

**STABLE TAMENESS OF AUTOMORPHISMS OF  
 $F\langle x, y, z \rangle$  FIXING  $z$**

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**Abstract.** It is proved that every  $z$ -automorphism ( $z$ -coordinates, respectively) of the free associative algebra  $F\langle x, y, z \rangle$  over an arbitrary field  $F$  is stably tame.

**1. Introduction and main results**

An  $F$ -automorphism of a free associative algebra  $F\langle x_1, \dots, x_n \rangle$  (a polynomial algebra  $F[x_1, \dots, x_n]$ ) is *elementary* if it fixes all variables except one. An  $F$ -automorphism is *tame* if it is product of elementary automorphisms. An  $F$ -automorphisms  $(f_1, \dots, f_n)$  of  $F\langle x_1, \dots, x_n \rangle$  ( $F[x_1, \dots, x_n]$ ) is *stably tame* if there exists a nonnegative integer  $m$ , the automorphism  $(f_1, \dots, f_n, x_{n+1}, \dots, x_{n+m})$  of  $F\langle x_1, \dots, x_{n+m} \rangle$  ( $F[x_1, \dots, x_{n+m}]$ ) is tame.

From now on all automorphisms are  $F$ -automorphisms.

Whether every automorphism of a free associative algebra (polynomial algebra) is stably tame, is a long-standing and interesting open question.

In [5], stable tameness of some special kind of automorphisms of polynomial and free associative algebras were obtained. In [6], it is proved that every fixing  $z$  automorphism of the polynomial algebra  $F[x, y, z]$

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over a field  $F$  of characteristic zero is stably tame, among other things. It is the first big step for attacking the stably tameness problem.

In this paper, based our recent result of the lifting problem, we prove the following

**Theorem 1.1.** *Every fixing  $z$  automorphism of the free associative algebra  $F\langle x, y, z \rangle$  over an arbitrary field  $F$  is stably tame and becomes tame after adding one variable.*

A polynomial  $f \in F\langle x_1, \dots, x_n \rangle$  is a *coordinate* if  $(f, f_2, \dots, f_n)$  is an automorphism for some  $f_2, \dots, f_n \in F\langle x_1, \dots, x_n \rangle$ . The coordinate  $f$  is stably tame if  $(f, f_2, \dots, f_n)$  is stably tame. A coordinate  $f$  is a  *$x_n$ -coordinate* if there exists an automorphism  $(f, f_2, \dots, f_{n-1}, x_n)$ .

As a direct consequence of Theorem 1.1, we obtain

**Theorem 1.2.** *Every  $z$ -coordinate of  $F\langle x, y, z \rangle$  is stably tame.*

## 2. Proofs

To prove Theorem 1.1, we only need to prove the following

**Theorem 2.1.** *For every automorphism  $(f, g)$  in  $\text{Aut}_{F[z]} F\langle x, y, z \rangle$ ,  $(f, g, t)$  is a tame automorphism in  $\text{Aut}_{F[z]} F\langle x, y, z, t \rangle$ .*

To prove Theorem 2.1, we need some preliminaries.

**Theorem 2.2.** *An automorphism  $(f, g)$  in  $\text{Aut}_{F[z]} F\langle x, y, z \rangle$ , can be canonically decomposed as product of the following type of automorphisms: i) Linear automorphisms in  $\text{Aut}_{F[z]} F\langle x, y, z \rangle$ ; ii) Automorphisms which can be obtained by an elementary automorphism in*

$$\text{Aut}_{F[z]} F\langle x, y, z \rangle,$$

*conjugated by a linear automorphism in*

$$\text{Aut}_{F(z)} F(z) *_F F\langle x, y \rangle$$

*Proof.* It is Theorem 3.4 in [1]. □

**Lemma 2.3.** *Suppose a polynomial  $f \in F\langle x, y, z \rangle$  is neither a left multiple nor a right multiple of any nontrivial polynomial in  $F[z]\setminus F$ . Then  $(F(z) * F[f]) \cap (F\langle x, y, z \rangle) = F(z) * F[f]$ .*

*Proof.* Suppose  $R \in F(z) * F[f] \setminus F\langle x, y, z \rangle = (F(z) * F[f]) \setminus F$ . We need to prove that  $R \in F(z) * F\langle x, y \rangle \setminus F\langle x, y, z \rangle$ , i.e.  $R \notin F\langle x, y, z \rangle$ .

Let  $\{M_i\}_{i \in I}$  be an arbitrary  $F$ -basis of  $F(z) * F[f]$ . As  $R \in F(z) * F[f]$ , it can be expressed uniquely

$$R = \sum_i \alpha_i f M_i,$$

where  $\alpha_i \in F(z)$ .

