ON GLOBAL RIGIDITY OF THE p-TH ROOT EMBEDDING

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ABSTRACT. We study bona fide holomorphic isometric embeddings of the unit disk Δ into polydisks Δ^p ($p \geq 2$) with sheeting number equals p and the assumption that all component functions of such embeddings are non-constant. We prove that all such embeddings are congruent to the p-th root embedding (cf. [8]).

1. INTRODUCTION

In 2010, Ng [8] has proven the global rigidity of the p-th root embedding when $p \geq 2$ is an odd integer or $p = 2$. However, the case when $p \geq 4$ is an even integer is still not known even for the case $p = 4$. Mok [7] has expected that the p-th root embedding is at least locally rigid for any integer $p \geq 2$. For the proof of the global rigidity of the p-th root embedding when $p \geq 2$ is odd, Ng [8] has relied on the bijectivity of certain rational function $R^\mu|_{\partial\Delta} : \partial\Delta \to \partial\Delta$ and the unimodular value of different branches of the holomorphic isometric embeddings f_l around a point on the boundary of the unit disk which is not a branch point of any component functions of f_l. However, for the case of 4-th root embedding, such rational function $R^\mu|_{\partial\Delta} : \partial\Delta \to \partial\Delta$ is neither injective nor surjective. This shows that the method in [8] does not apply to the case of $2q$-th root embedding, where $q \geq 2$ is an integer. In this article, all holomorphic isometric embeddings

$$f = (f^1, \ldots, f^p) : (\Delta, k ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$$

will be assumed to be *genuine*, i.e. all component functions of f are non-constant, as mentioned in [9], p. 7. Denote by $\mathbf{HI}_1(\Delta, \Delta^p; p)$ the set of all *genuine* holomorphic isometric embeddings $(\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ with the sheeting number $n = p$ and the isometric constant $k = 1$ as in [7]. We shall prove that the p-th root embedding is globally rigid as a map in $\mathbf{HI}_1(\Delta, \Delta^p; p)$ based on the theory developed in [8] and [6] as follows:

Theorem 1.1. (Global Rigidity of the p-th Root Embedding)
Let $p \geq 2$ be an integer. If $f : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ is a holomorphic isometric embedding with sheeting number $n = p$, then f is the p-th root embedding up to reparametrizations.

Remark. The theorem says that any map $f \in \mathbf{HI}_1(\Delta, \Delta^p; p)$ is congruent to the p-th root embedding for any integer $p \geq 2$ in the sense of [6], p. 1648.

2. PRELIMINARIES

In this article, we essentially follow the settings in [8], and basic results from [8] are provided as follows. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ be the Riemann sphere. The unit disk Δ is always equipped with the Bergman metric $ds^2_\Delta = 2\mathrm{Re}(g dz \otimes d\bar{z})$, where $g = -2\frac{\partial^2}{\partial z \partial \bar{z}} \log(1 - |z|^2)$. For integer $p \geq 2$, let $\Delta^p = \{(z_1, \ldots, z_p) \in \mathbb{C}^p \mid |z_j| < 1,\ 1 \leq j \leq p\}$ be the polydisk which is viewed as p copies of Δ. Moreover, Δ^p is equipped with the Kähler metric $ds^2_{\Delta^p}$, which is the product metric induced from the Poincaré metric ds^2_Δ. More precisely, we take the real analytic function $-2\sum_{j=1}^p \log(1 - |z_j|^2)$ as Kähler potential for $ds^2_{\Delta^p}$ (cf. [8], p. 2908).

From [6], any germ of holomorphic isometric embedding $f : U \to \Delta^p$ can be extended to a holomorphic isometric embedding $g : (\Delta, k ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, where $U \subset \Delta$ is some open neighborhood of 0 and $f(0) = \mathbf{0}$. For simplicity, we denote this extension also by f. Therefore, we can let $f = (f^1, \ldots, f^p) : \Delta \to \Delta^p$ be a holomorphic isometric embedding with isometric constant k and $f(0) = \mathbf{0}$, where k is an integer satisfying $1 \leq k \leq p$ by [8]. One can define a map h by

