Proper holomorphic mappings on flag domains of $SU(p,q)$-type on projective spaces

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Abstract

Rigidity for proper holomorphic mappings among $SU(p,q)$-type flag domains (also known as generalized balls) on projective spaces is obtained. We prove that the mappings are linear when the signature difference is not too large. One important ingredient of our proof is a linearity criterion of Feder about holomorphic immersions between projective spaces. Using extension techniques in several complex variables and carefully analyzing the structure of the moduli of linear subspaces in these flag domains, we are able to get holomorphic immersions between two projective subspaces of suitable dimensions and hence apply Feder's result.

1 Introduction

The objects of study in the present article are the domains in $\mathbb{P}^n$ defined by

$$\mathbb{D}_n^\ell = \left\{ [z_0, \ldots, z_n] \in \mathbb{P}^n : \sum_{j=0}^{\ell} |z_j|^2 > \sum_{j=\ell+1}^{n} |z_j|^2 \right\}$$

and the proper holomorphic mappings among them. They are examples of the so-called *flag domains* in $\mathbb{P}^n$ when the latter is regarded as a flag variety. More explicitly, they are open orbits of the real forms $SU(\ell+1, n-\ell)$ of the complex simple Lie group $SL(n+1,\mathbb{C})$ when both of which act on $\mathbb{P}^n$ as biholomorphisms.

The domain $\mathbb{D}_n^0$ is just the complex unit n-ball embedded in $\mathbb{P}^n$ and there has been an extensive literature in the study of their proper holomorphic mappings in the last couple of decades. For a survey, see [4]. In general, when the codimension is high, the set of proper holomorphic mappings between complex unit balls is large and difficult to determine. On the other hand, in the recent works of Baouendi-Huang [2] and Baouendi-Ebenfelt-Huang [1], the domains $\mathbb{D}_n^\ell$ with $\ell \geq 1$ and the associated holomorphic mappings are studied by methods in Cauchy-Riemann geometry. It appears that there is in general much more rigidity when $\ell \geq 1$. Indeed, there is one essential difference between the complex unit n-ball and the domains $\mathbb{D}_n^\ell$ with $\ell \geq 1$, for the latter contain linear subspaces of $\mathbb{P}^n$. Motivated by this the author of the present article studied in [6] the domains $\mathbb{D}_n^\ell$, $\ell \geq 1$, and their generalizations in Grassmannians by exploiting the structure of the moduli spaces of compact complex analytic subvarieties. Rigidity results analogous to those of [2] are obtained in a more geometric way.

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We will follow the terminologies in [2] and [1] to call ℓ the *signature* of the domain $\mathbb{D}_n^\ell$. As far as rigidity of holomorphic mappings among those domains is concerned, the determining factor should be the difference in signatures rather than the codimension. This is illustrated by, for instance, in [2] that when the domain and target are of the same signature, then any local proper holomorphic map is the restriction of a linear embedding between the ambient projective spaces. On the other hand, in [1], Baouendi-Ebenfelt-Huang studied the situations with a small signature difference. Together with other results, they proved that there is partial rigidity for local proper holomorphic mappings $h : U \subset \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ when $1 \leq \ell < n/2$, $1 \leq \ell' < m/2$ and $\ell' \leq 2\ell - 1$. Furthermore, simple examples can be constructed explicitly to demonstrate that their partial rigidity is best possible and in particular we cannot have full rigidity for local proper holomorphic mappings, i.e. there are local proper holomorphic maps which are not the restrictions of linear embeddings between projective spaces.

The main purpose of the current article is to prove the following theorem regarding the rigidity for *global* proper holomorphic mappings among $\mathbb{D}_n^\ell$ when the difference in signatures is small.

