clozel-Ullmo [CU] concerning a problem on dynamics in Arithmetic Geometry. To formulate it, let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free lattice, so that, endowing $X := \Omega/\Gamma$ with the Kähler metric g induced from the Bergman metric ds^2_Ω on Ω, which is invariant under automorphisms, we have a complete Kähler metric (X, ds^2_X) of finite volume. Thus, X is either compact, in which case it is projective-algebraic, or it is non-compact and of finite volume, in which case it admits a minimal projective-algebraic compactification $\overline{X}_{\min}$ according to Satake-Borel-Baily ([Sa1], [BB]) in the arithmetic case and Siu-Yau [SY] in the rank-1 non-arithmetic case, such that $X \subset \overline{X}_{\min}$ is a Zariski-open subset. In what follows by a correspondence on X, $\dim(X) = n$, in the projective case we mean a pure n-dimensional subvariety $Y \subset X \times X$ in which the canonical projection of each irreducible component of Y to either direct factor of $X \times X$ is of maximal rank at a general smooth point. In the general quasi-projective case we will impose the condition that the canonical projections are proper maps.

Let $g \in \mathrm{Aut}(\Omega)$ be such that Γ and $g^{-1}\Gamma g$ are commensurable, in other words $\big|[\Gamma : \Gamma \cap g^{-1}\Gamma g]\big| < \infty$. Define now $i_g : \Omega \to \Omega \times \Omega$ by $i_g(\tau) = (\tau, g \cdot \tau)$. Passing to quotients we have $\overline{i_g} : \Omega/\Gamma_g \to X \times X$. The subvariety $\overline{i_g}(\Omega/\Gamma_g) \subset X \times X := Y_g$ is an algebraic correspondence, to be called an irreducible modular correspondence (Hecke correspondence). Given an algebraic correspondence $Y \subset X \times X$, for $i = 1, 2$ we denote by $\pi_i : Y \to X$ the canonical projection onto the i-th direct factor of $X \times X$. A correspondence T_Y in the form of an operator is defined on X by $T_Y \cdot x = \pi_2(\pi_1^{-1}(x))$, and its adjoint is given by $T_Y^\star \cdot x = \pi_1(\pi_2^{-1}(x))$, T_Y and likewise $T_Y^\star$ defining thereby operators on spaces of functions. In the case of $Y = Y_g$, $T_g := T_{Y_g}$ is also referred to as the Hecke correspondence. A Hecke correspondence T_g on Ω is said to be *interior* if the subgroup $\langle \Gamma, g \rangle \subset \mathrm{Aut}(\Omega)$ generated by Γ and g is discrete. Otherwise it is said to be *exterior*. In [CU], Clozel-Ullmo posed the following problem.

Problem 2.2.1. *Let $T = \sum_i (T_{g_i} + T_{g_i^{-1}})$ be a self-adjoint modular correspondence such that at least one of the T_{g_i} is exterior.* (*We will say that T is exterior.*) *Let S be an algebraic correspondence on X which commutes with T. Is S necessarily modular?*

Clozel-Ullmo [CU] rendered the problem interesting from the perspective of Complex Geometry by reducing it to a local differential-geometric problem of characterizing measure-preserving germs of holomorphic maps $f : (\Omega; 0) \to (\Omega \times \cdots \times \Omega; 0)$, meaning that the Bergman volume form $d\mu_\Omega$ on Ω is up to an integral constant the pull-back by f of the sum of the pull-backs of Bergman volume forms $d\mu_{\Omega_\alpha}$ from the individual Cartesian factors Ω_α of $\Omega \times \cdots \times \Omega$ by the canonical projection maps $\pi_\alpha : \Omega \times \cdots \times \Omega \to \Omega_\alpha$. Moreover, in [CU] the latter problem is further reduced to the question of characterizing germs of holomorphic isometries up to normalizing constants $f : (\Omega; 0) \to (\Omega \times \cdots \times \Omega; 0)$ with respect to the Bergman metric, in which the normalizing constants are *a priori* only known to be positive real numbers λ. They proved

Theorem 2.2.1 (Clozel-Ullmo ([CU, 2003]). *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free lattice, $X := \Omega/\Gamma$. Let T be a self-adjoint exterior modular correspondence. Let S be an algebraic correspondence on X which commutes with T. Then S is necessarily modular provided that either* (*a*) *Ω is the unit disk Δ; or* (*b*) *Ω is of rank ≥ 2.*

After the reduction of the problem of characterizing commutators of the modular correspondences considered to that of characterizing germs of holomorphic isometries $f : (\Omega, \lambda\, ds_\Omega; 0) \to (\Omega, ds_\Omega; 0)^p$ up to normalizing constants, the proof of (b) follows from the work of Mok ([Mk1, 1987] and [Mk2, 1989]) on Hermitian metric rigidity. As to the case of (a) Clozel-Ullmo made use of real-analytic functional identities which arise from a special case of the identity on diastases of Calabi [Ca] for holomorphic isometries. A primary difficulty of the problem is to prove algebraic extension of the germ of variety Graph$(f) \subset \Delta \times \Delta^p$, i.e., to prove that Graph(f) is contained in an irreducible affine-algebraic curve $V \subset \mathbb{C} \times \mathbb{C}^p$. In Mok [Mk4, 2002] the author made use of holomorphic functional identities arising from polarizing the real-analytic functional identity to give a proof which generalizes readily to the remaining case of the complex unit ball B^n, $n \geq 2$, as follows.

Theorem 2.2.2 (Mok [Mk4]). *Let p and q be positive integers. Let $f : \big(B^n, q\, ds^2_{B^n}; 0\big) \to \big((B^n)^p, ds^2_{(B^n)^p}; 0\big)$ be a germ of holomorphic isometry, $f = (f^1, \cdots, f^p)$. Assume that each component $f^k : (B^n; 0) \to (B^n; 0); 1 \leq k \leq p;$ is of maximal rank at some point. Then, $q = p$ and f extends to a totally-geodesic embedding $F : (B^n, p\, ds^2_{B^n}) \to \Big((B^n)^p, ds^2_{(B^n)^p}\Big)$.*

Sketch of Proof. For $r > 0$ we denote by B^n_r the complex Euclidean n-ball centred at 0 and of radius r. In what follows we identify the germ of map f as a holomorphic map defined on B^n_ϵ for some sufficiently small $\epsilon > 0$. Following Clozel-Ullmo [CU] in the case of $n = 1$, we have from the hypothesis

$$-\sqrt{-1}\partial\overline{\partial} \log \sum_{k=1}^{p} \big(1 - \|f^k\|^2\big) = -q\sqrt{-1}\partial\overline{\partial} \log \big(1 - \|z\|^2\big)\,;$$

$$-\sum_{k=1}^{p} \log \big(1 - \|f^k\|^2\big) = -q \log \big(1 - \|z\|^2\big) + \mathrm{Re}\ h$$

for some holomorphic function h. Since $f(0) = 0$, by comparing Taylor coefficients it follows that $h \equiv 0$. Thus

$$\sum_{k=1}^{p} \log \big(1 - \|f^k\|^2\big) = q \log \big(1 - \|z\|^2\big)\ ;$$

$$\prod_{k=1}^{p} \big(1 - \|f^k\|^2\big) = \big(1 - \|z\|^2\big)^q\ .$$

We introduce now holomorphic functional identities by polarization, viz.

$$-\sum_{k=1}^{p} \log \Big(1 - \sum_{j=1}^{n} \overline{f^k_j(w)} f^k_j(z)\Big) = -q \log \Big(1 - \sum_{i=1}^{n} \overline{w}_i z_i\Big)\,.$$

Let $V_0 \subset \mathbb{C}^n \times \mathbb{C}^{pn}$ be the germ of graph of f at 0. Then we have

Proposition 2.2.1 (Mok [Mk4]). *With the same assumptions as in Theorem 2.2.1, except that $n \geq 1$ is an arbitrary positive integer, let $V_0 \subset \mathbb{C}^n \times (\mathbb{C}^n)^p$ be the germ of* *graph of f at 0. Then, there exists an irreducible n-dimensional affine-algebraic variety $V \subset \mathbb{C}^n \times (\mathbb{C}^n)^p$ extending V_0.*

We will give a proof of Proposition 2.2.1 for the case of $n = 1$. The general case follows with minor modifications, and we refer the reader to the original article Mok [Mk4] for details.

Proof for the case $n = 1$. For $z, w \in B^1_\epsilon$. Let $(z; \zeta^1, \dots, \zeta^p)$ be a point on the germ of graph V_0 of f at 0. Then, for each $w \in B^1_\epsilon$, $(z; \zeta^1, \dots, \zeta^p)$ satisfies

$$(\dagger) \qquad \prod_{k=1}^{p} (1 - \overline{f^k(w)}\zeta^k) = (1 - \overline{w}z)^q \ .$$

Let S_w be the affine-algebraic hypersurface in $\mathbb{C}^1 \times \mathbb{C}^p$, with coordinates $(z; \zeta)$, defined by $(\dagger)$ with w fixed, $w \in B^1_\epsilon$. We have $V_0 \subset \bigcap_{w \in B^1_\epsilon} S_w := S$. The germ of V_0 at 0 agrees with an irreducible branch of the germ of S at 0, if we can show that S is of dimension 1 at 0. This is the case if the fiber of the projection $\pi : S \to \mathbb{C}^1$ into the first factor of $\mathbb{C}^1 \times \mathbb{C}^p$ has finite fibers over any $z \in B^1_\epsilon$.

It suffices to find a uniform *a-priori* bound on $\|\zeta\|$ for all points $(z; \zeta) \in S$, $|z| < \epsilon$. Starting with $(\dagger)$, and expanding $f^k(w)$ in power series

$$f^k(w) = a^k_1 w + a^k_2 w^2 + a^k_3 w^3 + \cdots \ ,$$

where $a^k_1 \neq 0$ for $1 \leq k \leq p$, we have

$$\begin{aligned} \prod_{k=1}^{p} (1 - \overline{f^k(w)}\zeta^k) &= \prod_{k=1}^{p} \Big(1 - \Big(\sum_{m=1}^{\infty} \overline{a^k_m}\ \overline{w^m}\Big)\zeta^k\Big) \\ = 1 - \overline{w}(\overline{a^1_1}\zeta^1 + \cdots + \overline{a^p_1}\zeta^p) &+ \overline{w}^2\Big(\sum_{k<\ell} \overline{a^k_1 a^\ell_1}\zeta^k\zeta^\ell - \sum_k \overline{a^k_2}\zeta^k\Big) - \cdots \ . \end{aligned}$$

Generally,

$$\prod_{k=1}^{p} \Big(1 - \overline{f^k(w)}\zeta^k\Big) = 1 + \sum_{m=1}^{\infty} (-1)^m \tau_m(\zeta^1, \dots, \zeta^p)\overline{w}^m \ ,$$

where

$$\tau_m(\zeta) = \sigma_m(\eta) + P_{m-1}(\eta) \ ,$$

in which $\eta = (\eta^1, \cdots, \eta^p)$, $\eta^k = \overline{a^k_1}\zeta^k$, σ_m is the m-th symmetric polynomial, and $P_{m-1}(\eta)$ is a polynomial in $\eta = (\eta^1, \dots, \eta^p)$ of total degree $\leq m - 1$. For $(z; \zeta) \in V_0$, we have from $(\dagger)$

$$\begin{cases} \tau_m(\zeta) = \frac{q!}{m!(q-m)!} z^m & \text{for } 1 \leq m \leq q \\ \tau_m(\zeta) = 0 & \text{for } m \geq q + 1. \end{cases}$$

This implies readily that for $z \in B^1_\epsilon$, we have

$$\|\zeta\| \leq C \, .$$

It follows that for $S = \bigcap_{w \in B^1_\epsilon} S_w$, the fibers of the projection $S \subset \mathbb{C}^1 \times \mathbb{C}^p \to \mathbb{C}^1$ are finite for $z \in B^1_\epsilon$, $\dim(S) = 1$. The irreducible component V of S, which contains $V_0 = \mathrm{Graph}(f)$, gives the desired extension of V_0 as an affine-algebraic variety. $\square$

Proof of Theorem 2.2.2 continued. The proof of algebraic extension of Graph(f) generalized readily. We have

$$\prod_{k=1}^{p}(1 - \|f^k\|^2) = (1 - \|z\|^2)^q \ ,$$

which implies that at least for some k, at a good boundary point $b \in \partial B^n$, f^k can be regarded as a local holomorphic map preserving the boundary. By a result of Alexander stated below, f^k extends to a biholomorphism. Removing the factor f^k from the functional identity above, by induction we have established Theorem 2.2.1. $\square$

The result of Alexander's referred to in the above concerns the characterization of of automorphisms of the complex unit ball B^n, $n \geq 2$, by their boundary behavior, as follows.

Theorem 2.2.3 (Alexander [Al]). *Let $B^n \Subset \mathbb{C}^n$ be the complex unit ball of dimension $n \geq 2$. Let $b \in \partial B^n$, U_b be a connected open neighborhood of b in $\mathbb{C}^n$, and $f : U_b \to \mathbb{C}^n$ be a nonconstant holomorphic map such that $f(U_b \cap \partial B^n) \subset \partial B^n$. Then, there exists an automorphism $F : B^n \to B^n$ such that $F|_{U_b \cap B^n} \equiv f|_{U_b \cap B^n}$.*

A minor modification of the proof of Theorem 2.2.1 yields the same result with the integral constant q replaced by an arbitrary real constant $\lambda > 0$. As a consequence, combining with the result Theorem 2.2.1 here from Clozel-Ullmo [CU] we have

Corollary 2.2.1. *Let $Z \subset X \times X$ be a self-adjoint exterior irreducible modular correspondence on $X = \Omega/\Gamma$. Suppose $Y \subset X \times X$ is an algebraic correspondence which commutes with Z. Then, Y is necessarily a modular correspondence.*

(2.3) *Analytic continuation of holomorphic isometries up to normalizing constants with respect to the Bergman metric – extension beyond the boundary* Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains, and $\lambda > 0$ be a real constant. We are interested to prove extension theorems for holomorphic isometries $f : (D, \lambda\, ds^2_D; x_0) \to (\Omega, ds^2_\Omega; y_0)$ up to normalizing constants with respect to the Bergman metric. Extension results concerning the analytic continuation of f as a germ to D (or a maximal open subset of D) will be called interior extension results. Those that concern the analytic continuation of f beyond ∂D under certain assumptions on the Bergman kernels of D and Ω will be called boundary extension results.

