Coloring Digraphs with Forbidden Cycles

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Abstract

Let k and r be two integers with $k \geq 2$ and $k \geq r \geq 1$. In this paper we show that (1) if a strongly connected digraph D contains no directed cycle of length 1 modulo k, then D is k-colorable; and (2) if a digraph D contains no directed cycle of length r modulo k, then D can be vertex-colored with k colors so that each color class induces an acyclic subdigraph in D. The first result gives an affirmative answer to a question posed by Tuza in 1992, and the second implies the following strong form of a conjecture of Diwan, Kenkre and Vishwanathan: If an undirected graph G contains no cycle of length r modulo k, then G is k-colorable if $r \neq 2$ and $(k+1)$-colorable otherwise. Our results also strengthen several classical theorems on graph coloring proved by Bondy, Erdős and Hajnal, Gallai and Roy, Gyárfás, etc.

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1 Introduction

Digraphs considered in this paper contain no loops nor parallel arcs. By a *cycle* (resp. *path*) in a digraph we mean a simple and directed one throughout. Let D be a digraph. As usual, the *underlying graph* of D, denoted by G, is obtained from D by replacing each arc with an edge having the same ends. A *proper k-coloring* of D is simply a proper k-coloring of G. Thus D is k-colorable iff so is G, and the *chromatic number* $\chi(D)$ of D is exactly $\chi(G)$. An *acyclic k-coloring* of D is an assignment of k colors, $1, 2, \ldots, k$, to the vertices of D so that each color class induces an acyclic subdigraph in D. The *acyclic chromatic number* $\chi_a(D)$ of D is the minimum k for which D admits an acyclic k-coloring. Clearly, $\chi_a(D) \leq \chi(D)$; this inequality, however, need not hold equality in general.

Classical digraph coloring arises in a rich variety of applications, and hence it has attracted many research efforts. As it is NP-hard to determine the chromatic number of a given digraph, the focus of extensive research has been on good bounds. A fundamental theorem due to Gallai and Roy [8,17] asserts that the chromatic number of a digraph is bounded above by the number of vertices in a longest path. It is natural to further explore the connection between chromatic number and cycle lengths. To get meaningful results in this direction, a common practice is to impose strong connectedness on digraphs we consider. Bondy [3] showed that the chromatic number of a strongly connected digraph D is at most its *circumference*, the length of a longest cycle in D. In [18], Tuza proved that if an undirected graph G contains no cycle whose length minus one is a multiple of k, then G is k-colorable. He also asked whether or not similar results can be obtained for digraphs in terms of cycle lengths that belong to prescribed residue classes. One objective of this paper is to give an affirmative answer to his question, which strengthens, among others, all the theorems stated above.

Theorem 1. *Let $k \geq 2$ be an integer. If a strongly connected digraph D contains no directed cycle of length 1 modulo k, then $\chi(D) \leq k$.*

We point out that the bound is sharp for infinitely many digraphs, such as strongly connected tournaments with an even number of vertices.

The *odd circumference* of a graph G (directed or undirected), denoted by $l(G)$, is the length of a longest odd cycle (if any) in G. We set $l(G) = 1$ if G contains no odd cycle. A corollary of the above theorem is the following statement, which has interest in its own right and is in the same spirit as the above Bondy theorem [3].

Theorem 2. *For every strongly connected digraph D, we have $\chi(D) \leq l(D) + 1$.*

It was shown by Erdős and Hajnal [7] that $\chi(G) \leq l(G) + 1$ for any undirected graph G; the bound is achieved only when G contains a complete subgraph with $l(G) + 1$ vertices (see Kenkre and Vishwanathan [13]). So a natural question to ask is whether this characterization remains valid for the directed case. Interestingly, the answer is in the negative: Let D be obtained from the orientation

$$v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow \ldots \rightarrow v_{2k} \leftarrow v_{2k+1} \rightarrow \ldots \rightarrow v_{2n} \leftarrow v_{2n+1} \rightarrow v_1$$

of a $(2n+1)$-cycle $v_1 v_2 \ldots v_{2n+1} v_1$ by adding a new vertex v_{2n+2} and a pair of opposite arcs (v_{2n+2}, v_i) and (v_i, v_{2n+2}) for all $1 \leq i \leq 2n+1$. It is easy to see that D is a strongly connected digraph with $\chi(D) = 4$ and $l(D) = 3$. Nevertheless, D does not contain four pairwise adjacent vertices.

The concept of acyclic chromatic number was independently introduced by Neumann-Lara [16] and Mohar *et al.* [2, 15], and the theory of acyclic coloring provides an interesting way to extend theorems about coloring graphs to digraphs. In [5], Chen, Hu and Zang proved that it is NP-complete to decide if the acyclic chromatic number of a given digraph D is 2, even when D is restricted to a tournament. A tournament H is called a *hero* if there exists a constant $c(H)$ such that every tournament not containing H as a subtournament has acyclic chromatic number at most $c(H)$. In [1], Berger *et al.* obtained a complete characterization of all heroes. In a series of papers [2, 10–12, 15], Mohar and his collaborators proved that many interesting results on graph coloring can be naturally carried over to digraphs with respect to acyclic coloring.

As exhibited by Neumann-Lara [16], there also exist some intimate connections between acyclic chromatic numbers and cycle lengths: For any fixed integers k and r with $2 \leq r \leq k$, if a digraph D contains no cycle of length 0 or 1 modulo r, then $\chi_a(D) \leq k$. Recall the aforementioned Tuza theorem [18], if an undirected graph G has no cycle of length 1 modulo k, then $\chi(G) \leq k$. In [6], Diwan, Kenkre and Vishwanathan proved that $\chi(G) \leq k + 1$ if graph G contains no cycle of length 2 modulo k, and $\chi(G) \leq 2k$ if G contains no cycle of length 3 modulo k; they [6] further conjectured that for any fixed integer r with $1 \leq r \leq k$, if graph G contains no cycle of length r modulo k, then $\chi(G) \leq k + f(k)$, where $f(k) = o(k)$ (possibly a constant). The second objective of this paper is to confirm this conjecture by revealing further connection between acyclic chromatic numbers and cycle lengths.