Main Theorem. *Let $1 \leq \ell < n/2$, $1 \leq \ell' < m/2$ and $f : \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ be a proper holomorphic map. If $\ell' \leq 2\ell - 1$, then f extends to a linear embedding of $\mathbb{P}^n$ into $\mathbb{P}^m$.*

Remarks. (1) In [1], it has been proved under the same assumptions that the image of f is contained in a projective linear subspace of dimension $n + (\ell' - \ell)$. Our proof is independent of this result. **(2)** In the theorem, when $\ell = 1$, which also forces $\ell' = 1$, the above result is already obtained in [2] and a more geometric proof is given in [6]. **(3)** Without further assumptions, the condition $\ell' \leq 2\ell - 1$ is necessary to guarantee linearity. This is illustrated by the following non-linear mapping from $\mathbb{P}^3$ to $\mathbb{P}^5$ defined by

$$[z_0, z_1, z_2, z_3] \mapsto [z_0^2, \sqrt{2}z_0z_1, z_1^2, z_2^2, \sqrt{2}z_2z_3, z_3^2].$$

It is easy to see that this map restricts to a proper holomorphic map from $\mathbb{D}_3^1$ to $\mathbb{D}_5^2$. This example is take from [2].

We now discuss the scheme of proof for the Main Theorem. Our proof relies on the following linearity criterion of Feder [3].

Feder's Theorem. *Let $h : \mathbb{P}^\ell \to \mathbb{P}^{\ell'}$ be a holomorphic immersion. If $\ell' \leq 2\ell - 1$, then h is linear.*

In order to apply Feder's theorem, we have to show two things: (i) there exists some ℓ-dimensional projective linear subspace $L \subset \mathbb{D}_n^\ell$ on which the restriction of f is an immersion; (ii) the image $f(L)$ is contained in some ℓ'-dimensional projective linear subspace in $\mathbb{P}^m$.

We prove (i) by first showing that f extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$ and this is achieved by standard Hartogs' extension techniques in several complex variables. From that we can furthermore deduce the finiteness of f on $\mathbb{D}_n^\ell$. We then establish our key Proposition 3.4. Roughly speaking, it allows us to extract from a finite holomorphic mapping some holomorphic immersions of linear subspaces of sufficiently high dimension. The statement is obtained essentially by analyzing the kernel of the differential of f.

For (ii), we basically follow the same approach as in [6]. We first prove that ℓ-dimensional projective linear subspaces in the boundary $\partial\mathbb{D}_n^\ell$ are mapped to ℓ'-dimensional projective linear subspaces in the target space due to the properness of f. Then by analyzing the moduli space of these projective linear subspaces we prove that the boundary behaviour can be carried over to the interior and hence (ii).

After establishing (i) and (ii), Feder's Theorem now says that the restriction of f on some ℓ-dimensional projective linear subspace is linear and thus $\deg(f) = 1$.

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2 Linear subspaces of $\mathbb{D}_n^\ell$ and foliations

In [6], the structure of the set of projective linear subspaces contained in $\mathbb{D}_n^\ell$ is studied and it is crucial to the present article also. In order to make the article more self-contained, we will briefly recall some relevant facts in this section.

For a point $[\mathbf{z}] = [z_0, \ldots, z_n] \in \mathbb{P}^n$, we split its homogeneous coordinates as $[\mathbf{z}] = [\mathbf{z}', \mathbf{z}'']_\ell$, where $\mathbf{z}' = (z_0, \ldots, z_\ell)$ and $\mathbf{z}'' = (z_{\ell+1}, \ldots, z_n)$. We denote the closure of $\mathbb{D}_n^\ell$ in $\mathbb{P}^n$ by $\overline{\mathbb{D}}_n^\ell$. We first recall the definition of type-I irreducible bounded symmetric domains and its compact dual the complex Grassmannians.

Definition 2.1. *Let $M(p, q; \mathbb{C})$ be the set of $p \times q$ complex matrices. We identify $M(p, q; \mathbb{C})$ as $\mathbb{C}^{pq}$. The type-I irreducible bounded symmetric domain $\Omega_{p,q}$ is the domain in $\mathbb{C}^{pq}$ defined by $\Omega_{p,q} = \{A \in M(p, q; \mathbb{C}) : I - AA^H > 0\}$, where A^H denotes the Hermitian transpose of A. As a Hermitian symmetric space, the compact dual of $\Omega_{p,q}$ is the complex Grassmannian of p-dimensional linear subspaces of $\mathbb{C}^{p+q}$ and we denote it by $G_{p,q}$.*