Theorem 2.1.2 from Calabi [Ca] is applicable to give interior extension results, as follows. Recall that we have a canonical holomorphic embedding $\Phi_\Omega : \Omega \to \mathbb{P}(H^2(\Omega)^\star)$. Choosing any orthonormal basis $\big(h_i\big)_{i=0}^{\infty}$ of $H^2(\Omega)$, $\Phi_\Omega : \Omega \to \mathbb{P}^\infty \cong \mathbb{P}(H^2(\Omega)^\star)$ is given by $\Phi_\Omega(\zeta) = [h_0(\zeta), \cdots, h_i(\zeta), \cdots]$. The mapping $\Phi_\Omega \circ f : (D, ds^2_D; x_0) \to \big(\mathbb{P}(H^2(\Omega)^\star), \frac{1}{\lambda} ds^2_{FS}; \Phi_\Omega(y_0)\big)$ is a holomorphic isometry into a projective space of countably infinite dimension equipped with the Fubini-Study metric. Let $\mathbb{P}(\Lambda) \subset \mathbb{P}(H^2(\Omega)^\star)$ be the topological projective-linear span of the image of $\Phi_\Omega \circ f$, $\Lambda \subset H^2(\Omega)^\star$ being a

Hilbert subspace. By the Identity Theorem on holomorphic functions the Hilbert subspace Λ is independent of the choice of representative of the germ of holomorphic map, i.e., independent of the choice of open neighborhood U of 0 on which f is taken to be defined. If D is simply connected, by Theorem 2.1.1, $\Phi_\Omega \circ f$ extends into a holomorphic isometry $\Psi : D \to \mathbb{P}(\Lambda)$. Analytic continuation of $\Phi_\Omega \circ f$ implies analytic continuation of Graph(f) to a complex-analytic subvariety of $D \times \Omega$. If (Ω, ds^2_Ω) is complete as a Kähler manifold, it is easy to see that $\Psi(D) \subset \Phi_\Omega(\Omega)$ and we derive the existence of a holomorphic isometry $F : (D, \lambda\, ds^2_D) \to (\Omega, ds^2_\Omega)$ which extends the germ of map f. We argue using Theorem 2.1.2 that the same holds true without assuming that D is simply connected. By Theorem 2.1.2 to prove the extendibility of $\Phi_\Omega \circ f$ it suffices to check the univalence of $\delta_D(x, y)$ as a function of y whenever a base point $x \in D$ is fixed. Now on the bounded domain D the diastasis is given by

$$\delta_D(x, y) = \log \frac{K_D(x, x) K_D(y, y)}{K_D(x, y) K_D(y, x)}\,.$$

Thus, fixing $x \in D$, $\delta_D(x, y)$ is a single-valued function defined on all of D, as desired. Theorem 2.1.2 implies furthermore that F is injective, i.e., F is a holomorphic isometric embedding. To see this, $\delta_D(x, y) = 0$ if and only if $|K_D(x, y)|^2 = K_D(x, x)K_D(y, y)$. Choosing any orthonormal basis $(s_i)_{i=0}^\infty$ of $H^2(D)$ we have $K_D(x, y) = \sum s_i(x)\overline{s_i(y)}$, and it follows from the Cauchy-Schwarz inequality that $|K_D(x, y)|^2 \le K_D(x, x)K_D(y, y)$, with equality if and only if $\big(s_0(x), \cdots, s_i(x), \cdots\big)$ and $\big(s_0(y), \cdots, s_i(y), \cdots\big)$ are proportional to each other, equivalently if and only if there exists some non-zero $\alpha \in \mathbb{C}$ such that $s(y) = \alpha s(x)$ for any $s \in H^2(D)$. By considering $s = z_k, 1 \le k \le n$, it follows readily that equality holds if and only if $x = y$. By Theorem 2.1.2, $\Psi : D \to \mathbb{P}(H^2(\Omega)^\star)$ must necessarily be a holomorphic isometric embedding, hence $F : D \to \Omega$ is a holomorphic isometric embedding.

In Mok [Mk7] we adopt a direct approach to prove extension results for germs of holomorphic isometries $f : (D, \lambda\, ds^2_D; x_0) \to (\Omega, ds^2_\Omega; y_0)$ by directly working with the functional equations. We give a self-contained proof for interior extension results without resorting to Theorem 2.1.2 from [Ca], thus without resorting to local existence results on holomorphic isometries (using b-resolvability). Our proof can be adapted to give boundary extension results whenever $K_D(z, w)$ can be extended holomorphically in z and anti-holomorphically in w from $D \times D$ to neighborhood of $\overline{D} \times D$, and the analogous statement holds true for Ω. For simplicity we assume first of all that D and Ω are complete circular domains and $f : (D; 0) \to (\Omega; 0)$. The latter covers the case of bounded symmetric domains, which are at the centre of our investigation.

Let $G \subset \mathbb{C}^n$ be a bounded complete circular domain. Because of the invariance of the Bergman kernel K_G under the circle group action, i.e., $K_G(e^{i\theta}z, e^{i\theta}w)) = K_G(z, w)$ for $\theta \in \mathbb{R}$, it follows that $K_G(z, 0)$ is a constant. Denoting by $\delta_G(z, z)$ the diastasis on (G, ds^2_G) and by $\Delta_0(z, w)$ the polarization of the real-analytic function $\delta_G(z, 0)$. we have

$$\begin{aligned}\delta_G(z, 0) &= \log K_G(z, z) - \log K_G(z, 0) - \log K_G(0, z) + \log K_G(0, 0)\\ &= \log K_G(z, z) - \log K_G(0, 0)\,;\\ \Delta_0(z, w) &= \log K_G(z, w) - \log K_G(0, 0)\,.\end{aligned}$$

From the functional identity on diastases for holomorphic isometries we obtain

Proposition 2.3.1 (Mok [Mk7, Prop.1.1.1]). *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded complete circular domains. Let λ be any positive real number and $f : (D, \lambda\, ds^2_D; 0) \to (\Omega, ds^2_\Omega; 0)$ be a germ of holomorphic isometry at $0 \in D$, $f(0) = 0$. Then, there exists some real number $A > 0$ such that for $z, w \in D$ sufficiently close to 0 we have*

$$\begin{aligned} K_\Omega(f(z), f(z)) &= A \cdot K_D(z,z)^\lambda; \quad \text{and hence} \\ K_\Omega(f(z), f(w)) &= A \cdot K_D(z,w)^\lambda; \quad \text{where} \\ K_D(z,w)^\lambda &= Ae^{\lambda \log K_D(z,w)} \ , \end{aligned}$$

in which log *denotes the principal branch of logarithm.*

Proposition 2.3.2 (Mok [Mk7, Prop.1.1.2]). *In the notations of Proposition 2.3.1, for each $w \in D_\epsilon$, let $V_w \subset D \times \mathbb{C}^N$ be the set of all $(z, \zeta) \in D \times \Omega$ such that*

$$(\mathbf{I}_w) \qquad K_\Omega\big(\zeta, f(w)\big) = A \cdot K_D\big(z, w\big)^\lambda \ .$$

Define $V = \bigcap_{w \in D_\epsilon} V_w$. Suppose for $z \in D$, $\dim_{(z,f(z))}\big(V \cap (\{z\} \times \mathbb{C}^N)\big) \geq 1$. Then, there exists a family of holomorphic functions $h_\alpha \in H^2(\Omega)$, $\alpha \in \mathbf{A}$, such that

$$\mathrm{Graph}(f) \subset D_\epsilon \times E \ , \quad \text{where} \quad E := \bigcap_{\alpha \in \mathbf{A}} \mathrm{Zero}(h_\alpha) \ ,$$

and such that $\dim_{(z,f(z))}\big(V \cap (\{z\} \times E)\big) = 0$ for a general point $z \in D_\epsilon$.

Abstractly, if we consider the Hilbert subspace $\Lambda \subset H^2(\Omega)^\star$ such that $\mathbb{P}(\Lambda)$ is the topological projective-linear span of the image of $\Phi_\Omega \circ f$, then Proposition 2.3.2 holds true if we take h_α, $\alpha \in \mathbf{A}$, to consist of all elements lying in the annihilator $J \subset H^2(\Omega)$ of $\Lambda \subset H^2(\Omega)^\star$. But such an abstract proof will be useless for the purpose of proving results on analytic continuation beyond the boundary. The meaning of Proposition 2.3.2 lies therefore not in its statement but rather its proof, in which the functions $h_\alpha \in H^2(\Omega)$ can be described in terms of the Bergman kernel K_Ω. In fact, it is in general not necessary to cut down on $\mathbb{P}(H^2(\Omega)^\star)$ to $\mathbb{P}(\Lambda)$. For instance, in the proof of Theorem 2.1.1 for the case of $f : (B^n, q\, ds^2_{B^n}; 0) \to (B^n, ds^2_{B^n}; 0)^p$, $f = (f^1, \cdots, f^n)$, where each component f^k is of maximal rank at some point, it was shown that the canonical projection of $V \subset B^n \times (\mathbb{C}^n)^p$ to B^n has discrete fibers over a general point of B^n, and hence it is no longer necessary to cut down $\mathbb{P}(H^2(\Omega)^\star)$ any further by a family of linear sections consisting of square-integrable holomorphic functions, even though $\Lambda \neq H^2(\Omega)^\star$. In the case where V has discrete fibers over a general point of D, we say that the holomorphic functional identities $(\mathbf{I}_w)$ are sufficiently non-degenerate. The difficulty of proving Proposition 2.3.2 is to consider the case where the fibers are of positive dimension, in which case we find the functions $h_\alpha, \alpha \in \mathbf{A}$, by studying infinitesimal deformations of simultaneous solutions of the holomorphic functional identities $(\mathbf{I}_w)$.

In the latter case there exists a complex-analytic 1-parameter family of solutions of the functional identities valid for z belonging to some non-empty open subset U of D_ϵ

$$(\sharp) \qquad K_\Omega(f_t(z), \overline{f(w)}) = K_D(z,w)^\lambda$$

such that $f_0(z) = f(z)$. Assume that $\frac{\partial^k}{\partial t^k} f_t(z)\big|_{t=0} \equiv 0$ for $k < \ell$ and $\eta(z) := \frac{\partial^\ell}{\partial t^\ell} f_t(z)\big|_{t=0}$ $\not\equiv 0$. Let $(h_n)_{n=0}^\infty$ be an orthonormal basis of $H^2(\Omega)$. We have

$$K_\Omega\big(f_t(z), \overline{f(w)}\big) = \sum h_n(f_t(z))\overline{h_n(f(w))} = K_D(z,w)^\lambda$$

for every t. As a consequence,

$$\frac{\partial^\ell}{\partial t^\ell} K_\Omega(f_t(z), \overline{f(w)})\Big|_{t=0} \equiv 0 \ ; \quad \text{hence}$$
$$\sum_{i,n} \frac{\partial h_n}{\partial \zeta_i} \, \frac{\partial^\ell f_t^i}{\partial t^\ell}(z)\Big|_{t=0} \overline{h_n(f(w))} \equiv 0 \ .$$

Equivalently, we have proved the identity

$$(\flat) \qquad \sum_n \overline{dh_n(\eta(z))} h_n(f(w)) = 0 \ .$$

Let $\Phi_0 : \Omega \to \mathbb{C}^\infty$ be defined by $\Phi_0(\zeta) = (h_0(\zeta), \ldots, h_n(\zeta), \ldots)$. Here $\mathbb{C}^\infty$ is the separable infinite-dimensional Hilbert space which can identified with the Hilbert space ℓ^2 of square-integrable sequences with complex coefficients. In the identity $(\flat)$, $\eta(z)$ can now be interpreted as a holomorphic vector field along $\Phi_0(f(U)) := \Gamma_0$. For each vector field $\eta(z)$ arising from a choice of deformation (f_t), $f_0 = f$, of solutions of the holomorphic functional identities $(\sharp)$, and for each choice of base point $z_0 \in U \subset D_\epsilon$, the identity $(\flat)$ determines a hyperplane section of the Hilbert space $H^2(\Omega)^\star$ defined by the zero locus of a square-integrable function. Choosing an orthonormal basis of $H^2(\Omega)$ adapted to z_0 and $\eta(z_0)$ the identity $(\flat)$ translates into saying that the image of $\Phi_0 \circ f$ lies on the zero set of some $h \in H^2(\Omega)$ which is in some sense extremal with respect to z_0 and $\eta(z_0)$. Such extremal functions can be derived from the Bergman kernel $K_\Omega(\zeta, \xi)$. As a consequence, if K_Ω extends holomorphically in $(\zeta, \overline{\xi})$ from $\Omega \times \Omega$ to a neighborhood of $\overline{\Omega} \times \Omega$, the linear section defined as one varies over all possible deformations (f_t), $f_0 = f$, over some nonempty open subset $U \subset D_\epsilon$ gives a subvariety $E \subset \Omega$ which extends holomorphically to a neighborhood of $\overline{\Omega}$. Assuming that the Bergman kernel $K_D(z,w)$ on the domain D satisfies the regularity property analogous to K_Ω, we deduce from the proof of Proposition 2.3.2 that $\mathrm{Graph}(f) \subset D \times E$ extends holomorphically beyond ∂D to give

Theorem 2.3.1 (Mok [Mk7, Thm.1.1.1]). *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded complete circular domains. Denote by ds_D^2,* resp. *ds_Ω^2, the Bergman metrics on D,* resp. *Ω. Let $\epsilon_0 > 0$ be such that $D_{\epsilon_0} := B^n(0;\epsilon_0) \Subset D$ and $\delta_0 > 0$ be such that $\Omega_{\delta_0} := B^N(0;\delta_0) \Subset \Omega$. Let λ be any positive real number and $f : \big(D_{\epsilon_0}, \lambda\, ds_D^2\big|_{D_{\epsilon_0}}\big) \to (\Omega, ds_\Omega^2)$ be a holomorphic isometry such that $f(0) = 0$ and $f(D_{\epsilon_0}) \Subset \Omega_{\delta_0}$. Then, there exists an irreducible complex-analytic subvariety $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ of dimension $n = \dim(\mathrm{Graph}(f))$ such that* $\mathrm{Graph}(f) \subset S^\sharp$. *In other words,* $\mathrm{Graph}(f) \subset D_\epsilon \times \Omega$ *extends as a subvariety to $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$.*