$$h = \Psi \circ f \circ \psi$$

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for some $\psi \in \mathrm{Aut}(\Delta)$ and $\Psi \in \mathrm{Aut}(\Delta^p)$ such that $\Psi(f(\psi(0))) = \mathbf{0}$, then h is called a **reparametrization** of f as in [8], p. 2910. From [6], f can be extended continuously to a continuous map $\widetilde{f} = (\widetilde{f^1}, \dots, \widetilde{f^p}) : \overline{\Delta} \to \overline{\Delta^p}$ such that $\widetilde{f}|_\Delta = f$. In [8], we have the following functional equation

$$\prod_{\mu=1}^{p} \left(1 - |f^\mu(z)|^2\right) = (1 - |z|^2)^k$$

and also the polarized functional equation

$$\prod_{\mu=1}^{p} \left(1 - f^\mu(z)\overline{f^\mu(w)}\right) = (1 - z\overline{w})^k.$$

By Proposition 4.2 in [8], there is an irreducible 1-dimensional projective algebraic subvariety $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ such that V extends the graph of f. Moreover, the projection map $\pi : V \to \mathbb{P}^1$, $(z, w_1, \dots, w_p) \mapsto z$, is a finite branched covering map. Let π be n-sheeted for some positive integer n. Let $P_\mu : V \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection map $P_\mu(z, w_1, \dots, w_p) = (z, w_\mu)$ for $1 \le \mu \le p$. Then $V_\mu := P_\mu(V) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is again a 1-dimensional projective algebraic subvariety extends the graph of the component function f^μ. The the projection map $\pi_\mu : V_\mu \to \mathbb{P}^1$, $(z, w_\mu) \mapsto z$, is a finite branched covering which is s_μ-sheeted, where s_μ is an integer dividing n. By Lemma 4.3 in [8], if $(z, w_1, \dots, w_p), (\zeta, \xi_1, \dots, \xi_p) \in V$ are any two points, then

$$\prod_{\mu=1}^{p} \left(1 - w_\mu \overline{\xi_\mu}\right) = \left(1 - z\overline{\zeta}\right)^k.$$

Moreover, from Lemma 4.4 in [8], if $(z, w_1, \dots, w_p) \in V$ and $z \in \mathbb{C} \subset \mathbb{P}^1$, then $w_\mu \in \mathbb{C} \subset \mathbb{P}^1$ for $1 \le \mu \le p$. From [8], for $1 \le \mu \le p$, there is a rational function $R^\mu : \mathbb{P}^1 \to \mathbb{P}^1$ such that $z = R^\mu(f^\mu(z))$, $R^\mu(\partial\Delta) \subset \partial\Delta$ and $R^\mu\left(\frac{1}{\overline{w}}\right) = \frac{1}{\overline{R^\mu(w)}}$, where $\partial\Delta$ is the boundary of the unit disk Δ. The sheeting number of a component function f^μ is defined to be the degree of the rational function R^μ for $1 \le \mu \le p$.

Lemma 2.1 (Ng, [8])**.** *Let h be a component function of a holomorphic isometric embedding $\Delta \to \Delta^p$ and sheeting number of h be q. If h has exactly two distinct branch points, then h is a component function of the q-th root embedding up to reparametrizations.*

Now, s_μ is the sheeting number of f^μ. Moreover, from [8], we also have

$$\sum_{\mu=1}^{p} \frac{1}{s_\mu} = k$$

and $s_\mu | n$ for $1 \le \mu \le p$. Furthermore, the degree n of the branched covering π satisfies $\frac{p}{k} \le n \le 2^{p-1}$ by [8].

The terminology of ramification index follows [3], p. 217 while a ramification point of a map mentioned in this article is the same as a branch point mentioned in [3], p. 217. Moreover, given a finite branched covering map $\pi : S \to \mathbb{P}^1$, where S is a 1-dimensional projective algebraic variety, then $a \in \mathbb{P}^1$ is called a branch point of π if $a = \pi(c)$ for some ramification point c of π.

Definition 2.2 ([7], p.261)**.** Let $f : (\Delta, k ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ be a holomorphic isometric embedding, where k is the isometric constant. Then f is said to be **globally rigid** if and only if for any holomorphic isometric embedding $g : (\Delta, k ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$, we have $g = \Psi \circ f \circ \psi$ for some $\psi \in \mathrm{Aut}(\Delta)$ and $\Psi \in \mathrm{Aut}(\Delta^p)$.