Proposition 2.2. *$\mathbb{D}_n^\ell$ (resp. $\overline{\mathbb{D}}_n^\ell$) contains a family of ℓ-dimensional projective linear subspaces. They are maximal compact complex analytic subvarieties in $\mathbb{D}_n^\ell$ (resp. $\overline{\mathbb{D}}_n^\ell$). Moreover, the set of all such linear subspaces is parametrized by $\Omega_{\ell+1,n-\ell}$ (resp. $\overline{\Omega}_{\ell+1,n-\ell}$). Furthermore, if $\ell < n/2$, the boundary $\partial\mathbb{D}_n^\ell$ also contains a family of ℓ-dimensional projective linear subspaces and the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ parametrizes precisely those contained in the boundary.*

Proof. The complete proof is given in [6] (Proposition 2.2, Proposition 2.3 and Lemma 2.4 therein). Here we just give the explicit parametrization of the linear subspaces. Let $A \in M(\ell + 1, n - \ell; \mathbb{C})$. Consider the ℓ-dimensional linear subspace

$$\{[\mathbf{z}', \mathbf{z}'']_\ell \in \mathbb{P}^n : \mathbf{z}'' = \mathbf{z}'A\} \cong \mathbb{P}^\ell \subset \mathbb{P}^n.$$

Then as $\mathbf{z}'\mathbf{z}'^H > \mathbf{z}'AA^H\mathbf{z}'^H$ for all $\mathbf{z}'$ if and only if $I - AA^H > 0$, we see that such the above linear subspace is contained in $\mathbb{D}_n^\ell$ if and only if $A \in \Omega_{\ell+1,n-\ell}$. The parametrization extends to the respective closures in the natural way. Note that when $\ell < n/2$, the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ is just the set of all matrices A such that $AA^H = I$. For such an A, the above linear subspace will then be contained completely in $\partial\mathbb{D}_n^\ell$. $\square$

In the followings, by an ℓ-Grassmann bundle of a manifold M, denoted by $G_\ell TM$, we mean the bundle of Grassmannians of the ℓ-planes in each tangent space on M. We denote the Grassmannian of ℓ-planes in the tangent space at $p \in M$ by $G_\ell T_pM$.

Proposition 2.3. *Let $\pi : G_\ell T\mathbb{D}_n^\ell \to \mathbb{D}_n^\ell$ be the ℓ-Grassmann bundle of $\mathbb{D}_n^\ell$. There is an open set $V_n^\ell \subset G_\ell T\mathbb{D}_n^\ell$, $\pi(V_n^\ell) = \mathbb{D}_n^\ell$ such that V_n^ℓ is a trivial holomorphic $\mathbb{P}^\ell$-bundle over $\Omega_{\ell+1,n-\ell}$.*

Proof. Fix a point $p \in \mathbb{D}_n^\ell$. Since $\mathbb{P}^\ell$ is compact and $\mathbb{D}_n^\ell$ is open, we deduce that there is an open set $U_p \subset G_\ell T_p \mathbb{D}_n^\ell$ consisting of precisely all the tangent ℓ-planes which are tangent to some ℓ-dimensional projective linear subspace contained in $\mathbb{D}_n^\ell$. Since the tangent plane at a point uniquely determine the linear subspace, combining with Proposition 2.2, the statements in the proposition are immediate. $\square$

The above $\mathbb{P}^\ell$-foliation of V_n^ℓ is just the universal family of ℓ-dimensional projective linear subspaces in $\mathbb{D}_n^\ell$ and we denote it by $\Pi : V_n^\ell \to \Omega_{\ell+1,n-\ell}$. Furthermore, it is simply the restriction of the standard universal family of ℓ-dimensional projective linear subspaces in $\mathbb{P}^n$ and we also denote it by $\Pi : G_\ell T\mathbb{P}^n \to G_{\ell+1,n-\ell}$.