Theorem 3. *Let k and r be two integers with $k \geq 2$ and $k \geq r \geq 1$. If a digraph D contains no directed cycle of length r modulo k, then $\chi_a(D) \leq k$.*

Unlike Theorem 1, digraph D is not assumed to be strongly connected here, though (as we shall see) the assertion reduces to this case. Theorem 3 implies the following strong form of the above Diwan, Kenkre and Vishwanathan conjecture [6].

Theorem 4. *Let k and r be two integers with $k \geq 2$ and $k \geq r \geq 1$. If an undirected graph G contains no cycle of length r modulo k, then G is k-colorable if $r \neq 2$ and $(k+1)$-colorable otherwise.*

We have noticed that this bound is sharp in several cases, such as $r = 1$ or 2 (consider the complete graph with k or $k + 1$ vertices, respectively).

Let us digress to introduce some notations and terminology, which will be used repeatedly in our proofs. For a directed cycle (or a path) C, we use $|C|$ to denote its length and use xCy to denote the segment of C from x to y for any two vertices x, y on C. A digraph is called *strong* if it is strongly connected, and called *nontrivial* if it contains at least two vertices.

Let $D = (V, A)$ be a digraph, and let F be a subdigraph of D. An F-*ear* P in D is *either* a path in D whose two ends lie in F but whose internal vertices do not, *or* a cycle in D that contains precisely one vertex of F. Recall that if P is a path from u to v, then u and v are called the *origin* and *terminus* of P, respectively. If P is a cycle, then we view the common vertex of P and F as both the *origin* and *terminus* of P. A nested sequence $(D_0, D_1, \ldots, D_m)$ of subdigraphs of D is called an *ear decomposition* of D if the following conditions are satisfied:

- D_0 is a cycle;
- $D_{i+1} = D_i \cup P_{i+1}$, where P_{i+1} is a D_i-ear in D for $0 \leq i \leq m-1$;
- $D_m = D$.

As is well known, every nontrivial strong digraph admits an ear decomposition (see, for instance, [4]). For any function f defined on $V(D_i)$ (the vertex set of D_i) and any D_i-ear P with origin u and terminus v in D_i, define

$$f_i(P) = |P| - (f(v) - f(u)). \tag{1.1}$$

Observe that $f_i(P) = |P|$ if P is a cycle.

The remainder of this paper is organized as follows. In section 2, we establish Theorem 1 by developing the ear decomposition technique, and then deduce Theorem 2 as a corollary. In section 3, we prove Theorem 3 based on a more sophisticated ear decomposition, and also apply it to show Theorem 4. In section 4, we demonstrate that our theorems strengthen several classical theorems on graph coloring. In the last section, we conclude this paper with some remarks and open questions.

2 Classical Coloring

The purpose of this section is to prove two theorems concerning classical digraph coloring.

Proof of Theorem 1. Clearly, we may assume that D contains at least two vertices. We propose to construct an ear decomposition $(D_0, D_1, \ldots, D_m)$ of D (see the above description) and a function $f : V(D) \to \{0, 1, \ldots, k-1\}$, such that for $i = 0, 1, \ldots, m$, we have

(A) $f(u) \neq f(v)$ for any arc (u, v) of D_i;

(B) $f_i(P) \not\equiv 1 \pmod{k}$ (see (1.1)) for any D_i-ear P in D.

If successful, from (A) we see that f is a proper k-coloring of D_i for $0 \leq i \leq m$, and hence $\chi(D) = \chi(D_m) \leq k$.

For $1 \leq i \leq k$, let $\mathcal{C}_i$ be the set of all cycles of length i modulo k in D, which we call a *residue cycle class*. By hypothesis,

(1) $\mathcal{C}_1 = \emptyset$.

For convenience, we define a linear order on other residue cycle classes as follows:

(2) $\mathcal{C}_k > \mathcal{C}_{k-1} > \mathcal{C}_{k-2} > \ldots > \mathcal{C}_2$.

Let $\mathcal{C}_t$ be the first nonempty residue cycle class in this linear order. From this definition and (1) we deduce

(3) $\mathcal{C}_{t+1} = \emptyset$.

Let D_0 be a cycle in $\mathcal{C}_t$. Write D_0 as $v_0 \to v_1 \to \ldots \to v_n \to v_0$. For each integer r, we use $\bar{r}$ to denote the element of $\{0, 1, \ldots, k-1\}$ which is congruent to r modulo k throughout. Define $f : V(D_0) \to \{0, 1, 2, \ldots, k-1\}$ by $f(v_r) = \bar{r}$ for $0 \leq r \leq n$.

Claim 1. D_0 and f satisfy both (A) and (B).

To justify this, note first that the length of D_0 is $n+1$, so $n \not\equiv 0 \pmod{k}$ by (1). Hence $f(v_n) \neq f(v_0)$. From the definition of f, it follows that (A) is satisfied.

Suppose for a contradiction that (B) fails on D_0 and f. Then there exists a D_0-ear P with $f_0(P) \equiv 1 \pmod k$. Let v_i and v_j be the origin and terminus of P, respectively. By (1.1), we have $f_0(P) = |P| - (f(v_j) - f(v_i)) \equiv |P| - (j - i) \pmod k$. So $|P| \equiv (j-i)+1 \pmod k$. Observe that $i \neq j$, for otherwise P would be a cycle of length 1 modulo k, contradicting (1). From the definition of f, we see that $|v_i D_0 v_j| \equiv (j-i) \pmod k$ if $i < j$ and that $|v_j D_0 v_i| \equiv (i-j) \pmod k$ if $j < i$. Therefore $P \cup v_j D_0 v_i$ is a cycle in $\mathcal{C}_{t+1}$ if $i < j$ and in $\mathcal{C}_1$ otherwise. This contradiction to (3) or (1) justifies Claim 1.

Recall the definition of an ear decomposition of D, suppose we have already constructed a D_i and a function $f : V(D_i) \to \{0, 1, 2, ..., k-1\}$ that satisfy both (A) and (B) for some $i \geq 0$ (see Claim 1). If $D_i = D$, we are done by (A). So we assume that D_i is a proper subdigraph of D. Let us proceed to the construction of D_{i+1}.

As D is strong, it contains at least one D_i-ear. For $1 \leq j \leq k$, let $\mathcal{P}_j$ be the set of all D_i-ears P with $f_i(P) \equiv j \pmod k$. Since D_i and f satisfy (B),

(4) $\mathcal{P}_1 = \emptyset$.