Extension results can now be generalized to germs of holomorphic isometries up to normalizing constants $f : (D, \lambda ds_D^2; x_0) \to (\Omega, ds_\Omega^2; f(x_0))$ with respect to the Bergman metric between bounded domains $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$. To start with, writing $y_0 = f(x_0)$ we have the polarized functional identity on diastases in the form

$$\log \frac{K_\Omega\big(f(z), f(w)\big) K_\Omega\big(y_0, y_0\big)}{K_\Omega\big(f(z), y_0\big) K_\Omega\big(y_0, f(w)\big)} = \lambda \log \frac{K_D(z,w) K_D(x_0,x_0)}{K_D(z,x_0) K_D(x_0,w)} \ .$$

Consider the holomorphic functional identities

$$(\mathbf{J}_w) \quad \log\left(\frac{K_\Omega(\zeta, f(w)) K_\Omega\big(y_0,y_0\big)}{K_\Omega(\zeta, f(x_0)) K_\Omega(f(x_0), f(w))}\right) - \lambda \log\left(\frac{K_D(z,w)K_D(x_0,x_0)}{K_D(z,x_0)K_D(x_0,w)}\right) = 0 \ .$$

Writing the left hand-side as $H(z,\zeta;w)$ we observe that $H(z,\zeta;x_0) = 0$, and thus the system of functional identities $(\mathbf{J}_w)$ as w ranges over a neighborhood of x_0 is equivalent to the system of holomorphic identities

$$(\mathbf{K}^i_w) \qquad \frac{\partial}{\partial \overline{w_i}} H(z,\zeta;w)\bigg|_{w=x_0} = 0 \, ,$$

where $1 \le i \le n$, and w varies over a neighborhood D_ϵ of x_0 on D. By differentiating against $\overline{w_i}$ logarithms disappear, and the functional identities $(\mathbf{K}^i_w)$ are for instance rational in (z,ζ) whenever $K_D(z,w)$ is rational in $(z,\overline{w})$ and $K_\Omega(\zeta,\xi)$ is rational in $(\zeta,\overline{\xi})$. The argument of forcing analytic continuation of f by cutting down the image D_ϵ under $\Phi_\Omega \circ f$ by linear sections works equally well in the general case of bounded domains when one consider deformations (f_t), $f_0 = f$ of solutions $\zeta = f_t(z)$ in the holomorphic functional identities $(\mathbf{J}_w)$. Again the logarithms disappear after differentiating against t (a finite number of times) at $t = 0$. Our arguments on extensions of germs of graphs of holomorphic isometries now generalize to yield

Theorem 2.3.2 (Mok [Mk7, Thm.2.1.1]). *Let $D \Subset \mathbb{C}^n$, resp. $\Omega \Subset \mathbb{C}^N$, be bounded domains. Let $x_0 \in D$, λ be a positive real number and $f : (D, \lambda ds^2_D; x_0) \to (\Omega, ds^2_\Omega; f(x_0))$ be a germ of holomorphic isometry. Suppose furthermore that the Bergman kernel $K_D(z,w)$ extends as a meromorphic function in $(z,\overline{w})$ to a neighborhood of $\overline{D} \times D$ and $K_\Omega(\zeta,\xi)$ extends as a meromorphic function in $(\zeta,\overline{\xi})$ to a neighborhood of $\overline{\Omega} \times \Omega$. Then, there exists an open neighborhood $D^\sharp$ of $\overline{D}$ and an open neighborhood $\Omega^\sharp$ of $\overline{\Omega}$ such that the germ* Graph$(f) \subset D \times \Omega$ *at $(x_0, f(x_0))$ extends to an irreducible complex-analytic subvariety $S^\sharp$ of $D^\sharp \times \Omega^\sharp$. If in addition we assume (Ω, ds^2_Ω) to be complete as a Kähler manifold, then $S := S^\sharp \cap (D \times \Omega)$ is the graph of a holomorphic isometric embedding $F : (D, \lambda\, ds^2_D) \to (\Omega, ds^2_\Omega)$. If furthermore (D, ds^2_D) is complete, then $F : D \to \Omega$ is proper.*

Only the last two statements require some explanation. The univalence of F can be deduced from the holomorphic functional identities $(\mathbf{J}_w)$ directly from the Cauchy-Schwarz inequality, as follows. Suppose otherwise. Starting with the germ of f at x_0 and extending Graph(f) using S by lifting from neighborhoods of continuous paths issuing from x_0 we obtain two distinct branches $f_1(z)$ and $f_2(z)$ defined on some non-empty open subset $U \subset D$ such that the following holomorphic functional identities hold by analytic continuation.

$$\frac{K_\Omega\big(f_1(z), f_1(z)\big)}{K_\Omega\big(f_1(z), f(x_0)\big) K_\Omega\big(f(x_0), f_1(z)\big)} = \frac{K_\Omega\big(f_1(z), f_2(z)\big)}{K_\Omega\big(f_1(z), f(x_0)\big) K_\Omega\big(f(x_0), f_2(z)\big)}$$
$$= \frac{K_\Omega\big(f_2(z), f_2(z)\big)}{K_\Omega\big(f_2(z), f(x_0)\big) K_\Omega\big(f(x_0), f_2(z)\big)} \ .$$

Taking absolute values and comparing the first and last expressions with the square of the middle one we have

$$\left|K_\Omega\big(f_1(z), f_2(z)\big)\right|^2 = K_\Omega\big(f_1(z), f_1(z)\big) K_\Omega\big(f_2(z), f_2(z)\big) \ .$$

Recalling that $K_\Omega(\zeta, \xi) = \sum h_i(\zeta)\overline{h_i(\xi)}$ in terms of an orthonormal basis $(h_i)_{i=0}^\infty$ of $H^2(\Omega)$, the equality $f_1(z) = f_2(z)$ follows from the equality case of the Cauchy-Schwarz inequality on a Hilbert space and from the injectivity of the canonical map $\Phi : \Omega \hookrightarrow \mathbb{P}(H^2(\Omega)^\star)$, giving a contradiction and proving the univalence of the analytic continuation of f within D. If (Ω, ds^2_Ω) is complete as a Kähler manifold, then the domain of definition of f within D can be indefinitely enlarged by means of holomorphic functional identities on $D \times \Omega$. For the last statement assume that (D, ds^2_D) is complete. If F transforms a boundary point $b \in \partial D$ to an interior point $q \in \Omega$, by considering a neighborhood of (b, q) on $S^\sharp$ some path of infinite length on (D, ds^2_D) is mapped to a path of finite length on (Ω, ds^2_Ω) under the isometry F, yielding a contradiction.

(2.4) *Canonically embeddable Bergman manifolds and Bergman meromorphic compactifications* For further applications of extension results on holomorphic isometries it is desirable to consider the general context of complex manifolds. Here in place of square-integrable holomorphic functions we consider the notion of square-integrable holomorphic n-forms, which is well-defined independent of any choice of systems of local holomorphic coordinates. In other words, we consider the Hilbert space $H^2(X, \omega_X)$ of square-integrable sections of the canonical line bundle $\omega_X = \det(T^\star_X)$. We have correspondingly the notion of the Bergman kernel form $\mathcal{K}_X(z, w)$ defined in terms of an arbitrary orthonormal basis of $H^2(X, \omega_X)$. In the case of a bounded domain $D \Subset \mathbb{C}^n$, to each $f \in H^2(D)$ we can associate the holomorphic n-form $2^{-n} f dz^1 \wedge \cdots \wedge dz^n$. This gives an isometry between $H^2(D)$ and $H^2(D, \omega_D)$. The Bergman kernel function $K_D(z, w)$ and the Bergman kernel form $\mathcal{K}_D(z, w)$ are then related by

$$\mathcal{K}_D(z, w) = 2^{-n} \cdot K_D(z, w)\Big((\sqrt{-1})^{n^2} dz^1 \wedge \cdots dz^n \wedge \overline{dw^1} \wedge \cdots \wedge \overline{dw^n}\Big) .$$

We are interested in the class of Bergman manifolds, especially in a subclass called canonically embeddable Bergman manifolds, defined as follows.

Definition 2.4.1. *Let X be a complex manifold and denote by ω_X its canonical line bundle. Suppose the Hilbert space $H^2(X, \omega_X)$ of square-integrable holomorphic n-forms on X has no base points, and denote by $\mathcal{K}_X(z, w)$ the Bergman kernel form on X. Regarding $\mathcal{K}_X(z, z)$ as a Hermitian metric h on the anti-canonical line bundle $\omega^\star_X$, we denote by $\beta_X \geq 0$ the curvature form of the dual metric $h^\star$ on ω_X, and write ds^2_X for the corresponding semi-Kähler metric on X. We say that (X, ds^2_X) is a Bergman manifold whenever ds^2_X and equivalently β_X are positive definite. We call (X, ds^2_X) a canonically embeddable Bergman manifold if furthermore the canonical map $\Phi_X : X \to \mathbb{P}((H^2(X, \omega_X)^\star)$ is an embedding.*

The extension result Theorem 2.3.2 on holomorphic isometries between bounded domains can be readily adapted to yield the following general extension result for canonically embeddable Bergman manifolds realized as relatively compact domains on complex manifolds, including especially the case of bounded domains on Stein manifolds.

Theorem 2.4.1 (Mok [Mk7,Thm.2.2.1]). *Let D (resp. Ω) be a canonically embeddable Bergman manifold. Let $D \Subset M$ (resp. $\Omega \Subset Q$) be a realization of D (resp. Ω) as a relatively compact domain on a complex manifold M (resp. Q) such that the Bergman kernel form $\mathcal{K}_D(z,w)$ (resp. $\mathcal{K}_\Omega(\zeta,\xi)$) extends meromorphically in $(z,\overline{w})$ to $M \times D$ (resp. in $(\zeta,\overline{\xi})$ to $Q \times \Omega$). Then, the analogue of Theorem 2.3.2 holds true with M replacing $D^\sharp$ and Q replacing $\Omega^\sharp$.*

As suggested by the preceding theorem, the realization of a canonically embeddable Bergman manifold as a domain on a compact complex manifold Z is especially interesting whenever the Bergman kernel form $\mathcal{K}_X(z,w)$ extends meromorphically in $(z,\overline{w})$ from $X \times X$ to $Z \times Z$. We have more precisely the following notion of Bergman meromorphic compactifications. In what follows we note that while in the case of bounded domains $D \Subset \mathbb{C}^n$ we can define $K_{D,w}(z) = K_D(z,w)$ unambiguously, in the general case of a manifold X, fixing $w \in X$, we can analogously define $\mathcal{K}_{X,w}$ as an n-form in $z \in X$ only up to a multiplicative constant, depending on the choice of local holomorphic coordinates at w. When the notation $\mathcal{K}_{X,x}$ is used, we mean an arbitrary but fixed choice of such an n-form.

Definition 2.4.2. *Let (X, ds^2_X) be an n-dimensional canonically embeddable Bergman manifold, and $i : X \hookrightarrow Z$ be an open embedding of X into a compact complex manifold Z. Choose any base point $x_0 \in X$ and define $\sigma_0 := \mathcal{K}_{X,x_0}$, which is uniquely determined up to a multiplicative constant. Writing $\mathcal{K}_X(z,w) = K^\flat_X(z,w)\Big((\sqrt{-1})^{n^2}\sigma_0(z) \wedge \overline{\sigma_0(w)}\Big)$ on X, we say that $i : X \hookrightarrow Z$ is a Bergman meromorphic compactification if and only if (a) the function $K^\flat_X(z,w)$ extends meromorphically in $(z,\overline{w})$ from $X \times X$ to $Z \times Z$; and (b) there exists an open embedding $i : X \hookrightarrow Z'$ into a compact complex manifold Z' such that the identity map id_X extends to a (possibly) branched covering $\xi : Z' \to Z$ and such that $\xi^\star(\sigma_0)$ extends meromorphically to Z'.*

Here the extra condition (a) has been inserted in the definition in order to allow us to define the notion of a reduction of a Bergman meromorphic compactification (cf. Definition 2.4.3 and the paragraph after it). Evidently, condition (a) is independent of the choice of the base point $x_0 \in X$. Given $i : X \hookrightarrow Z$ as in the above, by means of a finite number of meromorphic n-forms $\mathcal{K}_{X,x}$, $x \in X$, one can define a generically finite meromorphic map $\Phi : X \to \mathbb{P}^N$ onto a subvariety $Z^\flat \subset \mathbb{P}^N$ such that Φ restricts to an open embedding on $X \subset Z$. The existence of Φ implies in particular that Z is Moishezon, i.e., bimeromorphic to a projective-algebraic manifold. Replacing $Z^\flat$ by a desingularized model $\tau : Z_1 \to Z^\flat$ of $Z^\flat$, where no modification is performed over $X \subset Z^\flat$, we have a Bergman meromorphic compactification $i_1 : X \to Z_1$ with the additional property that the corresponding map $\Phi_1 : X \to \mathbb{P}^{N_1}$ maps Z_1 birationally onto a subvariety $Z^\flat_1$. We have further the following notions of reduced and minimal Bergman meromorphic compactifications.