Lemma 2.4. *If $\ell < n/2$, then any germ of complex submanifold in $\partial\mathbb{D}_n^\ell$ must lie in an ℓ-dimensional projective linear subspace contained in $\partial\mathbb{D}_n^\ell$.*

Proof. In fact, by [8], the ℓ-dimensional projective linear subspace contained in $\partial\mathbb{D}_n^\ell$ are the *holomorphic arc components* or *boundary components* of $\partial\mathbb{D}_n^\ell$, whose defining properties are precisely the statement in the lemma. For a more elementary proof of the lemma, see [6]. $\square$

3 Finite rationality and immersion

We will first prove that every proper holomorphic map $f : \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$, $\ell \geq 1$, extends to a finite rational map. We begin with an elementary lemma in algebraic geometry.

Lemma 3.1. *Let $h : \mathbb{P}^n \to \mathbb{P}^m$ be a rational map. If $S \subset \mathbb{P}^n$ is a compact complex analytic subvariety in the domain of h and h is constant on S, then S is a finite set of points.*

Proof. By composing h with a linear transformation, we may assume that $h(S) = [1, 0, \ldots, 0]$. Let $h = [h_0, \ldots, h_m]$, where all h_j are polynomials of the same degree. By assumption, for $1 \leq j \leq m$, we have $h_j|_S \equiv 0$. If S is of positive dimension, then the zero set of h_0 must intersect S and hence S intersects the set of indeterminacy of h and this violates our initial assumption. Thus, S is finite set of points. $\square$

Proposition 3.2. *f extends to a rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$. Furthermore, f is a finite map.*

Proof. For each $j \in \{0, \ldots, n\}$, let $U_j \subset \mathbb{P}^n$ be the open set defined by $z_j \neq 0$. Note that the complement $\mathbb{P}^n \setminus \mathbb{D}_n^\ell$ is the domain defined by $\sum_{j=0}^{\ell} |z_j|^2 \leq \sum_{j=\ell+1}^{n} |z_j|^2$. In particular, we have $\mathbb{P}^n \setminus \mathbb{D}_n^\ell \subset \bigcup_{j=\ell+1}^{n} U_j$.

Hence, it suffices to establish the meromorphic extension of the component functions of f (as meromorphic functions) on U_j, for each $j \in \{\ell+1, \ldots, n\}$. Now fix $j \in \{\ell+1, , \ldots, n\}$, then in terms of the standard inhomogeneous coordinates $(w_1, \ldots, w_n)$ on U_j, the domain $\mathbb{D}_n^\ell \cap U_j$ is defined by the equation

$$\sum_{k=1}^{\ell+1} |w_k|^2 > \sum_{k=\ell+2}^{n} |w_k|^2.$$

If we decompose $U_j \cong \mathbb{C}^n = \mathbb{C}^{\ell+1} \times \mathbb{C}^{n-\ell-1}$, then for every relatively compact open set $V \Subset \mathbb{C}^{n-\ell-1}$ containing the origin, the component functions of f extend meromorphically over $\mathbb{C}^{\ell+1} \times V \subset \mathbb{C}^n$ by Hartogs' extension [7] since $\ell + 1 \geq 2$. In other words, f extends to a meromorphic map from U_j to $\mathbb{P}^m$. We have thereby established the meromorphic extension of f on each U_j, $j \in \{\ell+1, \ldots, n\}$ and hence f extends to a meromorphic and hence rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$.

Now since $f : \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ is proper and holomorphic, for every $p \in \mathbb{D}_m^{\ell'}$, the preimage $f^{-1}(p) \subset \mathbb{D}_n^\ell$ is a compact complex analytic subvariety in $\mathbb{P}^n$ and hence is a finite set by Lemma 3.1. Thus, f is a finite map. □

In the remaining of this section, we will prove the key proposition of the present article. It is by this proposition that we can extract from f holomorphic immersions of projective spaces of sufficiently high dimension. We need the following dimension formula in its proof.