Now let us define a linear order on other $\mathcal{P}_j$'s as follows:

(5) $\mathcal{P}_k > \mathcal{P}_{k-1} > \mathcal{P}_{k-2} > \ldots > \mathcal{P}_2$.

Let $\mathcal{P}_s$ be the first nonempty set in this linear order. By this definition and (4), we obtain

(6) $\mathcal{P}_{s+1} = \emptyset$.

Let P_{i+1} be a member of $\mathcal{P}_s$ and set $D_{i+1} = D_i \cup P_{i+1}$. Write P_{i+1} as $u_0 \to u_1 \to \ldots \to u_h$, where $\{u_0, u_h\} \subseteq V(D_i)$. We extend the previous function f to the domain $V(D_{i+1})$ by defining $f(u_r) = \overline{f(u_0) + r}$ for $1 \leq r \leq h-1$. Let us show that D_{i+1} and f are as desired.

Claim 2. D_{i+1} and f satisfy both (A) and (B).

To justify this, note first that $f(u_{h-1}) = \overline{f(u_0) + h - 1} \equiv f(u_0) + h - 1 \equiv f(u_0) + |P_{i+1}| - 1 \pmod k$. By (4), we have $f_i(P_{i+1}) \not\equiv 1 \pmod k$; that is, $|P_{i+1}| - (f(u_h) - f(u_0)) \not\equiv 1 \pmod k$ using (1.1). So $f(u_0) + |P_{i+1}| - 1 \not\equiv f(u_h) \pmod k$ and hence $f(u_{h-1}) \neq f(u_h)$. From the definition of f, we see that (A) is satisfied.

To establish property (B), assume the contrary: $f_{i+1}(P) \equiv 1 \pmod k$ for some D_{i+1}-ear P in D. Let a and b be the origin and terminus of P, respectively. Then $f_{i+1}(P) = |P| - (f(b) - f(a))$. So

(7) $|P| - (f(b) - f(a)) \equiv 1 \pmod k$.

It follows that $a \neq b$, for otherwise P would be a cycle of length 1 modulo k, contradicting (1). Since D_i and f satisfy (B), we may assume that at least one of a and b is in $P_{i+1} \backslash D_i$. Depending on the locations of a and b, we distinguish among four cases.

Case 1. $a = u_p$ and $b = u_q$ with $0 \leq q < p \leq h$. In this case, set $C = P \cup bP_{i+1}a$. If $a \neq u_h$, then C is a cycle in D with $|C| = |P| + p - q \equiv |P| - (f(b) - f(a)) \equiv 1 \pmod k$ by (7). Hence $C \in \mathcal{C}_1$, contradicting (1). If $a = u_h$, then $b \neq u_0$. Thus cycle C is a D_i-ear in D with $f_i(C) = |C| = |P| + |bP_{i+1}a| \equiv (f(b) - f(a) + 1) + (|P_{i+1}| - f(b) + f(u_0)) \equiv |P_{i+1}| - (f(u_h) - f(u_0)) + 1 \equiv f_i(P_{i+1}) + 1 \equiv s + 1 \pmod k$, contradicting (6).

Case 2. $a = u_p$ and $b = u_q$ with $0 \leq p < q \leq h$. In this case, set $Q_1 = u_0 P_{i+1} a \cup P \cup bP_{i+1} u_h$. If $b \neq u_h$, then Q_1 is a D_i-ear in D with $|Q_1| - |P_{i+1}| \equiv |P| - (f(b) - f(a)) \equiv 1 \pmod k$ by (7). As $P_{i+1} \in \mathcal{P}_s$, we get $Q_1 \in \mathcal{P}_{s+1}$, contradicting (6). If $b = u_h$, then $f_i(Q_1) = |Q_1| - (f(b) - f(u_0)) =$

$|P|+|u_0P_{i+1}u_p|-(f(b)-f(u_0)) \equiv |P|+(f(a)-f(u_0))-(f(b)-f(u_0)) \equiv |P|-(f(b)-f(a)) \equiv 1 \pmod{k}$ by (7), contradicting (4).

Case 3. $a \in D_i \backslash P_{i+1}$ and $b = u_p$ with $0 < p < h$. In this case, set $Q_2 = P \cup bP_{i+1}u_h$. Then Q_2 is a D_i-ear in D with $f_i(Q_2) = |Q_2|-(f(u_h)-f(a)) \equiv (|P|+|P_{i+1}|-f(b)+f(u_0))-(f(u_h)-f(a)) \pmod{k}$. From (7) it follows that $f_i(Q_2) \equiv 1 + |P_{i+1}| + f(u_0) - f(u_h) \equiv f_i(P_{i+1}) + 1 \equiv s+1 \pmod{k}$, which implies $Q_2 \in \mathcal{P}_{s+1}$, contradicting (6).

Case 4. $b \in D_i \backslash P_{i+1}$ and $a = u_p$ with $0 < p < h$. In this case, set $Q_3 = u_0P_{i+1}u_p \cup P$. Then Q_3 is a D_i-ear in D with $f_i(Q_3) = |Q_3| - (f(b)-f(u_0)) \equiv (|P|+f(a)-f(u_0))-(f(b)-f(u_0)) \equiv |P|-(f(b)-f(a)) \equiv 1 \pmod{k}$ by (7), which implies $Q_3 \in \mathcal{P}_1$, contradicting (4). So Claim 3 holds.

Repeating the above construction process, we shall eventually get an ear decomposition $(D_0, D_1, \ldots, D_m)$ of D and a function $f : V(D) \to \{0, 1, \ldots, k-1\}$ with properties (A) and (B) (see Claims 1 and 2). This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Let $k = l(D)+1$. Then k is an even integer with $k \geq 2$. Observe that D contains no cycle C whose length minus one is a multiple of k, for otherwise C is an odd cycle with $|C| \geq k+1 > l(D)$, contradicting the definition of $l(D)$. From Theorem 1, we thus deduce that $\chi(D) \leq k = l(D)+1$, as desired. ∎

3 Acyclic Coloring

Let us define a few terms before presenting the proof of Theorem 3. Let $D = (V, A)$ be a digraph and let $\prec$ be a linear order on V; that is, for any two vertices u and v, precisely one of the relations $u \prec v$ and $v \prec u$ holds. We say that u *precedes* v (also v *succeeds* u) in the order $\prec$ if $u \prec v$. An arc (u, v) of D is called *forward* if $u \prec v$ and *backward* otherwise. More generally, let F be a subdigraph of D. An F-ear P with origin u and terminus v is called *forward* if $u \prec v$, *backward* if $v \prec u$, and *cyclic* otherwise. A vertex pair $\{u, v\}$ of F is called a *backward pair* in F if there exists a backward F-ear between u and v in D.