Denote by $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\}$. Define a map $\rho_p : \mathcal{H} \to \mathcal{H}^p$ by

$$\rho_p(\tau) = \left(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \dots, \gamma^{p-1}\tau^{\frac{1}{p}}\right),$$

where $\gamma = e^{\frac{i\pi}{p}}$ and $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$ if $\tau = re^{i\theta}$, $0 < \theta < \pi$. From [6], the map ρ_p is a non-totally geodesic holomorphic isometric embedding. Then, the p-th root embedding $F_p : \Delta \to \Delta^p$ can be defined from ρ_p via the Cayley transform $\iota : \mathcal{H} \to \Delta$, $\tau \mapsto \frac{\tau - i}{\tau + i}$ and target automorphisms.

3. Boundary Behaviour of Holomorphic Isometric Embeddings

In this section, we are going to investigate how each component function $\widetilde{f^j}$ behave on the boundary $\partial\Delta$ of the unit disk Δ.

Lemma 3.1. *Let $f = (f^1, \ldots, f^p) : (\Delta, kds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a holomorphic isometric embedding. Suppose that f^j has a branch point $a_0 = e^{i\theta_0} \in \partial\Delta$ for some j, $1 \le j \le p$, where $\theta_0 \in [0, 2\pi)$ is a real number. Note that f can be extended continuously to $\tilde{f} = (\widetilde{f^1}, \ldots, \widetilde{f^p}) : \overline{\Delta} \to \overline{\Delta^p}$ by [6]. Then the component function $\widetilde{f^j}$ cannot map any arc $\{z = e^{i\theta} \in \mathbb{C} \mid \theta \in (\theta_0 - \delta, \theta_0 + \delta)\}$ into $\partial\Delta$ for $\delta > 0$. In particular, if $\widetilde{f^j}$ maps $\{z = e^{i\theta} \in \mathbb{C} \mid \theta_0 \le \theta < \theta_0 + \delta\}$ into $\partial\Delta$, then $\widetilde{f^j}$ maps $\{z = e^{i\theta} \in \mathbb{C} \mid \theta_0 - \delta' < \theta < \theta_0\}$ into Δ for some $\delta' > 0$.*

Proof. Suppose that $a_0 = e^{i\theta_0}$ is a branch point of $h := f^j$; more precisely, for the holomorphic isometry $f : \Delta \to \Delta^p$, $f|_{U_{a_0} \cap \Delta}$ cannot extend holomorphically to U_{a_0} for any neighborhood U_{a_0} of a_0 in $\mathbb{C}$. Consider h as a holomorphic map $\mathcal{H} \to \mathcal{H}$ and denote by $\widetilde{h}$ the extension of h to $\overline{\mathcal{H}}$, then we identify a_0 as a point a on the real line $\{z \in \mathbb{C} \mid \mathrm{Im} z = 0\}$. Let $\delta > 0$ so that $\delta < \min\{2\pi - \theta_0, \theta_0 - \pi\}$. Suppose that $\widetilde{h}$ maps $\{z \in \mathbb{C} \mid \mathrm{Im} z = 0,\ |z - a| < \delta_H\}$ into $\partial\mathcal{H}$, where $\delta_H > 0$ is some real number so that $\{z \in \mathbb{C} \mid \mathrm{Im} z = 0,\ |z - a| < \delta_H\}$ can be identified with $\{z = e^{i\theta} \in \mathbb{C} \mid \theta_0 - \delta < \theta < \theta_0 + \delta\}$ via Cayley transform. Take a neighborhood $U_a = \{z \in \mathbb{C} \mid |z - a| < \delta_H\}$ of a in $\mathbb{C}$ such that $\widetilde{h}$ maps $I_a := \{z \in U_a \mid \mathrm{Im} z = 0\}$ into $\partial\mathcal{H}$. Note that $\widetilde{h}|_{U_a \cap \overline{\mathcal{H}}}$ is continuous and $\widetilde{h}|_{U_a \cap \mathcal{H}}$ is holomorphic. By Schwarz Reflection Principle ([4], p. 211), there exists a holomorphic function $g : U_a \to \mathbb{C}$ such that

$$g(z) = \widetilde{h}(z) \quad \forall\, z \in \{z \in U_a \mid \mathrm{Im} z \ge 0\},$$

i.e. $\widetilde{h}|_{U_a \cap \mathcal{H}}$ can be extended holomorphically to U_a. However, a is a branch point of h, this leads to a contradiction.