Lemma 3.3. *Let V be an n-dimensional complex vector space and $G_V(\ell)$ be the Grassmannian of ℓ-dimensional vector subspaces of V. Fix a k-dimensional vector subspace $W \subset V$ and denote by $\mathcal{W} \subset G_V(\ell)$ the irreducible analytic subvariety consisting of elements having non-trivial intersection with W. Then*

$$\dim(\mathcal{W}) = \begin{cases} (k-1) + (\ell-1)(n-\ell) & \text{if } \; k \leq n-\ell \\ \ell(n-\ell) & \text{if } \; k > n-\ell. \end{cases}$$

Proof. When $k > n - \ell$, the lemma is trivial since in this case $\mathcal{W} = G_V(\ell)$ and $\dim(G_V(\ell)) = \ell(n-\ell)$.

Suppose now $k \leq n - \ell$, then $\mathcal{W}$ is simply the closure of a Schubert cell in $G_V(\ell)$ and one can follow the procedures in [5] (Chapter 1, Section 5) to calculate its dimension. For the convenience of the reader, we provide an elementary proof here.

We may simply take $V = \mathbb{C}^n$. Let $E_\ell = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_{\ell+1} = z_{\ell+2} = \cdots = z_n = 0\}$. Let $\pi : \mathbb{C}^n \to E_\ell$ be the canonical projection and $U = \{Q \in G_{\mathbb{C}^n}(\ell) : \pi(Q) = E_\ell\}$. Then U is a standard Euclidean coordinate chart in $G_{\mathbb{C}^n}(\ell)$ and $U \cong \mathbb{C}^{\ell(n-\ell)}$. Note that every $Q \in U$ can be represented by an $\ell \times n$ matrix of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & z_{1,1} & \cdots & z_{1,(n-\ell)} \\ 0 & 1 & \cdots & 0 & z_{2,1} & \cdots & z_{2,(n-\ell)} \\ & & \ddots & & & \vdots & \\ 0 & 0 & \cdots & 1 & z_{\ell,1} & \cdots & z_{\ell,(n-\ell)} \end{bmatrix},$$

in which the rows constitute a basis of Q. The $z_{j,k}$ are precisely the standard Euclidean coordinates on U.

Add a basis of W as rows to the above matrix and we get a $(k+\ell) \times n$ matrix. Now it is easy to see that the condition that $\dim(Q \cap W) \geq 1$ is given by the vanishing of $n - (k+\ell) + 1$ minors of size $(k+\ell) \times (k+\ell)$. Equivalently, $\mathcal{W} \cap U$ is defined by $n - (k+\ell) + 1$ independent algebraic equations and hence

$$\begin{aligned} \dim(\mathcal{W}) &= \dim(G_{\mathbb{C}^n}(\ell)) - [n - (k+\ell) + 1] \\ &= \ell(n-\ell) - [n - (k+\ell) + 1] \\ &= (k-1) + (\ell-1)(n-\ell). \end{aligned}$$

□

Proposition 3.4. *Let $g : \mathbb{P}^n \to \mathbb{P}^m$ be a finite rational map. Then for $\ell < n/2$, the restriction of g on a general ℓ-dimensional projective linear subspace is a holomorphic immersion.*

Proof. Let $X \subset \mathbb{P}^n$ be the indeterminacy of g and $U := \mathbb{P}^n \setminus X$. We still denote the restriction of g on U as g and thus $g : U \to \mathbb{P}^m$ is a finite holomorphic map.

Let $dg : TU \to T\mathbb{P}^m$ be the differential of g. Since g is finite, in particular not totally degenerate, dg induces naturally a meromorphic map $[dg]$ from $G_\ell TU$ (the ℓ-Grassmann bundle of U) to $G_\ell T\mathbb{P}^m$. Let $Z \subset G_\ell TU$ be set of the indeterminacy of $[dg]$. We are going to show that the complex analytic subvariety Z is of dimension less than $(\ell+1)(n-\ell)$. Assume this dimension estimate for the moment. Now let $\Pi : G_\ell T\mathbb{P}^n \to G_{\ell+1,n-\ell}$ be the universal family of ℓ-dimensional projective linear subspaces in $\mathbb{P}^n$ (see Proposition 2.3 and the paragraph thereafter). Note that Π is proper and hence $\Pi(Z) \subset G_{\ell+1,n-\ell}$ is a locally-closed complex analytic subvariety. But $\dim(G_{\ell+1,n-\ell}) = (\ell+1)(n-\ell)$ and therefore by our dimension estimate $\Pi(Z)$ is not dense in $G_{\ell+1,n-\ell}$. Thus, for a general point $q \in G_{\ell+1,n-\ell}$, the differential $[dg]$ is well defined on $\Pi^{-1}(q) \cong \mathbb{P}^\ell$. It is equivalent to saying that the restriction of g on the ℓ-dimensional projective linear subspace corresponding to $\Pi^{-1}(q)$ is an immersion and the proof is complete.