Proof of Theorem 3. For convenience, we shall treat r as an integer satisfying $0 \leq r \leq k-1$. It is easy to see that for any digraph D, we have

$$\chi_a(D) = \max\{\chi_a(F) : F \text{ is a strong subdigraph of } D\}.$$

So we may assume that D addressed in the theorem is strong. Clearly, we may also assume that D is nontrivial.

We propose to construct an ear decomposition $(D_0, D_1, \ldots, D_m)$ of D, a linear order $\prec$ on the vertices of D, and a function $f : V(D) \to \{0, 1, \ldots, k-1\}$, with the following properties for each $i = 0, 1, ..., m$:

(A) $f(u) \neq f(v)$ for any forward arc (u, v) of D_i;

(B) $f_i(P) \not\equiv 1 \pmod{k}$ (see (1.1)) for any forward D_i-ear P in D; and

(C) there exists an integer $\alpha = \alpha(u, v)$ for any backward pair $\{u, v\}$ with $u \prec v$ in D_i, such that $|P| \not\equiv \alpha \pmod{k}$ for any backward D_i-ear P from v to u in D.

If successful, from (A) we see that each color class induces a subdigraph in D_i which contains no forward arcs and hence is acyclic. It follows that f is an acyclic k-coloring of D_i for all $0 \le i \le m$. Therefore, $\chi_a(D) = \chi_a(D_m) \le k$.

Once again, we use $\bar{p}$ to denote the element of $\{0, 1, \dots, k-1\}$ which is congruent to p modulo k for any integer p; and we use $\mathcal{C}_p$ to denote the residue cycle class consisting of all cycles of length p modulo k in D for $0 \le p \le k-1$. By hypothesis, we have

(1) $\mathcal{C}_r = \emptyset$.

We define a linear order on other residue cycle classes by

(2) $\mathcal{C}_{r-1} > \mathcal{C}_{r-2} > \dots > \mathcal{C}_0 > \mathcal{C}_{k-1} > \mathcal{C}_{k-2} > \dots > \mathcal{C}_{r+1}$.

Let $\mathcal{C}_t$ be the first nonempty residue cycle class in this linear order. In view of (1), we obtain

(3) $\mathcal{C}_{t+1} = \emptyset$.

Let D_0 be a cycle in $\mathcal{C}_t$ and write $D_0 = v_0 \to v_1 \to \dots \to v_n \to v_0$. We define a linear order $\prec$ on $V(D_0)$ by $v_0 \prec v_1 \prec v_2 \prec \dots \prec v_n$, and define a function $f : V(D_0) \to \{0, 1, \dots, k-1\}$ by $f(v_p) = \bar{p}$ for $0 \le p \le n$.

Claim 1. D_0, $\prec$ and f satisfy all of (A), (B) and (C).

Indeed, since the arc (v_n, v_0) is backward, property (A) follows instantly from the definition of f.

Assume on the contrary that property (B) fails. Then there exists a forward D_0-ear P from some v_i to v_j with $f_0(P) \equiv 1 \pmod{k}$. By (1.1), we obtain $|P| \equiv f(v_j) - f(v_i) + 1 \equiv |v_i D_0 v_j| + 1 \pmod{k}$ as $v_i \prec v_j$. Thus the cycle $P \cup v_j D_0 v_i$ has length $|P| + |D_0| - |v_i D_0 v_j| \equiv |D_0| + 1 \equiv t+1 \pmod{k}$ and hence belongs to $\mathcal{C}_{t+1}$, contradicting (3).

To establish property (C), set $\alpha(v_i, v_j) = i - j + r$ for each vertex pair $\{v_i, v_j\}$ of D_0 with $i < j$. If there exists a backward D_0-ear P in D from v_j to v_i with $|P| \equiv \alpha(v_i, v_j) \pmod{k}$, then the cycle $P \cup v_i D_0 v_j$ would belong to $\mathcal{C}_r$, because $|P \cup v_i D_0 v_j| \equiv \alpha(v_i, v_j) + |v_i D_0 v_j| \equiv (i - j + r) + (j - i) \equiv r \pmod{k}$; this contradiction to (1) justifies Claim 1.

Suppose we have already constructed a nontrivial strong D_i, a linear order $\prec$ on $V(D_i)$, and a function $f : V(D_i) \to \{0, 1, 2, ..., k-1\}$ that satisfy all of (A), (B) and (C) for some $i \ge 0$ (see Claim 1). If $D_i = D$, we are done by (A). So we may assume that D_i is a proper subdigraph of D. Let us proceed to the construction of D_{i+1} and first consider the situation when

(4) there exists at least one forward or cyclic D_i-ear in D.

For $0 \le j \le k-1$, let $\mathcal{P}_j$ (resp. $\mathcal{Q}_j$) be the set of all forward (resp. cyclic) D_i-ears P with $f_i(P) \equiv j \pmod{k}$. Observe that

(5) $\mathcal{P}_1 = \emptyset$ and $\mathcal{Q}_r = \emptyset$,

where the first equality follows from property (B) with respect to i, and the second from (1). We define a linear order on other $\mathcal{P}_j$'s and $\mathcal{Q}_j$'s as follows:

(6) $\mathcal{P}_0 > \mathcal{P}_{k-1} > \mathcal{P}_{k-2} > \dots > \mathcal{P}_2 > \mathcal{Q}_{r-1} > \mathcal{Q}_{r-2} > \dots >$
$\mathcal{Q}_0 > \mathcal{Q}_{k-1} > \mathcal{Q}_{k-2} > \dots > \mathcal{Q}_{r+1}$.