Now, going back to the original holomorphic map $h : \Delta \to \Delta$, then the extension $\widetilde{h}$ of h cannot maps $\{z = e^{i\theta} \in \mathbb{C} \mid \theta_0 - \delta < \theta < \theta_0 + \delta\}$ into $\partial\Delta$. □

Let $f = (f^1, \ldots, f^p) : \Delta \to \Delta^p$ be a holomorphic isometric embedding with the isometric constant k and $f(0) = \mathbf{0}$. Choose an arbitrary component function f^j of f, suppose that $\{a_1, \ldots, a_m\} \subset \partial\Delta$ is the set of all distinct branch points of the finite branch covering $\pi_j : V_j \to \mathbb{P}^1$.

Lemma 3.2. *With the same settings as above, we suppose that $z_0 \in \partial\Delta$ is not a branch point of f^j, i.e. $z_0 \in A$, where $A \subset \partial\Delta \smallsetminus \{a_1, \ldots, a_m\}$ is some connected component. If $|\widetilde{f^j}(z_0)| = 1$, then $|\widetilde{f^j}(z)| = 1$ for all $z \in \overline{A}$, where $\overline{A}$ is the closure of A in $\partial\Delta$. Denote by $\tilde{f} = (\widetilde{f^1}, \ldots, \widetilde{f^p}) : \overline{\Delta} \to \overline{\Delta^p}$ the continuous extension of f. In particular, if the set B of all distinct branch points of the finite branched coverings π, π_μ, $1 \le \mu \le p$, are the same, say $B = \{a_1, \ldots, a_m\}$, and isometric constant of f equals $k = 1$, then for each connected component $A' \subset \partial\Delta \smallsetminus \{a_1, \ldots, a_m\}$, there is a unique $j = j(A') \in \{1, \ldots, p\}$ such that $\widetilde{f^j}(\overline{A'}) \subset \partial\Delta$.*

Proof. Note that $A \subset \partial\Delta$ is an open subset. Since z_0 is not a branch point, $\exists$ a neighborhood U_{z_0} of z_0 in $\mathbb{C}$ such that $f^j|_{U_{z_0} \cap \Delta}$ can be extended holomorphically to U_{z_0}. More precisely, $\exists$ a holomorphic function $g : U_{z_0} \to \mathbb{C}$ such that $g(z) = f^j(z)\ \forall\ z \in U_{z_0} \cap \Delta$. Note that f^j is non-constant, so g is a non constant holomorphic function on U_{z_0}; otherwise, if g is a constant function, then $f^j|_{U_{z_0} \cap \Delta} \equiv C$ for some constant C, this implies that $f^j \equiv C$ by the identity theorem because $U_{z_0} \cap \Delta$ is an open subset. By the same procedure, for each $z \in A$, f^j can be extended holomorphically to some open neighborhood U_z of z in $\mathbb{C}$, so f^j can be extended holomorphically to $U := \bigcup_{z \in A} U_z$, which is an open subset in $\mathbb{C}$. Denote also by $g = f^j$ the extension of $f^j|_{U \cap \Delta}$ to U. Note that U does not contain any branch point of f^j and $|f^j(z)|^2$ is real analytic on U.

By open mapping theorem, under the assumption that each f^j is non-constant so that the extension $g : U \to \mathbb{C}$ is non-constant, so for any open subset $V \subset U$, $g(V) \subset \mathbb{C}$ is open. Let $A' = f^j(A)$. If $|f^j(z_0)| = 1$ for some $z_0 \in A$, then $(f^j)^{-1}(A')$ contains some nonempty smooth real-analytic curve, actually $A \subset (f^j)^{-1}(A')$. For some open neighborhood U_0 of z_0 in U, $g(U_0) \subset \mathbb{C}$ is an open set containing the point $f^j(z_0) =: e^{i\phi_0}$ by open mapping theorem. In particular, $\exists\ \delta > 0$ such that $A_0 = \{e^{i\phi} \in \partial\Delta \mid \phi \in (\phi_0 - \delta, \phi_0 + \delta)\} \subset g(U_0)$, i.e. for each $e^{i\phi} \in A_0$, $\exists\ \zeta \in U$ such that $g(\zeta) = e^{i\phi} \in \partial\Delta$. By the functional equation, we have $|f^j(z)| \ne 1$ whenever $z \notin \partial\Delta$, so $|f^j(z)|^2 = 1$ for $z \in I$ for some non-empty open subset $I \subset A$. By the Identity Theorem for real-analytic functions (see [5], Corollary 1.2.7), we have $|f^j(z)| = 1 \ \forall\, z \in A$. The rest follows from the functional equation