We now prove the dimension estimate.

For $k \in \{1, \ldots, n\}$, let $I_k \subset U$ be the set of points where the kernel of dg (as a linear map at each individual point) is of dimension at least k. As g is finite, $I_k \subset U$ is a complex analytic subvariety of dimension at most $n-k$ and $I_n \subset \cdots \subset I_1 = \pi(Z)$, where $\pi : G_\ell TU \to U$ is the canonical projection. Now for every $p \in U$, the fibre of Z over p (i.e. $Z \cap G_\ell T_p U$) is the set of ℓ-planes in T_pU having a non-trivial intersection with the kernel of dg at p. By Lemma 3.3,

$$\dim(Z \cap G_\ell T_p U) = \begin{cases} (k-1) + (\ell-1)(n-\ell) & \text{if } \ p \in I_k \setminus I_{k+1}, \ \ k \in \{1, \ldots, n-\ell\} \\ \ell(n-\ell) & \text{if } \ p \in I_{n-\ell+1}. \end{cases} \tag{$*$}$$

Let $Z_k = \pi^{-1}(I_k) \subset Z$, where $1 \le k \le n$. It is clear that each Z_k is also a complex analytic subvariety of $G_\ell TU$ and $Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 = Z$. We start from Z_n. Considering the projection π, we deduce that

$$\dim(Z_n) \le \dim(\text{fibre}) + \dim(\text{base}) = \dim(Z \cap G_\ell T_p U) + \dim(I_n),$$

where $p \in I_n$ is arbitrary. Consequently, we have

$$\dim(Z_n) \le \ell(n-\ell) + 0 < (\ell+1)(n-\ell)$$

by $(*)$. Next, $Z_{n-1} \setminus Z_n$ is a locally closed complex analytic subvariety and its dimension, by similar reasoning, is at most equal to

$$\ell(n-\ell) + 1 < \ell(n-\ell) + (n-\ell) = (\ell+1)(n-\ell)$$

as $\ell < n/2$. Hence

$$\dim(Z_{n-1}) < (\ell+1)(n-\ell).$$

With $\ell < n/2$, we have analogously for every $k \in \{1, \ldots, \ell-1\}$,

$$\ell(n-\ell) + k < \ell(n-\ell) + (n-\ell) = (\ell+1)(n-\ell).$$

Thus we can repeat the above argument to conclude that

$$\dim(Z_n) \le \dim(Z_{n-1}) \le \cdots \le \dim(Z_{n-\ell+1}) < (\ell+1)(n-\ell).$$

Now for $Z_{n-\ell} \setminus Z_{n-\ell+1}$, it is a locally closed complex analytic subvariety and

$$\begin{aligned}
\dim(Z_{n-\ell} \setminus Z_{n-\ell+1}) &\leq \dim(Z \cap G_\ell T_p U) + \dim(I_{n-\ell}) \qquad (p \in I_{n-\ell}) \\
&\leq [(n-\ell-1) + (\ell-1)(n-\ell)] + \ell \\
&= (n-1) + (\ell-1)(n-\ell) \\
&< (\ell+1)(n-\ell),
\end{aligned}$$

where the term in the square bracket is by $(*)$ and the last inequality is again due to the assumption that $\ell < n/2$. Consequently,

$$\dim(Z_{n-\ell}) < (\ell+1)(n-\ell).$$

By repeating the argument, we get for every $k \in \{1, \ldots, n-\ell\}$ that $\dim(Z_k) < (\ell+1)(n-\ell)$ and hence we have established $\dim(Z) < (\ell+1)(n-\ell)$. □

It is for the sake of notational simplicity that we work with global mappings in Proposition 3.4. Indeed, we can simply restrict the whole argument on open subsets and obtain the following local version.