Let $\mathcal{A}$ denote the first nonempty set in this linear order. Then $\mathcal{A}$ is $\mathcal{P}_s$ or $\mathcal{Q}_s$ for some subscript s. From the definition of $\mathcal{A}$ and (5), we deduce that

(7) $\mathcal{P}_{s+1} = \emptyset$ in any case, and $\mathcal{Q}_{s+1} = \emptyset$ if $\mathcal{A} = \mathcal{Q}_s$.

Let P_{i+1} be an element of $\mathcal{A}$ (so we always have $f_i(P_{i+1}) \equiv s \pmod{k}$) and set $D_{i+1} = D_i \cup P_{i+1}$. Write $P_{i+1} = u_0 \to u_1 \to \dots \to u_h$, where $\{u_0, u_h\} \subseteq V(D_i)$. If $\mathcal{A} = \mathcal{P}_s$, then P_{i+1} is a forward

D_i-ear, implying that

(8) $u_0 \prec u_h$ when $u_0 \neq u_h$.

Let u_0^+ be the vertex of D_i that succeeds u_0 immediately in the order $\prec$. We extend the linear order $\prec$ from $V(D_i)$ to $V(D_{i+1})$ by inserting all u_j, with $1 \le j \le h-1$, between u_0 and u_0^+, such that

(9) $u_0 \prec u_1 \prec \ldots \prec u_{h-1} \prec u_0^+$.

Moreover, we extend the function f from the domain $V(D_i)$ to the domain $V(D_{i+1})$ by defining $f(u_j) = \overline{f(u_0)+j}$ for $1 \le j \le h-1$. Let us now establish correctness of this construction.

Claim 2. D_{i+1}, $\prec$ and f satisfy both (A) and (B).

To justify this, note from (8) and (9) that (u_j, u_{j+1}) is a forward arc for $0 \le j \le h-2$, and that (u_{h-1}, u_h) is a forward arc if $u_0 \neq u_h$ and a backward arc otherwise. Clearly, $f(u_{h-1}) = \overline{f(u_0)+h-1} \equiv f(u_0)+h-1 \equiv f(u_0)+|P_{i+1}|-1 \pmod k$. If $u_0 \neq u_h$, then $f_i(P_{i+1}) \not\equiv 1 \pmod k$ by (5), which implies $|P_{i+1}| - (f(u_h) - f(u_0)) \not\equiv 1 \pmod k$ using (1.1). So $f(u_0) + |P_{i+1}| - 1 \not\equiv f(u_h) \pmod k$ and hence $f(u_{h-1}) \neq f(u_h)$. From the definition of f, we see that (A) is satisfied.

Suppose for a contradiction that (B) fails. Then there exists a forward D_{i+1}-ear P from a to b with $f_{i+1}(P) \equiv 1 \pmod k$. Thus

(10) $a \prec b$ and $|P| - (f(b) - f(a)) \equiv 1 \pmod k$.

As (B) holds for D_i, $\prec$ and f, we may assume that at least one of a and b is in $P_{i+1} \backslash D_i$. Depending on the locations of a and b, we consider three cases.

Case 1. $a, b \in P_{i+1}$. By (8), (9) and (10), we have $a = u_p$ and $b = u_q$ for some p and q with $0 \le p < q \le h$. Set $Q_1 = u_0 P_{i+1} a \cup P \cup b P_{i+1} u_h$. If $b \neq u_h$, then Q_1 is a D_i-ear from u_0 to u_h in D with $|Q_1| - |P_{i+1}| \equiv |P| - (f(b) - f(a)) \equiv 1 \pmod k$ by (10). It follows that $Q_1 \in \mathcal{P}_{s+1}$ if $P_{i+1} \in \mathcal{P}_s$ and that $Q_1 \in \mathcal{Q}_{s+1}$ if $P_{i+1} \in \mathcal{Q}_s$, contradicting (7) in either subcase. If $b = u_h$, then $u_0 \neq u_h$ by (9) and (10). Thus Q_1 is a forward D_i-ear from u_0 to u_h in D with $f_i(Q_1) = |Q_1| - (f(b) - f(u_0)) = |P| + |u_0 P_{i+1} u_p| - (f(b) - f(u_0)) \equiv |P| + (f(a) - f(u_0)) - (f(b) - f(u_0)) \equiv |P| - (f(b) - f(a)) \equiv 1 \pmod k$ by (10), and hence $Q_1 \in \mathcal{P}_1$, contradicting (5).

Case 2. $a \in P_{i+1} \backslash D_i$ and $b \in D_i \backslash P_{i+1}$. By (9) and (10), we have $u_0 \prec a \prec b$. Set $Q_2 = u_0 P_{i+1} a \cup P$. Then Q_2 is a forward D_i-ear from u_0 to b in D with $f_i(Q_2) \equiv |u_0 P_{i+1} a| + |P| - (f(b) - f(u_0)) \equiv (f(a) - f(u_0)) + (f(b) - f(a) + 1) - (f(b) - f(u_0)) \equiv 1 \pmod k$, where the second equality follows from (10). Hence $Q_2 \in \mathcal{P}_1$, contradicting (5).

Case 3. $a \in D_i \backslash P_{i+1}$ and $b \in P_{i+1} \backslash D_i$. By (8), (9) and (10), we have $a \prec b \prec u_h$ if $u_0 \neq u_h$ and $a \prec u_0 \prec b$ if $u_0 = u_h$. Set $Q_3 = P \cup b P_{i+1} u_h$. Then Q_3 is a forward D_i-ear from a to u_h in D with $f_i(Q_3) = |Q_3| - (f(u_h) - f(a)) \equiv (|P| + |P_{i+1}| - f(b) + f(u_0)) - (f(u_h) - f(a)) \pmod k$. From (10) we see that $f_i(Q_3) \equiv 1 + |P_{i+1}| - (f(u_h) - f(u_0)) \equiv f_i(P_{i+1}) + 1 \equiv s + 1 \pmod k$. So $Q_3 \in \mathcal{P}_{s+1}$; this contradiction to (7) establishes Claim 2.

Claim 3. D_{i+1}, $\prec$ and f satisfy (C).

We aim to show that for any backward pair $\{a, b\}$ in D_{i+1} with $a \prec b$, the integer $\alpha(a,b)$ as described in (C) (with $i+1$ in place of i) exists. Since (C) holds for D_i, $\prec$ and f, we may assume that at least one of a and b is in $P_{i+1} \backslash D_i$. Depending on the locations of a and b, we consider four cases.