Definition 2.4.3. *A Bergman meromorphic compactification $i_0 : X \hookrightarrow Z$ is said to be reduced if and only if there exists a finite number of points $x_i \in X, 0 \le i \le m$, such that the meromorphic map $\Psi_m : X \to \mathbb{P}^m$ defined by $\Psi(y) = [\mathcal{K}_{X,x_0}(y), \cdots, \mathcal{K}_{X,x_m}(y)]$ extends to a generically injective meromorphic map $\Psi^\sharp : Z \to \mathbb{P}^m$. It is said to be minimal if and only if, given any Bergman meromorphic compactification $i : X \hookrightarrow Z$,* *the biholomorphism $h : i(X) \to i_0(X)$ corresponding to the identity map id_X extends to a meromorphic map $\eta : Z \to Z_0$.*

Given a Bergman meromorphic compactification $i : X \hookrightarrow Z$, the map $i_1 : X \hookrightarrow Z_1$ constructed in the above gives according to Definition 2.4.3 a reduced Bergman meromorphic compactification, which we will call the reduction of $i : X \hookrightarrow Z$. From the construction of $i_1 : X \hookrightarrow Z_1$ it can readily be proved that the function field $\mathcal{M}(Z_1)$ of Z_1 agrees with the subfield $\mathcal{F} \subset \mathcal{M}(Z)$ of the function field $\mathcal{M}(Z)$ of Z generated by quotients of meromorphic n-forms $\mathcal{K}_x$, $x \in X$. However, it is not apparent whether $\mathcal{K}_x$ descends to a meromorphic n-form on Z_1, and it is for this reason that we introduce condition (a) in the definition of Bergman meromorphic compactifications, so that the notion of reduction can be defined within this class of compactifications. Regarding minimal compactifications we have

Theorem 2.4.2. *Let (X, ds^2_X) be a canonically embeddable Bergman manifold admitting a Bergman meromorphic compactification $i : X \hookrightarrow Z$. Then, X admits a minimal meromorphic Bergman compactification $i_0 : X \hookrightarrow Z_0$. Furthermore, any two minimal Bergman meromorphic compactifications of X are equivalent.*

Sketch of Proof. From Definition 2.4.3 the notion of minimal Bergman meromorphic compactifications is defined in terms of a universal property of such compactifications, and it follows readily that such a compactification is unique. Existence is proved by showing that the reduction of any Bergman meromorphic compactification is minimal. Given any two Bergman meromorphic compactifications $i : X \hookrightarrow Z$ and $i' : X \hookrightarrow Z'$, from the fact that id_X is a holomorphic isometry with respect to Bergman metrics we have holomorphic functional identities, and it follows from the extension result Theorem 2.3.2 that the identity map id_X extends to an algebraic correspondence on $Z \times Z'$. When Z' is reduced, the meromorphic n-forms $\mathcal{K}_{Z',x}$, $x \in X$; separate any generic pair of distinct points, and it follows from the analytic continuation of the defining functional identities that the extension of id_X is univalent (cf. the paragraph after Theorem 2.3.2), i.e., id_X extends to a meromorphic map from Z onto Z', so that $i' : X \hookrightarrow Z'$ is minimal in the sense of Definition 2.4.3, as desired. $\square$

Borel embeddings $D \Subset \mathbb{C}^n \subset M$ of bounded symmetric domains D into their dual compact manifolds M give first examples of Bergman meromorphic compactifications. In fact, in terms of Harish-Chandra coordinates the Bergman kernel on D is of the form $K_D(z, w) = \frac{1}{Q_D(z,w)}$, where $Q_D(z, w)$ is a polynomial in $(z, \overline{w})$, cf. Faraut-Korányi [FK, pp.76-77, especially Eqns.(3.4) and (3.9)]. A more general class of examples is given by the canonical bounded realizations D of bounded homogeneous domains of Pyatetskii-Shapiro [Py]. The rationality of K_D for the canonical bounded realizations $D \Subset \mathbb{C}^n$ can be found in [Xu, Chapter 4]. In the case where a bounded homogeneous domain is a bounded symmetric domain, its canonical bounded realization is the same as the Harish-Chandra realization. It can readily be checked from [Xu, loc. cit.] that the Bergman meromorphic compactification $D \Subset \mathbb{P}^n$ is minimal.

§3 Holomorphic isometries of the Poincaré disk into bounded symmetric domains

(3.1) *Structural equations on the norm of the second fundamental form and asymptotic* *vanishing order* In the study of holomorphic isometries up to normalizing constants between bounded symmetric domains, in the case where the domain is of rank ≥ 2 rigidity is derived from the existence of zeros of holomorphic bisectional curvatures and from the rigidity of such zeros under holomorphic isometries. In the case where the domain is B^n, $n \geq 2$, rigidity of holomorphic isometries arising from algebraic correspondences on finite-volume quotients are derived from functional identities and from boundary behavior of the maps after analytic continuation beyond the boundary sphere. In the remaining case of the unit disk Δ, for which examples of non-standard holomorphic isometries do exist by the examples in (1.2) and (1.3), in order to study the set of holomorphic isometries we resort also to the study of boundary behavior of the maps after analytic continuation. Here for a holomorphic isometry f of the Poincaré disk into a bounded symmetric domain Ω, the norm of the second fundamental form is a basic scalar function associated to the map. With respect to Harish-Chandra coordinates coefficients of the Bergman metric are rational functions, and from the fact that Graph(f) extends algebraically it follows that $\varphi := \|\sigma\|^2$ extends real-analytically to a neighborhood U_b of a general point b on the unit circle. Concerning the second fundamental form we have first of all the following result on the vanishing order of $\|\sigma\|$ assuming first of all that f is asymptotically geodesic at a general point of the boundary circle, a condition which we are now able to remove (cf. (3.5)). In what follows we will say that $\|\sigma\|$ vanishes to the order β at $b \in \partial\Delta$ to mean that φ vanishes to the order 2β at $b \in \partial\Delta$.

Theorem 3.1.1 (Mok-Ng [MN1, Thm.1]). *Let λ be a positive real number and let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, g_\Omega)$ be a holomorphic isometry of the Poincaré disk into a bounded symmetric domain Ω. Suppose f is not totally geodesic and it is asymptotically totally geodesic at a general boundary point. Then, the length of the second fundamental form $\|\sigma\|$ must vanish to the order 1 or $\frac{1}{2}$ at a general boundary point b on the unit circle $S^1 = \partial\Delta$.*

We will say that the non-standard holomorphic isometry f is of the first kind whenever $\beta = 1$, and of the second kind when $\beta = \frac{1}{2}$.

Sketch of Proof. Write $S = f(\Delta) \subset \Omega$. At $x \in S$ denote by β a unit $(1,0)$-vector tangent in $T_x(S)$. Extend β to a holomorphic vector field on S on a neighborhood of x. By the Gauss equation, for any $x \in S$, $\beta \in T_x(X)$, we have the structural equation

$$(\sharp) \qquad R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} - \|\sigma\|^2\|\alpha\|^4 = -\frac{1}{\lambda}\|\alpha\|^4\,,\ R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} = \Big(-\frac{1}{\lambda} + \varphi\Big)\|\alpha\|^2\ .$$

Since (Ω, g_Ω) is locally symmetric, the curvature tensor R is parallel, i.e., $\nabla R = 0$. In what follows we will identify the unit disk Δ with the upper half-plane $\mathcal{H}$, with Euclidean coordinate $\tau = t + is$, by means of the inverse Cayley transform. At $f(x) \in \Omega$ we write μ for the orthogonal lifting of $\sigma(\alpha, \alpha) \in N_{S|\Omega, f(x)}$ to $T_{f(x)}(\Omega)$. Applying covariant differentiation two times to the structural equation $(\sharp)$ we obtain in Mok [Mk6, (1.2), Eqn.(5)] the identity

$$\frac{2\lambda}{t^2}R_{\mu\overline{\mu}\alpha\overline{\alpha}} - \frac{\partial^2\varphi}{\partial\tau\partial\overline{\tau}} = \frac{\lambda}{t^2}\left(R(R_{\alpha\overline{\alpha}\alpha}, \overline{\alpha}; \alpha, \overline{\alpha}) - \frac{1}{\lambda^2} + \frac{\varphi}{\lambda}\right).$$

Let q be the vanishing order of φ at b and write $\varphi(\tau) = t^q u(\tau)$. In the special case where Ω is irreducible and of rank 2 and where $\lambda = 1$, we have the exact formula

$$R_{\mu\overline{\mu}\alpha\overline{\alpha}} = \frac{t^2}{2}\Big(\frac{\partial^2\varphi}{\partial\tau\partial\overline{\tau}} - \frac{\varphi}{2t^2}\Big) = \frac{1}{8}\big(q(q-1)-2\big)t^q u + \frac{qt^{q+1}}{4}\frac{\partial u}{\partial t} + \frac{t^{q+2}}{8}\Big(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial s^2}\Big).$$

The left-hand side is nonpositive by the nonpositivity of holomorphic bisectional curvatures on (Ω, g_Ω). If $q \geq 3$ the right-hand side is positive for $t > 0$ sufficiently small, yielding a contradiction and proving Theorem 3.1.1 in the special case. In general we have the inequality

$$R_{\mu\overline{\mu}\alpha\overline{\alpha}} \geq \frac{1}{8\lambda}(q^2-q-4)t^q u + \frac{qt^{q+1}}{4\lambda}\frac{\partial u}{\partial t} + \frac{t^{q+2}}{8\lambda}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial s^2}\right) + \frac{\varphi^2}{2}.$$

(Mok-Ng [MN1, (1.1), Proof of Theorem 1]) and the same follows. $\square$

(3.2) *Holomorphic isometries of the Poincaré disk into polydisks: structural results* Let $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$ be the unit disk, p and k be positive integers, $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a holomorphic isometry, and $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ be the irreducible projective-algebraic variety which contains Graph(f) as an open subset. Equip $\mathcal{O} := \mathbb{P}^1 - \overline{\Delta}$ with the Hermitian metric $ds^2_{\mathcal{O}}$ which on $\mathbb{C}^1 - \overline{\Delta}$ is given by $ds^2_{\mathcal{O}} = \frac{4\mathrm{Re}(dz\otimes d\overline{z})}{(|z|^2-1)^2}$. Let G be any connected component of $(\mathbb{P}^1 - \partial\Delta) \times (\mathbb{P}^1 - \partial\Delta)^p$, and write $G = W \times G'$, where W is either Δ or $\mathcal{O} = \mathbb{P}^1 - \overline{\Delta}$ and G' is a Cartesian product of p domains each of which is either Δ or $\mathcal{O}$. Equip G' with the Kähler metric $ds^2_{G'}$ which is the product metric of the Hermitian metrics ds^2_Δ, resp. $ds^2_{\mathcal{O}}$, for each Cartesian factor equal to Δ resp. $\mathcal{O}$. For a boundary $b \in \partial\Delta$, the holomorphic isometry f extends holomorphically to a neighborhood U_b of b in $\mathbb{C}$. Denoting the extended map on U_b still by f, we have $f|_{U_b\cap\mathcal{O}} : U_b \cap \mathcal{O} \to G'$ for some connected component G' of $(\mathbb{P}^1 - \partial\Delta)^p$ equipped with the Kähler metric $ds^2_{G'}$, and f remains a holomorphic isometry on $U_b \cap \mathcal{O}$ when the latter is equipped with the Poincaré metric $k\, ds^2_{\mathcal{O}}$. Thus, for the curvature identities derived from the structural equation ($\sharp$) in the proof of Theorem 3.1.1 (arising from the Gauss equation applied to holomorphic isometries), after formally extending the identities to U_b, the curvature terms actually are *bona fide* curvatures in the sense of Kähler geometry with respect to a product of Poincaré metrics. In particular, they remain nonpositive outside of the unit circle. From this we deduce the following property on the boundary behavior of the second fundamental form for holomorphic isometries of the Poincaré disk into polydisks.

Theorem 3.2.1 ([Mok-Ng [MN1, Theorem 2]). *Let $p \geq 2$ be an integer, k be a positive integer, and $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a non-standard holomorphic isometry, necessarily of the first kind. Write $S := f(\Delta) \subset \Delta^p$, and denote by φ the square of the norm of the second fundamental form σ of $\big(S, ds^2_{\Delta^p}\big|_S\big) \hookrightarrow (\Delta^p, ds^2_{\Delta^p})$ as a Kähler submanifold. Identify Δ with the upper half-plane $\mathcal{H}$, with Euclidean coordinate τ, via a Cayley transform, and write $\varphi(f(\tau)) = \|\sigma(f(\tau))\|^2 = t^2 u(\tau)$, where $t = \mathrm{Im}(\tau)$ and $u(\tau)$ is real-analytic at a general point on ∂H. Then, we have $\dfrac{\partial u}{\partial t} = 0$ over $U_b \cap \partial\mathcal{H}$ for an open neighborhood U_b of a general point $b \in \partial\mathcal{H}$.*

For the case $p = 2$, Theorem 3.2.1 follows from the exact curvature formula for $R_{\mu\overline{\mu}\alpha\overline{\alpha}}$. For $p \geq 3$ a modification of the formula holds true with error terms which are of order ≥ 4 in t, and Theorem 3.2.1 follows.

The interpretation of analytic continuations beyond the unit circle again as holomorphic isometries into polydisks is special to the case where the target domain is a polydisk. In this case, the same observation leads to the following characterization of totally geodesic holomorphic embeddings.