It is easy to see that if  $\alpha_k \in F(z) \setminus F[z]$  for some  $k$ , then  $R \notin F\langle x, y, z \rangle$ . Hence we may assume that  $\alpha_k \in F[z]$  for all  $k$ . Then  $R$  has the form  $R = \sum_{k=0}^n \gamma_k z^k f N_k$ ,  $\gamma_k \in F$ . The set  $\{z^k\}_{k=0}^\infty$  can be completed to a basis  $\{e_i\}$  of  $F(z)$  as a vector space over  $F$ , and every element  $x$  in the coproduct  $F(z) * F[f]$  can be expressed as  $x = e_i f h_i$  in canonical way. In our situation  $h_i \neq 0$  only for  $e_i \in \{z^k\}_{k=0}^\infty$ . Hence all  $N_k \in F(z) * F[f]$ . On the other hand, if some of  $N_k \notin F\langle x, y, z \rangle$ , then  $f N_k \notin F\langle x, y, z \rangle$ , because  $f$  is not right-divisible by any polynomial from  $F[z] \setminus F$ . In this situation  $R = \sum_{k=0}^n \gamma_k z^k f N_k \notin F\langle x, y, z \rangle$ . Hence for all  $k$

$$(N_k \in F(z) * F[f]) \cap (F\langle x, y, z \rangle) = F(z) * F[f]$$

and  $\deg(N_k) < \deg(R)$ . We conclude by induction on the degree of  $R$ .  $\square$

Let  $z_l$  denote the left multiplication operator on  $z$ ,  $z_r$  the right multiplication operator. An automorphism  $\psi$  in  $\text{Aut}_{F[z]} F\langle x, y, z \rangle$  linear in both  $x$  and  $y$  has the following form:  $\psi : x \rightarrow a_{11}x + a_{12}y, y \rightarrow a_{21}x + a_{22}y$  where  $a_{ij} \in F[z_l, z_r]$ . It should be pointed out, the study of such automorphisms is equivalent to the study of invertible  $2 \times 2$  matrices over the polynomial ring of two commuting variables  $z_l$  and  $z_r$  over a field.

**Lemma 2.4.** *Let  $\psi : x \rightarrow a_{11}x + a_{12}y, y \rightarrow a_{21}x + a_{22}y$  be a linear automorphism of  $F(z_l, z_r)\langle x, y \rangle$  where  $a_{ij} \in F(z_l, z_r)$  and let  $\varphi : x \rightarrow$*

$x, y + Q(x)$  be an elementary  $z$ -automorphism of  $F\langle x, y, z \rangle$ . Suppose  $\phi = \psi \circ \varphi \circ \psi^{-1} \in \text{Aut}_{F[z]} F\langle x, y, z \rangle$ . Then it has the following form

$$\phi : x \rightarrow x + bh(ax + by), \quad y \rightarrow y - ah(ax + by); \quad a, b \in F[z_l, z_r],$$

where  $h(t) \in F\langle z, t \rangle$ .

*Proof.* Let  $\alpha \in F[z_l, z_r]$  be the least common multiple of the denominators of  $a_{21}, a_{22}$ ,  $a = a_{21}\alpha, b = a_{22}\alpha$ . Then

$$\phi : x \rightarrow x + bQ((ax + by)/\alpha), \quad y \rightarrow y - aQ((ax + by)/\alpha);$$

where  $a, b \in F[z_l, z_r]$ .

As  $a, b$  are relatively prime,  $Q((ax + by)/\alpha) \in F\langle x, y, z \rangle$ . Hence by Lemma 2.3, the coefficients of  $Q((ax + by)/\alpha) \in F\langle z, ax + by \rangle$ . Therefore  $Q((ax + by)/\alpha)$  must have the form  $h(ax + by)$  for some  $h(t) \in F\langle z, t \rangle$ .  $\square$

**Lemma 2.5.** *Let  $\psi$  be a  $z$ -automorphism of  $F\langle x, y, z \rangle$  in the form*

$$x \rightarrow x + bh(ax + by), \quad y \rightarrow y - ah(ax + by); \quad a, b \in F[z_l, z_r]$$

*for some polynomial  $h(t) \in F\langle z, t \rangle$ . Then it is stably tame and becomes tame after adding one variable.*

*Proof.* Based on the method of Smith [5],

$$(x + bh(ax + by), y - ah(ax + by), t) = \\ (x, y, t - h(ax + by))(x - bt, y + at, t)(x, y, z, t + h(ax + by))(x + bt, y - at, t).$$

$\square$

**Remark 2.6.** The general Anick type automorphisms (see [3, 4])

$$(x + zg(xz - zy, z), y + g(xz - zy, z)z),$$

in  $\text{Aut}_{F[z]} F\langle x, y, z \rangle$ , with an arbitrary polynomial  $g(t, s) \in F\langle t, s \rangle$ , are obviously covered by Lemma 2.5.

**Proof of Theorem 1.1.** First, an automorphism of Type i) in Theorem 2.1 is stably tame and becomes tame after adding one variable, according to [2].

Now, suppose we have an automorphism of Type ii) in Theorem 2.1, we are done by Lemma 2.5.  $\square$

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### REFERENCES

- [1] A.Belov-Kanel and Jie-Tai Yu, *On the lifting of the Nagata automorphism*, arXiv:1011.3349; Selecta Math (New Series), to appear.
- [2] V.Drensky and J.-T.Yu, *Automorphisms fixing a variable of  $K\langle x, y, z \rangle$* , J. Algebra **291** (2005) 250-258.
- [3] V.Drensky and J.-T.Yu, *The strong Anick conjecture is true*, J. Eur. Math. Soc. (JEMS) **9** (2007) 659-679.
- [4] V.Drensky and J.-T.Yu, *The strong Anick conjecture*, Proc. Natl. Acad. Sci. USA (PNAS) **103** (2006) 4836-4840.
- [5] M.Smith, *Stably tame automorphisms*, J. Pure Appl. Algebra **58** (1989) 209-212.
- [6] J.Berson, A. van den Essen and David Wright, *Stable tameness of two-dimensional polynomial automorphisms over a regular ring*, arXiv:0707.3151.

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