Case 1. $a, b \in P_{i+1}$. In this case, set $\alpha(a,b) = r - |aP_{i+1}b|$. Suppose on the contrary that there exists a backward D_{i+1}-ear P from b to a in D with $|P| \equiv \alpha(a,b) \pmod k$. Let $C = P \cup aP_{i+1}b$. Then C is a directed cycle of length $|C| = |P| + |aP_{i+1}b| \equiv \alpha(a,b) + |aP_{i+1}b| \equiv r \pmod k$, so $C \in \mathcal{C}_r$, contradicting (1).

Case 2. $a \in P_{i+1} \backslash D_i$ and $b \in D_i \backslash P_{i+1}$ with $u_h \prec b$. In this case, by (8) and (9), we have $u_0 \prec a \prec u_h \prec b$ if $u_0 \neq u_h$ and $u_0 \prec a \prec b$ if $u_0 = u_h$. Let P be an arbitrary backward D_{i+1}-ear from b to a in D. Then $Q_1 = P \cup aP_{i+1}u_h$ is a backward D_i-ear from b to u_h. Since (C) holds for D_i, $\prec$ and f, there exists an integer $\alpha(b, u_h)$ such that no backward D_i-ear from b to u_h in D has length $\alpha(b, u_h)$ modulo k. In particular, $|P| + |aP_{i+1}u_h| = |Q_1| \not\equiv \alpha(b, u_h) \pmod k$. Therefore, $|P| \not\equiv \alpha(b, u_h) - |aP_{i+1}u_h| \pmod k$. So $\alpha(a,b) = \alpha(b, u_h) - |aP_{i+1}u_h|$ is as desired.

Case 3. $a \in P_{i+1} \backslash D_i$ and $b \in D_i \backslash P_{i+1}$ with $b \prec u_h$. In this case, by (8) and (9), we obtain $u_0 \neq u_h$ and $u_0 \prec a \prec b \prec u_h$. Let us show that $\alpha(a,b) = f(a) - f(b) + 1$ will do. Assume the contrary: some backward D_{i+1}-ear P from b to a in D has length $\alpha(a,b)$ modulo k. Let $Q_2 = P \cup aP_{i+1}u_h$. Then Q_2 is a forward D_i-ear from b to u_h in D with $f_i(Q_2) = |P| + |aP_{i+1}u_h| - (f(u_h) - f(b)) \equiv \alpha(a,b) + |P_{i+1}| - (f(a) - f(u_0)) - (f(u_h) - f(b)) \equiv 1 + f_i(P_{i+1}) \equiv s + 1 \pmod k$, so $Q_2 \in \mathcal{P}_{s+1}$, contradicting (7).

Case 4. $a \in D_i \backslash P_{i+1}$ and $b \in P_{i+1} \backslash D_i$. In this case, by (9) we have $a \prec u_0 \prec b$. Since (C) holds for D_i, $\prec$ and f, there exists an integer $\alpha(a, u_0)$ such that no backward D_i-ear from u_0 to a has length $\alpha(a, u_0)$ modulo k. Set $\alpha(a,b) = \alpha(a, u_0) - f(b) + f(u_0)$. Then there is no backward D_{i+1}-ear P from b to a in D with $|P| \equiv \alpha(a,b) \pmod k$, for otherwise, $Q_3 = u_0P_{i+1}b \cup P$ would be a backward D_i-ear from u_0 to a in D with $|Q_3| \equiv (f(b) - f(u_0)) + |P| \equiv \alpha(a, u_0) \pmod k$; this contradiction finishes the proof of Claim 3.

It remains to consider the situation when (4) does not occur; that is,

(11) there exists neither forward nor cyclic D_i-ear in D.

Since D is strong, D_i contains at least one backward pair. Among all such backward pairs, we choose a pair $\{x, y\}$ with $x \prec y$ such that

(12) the set $[x, y]_i = \{z \in V(D_i) : x \preceq z \preceq y\}$ has the smallest size.

For $0 \leq j \leq k-1$, let $\mathcal{R}_j$ be the set of all backward D_i-ears P from y to x in D with $|P| \equiv j \pmod k$. Since property (C) holds on D_i, $\prec$ and f, there exists an integer $\alpha = \alpha(x,y)$ such that

(13) $\mathcal{R}_\alpha = \emptyset$.

We define a linear order on other $\mathcal{R}_j$'s as follows:

(14) $\mathcal{R}_{\alpha-1} > \mathcal{R}_{\alpha-2} > \ldots > \mathcal{R}_0 > \mathcal{R}_{k-1} > \mathcal{R}_{k-2} > \ldots > \mathcal{R}_{\alpha+1}$.

Let $\mathcal{R}_s$ be the first nonempty set in this linear order. By (13), we obtain

(15) $\mathcal{R}_{s+1} = \emptyset$.

Let P_{i+1} be a path in $\mathcal{R}_s$ and set $D_{i+1} = D_i \cup P_{i+1}$. Write $P_{i+1} = u_h \to u_{h-1} \to \ldots \to u_1 \to u_0$. Then

(16) $u_0 = x \prec y = u_h$.

Let u_0^- be the vertex of D_i that precedes u_0 immediately in the order $\prec$. We extend the linear order $\prec$ from $V(D_i)$ to $V(D_{i+1})$ by inserting all u_j, with $1 \leq j \leq h-1$, between u_0^- and u_0, such that

(17) $u_0^- \prec u_{h-1} \prec u_{h-2} \prec \ldots \prec u_1 \prec u_0$.

Moreover, we extend the function f from the domain $V(D_i)$ to the domain $V(D_{i+1})$ by defining $f(u_j) = \overline{f(u_0)} - j$ for $1 \leq j \leq h-1$. Let us now show correctness of this construction.

Claim 4. D_{i+1}, $\prec$ and f satisfy both (A) and (B).

To justify this, note that (u_j, u_{j-1}) is a forward arc for $1 \leq j \leq h-1$ while (u_h, u_{h-1}) is a backward arc by (16) and (17). From the definition of f, we see that (A) is satisfied.

To establish property (B), assume the contrary: there exists some forward D_{i+1}-ear P from a to b with $f_{i+1}(P) \equiv 1 \pmod{k}$. Then

(18) $a \prec b$ and $|P| \equiv f(b) - f(a) + 1 \pmod{k}$.