Theorem 3.2.2 (Mok-Ng [MN1, Theorem 3]). *Let $f : (\Delta, \lambda ds^2_\Delta; 0) \to (\Delta^p, ds^2_\Delta; 0)$ be a germ of holomorphic isometry. Suppose f extends holomorphically to some neighborhood of $\overline{\Delta}$. Then, f is totally geodesic.*

Sketch of Proof. Graph$(f) \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ extends holomorphically to a projective-algebraic subvariety $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$, and λ is a positive integer k, $1 \leq k \leq p$. Each local branch $f = (f_1, \cdots, f_p)$ of f satisfies the real-analytic functional identity

$$(\dagger) \qquad \prod_{i=1}^{p} \frac{1}{1 - |f_i(z)|^2} = \frac{1}{(1 - |z|^2)^k} \ .$$

Since the extended map remains a holomorphic isometry when one exits through the unit circle and by assumption no singularities are developed along the unit circle, we have actually a global holomorphic map $f : \mathbb{P}^1 \to (\mathbb{P}^1)^p$, still denoted by the same symbol f, such that f restricts to $\mathcal{O}$ to give a holomorphic isometry $f : (\mathcal{O}, ds^2_{\mathcal{O}}) \to (G', ds^2_{G'})$ into one of the connected components G' of $(\mathbb{P}^1 - \partial\Delta)^p$. From the functional identity $(\dagger)$ each component f_i either maps the unit circle to the unit circle, or f_i maps a neighborhood of $\overline{\Delta}$ to Δ. In the latter case, $f_i : \mathbb{P}^1 \to \Delta$ and hence f_i is a constant by the Maximum Principle. Removing the constant components f_i if necessary we have in the functional equation $p = k$ (by checking the vanishing orders of factors in $(\dagger)$ along the unit circle), in which case it follows easily that f must be totally geodesic by the Schwarz Lemma. $\square$

(3.3) *Calculated examples on the norm of the second fundamental form* Let λ be a positive constant and $F : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometry of the Poincaré disk into a bounded symmetric domain. By Theorem 2.3.2, F is necessarily proper and its graph extends algebraically. The qualitative behavior of F along the boundary circle is invariant under re-parametrization on the domain disk and automorphisms of the target domain Ω. These properties furnish invariants which can be used to distinguish holomorphic isometries modulo re-parametrizations and target automorphisms. For instance, one can see on the one hand that the p-th root map is singular precisely at two boundary points (which are 0 and ∞ when the domain is taken to be the upper half-plane) and that they are transformed to the Shilov boundary (distinguished boundary) of the target domain (which is the torus $S^1 \times \cdots \times S^1$ if the target domain is identified with the polydisk), and on the other hand that the map $G : \big(\mathcal{H}, 2ds^2_{\mathcal{H}}\big) \to \big(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\big)$ defined in (1.3) is also precisely singular at two boundary points, but they are transformed to a stratum of the boundary of the Siegel upper half-plane $\mathcal{H}_3$ (in its compact dual) other than the Shilov boundary. Thus, identifying $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ with a maximal polydisk of $\mathcal{H}_3$ and thus regarding the cube root map ρ_3 as a map into $\mathcal{H}_3$, it is not possible through re-parametrization and a target automorphism to transform ρ_3 into G. Using the classification of holomorphic isometries of the Poincaré disk into the 3-disk obtained by Ng [Ng1] (cf. (3.4) of the current article), we show in Mok [Mk7, Proposition 3.3.2] along the same line of argument that the image of G is not contained in any maximal polydisk.

There are other invariants arising from differential-geometric considerations. First of all, the norm of its second fundamental form σ is invariant under automorphisms of the target domain Ω, and the asymptotic behavior of $\|\sigma\|$ along the boundary circle is further invariant under re-parametrization. Thus, among other things the computation of second fundamental forms serve the purpose of distinguishing known examples of holomorphic isometries. In this direction we compute in Mok-Ng [MN1] first of all the second fundamental form of the p-th root map, as follows.

Theorem 3.3.1 (Mok-Ng [MN1, Thm.4]). *Let p be a positive integer. Denote by $ds^2_{\mathcal{H}}$ the Poincaré metric on the upper half-plane $\mathcal{H}$ of constant Gaussian curvature -1 and correspondingly by $ds^2_{\mathcal{H}_p}$ the product metric on the Cartesian product $\mathcal{H}^p$ of p copies of the upper half-plane. Write $\rho_p : (\mathcal{H}, ds^2_{\mathcal{H}}) \to (\mathcal{H}^p, ds^2_{\mathcal{H}^p})$ for the p-th root map given by $\rho_p(\tau) = \big(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}\big)$, where $\gamma = e^{\frac{\pi i}{p}}$, and $\tau = |\tau|e^{i\theta}$. Then, the second fundamental form σ_p of ρ_p is given by $\|\sigma_p\|^2 = \frac{2(p^2-1)}{3p^2}\sin^2\theta$. In particular, $-\log\|\sigma_p\|^2$ is a potential function for $ds^2_{\mathcal{H}}$. In other words, denoting by $\omega_{\mathcal{H}}$ stands for the Kähler form of $ds^2_{\mathcal{H}}$, we have $\sqrt{-1}\partial\overline{\partial}(-\log\|\sigma_p\|^2) = \omega_{\mathcal{H}}$.*

Using curvature formulas in symplectic geometry of Siegel [Si], we have studied in Mok [Mk6] the asymptotic behavior of the second fundamental form for the holomorphic isometry $G : \big(\mathcal{H}, 2ds^2_{\mathcal{H}}\big) \to \big(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\big)$, showing that it must vanish to the order 1 at a general boundary point (i.e., G is of the first kind), and have more explicitly computed $\|\sigma\|$ in Mok-Ng [MN1], deducing thereby that the image of G is not contained in any maximal disk by a classification-free proof, and obtaining a more refined result, as follows. (The formulation below incorporates the computation preceding the statement of the result in [MN1].)

Proposition 3.3.1(Mok-Ng [MN1, Prop.3]). *Let $G : \big(\mathcal{H}, 2ds^2_{\mathcal{H}}\big) \to \big(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\big)$ be the holomorphic isometry of the Poincaré disk into the Siegel upper half-plane $\mathcal{H}_3$ of genus 3 given by*

$$G(\tau) = \begin{bmatrix} \frac{\sqrt{3}}{2}\tau^{\frac{2}{3}} & \tau^{\frac{1}{3}} & 0 \\ \tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & \gamma\tau^{\frac{1}{3}} \end{bmatrix}.$$

Then, writing $\tau = |\tau|e^{i\theta}$, the norm of the second fundamental form σ_G of G is given by

$$\|\sigma_G\|^2 \;=\; \frac{4}{27}\sin^2\theta\left[3 - \cos\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right].$$

As a consequence, writing $G_m : \big(\mathcal{H}, 2m\,ds^2_{\mathcal{H}}\big) \to \big((\mathcal{H}_3)^m, ds^2_{(\mathcal{H}_3)^m}\big)$ for the holomorphic isometry given by $F_m(\tau) = (F(\tau), \cdots, F(\tau))$, m being any positive integer, and identifying the Cartesian product $(\mathcal{H}_3)^m$ as a totally geodesic complex submanifold of $\mathcal{H}_{3m}$ as usual, the image $G_m(\mathcal{H}) \subset \mathcal{H}_{3m}$ is not contained in any maximal polydisk of $\mathcal{H}_{3m}$.

The main distinction between $\|\sigma_G\|^2$ and $\|\sigma\|^2$ for the second fundamental form of a holomorphic isometry into a polydisk lies in the fact that, writing $\|\sigma_G\| = t^2 u_G$, we have $\frac{\partial u}{\partial t}\big|_{\partial\mathcal{H}} \not\equiv 0$, violating a necessary condition for the image of a holomorphic isometry to be contained in a maximal polydisk according to Theorem 3.1.1. The same applies to G_m in place of G obtained by duplicating the map G a number of times.

(3.4) *Holomorphic isometries of the Poincaré disk into polydisks: uniqueness results* In the case where Ω is the polydisk of dimension ≥ 3 one can construct continuous families of holomorphic isometries which are incongruent to one another (cf. (5.1)). It is an interesting problem to identify the structure of the space of holomorphic isometries of the Poincaré disk up to a normalizing constant into a polydisk. We do not know whether the space of such holomorphic isometries are generated by the set of p-th root maps (cf. (1.2)). In this direction Ng [Ng1] has started to study the space of holomorphic isometries of the Poincaré disk into polydisks as a whole by examining the explicit forms of the functional identities arising from holomorphic isometries, obtaining the following preliminary results. To start with let k be a positive integer, $1 \leq k \leq p$ and $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a a holomorphic isometry (necessarily a proper holomorphic isometric embedding by Theorem 2.3.2). The mapping f defines a branched covering $\Phi : X \to \mathbb{P}^1$ such that f is a branch of Φ^{-1} restricted to the unit disk. Ng [Ng1] has proved the following general result.

Theorem 3.4.1 (Ng [Ng1]). *Let $f : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a holomorphic isometry. The sheeting number n of $\Phi : X \to \mathbb{P}^1$ satisfies $p \leq n \leq 2^{p-1}$. Furthermore,*

(1) *If p is odd and the sheeting number is equal to p, then up to re-parametrization F is the p-th root map.*

(2) *Writing $F = (f_1, \cdots, f_p)$, and s_i for the sheeting number of the branched cover $\Phi_i : X_i \to \mathbb{P}^1$ associated to f_i. Assume $s_1 \leq \cdots \leq s_p$; $s_p = 2^{p-1}$. Then, $n = 2^{p-1}$, and F can be factored as $F = F_{p-1} \circ F_{p-2} \circ \cdots \circ F_1$, where each intermediate $F_i : \Delta^i \to \Delta^{i+1}$ is given by $F_i(z_1, \cdots, z_i) = (z_1, \cdots, z_{i-1}, \alpha_i(z_i), \beta_i(z_i))$; where $z \mapsto (\alpha_i(z), \beta_i(z))$ is up to re-parametrization the square root map Δ into Δ^2.*

In the case of low dimensions Ng has proved

Theorem 3.4.2 (Ng [Ng1]). *For $p = 2$ any* bona fide *non-standard holomorphic isometric embedding $F : \Delta \to \Delta^2$ is up to re-parametrization the square root map. For $p = 3$, any such map $F : \Delta \to \Delta^3$ is either up to re-parametrization the cube root map, or it is of the form $F_2 \circ F_1$ as in* Theorem 3.4.1. *Furthermore, all non-standard holomorphic isometric embeddings $F : \Delta \to \Delta^2$ (with* a priori *arbitrary normalizing constant λ) must be bona fide isometries. For $F : \Delta \to \Delta^3$, we must have $\lambda = 1$ or 2, and for $\lambda = 2$ the map is up to and re-parametrization permutation of the Cartesian factors of Δ^3 given by $F(z) = (z, \alpha(z), \beta(z))$, where $z \mapsto (\alpha(z), \beta(z))$ is the square root map.*

(3.5) *Asymptotic total geodesy and applications* We consider holomorphic isometries of the Poincaré disk into bounded symmetric domains equipped with the Bergman metric. We prove (in an article under preparation)

Theorem 3.5.1. *Let (Ω, ds^2_Ω) be a bounded symmetric domain equipped with the Bergman metric ds^2_Ω, λ be a positive constant, and $F : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometry. Then, F is asymptotically totally geodesic at a general point $b \in \partial\Delta$ on the boundary circle.*

A key ingredient of the proof of the theorem above is the use of the Poincaré-Lelong equation as in Mok [Mk3]. As an immediate consequence of Theorem 3.5.1, we have the following result which says that, in contrast to the case of Hermitian symmetric manifolds of the compact type, analogues of maps such as the Veronese embedding or the Segre embedding cannot possibly exist in the dual case of bounded symmetric domains.

Theorem 3.5.2. *Let D and Ω be bounded symmetric domains, $\Phi : \mathrm{Aut}_0(D) \to \mathrm{Aut}_0(\Omega)$ be a group homomorphism, and $F : D \to \Omega$ be a Φ-equivariant holomorphic map. Then, F is totally geodesic.*

Deduction from Theorem 3.5.1. The proof of Theorem 3.5.2 can be easily reduced to the case where Ω is irreducible. By slicing by totally geodesic Poincaré disks on D the problem can be further reduced to the case where the domain D is the unit disk Δ. Any Φ-equivariant holomorphic map $F : \Delta \to \Omega$ is necessarily a holomorphic isometry up to a normalizing constant, and it follows from Theorem 3.5.1 that F is asymptotically totally geodesic at a general boundary point $b \in \partial\Delta$. On the other hand, by Φ-equivariance it follows that $\|\sigma\|$ is constant on Δ, which gives a contradiction unless $\sigma \equiv 0$, i.e., unless $F : \Delta \to \Omega$ is totally geodesic. $\square$

Theorem 3.5.2 in the special case where none of the irreducible direct factors of Ω is exceptional was established by Clozel [Cl] by Lie-theoretic methods. Another application of asymptotic total geodesy is to give a characterization of totally geodesic compact complex-analytic subvarieties of quotients of bounded symmetric domains in terms of local symmetry, as follows.

Theorem 3.5.3. *Let (Ω, ds^2_Ω) be a bounded symmetric domain equipped with the Bergman metric ds^2_Ω. Let $\Gamma \subset \mathrm{Aut}_0(\Omega)$ be a torsion-free discrete subgroup, $X := \Omega/\Gamma$. Denote by h the Kähler metric on X induced from g. Suppose $Z \subset X$ is a compact complex-analytic subvariety and $\big(\mathrm{Reg}(Z), h|_{\mathrm{Reg}(Z)}\big)$ is locally symmetric. Then, $Z \subset X$ is a totally geodesic subset.*

Since Z is locally symmetric at a smooth point as a germ of Kähler submanifold, we have a germ of holomorphic map $f_0 : (D; 0) \to (X, x_0)$ from a bounded symmetric domain D into the germ of X at a smooth point x_0 of Z, which lifts to a germ of holomorphic isometry $f : (D; 0) \to (\Omega; 0)$ via a local inverse of the universal covering map $\pi : \Omega \to X$. To prove total geodesy of f it suffices to show that $f|_\Delta : \Delta \to \Omega$ is totally geodesic for every totally geodesic Poincaré disk Δ on D. This follows readily from the asymptotic total geodesy of the holomorphic extension of $f|_\Delta$ to Δ and the existence of a compact fundamental domain for Z.