$$\prod_{\mu=1}^{p} \left(1 - |f^\mu(z)|^2\right) = 1 - |z|^2,$$

Lemma 6.1 in [8], and the above results. □

4. The Minimal Case

Let $f = (f^1, \ldots, f^p) : \Delta \to \Delta^p$ be a holomorphic isometric embedding with isometric constant $k = 1$, sheeting number $n = p$ and $f(0) = \mathbf{0}$. From the settings in the introduction section, we have $s_\mu \le p$ and $\sum_{\mu=1}^{p} \frac{1}{s_\mu} = 1$ so that $s_\mu = p$ for $1 \le \mu \le p$. Denote the p branches of f by $f_l(z) = (f_l^1(z), \ldots, f_l^p(z))$ defined on Δ for $l = 1, \ldots, p$, then we have the polarized functional equation

$$\prod_{j=1}^{p} \left(1 - f_l^j(z)\overline{f_k^j(w)}\right) = 1 - z\overline{w}$$

for $z, w \in \Delta$ and $1 \le l, k \le p$. Let $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ be an irreducible projective algebraic curve containing Graph(f), then $\pi : V \to \mathbb{P}^1$, $(z, \xi_1, \ldots, \xi_p) \mapsto z$, is a finite p-sheeted branched covering over $\mathbb{P}^1$.

Lemma 4.1 (cf. [8])**.** *Note that all branch points of f_l are lying on $\partial\Delta$, so $\infty \in \mathbb{P}^1$ is not a branch point of the branched covering $\pi : V \to \mathbb{P}^1$. Then for each $l = 1, \ldots, p$, the set $\{f_l^j(\infty) : 1 \le j \le p\}$ contains exactly one infinite value. Moreover, for each $j = 1, \ldots, p$, the set $\{f_l^j(\infty) : 1 \le l \le p\}$ contains exactly one infinite value.*

Remark. A general version of this result has been mentioned implicitly in the proof of Proposition 5.3 in [8], p. 2914.

Proof. Consider the polarized functional equation

$$\prod_{j=1}^{p} \left(1 - f_l^j(z)\overline{f_l^j(w)}\right) = 1 - z\overline{w}$$

for some fixed $w \in B^1(0; \varepsilon)$. Note that the order of pole at $z = \infty$ is 1 on the right-hand side, and so is the pole order on the left-hand side, so for each $l = 1, \ldots, p$, the set $\{f_l^j(\infty) : 1 \le j \le p\}$ contains exactly one infinite value.
Let $V_j \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the projective-algebraic subvariety extending Graph (f^j). Since $f^j(0) = \mathbf{0}$, we have $(0, 0) \in V_j$ so that by Corollary 4.7 in [8], $(\infty, \infty) \in V_j \subset \mathbb{P}^1 \times \mathbb{P}^1$. Hence, for each $j = 1, \ldots, p$, the set $\{f_l^j(\infty) : 1 \le l \le p\}$ contains at least one infinite value. Combining with the first result, we prove that the set $\{f_l^j(\infty) : 1 \le l \le p\}$ contains exactly one infinite value for each $j = 1, \ldots, p$. □

4.1. **Unimodular Values at Branch Points.** Let $f = (f^1, \ldots, f^p) : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ be a holomorphic isometric embedding. Let $\pi : V \to \mathbb{P}^1$ be the finite branched covering map, where $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ is an irreducible projective-algebraic subvariety which extends the graph of f. Suppose that the degree of π is $n = p$, we say that the sheeting number of f is $n = p$. Note that $\pi^{-1}(\Delta) = \bigsqcup_{l=1}^{p} U_l$, where for $1 \le l \le p$, $U_l = \text{Graph}(f_l)$ for some holomorphic isometric embedding $f_l = (f_l^1, \ldots, f_l^p) : (\Delta, ds_\Delta^2) \to (G_l', ds_{G_l'}^2)$, where $G_l' \subset (\mathbb{P}^1 \smallsetminus \partial\Delta)^p$ is some connected component. More precisely, if we denoted by $\Delta^+ = \Delta$ and $\Delta^- = \mathbb{P}^1 \smallsetminus \overline{\Delta}$, then G_l' can be written as