By (11), at least one of a and b is in $P_{i+1} \backslash D_i$. Depending on the locations of a and b, we distinguish among three cases.

Case 1. $a, b \in P_{i+1}$. If $b = y$, then $a \neq x$, and thus $C = P \cup bP_{i+1}a$ is a cyclic D_i-ear in D, contradicting (11). So $b \neq y$. By (16), (17) and (18), we have $a = u_p$ and $b = u_q$ with $0 \leq q < p < h$. Let $Q_1 = yP_{i+1}a \cup P \cup bP_{i+1}x$. Then Q_1 is a backward D_i-ear from y to x in D with $|Q_1| \equiv |P| + |P_{i+1}| - (f(b) - f(a)) \equiv |P_{i+1}| + 1 \equiv s+1 \pmod{k}$, where the second equality follows from (18). So $Q_1 \in \mathcal{R}_{s+1}$, contradicting (15).

Case 2. $a \in D_i \backslash P_{i+1}$ and $b \in P_{i+1} \backslash D_i$. By (17) and (18), we have $a \prec b \prec x$. Thus $Q_2 = P \cup bP_{i+1}x$ is a forward D_i-ear in D, contradicting (11).

Case 3. $a \in P_{i+1} \backslash D_i$ and $b \in D_i \backslash P_{i+1}$. By (17) and (18), we have $a \prec x \prec b$. Let $Q_3 = yP_{i+1}a \cup P$. By (11), Q_3 must be a backward D_i-ear in D. So $b \prec y$ and thus $\{b, y\}$ is a backward pair in D_i with $[b, y]_i \subsetneq [x, y]_i$, contradicting (12). This proves Claim 4.

Claim 5. D_{i+1}, $\prec$ and f satisfy (C).

We aim to show that for any backward pair $\{a, b\}$ in D_{i+1} with $a \prec b$, the integer $\alpha(a, b)$ as described in (C) (with $i+1$ in place of i) exists. Since (C) holds for D_i, $\prec$ and f, we may assume that at least one of a and b is in $P_{i+1} \backslash D_i$. Depending on the locations of a and b, we consider four cases.

Case 1. $a \in P_{i+1} \backslash D_i$ and $b = y$. By (16) and (17), we have $a \prec x \prec y = b$. Define $\alpha(a, b) = \alpha - f(x) + f(a)$ (see (13) for the definition of α). Then there is no backward D_{i+1}-ear P from b to a in D with $|P| \equiv \alpha(a, b) \pmod{k}$, for otherwise $Q_1 = P \cup aP_{i+1}x$ would be a backward D_i-ear from y to x in D of length $|P| + |aP_{i+1}x| \equiv |P| + f(x) - f(a) \equiv \alpha(a, b) + f(x) - f(a) \equiv \alpha \pmod{k}$, contradicting (13).

Case 2. $a, b \in P_{i+1}$ with $b \neq y$. Since $a \prec b$, by (16) and (17) we have $a = u_p$ and $b = u_q$ with $0 \leq q < p < h$. Set $\alpha(a, b) = f(a) - f(b) + r$. Then there exists no backward D_{i+1}-ear P from b to a in D with $|P| \equiv \alpha(a, b) \pmod{k}$, for otherwise $C = P \cup aP_{i+1}b$ would be a cycle of length $|P| + |aP_{i+1}b| \equiv \alpha(a, b) + f(b) - f(a) \equiv r \pmod{k}$, so $C \in \mathcal{C}_r$, contradicting (1).

Case 3. $a \in P_{i+1} \backslash D_i$ and $b \in D_i \backslash P_{i+1}$. Since $a \prec b$, by (16) and (17) we have $a \prec x \prec b$. Since (C) holds for D_i, $\prec$ and f, there exists an integer $\alpha(b, x)$ such that no backward D_i-ear from b to x in D has length $\alpha(b, x)$ modulo k. Define $\alpha(a, b) = \alpha(b, x) - f(x) + f(a)$. Then there exists no backward D_{i+1}-ear P from b to a with $|P| \equiv \alpha(a, b) \pmod{k}$, for otherwise $Q_2 = P \cup aP_{i+1}x$ would be a backward D_i-ear from b to x with length $|Q_2| \equiv |P| + f(x) - f(a) \equiv \alpha(b, x) \pmod{k}$, a contradiction.

Case 4. $a \in D_i \backslash P_{i+1}$ and $b \in P_{i+1} \backslash D_i$. In this case, by (16) and (17) we have $a \prec b \prec x \prec y$.

Since (C) holds for D_i, $\prec$ and f, there exists an integer $\alpha(a, y)$ such that no backward D_i-ear from y to a in D has length $\alpha(a, y)$ modulo k. Define $\alpha(a, b) = \alpha(a, y) - |P_{i+1}| + f(x) - f(b)$. Then there is no backward D_{i+1}-ear P from b to a in D with $|P| \equiv \alpha(a, b) \pmod{k}$, for otherwise $Q_3 = yP_{i+1}b \cup P$ would be a backward D_i-ear from y to a in D of length $|P| + |P_{i+1}| - (f(x) - f(b)) \equiv \alpha(a, y)$, a contradiction. So Claim 5 is true.

Repeating this construction process, we shall eventually get an ear decomposition $(D_0, D_1, \ldots, D_m)$ of D, a linear order $\prec$ on the vertices of D, and a function $f : V(D) \to \{0, 1, \ldots, k-1\}$ with properties (A), (B) and (C) (see Claims 1-5). This completes the proof of Theorem 3. ∎

Proof of Theorem 4. Recall that if G contains no odd cycle, then G is a bipartite graph, and that if G contains no even cycle, then each block of G other than an edge is an odd cycle, so the assertion holds trivially for $k = 2$.

Consider the case when $k \geq 3$. As shown by Diwan, Kenkre and Vishwanathan [6] (see its Corollary 2), if $r = 2$, then $\chi(G) \leq k + 1$. So we assume $r \neq 2$ hereafter.