Proposition 3.5. *Let $D \subset \mathbb{P}^n$ be an open set, M be a complex manifold and $g : D \to M$ be a finite holomorphic map. Let $\ell < n/2$ and $L \subset \mathbb{P}^n$ be an arbitrary ℓ-dimensional projective linear subspace intersecting D. Then for a general choice of L, the restriction of g on $L \cap D$ is a holomorphic immersion.*

4 Proof of the Main Theorem

Throughout this section, we let $f : \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$ be a proper holomorphic map, $\ell \geq 1$.

Proposition 4.1. *If $\ell < n/2$ and $\ell' < m/2$, then for each ℓ-dimensional projective linear subspace $L \subset \mathbb{D}_n^\ell$ (as described in Proposition 2.2), we have $f(L) \subset L'$, where L' is some ℓ'-dimensional linear subspace in the target $\mathbb{P}^m$.*

Proof. By Proposition 3.2, f extends as a rational map and hence the induced meromorphic map $[df] : G_\ell T\mathbb{D}_n^\ell \to G_\ell T\mathbb{P}^m$ extends to an open neighborhood of $G_\ell T\mathbb{D}_n^\ell$ and in particular to an open neighborhood of the universal family $\Pi : V_n^\ell \to \Omega_{\ell+1,n-\ell}$ (see the paragraph after Proposition 2.3). More precisely, we mean $[df]$ extends to an open neighborhood $W \supset \overline{V}_n^\ell$ and $\Pi(W) = U$ is some open neighborhood of $\overline{\Omega}_{\ell+1,n-\ell}$.

Now we consider the composition $f^\sharp := \pi \circ [df]$, where $\pi : G_\ell T\mathbb{P}^m \to \mathbb{P}^m$ is the canonical projection. Take a general point b in the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ so that $[df]$ and hence $f^\sharp$ is defined on the ℓ-dimensional projective linear subspace over the point b (i.e. $\Pi^{-1}(b)$). By the properness of f and Lemma 2.4, we have $f^\sharp(\Pi^{-1}(b)) \subset \partial\mathbb{D}_m^\ell$ and hence $f^\sharp(\Pi^{-1}(b)) \subset L'_b$ for some ℓ'-dimensional projective linear subspace $L'_b \subset \mathbb{P}^m$. In other words, on the holomorphic $\mathbb{P}^\ell$-bundle $\Pi : W \to U \supset \overline{\Omega}_{\ell+1,n-\ell}$, the map $f^\sharp$ maps the general fibres over the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ to ℓ'-dimensional projective linear subspaces in $\mathbb{P}^m$. Note that this is an analytic condition, i.e. it can be expressed in terms of the vanishing of a set of holomorphic functions in local coordinates (e.g. some degeneracy conditions on a set of vertical derivatives on the base). Now we have a set of holomorphic functions vanish on the intersection of an open set and the Shilov boundary of $\Omega_{\ell+1,n-\ell}$ and therefore they must vanish on the entire open set. (For a proof of this, see [6], Lemma 2.9 therein.) Hence, we conclude this degeneracy property also holds for the general fibres in the interior, i.e. $f^\sharp$ maps general fibres to ℓ'-dimensional projective linear subspaces in $\mathbb{P}^m$. This precisely means that f maps general and hence all ℓ-dimensional projective linear subspaces into ℓ'-dimensional projective linear subspaces. □

We are now ready to prove the Main Theorem.

Proof of the Main Theorem. By Proposition 3.2 together with Proposition 3.4, there exists an ℓ-dimensional projective linear subspace $L_0 \subset \mathbb{D}_n^\ell$ on which the restriction of f is an immersion. However, by Proposition 4.1, $f(L_0)$ is contained in some ℓ'-dimensional projective linear subspace in $\mathbb{P}^m$. As $\ell' \leq 2\ell - 1$, we have by Feder's Theorem that the restriction of f on L_0 is linear. Therefore, we can now conclude that as a rational map the degree of f is one. Thus, f is linear. □

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