§4 Measure-preserving algebraic correspondences on irreducible bounded symmetric domains

(4.1) *Statements of results* Let Ω be an irreducible bounded symmetric domain and $X := \Omega/\Gamma$ be the quotient of Ω by a torsion-free lattice Γ. In relation to the characterization problem of Clozel-Ullmo on commutators of modular correspondences on X, in a recent work of the author with S.-C, Ng [MN2], we have settled the problem by characterizing germs of measure-preserving holomorphic maps from Ω to its Cartesian products. When Ω is the unit disk, a measure-preserving map $f : (\Delta; 0) \to (\Delta; 0) \times \cdots \times (\Delta; 0)$ is a holomorphic isometry up to an integral normalizing constant, and the problem was settled by Clozel-Ullmo [CU], who proved algebraic extension of Graph(f), and deduced total geodesy from the action of the lattice Γ. When Ω is of dimension ≥ 2 we have proved

Theorem 4.1.1 (Mok-Ng [MN2, Main Theorem]). *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of complex dimension ≥ 2, and $d\mu_\Omega$ be the volume form of the Bergman metric on Ω. Suppose d_1 and d_2 are positive integers. For $1 \leq \alpha \leq d_2$ let $\pi_\alpha : \Omega^{d_2} \to \Omega$ be the canonical projection onto the $\alpha-th$ direct factor $\Omega_\alpha = \Omega$. Suppose $f = (f_1, \cdots, f_{d_2}) : (\Omega, d_1\, d\mu_\Omega; 0) \to (\Omega^{d_2}, \pi_1^\star d\mu_\Omega + \cdots + \pi_{d_2}^\star d\mu_\Omega; 0)$ is a germ of measure-preserving holomorphic map such that each $f_\alpha, 1 \leq \alpha \leq d_2$, is of maximal rank at some point. Then, $d_1 = d_2$ and f extends to a totally geodesic holomorphic embedding $f : \Omega \to \Omega^{d_2}$.*

Combining with the result of [CU] we have established

Theorem 4.1.2 (Mok-Ng [MN2, Thm.1.1.2]). *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain, and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free lattice. Write $X := \Omega/\Gamma$ and let $Y \subset X \times X$ be a measure-preserving algebraic correspondence with respect to the canonical measure $d\mu_\Omega$ on Ω. Then, Y is necessarily a modular correspondence.*

(4.2) *Extension results on strictly pseudoconvex algebraic hypersurfaces* We solve the characterization problem on measure-preserving maps essentially by means of extension and rigidity results in Several Complex Variables. The starting point is the following well-known extension result for germs of holomorphic maps on strongly pseudoconvex real algebraic hypersurfaces.

Theorem 4.2.1 (Huang [Hu]). . *Let $M_1 \subset \mathbb{C}^m$ and $M_2 \subset \mathbb{C}^{m+k}$ be real algebraic hypersurfaces with $m > 1$ and $k \geq 0$. Let $p \in M_1$ be a strongly pseudoconvex point. Suppose that h is a holomorphic mapping from a neighborhood U_p of p to $\mathbb{C}^{m+k}$ so that $h(U_p \cap M_1) \subset M_2$ and $h(p)$ is also a strongly pseudoconvex point, then h is algebraic.*

For the problem at hand, we have an irreducible bounded symmetric domain Ω of complex dimension at least 2, a germ of holomorphic map $f : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$, $f = (f_1, \cdots, f_{d_2})$, where each component f_α is of maximal rank at some point, which is measure-preserving in the sense that $f_1^\star(d\mu_\Omega) + \cdots f_{d_2}^\star(d\mu_\Omega) = d_1\, d\mu_\Omega$ (cf. statement of Theorem 4.1.1). Let (L, g) be the Hermitian anticanonical line bundle on Ω equipped with the $\mathrm{Aut}(\Omega)$-invariant metric g given by the volume form $d\mu_\Omega$. Likewise for $1 \leq \alpha \leq d_2$ denote by (L_α, g_α) the corresponding Hermitian anticanonical line bundle on the α-th direct factor Ω_α of $\Omega \times \cdots \times \Omega$, and define $(\mathcal{L}, \mathfrak{g})$ to be the direct sum $\bigoplus_{\alpha=1}^{d_2} (\pi_\alpha^\star L_\alpha, g_\alpha)$. Recall here that $\pi_\alpha : \Omega \times \cdots \times \Omega \to \Omega_\alpha$ denotes the canonical projection onto the α-th factor. By considering Jacobian determinants of the differential df_α of each component $f_\alpha : \Omega \to \Omega_\alpha$, from the hypothesis that f is measure-preserving we have an induced holomorphic map $\widetilde{f} : L \to \mathcal{L}$ over Ω which sends the unit circle bundle $S_L \subset L$ into the unit sphere bundle $S_{\mathcal{L}} \subset \mathcal{L}$. By regarding $L \cong \Omega \times \mathbb{C} \subset \mathbb{C}^n \times \mathbb{C}$ as an open subset of the Euclidean space $\mathbb{C}^{n+1}$, the hypersurface $S_L \subset L \subset \mathbb{C}^{n+1}$ is algebraic since the Bergman kernel $K_\Omega(z, w)$ is rational in $(z, \overline{w})$. Likewise $S_{\mathcal{L}} \subset (\mathbb{C}^n)^{d_2} \times \mathbb{C}^{d_2}$ is a real algebraic hypersurface.

The Hermitian holomorphic line bundle (L, g) is of strictly negative curvature. The Hermitian holomorphic vector bundle $(\mathcal{L}, \mathfrak{g})$, which is the direct sum of the Hermitian holomorphic line bundle (L_α, g_α) of strictly negative curvature, is of seminegative curvature in the sense of Griffiths. By curvature considerations going back to Grauert [Gr, 1962], the real hypersurface $S_L \subset L$ is strongly pseudoconvex, and the real hypersurface $S_{\mathcal{L}} \subset \mathcal{L}$ is weakly pseudoconvex and strictly pseudoconvex on a dense open subset, where the structure of the weakly pseudoconvex points on $S_{\mathcal{L}}$ is easily described in terms of the direct sum decomposition, viz., a point $(u_1, \cdots, u_{d_2}) \in S_{\mathcal{L}}$ is a weakly pseudoconvex point if and only if $u_\alpha = 0$ for at least one of the components u_α, $1 \le \alpha \le d_2$. The assumption that each f_α is of maximal rank at some point implies that $\widetilde{f}(u)$ is strongly pseudoconvex point on $S_{\mathcal{L}}$ for a dense open set of points u on S_L. Thus, the requirement on strong pseudoconvexity in Huang's result above is satisfied for the map $\widetilde{f} : S_L \to S_{\mathcal{L}}$, and the latter result applies to yield the algebraic extension of Graph(f). In other words, there exists an irreducible affine-algebraic subvariety $V \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$ of complex dimension $n = \dim(\text{Graph}(f))$ which contains Graph(f).

(4.3) *Alexander-type extension results in the higher-rank case* In the case of the complex unit ball B^n, $n \ge 2$, for a germ of measure-preserving map $f : (B^n; 0) \to (B^n; 0) \times \cdots \times (B^n; 0)$ the functional equation defining the measure-preserving property translates into

$$(\dagger) \qquad \sum_{\alpha=1}^{d_2} \frac{|\det(Jf_\alpha(z))|^2}{(1 - |f_\alpha(z)|^2)^{n+1}} = \frac{d_1}{(1 - |z|^2)^{n+1}} \,.$$

Having established the algebraic extension of Graph(f), at a general point $b \in \partial B^n$ the germ of map $f = (f_1, \cdots, f_{d_2})$ can be analytically continued to a neighborhood U_b of b, still denoted by f, such that each f_α remains unramified on U_b and the functional identity $(\dagger)$ remains valid. Since each summand of the left-hand side of $(\dagger)$ must be infinite when the corresponding component f_α exits the boundary sphere it follows from the process of analytic continuation that we must have $f_\alpha(U_b \cap B^n) \subset B^n$. Since the right-hand side of $(\dagger)$ is infinite along $U_b \cap \partial B^n$ at least one of the components, say f_{d_2}, must be such that the corresponding summand on the left-hand side is infinite along $U_b \cap \partial B^n$, i.e., $f_{d_2}(U_b \cap \partial B^n) \subset \partial B^n$. By Alexander's result, f_{d_2} extends to an automorphism of B^n, which itself preserves the Bergman volume form. Thus, $(\dagger)$ can be reduced by removing f_{d_2} on the left-hand side and replacing d_2 by $d_2 - 1$ on the right-hand side, and the total geodesy of f follows by induction.

When $\text{rank}(\Omega) \ge 2$ we use a well-known Alexander-type result of Henkin-Tumanov [TK1] involving the Shilov boundary to complete the proof of Theorem 4.1.1.

Theorem 4.3.1 (Henkin-Tumanov [TK1]). *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization, and denote by $Sh(\Omega) \subset \partial\Omega$ its Shilov boundary. Suppose $b \in Sh(\Omega)$. Let $U_b \subset \mathbb{C}^n$ be a connected open neighborhood of b in $\mathbb{C}^n$, and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap \Omega) = f(U_b) \cap \Omega$ and $f(U_b \cap Sh(\Omega)) = f(U_b) \cap Sh(\Omega)$. Then, there exists an automorphism $F : \Omega \to \Omega$ such that $F|_{U_b\cap\Omega} \equiv f|_{U_b\cap\Omega}$.*

Theorem 4.3.1 is first stated in Henkin-Tumanov [TK1] and proved in a special case. A simplified proof is given in Henkin-Tumanov [TK2, §4] on the same special case basing on the use of cone structures of Goncharov [Go]. Equivalently this follows from results on the geometric structures defined by irreducible Hermitian symmetric spaces of the compact type of rank ≥ 2 due already to Ochiai ([Oc, 1970]), reformulated as follows in terms of minimal rational tangents on the Hermitian symmetric manifold M of the compact type dual to Ω. Here by definition at $x \in M$ the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ is the space of all projectivizations of vectors tangent to minimal rational curves on M passing through x, where equivalently a minimal rational curve is a projective line with respect to the first canonical projective embedding, cf. Hwang-Mok [HM].

Theorem 4.3.2 (Ochiai [Oc]). *Let M be an irreducible compact Hermitian symmetric manifold of the compact type and of rank ≥ 2; $U, V \subset M$ be connected open subsets, and $f : U \to V$ be a biholomorphism. Suppose for every $x \in U$ the projectivization $[df(x)]$ of $df(x) : T_x(M) \to T_{f(x)}(M)$ satisfies $[df(x)](\mathcal{C}_x) = \mathcal{C}_{f(x)}$. Then, there exists an automorphism $F \in \mathrm{Aut}(M)$ such that $F\big|_U \equiv f$.*

For the purpose of giving an alternative proof parallel to the rank-1 case, we prove as follows an alternative Alexander-type result concerning the smooth locus of $\partial\Omega$ rather than the Shilov boundary, which is the most singular stratum of $\partial\Omega$ in its decomposition as the disjoint union of a finite number of $\mathrm{Aut}(\Omega)$-orbits.

Theorem 4.3.3. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. Suppose b is a smooth point on $\partial\Omega$. Let $U_b \subset \mathbb{C}^n$ be an open neighborhood of b in $\mathbb{C}^n$ and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap \Omega) \subset \Omega$ and $f(U_b \cap \partial\Omega) \subset \partial\Omega$. Then, there exists an automorphism $F : \Omega \to \Omega$ such that $F|_{U_b\cap\Omega} \equiv f|_{U_b\cap\Omega}$.*

Theorem 4.3.3 and its proof are of independent interest and relevant to the study of proper holomorphic maps between higher-rank bounded symmetric domains. The proof of Theorem 4.3.3 relies also on Ochiai's result above on G-structures in conjunction with the method of Mok-Tsai [MT] and Tsai [Ts] of considering non-tangential boundary values of proper holomorphic maps on totally geodesic complex submanifolds which are product domains. Conceptually, there is a decomposition of $\mathrm{Reg}(\partial\Omega)$ into boundary components (cf. Wolf [Wo]), and they should be transformed to boundary components when one takes boundary values on associated product domains. Such properties on boundary values can then be translated into properties at interior points, i.e., points on Ω. When applied to all relevant product domains the condition at appropriate interior points translates into a property on the differential of the holomorphic map f which implies that f preserves varieties of minimal rational tangents, from which Theorem 4.3.3 follows from Ochiai's result.

(4.4) *Total geodesy of germs of measure-preserving holomorphic map from an irreducible bounded symmetric domain of dimension* ≥ 2 *into its Cartesian products* The proof of Theorem 4.1.1 in the remaining case where the irreducible bounded symmetric domain Ω is of rank ≥ 2 will now follow from the use of Alexander-type theorems. We have given in [MN2] two proofs. To start with we have the functional equation

$$(\dagger) \qquad \sum_{\alpha=1}^{d_2} K_\Omega(f_\alpha(z), f_\alpha(z))|\det(Jf_\alpha(z))|^2 = d_1 K_\Omega(z,z)\,.$$

where $K_\Omega(z,z) = \frac{1}{Q_\Omega(z,z)}$ is an exhaustion function on Ω. For the first proof we make use of the result of Henkin-Tumanov [TK1, 2] (Theorem 4.3.1 here) to replace Alexander's theorem after having proved algebraic extension of Graph(f). Imitating the rank-1 case, one has to prove that for the measure-preserving map $f : \Omega \to \Omega \times \cdots \times \Omega$, $f = (f_1, \cdots, f_{d_2})$, one of the components, say f_{d_2}, must be an automorphism, which must itself preserve the Bergman volume form. One can then simplify the functional equation ($\dagger$) arising from the measure-preserving property of f to an analogous equation where the image Ω^{d_2} is reduced to Ω^{d_2-1} by removing the last factor, and conclude as in the rank-1 case by induction that indeed f is totally geodesic.

Write $\Omega \Subset \mathbb{C}^n \subset M$ simultaneously for the Harish-Chandra embedding and for the Borel embedding $\Omega \subset M$ into the compact dual of Ω and write $V \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$ for the algebraic extension of Graph(f) to an affine-algebraic variety. Write $\pi_0 : \overline{V} \to M$ for the natural projection into the factor M containing the domain Ω and $\pi_\alpha : \overline{V} \to M$, $1 \leq \alpha \leq d_2$, for the natural projection into the α-th direct factor of M^{d_2}. Let $E \subset \overline{V} \times \overline{V}^{d_2}$ be the smallest subset outside of which each π_i, $0 \leq i \leq d_2$, takes finite values and is unramified, and write $R \subset \mathbb{C}^n$ for the subvariety $\pi_0(E) \cap \mathbb{C}^n$. To carry through the first proof using Theorem 4.3.1, which involves the Shilov boundary, we observed first of all that the bad set $R \subset V$ does not contain the Shilov boundary $Sh(\Omega)$, so that each f_α, $1 \leq \alpha \leq d_2$, admits an analytic continuation to any point $b \in Sh(\Omega) - R \neq \emptyset$ to give a biholomorphism on some neighborhood U of b onto an open subset of $\mathbb{C}^n$.