Let D be the digraph obtained from G by replacing each edge uv of G with a pair of opposite arcs (u, v) and (v, u). Clearly, D has a directed cycle of length n iff G has a cycle of length n for any $n \geq 3$. Thus, it follows from hypothesis that D has no directed cycle of length r modulo k. By Theorem 3, $V(D)$ can be partitioned into k sets $V_1, V_2, ..., V_k$ such that each V_i induces an acyclic subdigraph $D[V_i]$ in D. Therefore $D[V_i]$ contains no arc (u, v) of D, for otherwise $u \to v \to u$ would be a directed cycle in $D[V_i]$, a contradiction. So, by definition, $G[V_i]$ is an independent set for all $1 \leq i \leq k$, and hence $\chi(G) \leq k$. ∎

4 Implications

In this section we show that the results established in the preceding sections strengthen several classical theorems proved by various researchers.

Theorem 5. (Erdős and Hajnal [7]) *For any undirected graph G, there holds $\chi(G) \leq l(G) + 1$.*

Proof. Let D be the digraph obtain from G by replacing each edge uv with a pair of opposite arcs (u, v) and (v, u). Then the odd circumference $l(D)$ of D is precisely $l(G)$. By Theorem 2, we have $\chi(D) \leq l(D) + 1$ and hence $\chi(G) \leq l(G) + 1$. ∎

The following result can be deduced from Theorem 1 by using the same proof technique as above, and is also contained in Theorem 4 as a special case.

Theorem 6. (Tuza [18]) *Let $k \geq 2$ be an integer. If an undirected graph G contains no cycle whose length minus one is a multiple of k, then $\chi(G) \leq k$.* ∎

Theorem 7. (Gyárfás [9]) *For an undirected graph G, let $L_o(G)$ be the set of odd cycle lengths in G. Then $\chi(G) \leq 2|L_o(G)| + 2$.*

Proof. Let $|L_o(G)| = k$ and let $\mathcal{C}_i$ be the set of all cycles of length i modulo $2k + 2$ in G. Since G has k distinct odd cycle lengths in total, at least one of $\mathcal{C}_1, \mathcal{C}_3, \ldots, \mathcal{C}_{2k+1}$ is empty. By Theorem 4, we obtain $\chi(G) \leq 2k + 2$. ∎

Theorem 8. (Mihók and Schiermeyer [14]) *For an undirected graph G, let $L_e(G)$ be the set of even cycle lengths in G. Then $\chi(G) \le 2|L_e(G)| + 3$.*

Proof. Let $|L_e(G)| = k$ and let $\mathcal{C}_i$ be the set of all cycles of length i modulo $2k+2$ in G. Then at least one of $\mathcal{C}_0, \mathcal{C}_2, \dots, \mathcal{C}_{2k}$ is empty. From Theorem 4 we deduce that $\chi(G) \le 2k+3$, as desired. ∎

We remark that the bound in Theorem 7 (resp. Theorem 8) is achieved only when G contains a complete graph with $2|L_o(G)|+2$ (resp. $2|L_e(G)|+3$) vertices, as shown by Gyárfás [9] (resp. by Mihók and I. Schiermeyer [14]).

Theorem 9. (Bondy [3]) *The chromatic number of every strongly connected digraph is at most its circumference.*

Proof. Let k be the circumference of a strongly connected digraph D. Then D contains no cycle whose length minus one is a multiple of k. From Theorem 1 it follows that $\chi(D) \le k$. ∎

Theorem 10. (Gallai-Roy [8,17]) *The chromatic number of every digraph is at most the number of vertices in a longest path.*

Proof. Let k be the number of vertices in a longest path in a digraph $D = (V, A)$. To show that $\chi(D) \le k$, we construct a digraph D' from D by adding a new vertex u and a pair of opposite arcs (u, v) and (v, u) for each $v \in V$. Clearly, D' is strongly connected and $\chi(D') = \chi(D) + 1$. Observe that D' contains no cycle C whose length minus one is a multiple of $k+1$, for otherwise $C \backslash u$ and hence D would contain a path with at least $k+1$ vertices. By Theorem 1, we have $\chi(D') \le k+1$. So $\chi(D) \le k$. ∎

5 Concluding Remarks

In this paper we have established bounds on chromatic numbers and acyclic chromatic numbers of digraphs in terms of cycle lengths. In particular, $\chi(D) \le l(D) + 1$ for any strong digraph D, where $l(D)$ is the odd circumference of D. An interesting open problem is to characterize all strong digraphs D for which $\chi(D) = l(D) + 1$. We believe that the following lemma will play a certain role in the study of such extremal digraphs.

Lemma 11. *Let $D = (V, A)$ be a strong digraph and let U be a subset of pairwise adjacent vertices of D. Then there exists a cycle C in D that contains all vertices in U. (So $|C| \ge |U| + 1$ if $D[U]$ is not strongly connected.)*

Proof. Partition U into disjoint subsets $U_1, U_2, ..., U_t$ such that
- for each i, either $|U_i| = 1$ or U_i induces a strong subdigraph in D, and
- for any $i < j$, each arc between U_i and U_j is directed from U_i to U_j.

Since D is strongly connected, there exists a path P from some vertex in U_t to a vertex in U_1; we choose such a shortest P. Let $P_1, P_2, ..., P_s$ be all sub-paths of P, each of which is internally vertex-disjoint from U and has at least two arcs. Let x_i and y_i be the origin and terminus of P_i, respectively. From the choice of P, we deduce that (y_i, x_i) is an arc in D. Let H be obtained from $D[U]$ by replacing each arc (y_i, x_i) with (x_i, y_i). Then H is strongly connected and hence contains a Hamiltonian cycle C. Let Q be obtained from C by replacing each arc (x_i, y_i) with P_i. Clearly, Q is a cycle in D containing all vertices in U. ∎

Given a strong digraph D with no cycle of length r modulo k, our theorems assert that $\chi_a(D) \leq k$ for a general r and $\chi(D) \leq k$ for $r = 1$. Can we establish a good bound on $\chi(D)$ in terms of k for a general r? This question is clearly worth some research effort.

In [18], Tuza came up with a linear-time algorithm for finding a proper k-coloring of a graph with no cycle of length 1 modulo k. In [6], an efficient algorithm for finding a proper $(k+1)$-coloring of a graph with no cycle of length 2 modulo k was also given. We close this paper with the following question: Is it true that there also exist efficient combinatorial algorithms for the coloring problems addressed in Theorems 1, 3 and 4?

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