$$G_l' = \Delta^{\chi_l^1} \times \cdots \times \Delta^{\chi_l^p},$$

for some $\chi_l^j \in \{+, -\}$, $1 \le j, l \le p$.
Note that all R^μ $(1 \le \mu \le p)$ have the same set of branch points by arguments after Lemma 6.3 in [8] (p. 2916). More precisely, the branching loci of π and π_μ, $1 \le \mu \le p$, are the same (by Lemma 6.3, [8]). Moreover, in [8], the ramification order of π at the point $(z, w_1, \ldots, w_p) \in V$ can be defined as the ramification order of any R^μ at w_μ, $1 \le \mu \le p$. Now, we define the ramification

index of π at some point in V as in [3], p. 217. Now, we look for the number of unimodular elements in the set

$$\{f_l^\mu(a_i) \mid 1 \le \mu \le p\}$$

for each branch point a_i of π and $1 \le l \le p$. The following lemma shows that the number is actually the ramification index of π at $(a_i, f_l^1(a_i), \dots, f_l^p(a_i)) \in V$.

Lemma 4.2. *Fixing $j \in \{1, \dots, p\}$. Let $\{a_1, \dots, a_m\} \subset \partial\Delta$ be the set of distinct branch points of R^j and let the branching order of R^j at a_i be b_i for $1 \le i \le m$, which is independent of the choice of j $(1 \le j \le p)$. For $1 \le i \le m$, let $v = (a_i, f_l^1(a_i), \dots, f_l^p(a_i)) \in \pi^{-1}(a_i)$ be a ramification point of π with ramification index $s \ge 2$. Then $\exists$ distinct $j_1, \dots, j_s \in \{1, \dots, p\}$ such that $|f_l^{j_\mu}(a_i)| = 1$ for $1 \le \mu \le s$. Furthermore, if $2 \le s < p$, then $|f_l^j(a_i)| \ne 1$ for $j \notin \{j_1, \dots, j_s\}$.*

Proof. Choose an arbitrary a_i in the set of all distinct branch points of π. Suppose that the ramification index of R^j at $f_l^j(a_i)$ is equal to s for some $j, l, s \in \{1, \dots, p\}$, then the ramification index of R^μ at $f_l^\mu(a_i)$ is also equal to s for $1 \le \mu \le p$ (by Lemma 6.3 in [8]). Now, we fix $l \in \{1, \dots, p\}$. In particular, after shrinking the ball $B^1(a_i, \varepsilon)$ if necessary, for $1 \le \mu \le p$, a Puiseux series for f_l^μ around the branch point a_i can be written as

$$f_l^\mu(z) = \varphi_l^\mu\left((z - a_i)^{\frac{1}{s}}\right) \quad \forall\, z \in B^1(a_i, \varepsilon),$$

where φ_l^μ is a holomorphic function on $B^1(0, \varepsilon^{\frac{1}{s}})$ for $1 \le \mu \le p$ and $\varepsilon > 0$ is some constant by [1]. Note that $\varphi_l^\mu(0) = f_l^\mu(a_i)$ for $1 \le \mu \le p$ and we have the functional equation

$$\prod_{\mu=1}^{p}\left(1 - f_l^\mu(z)\overline{f_l^\mu(a_i)}\right) = 1 - z\overline{a_i}$$

for such fixed l. Writing $z = a_i + (\zeta - a_i)^s$, then for $1 \le \mu \le p$, we have

$$f_l^\mu(a_i + (\zeta - a_i)^s) = \varphi_l^\mu\,(\zeta - a_i) \quad \forall\, \zeta \in B^1(a_i, \varepsilon^{\frac{1}{s}}),$$

and thus

$$\prod_{\mu=1}^{p}\left(1 - \varphi_l^\mu\,(\xi)\,\overline{\varphi_l^\mu(0)}\right) = -\overline{a_i}\xi^s$$

for $\xi \in B^1(0, \varepsilon^{\frac{1}{s}})$.

Suppose that $|f_l^j(a_i)| = |\varphi_l^j(0)| = 1$ for some $j \in \{1, \dots, p\}$. Consider the rational function $R^j : \mathbb{P}^1 \to \mathbb{P}^1$, then the holomorphic function $w(\xi) = \varphi_l^j(\xi)$ defined on $B^1(0, \varepsilon^{\frac{1}{s}})$ give a local parametrization of some branch of $\mathbb{P}^1$ around $f_l^j(a) \in \mathbb{P}^1$, namely $R^j(w(\xi)) = \xi^s + a$, so $\frac{\partial \varphi_l^j}{\partial \xi}(0) = w'(0) \ne 0$.