In order to apply the result of Henkin-Tumanov, we have to show that $f(U \cap \partial\Omega) \subset \partial\Omega$ and that furthermore $f(U \cap Sh(\Omega)) \subset Sh(\Omega)$. To prove the first we checked that $b \in \partial\Omega$ admits a fundamental system of neighborhoods Q_b on $\partial\Omega$ such that $Q_b \cap \mathrm{Reg}(\Omega)$ is connected. (Connectedness is needed to avoid the possibility that some connected component of $Q_b \cap \partial\Omega$ is mapped to the boundary, while some other connected component of $Q_b \cap \partial\Omega$ is mapped to the interior, a phenomenon that does occur in the case of the unit disk, as exemplified by the p-th root map). The latter can be checked using the structure of $\partial\Omega$ as a semi-analytic set. (For general references the reader may consult Łojasiewicz [Ło] on semi-analytic sets, and Wolf [Wo] on the fine structure of $\partial\Omega$). In its place we give a simpler and more direct argument exploiting the realization of $\partial\Omega$ as a Siegel domain $\mathcal{D}$ of the first or second kind via the Cayley transform of Korányi-Wolf [KW], with $b \in \partial\Omega$ being transformed to $0 \in \partial\mathcal{D}$, by means of which neighborhoods P_0 such that $P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$ is path-connected can be found by using the connectedness of $\mathrm{Reg}(\partial\mathcal{D})$ and linear contractions at 0 which restrict to automorphisms of $\mathcal{D}$.

To check that $f(U \cap Sh(\Omega)) \subset Sh(\Omega)$ we exploit the fine structure of irreducible bounded symmetric domains given in Wolf [Wo]. Write $r = \mathrm{rank}(\Omega)$ and $G = \mathrm{Aut}_0(\Omega)$. Let $P \subset \Omega$ by a maximal polydisk on Ω, of complex dimension r and identified with the Euclidean polydisk $\Delta^r \subset \Omega \Subset \mathbb{C}^n$. For $1 \leq k \leq r$ write $\epsilon_k = (1, \cdots, 1, 0, \cdots 0) \in \partial P$, where precisely the first k coefficients of ϵ_k are equal to 1. Writing $E_k = G\epsilon_k$, the topological boundary $\partial\Omega$ decomposes into the disjoint union of precisely r orbits $\partial\Omega = E_1 \cup \cdots \cup E_r$, where E_r is the Shilov boundary $Sh(\Omega)$. For $1 \leq \ell \leq r$ we define $K_\ell = E_\ell \cup \cdots \cup E_r$. By means of the proof of the Herman Convexity Theorem [Hr] we show that the smooth locus of the semi-analytic set K_ℓ is precisely equal to E_ℓ. Given this, for a point $b \in Sh(\Omega)$ and a biholomorphism $h : U_b \to \mathbb{C}^n$ of some open neighborhood U_b of b onto an open subset of $\mathbb{C}^n$ such that $h(U_b \cap \Omega) \subset \Omega$ and $h(U_b \cap \partial\Omega) \subset \partial\Omega$, $h\big|_{U_b \cap \partial\Omega}$ must respect the stratification $\partial\Omega = E_1 \cup \cdots \cup E_r$, mapping singular points to singular points, thereby mapping $U_b \cap Sh(\Omega)$ into $Sh(\Omega)$ by induction, as desired. With both topological difficulties removed, Theorem 4.1.1 for the case of rank ≥ 2 follows through as in the rank-1 case when Alexander's theorem is replaced by the result of Henkin-Tumanov.

An alternative proof for the case where $\mathrm{rank}(\Omega) \geq 2$ is to make use of Theorem 4.3.3, which is an Alexander-type theorem in the case an irreducible bounded symmetric domain Ω of rank ≥ 2 concerning the smooth part of $\partial\Omega$ in place of the Shilov boundary $\partial\Omega$, the most singular part $\partial\Omega$. Using Theorem 4.3.3 we can completely avoid the topological difficulties in the application of the result of Henkin-Tumanov, at the expense of course of having to prove a new Alexander-type theorem in place of quoting the well-known result of Henkin-Tumanov.

Finally the special case of the unit disk Δ can be incorporated as follows. The proof of algebraic extension using the result of Huang [Hu] on strictly pseudoconvex real algebraic hypersurfaces goes through in the case of Δ to give an affine-algebraic variety $V \subset \mathbb{C} \times \mathbb{C}^{d_2}$, noting that the extension result is applied to the unit circle bundle S_L of $L = T_\Delta$ with respect to the Bergman metric, which is of real dimension 3. Given a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\Delta)$ and a germ of holomorphic measure-preserving map $f : (\Delta; 0) \to (\Delta; 0) \times \cdots \times (\Delta; 0)$, at a general point $b \in \partial\Delta$ arising from an algebraic correspondence on $X = \Delta/\Gamma$, we have by algebraic extension of $\mathrm{Graph}(f)$ a holomorphic isometry defined on $U_b \cap \Delta$ for some open neighborhood U_b of b. The norm of the second fundamental form σ must by computation be asymptotically zero at b while it is invariant under some subgroup of Γ of finite index acting on $V \cap (\Delta \times \Delta^{d_2})$, implying that σ must vanish identically, i.e., f is totally geodesic.

§5 Open problems

(5.1) *On the structure of the space of holomorphic isometries of the Poincaré disk into polydisks* The p-th root map $\rho_p : \mathcal{H} \to \mathcal{H}^p$ as in Proposition 1.2.1 gives via the inverse Cayley transform a holomorphic isometry $f_p : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$. Here for the domain disk we use the inverse Cayley transform $\iota : \mathcal{H} \to D$ given by $z = \iota(\tau) = \frac{\tau - i}{\tau + i}$, and likewise the same map for each component of the target polydisk Δ^p. We have $f_p(0) = 0$, and f is singular exactly at two points $1, -1 \in \partial\Delta$ on the boundary circle, with images $f_p(1) = (1, \cdots, 1)$ and $f_p(-1) = (-1, \cdots, -1)$. We say that two holomorphic isometries of the Poincaré disk Δ into the polydisk Δ^p are congruent if and only if they are obtained from each other by re-parametrization, i.e., composing with automorphisms of the unit disk Δ in the domain, and by composing with target automorphisms, i.e., automorphisms of the target polydisk Δ^p. Continuous families of holomorphic isometries from the Poincaré disk into polydisks which are incongruent to one another can be constructed from p-th root maps (cf. [Mk7, (3.2)]), as follows. Write $f_2(z) = \big(\alpha(z), \beta(z)\big)$. Given any two distinct points $a, b \in \partial\Delta - \{1, -1\}$, we can choose $\varphi \in \mathrm{Aut}(\Delta)$ such that $\varphi(a) = 1$ and $\varphi(b) = -1$. Define $h : \Delta \to \Delta^3$ by $h(z) = \big(\alpha(z), \alpha(\beta(\varphi(z))), \beta(\beta(\varphi(z)))\big)$. Then, h is singular precisely at the 4 distinct points $1, -1, a, b$. Fixing a and varying b we get holomorphic isometries $h_b : (\Delta, ds^2_\Delta) \to (\Delta^3, ds^2_{\Delta^3})$ depending on b. For $b_1 \neq b_2$, h_{b_1} cannot be congruent to h_{b_2} since any automorphism of the unit disk fixing three distinct boundary points must be the identity map. This and similar examples can serve as first examples for the study of deformations of holomorphic isometries.

Given any two bounded symmetric domain $D \Subset \Omega$ and $\Omega \Subset \mathbb{C}^N$, denote by $\mathbf{HI}(D, \Omega)$ the space of holomorphic isometries $f : (D, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ where $\lambda > 0$ is an arbitrary normalizing constant. It is known from the functional identities and extension results for holomorphic isometries that λ is a rational number which can only take a finite number of possible values. For a fixed normalizing constant $\lambda > 0$ we write $\mathbf{HI}_\lambda(D, \Omega) \subset \mathbf{HI}(D, \Omega)$ for the subset consisting of those maps for which the normalizing constant is λ. A map $f \in \mathbf{HI}_\lambda(D, \Omega)$ is said to be (globally) rigid if and only if any $h \in \mathbf{HI}_\lambda(D, \Omega)$ is of the form $h = \Phi \circ f \circ \varphi$ for some $\varphi \in \mathrm{Aut}(D)$ and for some $\Phi \in \mathrm{Aut}(\Omega)$. We say that $f \in \mathbf{HI}_\lambda(D, \Omega)$ is locally rigid if and only if the following holds true:

(†) Given any relatively compact non-empty open subset $U \Subset D$ there exists a positive constant $\epsilon = \epsilon_U$ such that, for any $h \in \mathbf{HI}_\lambda(D, \Omega)$ satisfying $\|h(z) - f(z)\| < \epsilon_U$ for any $z \in U$, there exists some $\varphi \in \mathrm{Aut}(D)$ and some $\Phi \in \mathrm{Aut}(\Omega)$ such that $h = \Phi \circ f \circ \varphi$.

It is not difficult to prove that the space $\mathbf{HI}_\lambda(D, \Omega)$ can be naturally given the structure of a real-analytic variety. Given this, it follows readily that the defining condition (†) for local rigidity is satisfied whenever it is satisfied for a single nonempty open subset $U \Subset D$.

The only known examples of non-standard holomorphic isometries between bounded symmetric domains are those defined on polydisks, which reduce to holomorphic isometries defined from the Poincaré disk into a bounded symmetric domain. For this reason we focus on the latter class of maps, and in what follows we pose a number of problems on such maps.

Problem 5.1.1. *Study deformations of holomorphic isometries from the Poincaré disk into a bounded symmetric domain.*

Let $\Omega \Subset \mathbb{C}^N$ be any bounded symmetric domain in its Harish-Chandra realization. To study equivalence classes of holomorphic isometries $f : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega), \lambda > 0$, up to target automorphisms, without loss of generality we may assume that $f(0) = 0$. For $p \geq 2$ the p-th root map $\rho_p : \mathcal{H} \to \mathcal{H}^p$ is asymptotically totally geodesic at a general boundary point $b \in \partial\mathcal{H}$. Thus it is not equivariant with respect to $\mathrm{SL}(2, \mathbb{R})$. However, from the definition of ρ_p we have readily $\rho_p(A\tau) = A^{\frac{1}{p}} \rho_p(\tau)$ for any real number $A > 0$, thus ρ_p is equivariant with respect to a 1-parameter subgroup. Identifying the unit disk with the upper half-plane through the inverse Cayley transform, by re-parametrization one derives readily a real 1-parameter family of holomorphic isometries $\mu_t : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ such that $\mu_t(0) = 0$ which are mutually inequivalent under target automorphisms. (Consider maps corresponding to $f_t(\tau) = \rho_p(\tau + t)$.) Furthermore, by composing with equivalents of q-th root maps of individual factor disks we have examples of continuous families of holomorphic isometries which are mutually incongruent, i.e, inequivalent under re-parametrization and target automorphisms (cf. example in the first paragraph of (5.1). These provide initial examples for the study of deformation of holomorphic isometries of the Poincaré disk into polydisks. In Mok-Ng [MN1, (2.2), Proposition 2] the maps from Poincaré disk into the Siegel upper half-plane of genus 3 also as a continuous family of holomorphic isometries of the Poincaré disk to a bounded symmetric domain which are not apparently equivalent to one another, but which turn out to be equivalent to one another under target automorphisms, i.e., under symplectic transformations when the bounded symmetric domain is realized in the standard way as a Siegel domain.

Given a bounded symmetric domain Ω equipped with the Bergman metric ds^2_Ω, it is interesting to study the deformation theory of holomorphic isometries $f_t : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$, by a study of the functional identities satisfied by these holomorphic isometries. Ideally one should be able to identify holomorphic vector fields which potentially arise as infinitesimal variations of such families of holomorphic isometries, and identify the obstruction to realize such candidates of infinitesimal variations as *bona fide* infinitesimal variations of germs of holomorphic isometries. By the extension theorem of Mok [Mk7] (cf. Theorem 2.4.1 of the current article), once we have a 1-parameter family $f_t, t \in (-1, 1)$, of germs of holomorphic isometries, the family extends necessarily to a 1-parameter family of global proper holomorphic isometries $F_t : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$.

Problem 5.1.2. *Study the structure of all holomorphic isometries from the Poincaré disk into the polydisk. Are the known examples rigid modulo the obvious parameters?*

Currently our limited knowledge on the classification of holomorphic isometries $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, where k is a positive integer, is based on Ng [Ng1]. The known results are those for small dimensions $p = 2, 3$ and for certain extreme cases. Writing $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ for the extension of $\mathrm{Graph}(f) \subset \Delta \times \Delta^p$ to $\mathbb{P}^1 \times (\mathbb{P}^1)^p$ as a subvariety, and denoting by $V_i \subset \mathbb{P}^1 \times \mathbb{P}^1$ the extension of $\mathrm{Graph}(f_i) \subset \mathbb{P}^1 \times \mathbb{P}^1$ as a subvariety, by normalization we have maps on compact Riemann surfaces $\tau_i : X_i \to \mathbb{P}^1$, where X_i is a non-singular model of V_i, given by the normalization $\nu_i : X_i \to V_i$; $\tau_i = \pi_0 \circ \nu_i$, where $\pi_0 : \mathbb{P}^1 \times \mathbb{P}^1$ is the canonical projection onto the first factor. Similarly, from the normalization $\nu : X \to V$ we obtain a holomorphic map $\tau : X \to \mathbb{P}^1$, $\tau = \pi_0 \circ \nu$, where again $\pi_0 : \mathbb{P}^1 \times (\mathbb{P}^1)^p$ is the canonical projection onto the first factor. From this we have the i-th sheeting number s_i, $1 \le i \le p$ of $f : \Delta \to \Delta^p$, defined as the sheeting number of $\tau_i : X_i \to \mathbb{P}^1$, and the global sheeting number s of f, defined as the sheeting number of $\tau : X \to \mathbb{P}^1$. Thus to any $f \in \mathbf{HI}(\Delta, \Delta^p)$ we can attach the global sheeting number s, which is a topological invariant. We write $\mathbf{HI}_\lambda(\Delta, \Delta^p; s) \subset \mathbf{HI}_\lambda(\Delta, \Delta^p)$ for the subset consisting of those maps f for which furthermore the global sheeting number is s. We can further attach the p-tuple $(s_1, \cdots, s_p)$ where s_i stands for the i-th sheeting number. If we permute the components f_i so that $s_1 \leq \cdots \leq s_p$, then the p-tuple $(s_1, \cdots, s_p)$ is uniquely determined, but we will refrain from doing so. Given s and given $(s_1, \cdots, s_p)$, we will denote by $\mathbf{HI}_\lambda(\Delta, \Delta^p; s; s_1, \cdots, s_p) \subset \mathbf{HI}_\lambda(\Delta, \Delta^p; s)$ consisting of holomorphic isometries for which furthermore the i-th sheeting number is s_i for $1 \leq i \leq p$.