For $1 \le \mu \le p$, either $1 - \varphi_l^\mu(\xi)\overline{f_l^\mu(a_i)}$ has a zero of order 1 at $\xi = 0$ or $1 - \varphi_l^\mu(0)\overline{f_l^\mu(a_i)} = 1 - |f_l^\mu(a_i)|^2 \ne 0$.

Since the right hand side vanish to the order s at $\zeta = a_i$, $\exists$ distinct $j_1, \dots, j_s \in \{1, \dots, p\}$ such that $|f_l^{j_k}(a_i)| = |\varphi_l^{j_k}(0)| = 1$ for $1 \le k \le s$. Moreover, if $1 \le s < p$, then $|f_l^\mu(a_i)| = |\varphi_l^\mu(0)| \ne 1$ for $\mu \in \{1, \dots, p\} \setminus \{j_1, \dots, j_s\}$. ☐

4.2. **Proof of Theorem 1.1.** Now, we look for structures of the branched covering map $\pi : V \to \mathbb{P}^1$ from the functional equation, which provide further relations between different branches. The following proposition shows that for each distinct points $x, y \in \pi^{-1}(a_i)$, the ramification index of π at x is the same as that of π at y for each $i = 1, \dots, m$.

Proposition 4.3. *Let $\pi : V \to \mathbb{P}^1$ be the n-sheeted branched covering map as before, and $\{a_1, \dots, a_m\}$ be the set of all distinct points of π. Suppose that $n = p$. If $v \in \pi^{-1}(a_i)$ is a ramification point of π with ramification index $s \ge 2$, then $s \cdot |\pi^{-1}(a_i)| = p$, where $|\pi^{-1}(a_i)|$ denotes the cardinality of the set $\pi^{-1}(a_i)$. Moreover, we have $2 \le m \le 3$.*

Proof. Choosing an arbitrary branch point a_i of π. Note that in this case, ramification index of π_μ at $(a_i, f_l^\mu(a_i))$ is the same as ramification index of R^μ at $f_l^\mu(a_i)$ for $1 \le \mu \le p$.

Now, we choose a ramification point $f_l^1(a_i)$ of R^1 with ramification index s $(1 < s \le p)$, then

$f_l^\mu(a_i)$ is a ramification point of R^μ with ramification index s for $1 \leq \mu \leq p$. As in the proof of Lemma 4.2, one has

$$f_l^\mu(z) = \varphi_l^\mu\left((z-a_i)^{\frac{1}{s}}\right) \quad \forall\, z \in B^1(a_i, \varepsilon)$$

for some $\varepsilon > 0$ and some holomorphic function φ_l^μ defined on $B^1(0, \varepsilon^{\frac{1}{s}})$. Consider the functional equation

$$\prod_{\mu=1}^{p}\left(1 - f_l^\mu(z)\overline{f_k^\mu(a_i)}\right) = 1 - z\overline{a_i}.$$

for arbitrary $k \in \{1, \ldots, p\}$. Rewriting the above equation as

$$\prod_{\mu=1}^{p}\left(1 - \varphi_l^\mu(\xi)\,\overline{f_k^\mu(a_i)}\right) = 1 - (\xi^s + a_i)\overline{a_i} = -\overline{a_i}\xi^s. \tag{4.1}$$

Note that $\varphi_l^\mu(0) = f_l^\mu(a_i)$. Since there is a rational function R^μ such that $z = R^\mu(f_l^\mu(z))$, each f_l^μ is one-to-one on $\overline{\Delta}$ (note that f_l^μ extends continuously on $\overline{\Delta}$ by [6]). Suppose that $f_l^j(a_i) = \varphi_l^j(0) = \frac{1}{\overline{f_k^j(a_i)}}$ for some $j \in \{1, \ldots, p\}$, then follows from the same arguments in the proof of Lemma 4.2, we have $\frac{\partial \varphi_l^j}{\partial \xi}(0) \neq 0$. Hence, for $1 \leq \mu \leq p$, either $1 - \varphi_l^\mu(\xi)\overline{f_k^\mu(a_i)}$ has a zero of order 1 at $\xi = 0$ or $1 - \varphi_l^\mu(0)\overline{f_k^\mu(a_i)} \neq 0$.
Therefore, by comparing the vanishing order of both sides of the above functional equation (4.1) as $\xi \to 0$, we see that $\exists$ distinct $\mu_1, \ldots, \mu_s \in \{1, \ldots, p\}$ such that

$$f_l^{\mu_\nu}(a_i) = \varphi_l^{\mu_\nu}(0) = \frac{1}{\overline{f_k^{\mu_\nu}(a_i)}}, \quad 1 \leq \nu \leq s. \tag{4.2}$$