At this point a complete description of $\mathbf{HI}(\Delta, \Delta^p)$ appears to be difficult. For the extremal cases the structure of $\mathbf{HI}_1(\Delta, \Delta^p; p)$ is only known when p is odd, in which case there is only 1 map in the space up to re-parametrization and target automorphisms, viz., a map congruent to the p-th root map ρ_p. When $s = 2^{p-1}$, the structure $\mathbf{HI}_1(\Delta, \Delta^p; 2^{p-1}; s_1, \cdots, s_p)$ is known only in the special case when the maximum of s_i is 2^{p-1}. In this case, by Ng [Ng1] it was shown that any $f \in \mathbf{HI}_1(\Delta, \Delta^p; 2^{p-1}; s_1, \cdots, s_p)$ is obtained by successively applying a map congruent to the square-root map to a direct factor of the polydisk (starting with the unit disk). Here we no longer have rigidity but the parameters of the latter space are completely known.

A problem on the structure of $\mathbf{HI}(\Delta, \Delta^p)$ which can be a testing ground for a deformation theory of holomorphic isometries is the following. By Ng [Ng1], whenever p is odd, any *bona fide* holomorphic isometry $f : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ of global sheeting number p is necessarily congruent to the p-th root map (under Cayley transformations and up to re-parametrization and target automorphisms). Thus, the expectation is that the p-th root map $\rho_p : \mathcal{H} \to \mathcal{H}^p$ is at least locally rigid for all integers p (and for p odd we know by Ng [Ng1] that ρ_p is globally rigid). It will thus be interesting to compute the space of possible infinitesimal and higher order variations of holomorphic isometries for the p-th root map ρ_p and to prove along this line that ρ_p is locally rigid for all p. In the case of $\mathbf{HI}_1(\Delta, \Delta^p; 2^{p-1}; s_1, \cdots, s_p)$, where $\max(s_i) = 2^{p-1}$, which can be completely described, a first problem that can be posed is to check whether the infinitesimal deformation of holomorphic isometries at a given map f in the space correspond exactly to those described by the known parameters.

Problem 5.1.3. *Do there exist non-standard holomorphic isometries of the complex unit ball B^n, $n \geq 2$, into some bounded symmetric domain Ω?*

For the unit disk Δ, composition of maps congruent to p-th root maps define continuous families of non-standard holomorphic isometries $f : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$. For the complex unit ball, B^n, $n \geq 2$, such analogues do not exist. In fact, by Mok [Mk4], for $n \geq 2$, any holomorphic isometry $f : (B^n, \lambda\, ds^2_{B^n}) \to (B^n,\, ds^2_{B^n}) \times \cdots \times (B^n,\, ds^2_{B^n})$ is necessarily totally geodesic (cf. Ng [Ng2] for rigidity results in some special cases where the image there are two Cartesian factors which are B^m with $m > n$). We have the rigidity phenomenon that all holomorphic isometries in $\mathbf{HI}(D, \Omega)$ are totally geodesic whenever D is irreducible and of rank ≥ 2, as a consequence of the proof of Hermitian metric rigidity (Mok [Mk1, 2]). It is tempting to believe that such a rigidity phenomenon persists when D is of rank 1 and of dimension ≥ 2, i.e., when $D = B^n$, $n \geq 2$. In this direction one may consider restrictions of $f \in \mathbf{HI}(B^n, \Omega)$ to totally geodesic holomorphic disks on B^n, thereby obtaining families of holomorphic isometries of the Poincaré disk into Ω, and examine the (im)possibility of such deformations in relation to the deformation theory of holomorphic isometries of the Poincaré disk. In this sense Problem 5.1.1 and Problem 5.1.3 may be related to each other.

(5.2) *On the second fundamental form and asymptotic behavior of holomorphic isometries of the Poincaré disk into bounded symmetric domains* In Mok [Mk6] and more recently in Mok-Ng [MN1] we have posed open problems regarding the second fundamental form and asymptotic behavior of holomorphic isometries of the Poincaré disk into bounded symmetric domains. We collect these problems in this survey, and reformulate some of the problems in view of new developments. We will only make brief comments on the questions and refer the reader to the discussion in [Mk6] and [MN1]. To start with we have the following reformulation of [Mk6, (2.3), Question 1].

Problem 5.2.1. *Let Ω be a bounded symmetric domain which is either irreducible or a Cartesian product of identical irreducible bounded symmetric domains, λ be a positive constant, and $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a non-standard holomorphic isometry of the Poincaré disk. Prove that F is of the first kind.*

Recall that a non-standard holomorphic isometry $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ is said to be of the first kind if and only if, denoting by φ the square of the norm of the second fundamental form σ, at a general point of $\partial\Delta$, extension of φ to a neighborhood of b vanishes precisely to the order 2. The calculation of Mok [Mk6, (2.2)] and the more precise calculations of Mok-Ng [MN2, (2.3)] of $\varphi = \|\sigma\|^2$ for the map $G : (\mathcal{H}, ds^2_{\mathcal{H}}) \to (\mathcal{H}_3, ds^2_{\mathcal{H}_3})$ shows that it is of the first kind (cf. Proposition 3.3.1 of the current article).

Concerning the algebraic extension of Graph(F) for a holomorphic isometry $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ we have the following problem, which is a reformulation of [Mk6, (2.3), Question 2].

Problem 5.2.2. *Let Ω be a bounded symmetric domain λ be a positive constant, and $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a non-standard holomorphic isometry of the Poincaré disk. Prove that F must develop a singularity at some boundary point $b \in \partial\Delta$, i.e., F does not extend holomorphically to a neighborhood D of the closed unit disk $\overline{\Delta}$.*

In [MN1] we have now shown that singularities must indeed develop in the case where the target domain is the polydisk. The proof makes use of the very special fact that, in the case of the polydisk, the extension beyond the boundary circle remains a holomorphic isometry when the exterior of the closed unit disk in the domain, and likewise the exterior of each of the factor unit disks in the target is also equipped with the Poincaré metric. In the general case where the target space is an arbitrary bounded symmetric domain Ω in its Harish-Chandra realization, when one exits a general boundary point on $\partial\Delta$, the image under the analytic continuation of the holomorphic isometry F traverses a region equipped naturally with a pseudo-Kähler metric, i.e., associated to a closed (1,1)-form which is non-degenerate but not necessarily positive definite. Although the extended map is still a holomorphic isometry in a formal sense, the behavior of such maps have not yet been understood.

In [MN1] we have formulated a number of questions motivated by the study of the asymptotic behavior of second fundamental forms and also by the explicit calculation of second fundamental forms of the p-th root maps ρ_p and of the map $G : \mathcal{H} \to \mathcal{H}_3$ into the Siegel upper half-plane of genus 3. We recall these problems. They correspond to [MN1, (2.4), Problems 1-3] in the same order, with a reformulation on the last problem. We refer the reader to [MN1] for details.

Problem 5.2.3. *Among holomorphic isometric embeddings of the Poincaré disk into polydisks characterize in terms of second fundamental forms of the embeddings those that are equivalent to the p-th root map up to re-parametrization on the Poincaré disk and up to automorphisms of the target polydisk.*

Here the calculations in [MN1] show that for the p-th root map ρ_p with second fundamental form σ_p, writing $\varphi_p(\tau) := \|\sigma_p(\tau)\|^2 = t^2 u_p(\tau)$, the function $\log u_p(\tau)$ is harmonic. Any non-standard holomorphic isometry of the Poincaré disk into the polydisk is known to be of the first kind (cf. Mok [Mk6, (2.1), Thm.2]). Writing in general $\|\sigma\|^2 = t^2 u$ the question is whether the harmonicity of $\log u$ characterizes the p-th root map.

Problem 5.2.4. *Among holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains characterize in terms of second fundamental forms of the embeddings those that are given by holomorphic isometric embeddings into polydisks.*

Assume that the answer to Problem 5.2.1 is positive, i.e., any non-standard holomorphic isometry $F : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ is necessarily of the first kind. Then, identifying the unit disk in the domain with the upper half-plane $\mathcal{H}$ via the inverse Cayley transform, and denoting by σ the second fundamental form of F we have $\|\sigma(\tau)\|^2 = t^2 u(\tau)$ in terms of the Euclidean coordinate $\tau = t + is$ on $\mathcal{H}$, where $u\big|_{\partial\mathcal{H}} \not\equiv 0$. By Theorem 3.2.1, if the image of F is contained in a totally geodesic polydisk we must have $\frac{\partial u}{\partial t}\big|_{\partial\mathcal{H}} \equiv 0$. The first question is whether the latter boundary differential equation is enough to characterize holomorphic isometries whose images are contained in totally geodesic polydisks. Recall that the map $G : (\mathcal{H}, ds^2_{\mathcal{H}}) \to (\mathcal{H}_3, ds^2_{\mathcal{H}_3})$ can be distinguished from holomorphic isometries into polydisks precisely because $\frac{\partial u}{\partial t}\big|_{\partial\mathcal{H}} \not\equiv 0$. At this stage the problem is experimental in nature and there is no overwhelming evidence for or against the statement. It can however be noted that universal higher order boundary differential equations on u may no longer exist because of error terms in the curvature expansion arising from the structural equation as given in ($\sharp$) in the proof of Theorem 3.1.1.

Problem 5.2.5. *Let Ω be an irreducible bounded symmetric domain of rank equal to 2, λ be a positive constant, and $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometry. Does there exist a totally geodesic bidisk $P \subset \Omega$ such that the image of f is contained in P?*

When the target domain Ω is assumed to be of rank 2, up to congruence so far there are no examples of non-standard holomorphic isometries $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ other than the square root map. It is tempting to believe that there is more rigidity in the case where Ω is of rank 2 since the curvature expansion arising from the structural equation as given in [Thm 3.1.1., ($\dagger$)] is exact.

(5.3) *On germs of holomorphic maps preserving invariant (p,p)-forms* We have tackled problems on germs of holomorphic maps $f : D \to \Omega$ between bounded symmetric domains which are either holomorphic isometries up to normalizing constants or measure-preserving maps. They are special cases of the following general problem.

Problem 5.3.1. *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded symmetric domain in their Harish-Chandra realizations. Let $\mu^{p,p}_D$ be an* $\mathrm{Aut}(D)$*−invariant (p,p)-form on D and*

$\nu_\Omega^{p,p}$ *be an* $\mathrm{Aut}(\Omega)-$*invariant* (p,p)*-form on* Ω. *Let* $\lambda > 0$ *be a real number, and* f : $(D;0) \to (\Omega;0)$ *be a germ of holomorphic map such that* $f^\star \nu_\Omega^{p,p} = \lambda \mu_D^{p,p}$. *Under what conditions is* f *necessarily totally geodesic.*

While the problems dealt with in this survey arise from dynamical problems in Arithmetic Geometry, the general problem above may not be directly related to such problems, and one motivation is rather the links of special cases of Problem 5.3.1 to problems on geometric structures related to irreducible bounded symmetric domains. The work of Mok-Ng [MN2] in the case of germs of measure-preserving maps involves a result on algebraic extension, a verification of the necessary non-degeneracy condition by curvature considerations, and evoking Alexander-type results, which are in the case of rank ≥ 2 related to Ochiai's results.

To formulate an example where a resolution of Problem 5.3.1 is related to the study of geometric structures recall that by Tsai [Ts], given D an irreducible bounded symmetric domain of rank $r \geq 2$ and Ω of rank at most equal to r it is known that, without additional differential-geometric conditions, any proper holomorphic map $F : D \to \Omega$ is necessarily totally geodesic (in which case Ω must also be of rank r). Recently, Mok [Mk] has given a scheme of proof of Tsai's result by exploiting the intermediate result that f must transform varieties of minimal rational tangents into minimal rational tangents, and proving a non-equidimensional analogues of Ochiai's Theorem (cf. Hong-Mok [HoM] for a general formulation of such analogues and their verifications for pairs of irreducible bounded symmetric domains, equivalently for pairs of irreducible Hermitian symmetric manifolds of the compact type). In place of the global requirement that $F : D \to \Omega$ is a proper holomorphic map, one may instead consider germs of holomorphic maps $f : (D;0) \to (\Omega;0)$ satisfying for instance $f^\star \omega_\Omega^p = \lambda \omega_D^p$, where ω_D, resp. ω_Ω, stands for the Kähler form of the Bergman metric on D, resp. Ω, and p is a positive integer. When $p = 1$ we have a germ of holomorphic isometry up to a normalizing constant, and the problem is solved from completely local considerations from Hermitian metric rigidity. On the other hand, when $2 \leq p \leq \dim(D)$, one can follow the scheme of proving algebraic extension of $\mathrm{Graph}(f)$ and deducing properness at a general boundary point by means of functional identities, as is done in Mok-Ng [MN2]. Here one encounters first of all the difficulty of verifying the non-degeneracy condition there needed for applying Huang's result. More precisely, the Hermitian holomorphic vector bundles $(\Lambda^p T_D, h_D)$ and $(\Lambda^p T_\Omega, h_\Omega)$, with the Hermitian metric h_D, resp. h_Ω, induced from the Bergman metric ds_D^2, resp. ds_Ω^2, is of seminegative and in general not of strictly negative curvature in the sense of Griffiths, and one has to study the locus of weakly pseudoconvex points on their respective unit sphere bundles to exclude the possibility that the image of the induced map on unit sphere bundles lie entirely on the locus of weakly pseudoconvex points. The problem at hand can also be considerably generalized when we allow Ω to be a Cartesian product where each individual factor is or rank $\leq r$, or furthermore to the case where the rank condition on the target domain is weakened, since it is expected that the hypothesis $f^\star \omega_\Omega^p = \lambda \omega_D^p$ is sufficiently rigid, so that, once analytic continuation and thereby properness at a good boundary point can be established, the resulting global map is likely to be more rigid than what one can deduce solely from properness.

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