Moreover, if $s < p$, then $f_l^\mu(a_i) \neq \frac{1}{\overline{f_k^\mu(a_i)}}$ for $\mu \notin \{\mu_1, \ldots, \mu_s\}$. Similarly, for the chosen arbitrary $k \in \{1, \ldots, p\}$ in above argument, let the ramification index of R^μ at $f_k^\mu(a_i)$ be s' for some $1 \leq s' \leq p$ and $\forall\, \mu$, $1 \leq \mu \leq p$ (here $s' = 1$ means that R^μ is unramified at $f_k^\mu(a_i)$). Then one can write

$$f_k^\mu(z) = \psi_k^\mu\left((z-a_i)^{\frac{1}{s'}}\right) \quad \forall\, z \in B^1(a_i, \varepsilon')$$

for some $\varepsilon' > 0$ and some holomorphic function ψ_k^μ defined on $B^1(0, \varepsilon'^{\frac{1}{s'}})$. Consider the functional equation

$$\prod_{\mu=1}^{p}\left(1 - f_k^\mu(z)\overline{f_l^\mu(a_i)}\right) = 1 - z\overline{a_i} \tag{4.3}$$

as above. Similar to above arguments, we compare the vanishing order of both sides of the above functional equation (4.3) as $z \to a_i$, then $\exists$ distinct $j_1, \ldots, j_{s'} \in \{1, \ldots, p\}$ such that

$$f_k^{j_\nu}(a_i) = \frac{1}{\overline{f_l^{j_\nu}(a_i)}}, \quad 1 \leq \nu \leq s'. \tag{4.4}$$

Moreover, if $s' < p$, then $f_k^j(a_i) \neq \frac{1}{\overline{f_l^j(a_i)}}$ for $j \notin \{j_1, \ldots, j_{s'}\}$. Combining (4.2) and (4.4), we see that $s = s'$. Since $k \in \{1, \ldots, p\}$ is chosen arbitrarily, the ramification index of R^μ at $f_k^\mu(a_i)$ is precisely s for $1 \leq \mu, k \leq p$. Hence, we have

$$|\pi^{-1}(a_i)| \cdot s = p$$

and thus $s|p$ and $|\pi^{-1}(a_i)||p$. Moreover, since $2 \leq s \leq p$, we have

$$p = |\pi^{-1}(a_i)| \cdot s \geq 2|\pi^{-1}(a_i)|$$

so that $|\pi^{-1}(a_i)| \leq \frac{p}{2}$. Since a_i is chosen arbitrarily, we have $|\pi^{-1}(a_i)| \leq \frac{p}{2}$ for $i = 1, \ldots, m$. Now, from the Riemann-Hurwitz formula, we have

$$2p - 2 = \sum_{i=1}^{m} b_i = \sum_{i=1}^{m}\left(p - |\pi^{-1}(a_i)|\right) \geq \sum_{i=1}^{m}\frac{p}{2} = \frac{mp}{2}$$

and thus $m \leq \frac{4(p-1)}{p} < 4$, i.e. $m \leq 3$. We already know that $m \geq 2$ in [8], so we conclude that $2 \leq m \leq 3$.

□

Remark. We also have the Riemann-Hurwitz formula for the p-sheeted branched covering map π as follows:

$$2p - 2 = \sum_{i=1}^{m} p\left(1 - \frac{1}{v_i}\right),$$

where $v_i \cdot |\pi^{-1}(a_i)| = p$ for $1 \leq i \leq m$.

Corollary 4.4. (Global Rigidity of the $(2q+1)$-th Root Embedding) *Let $p = 2q+1$ be an odd integer, where $q \geq 1$ is an integer. Let $f : \Delta \to \Delta^p$ be a holomorphic isometric embedding with isometric constant $k = 1$ and $n = p$, then f is the p-th root embedding up to reparametrization.*

Remark. This corollary has been proven by Ng (cf. Theorem 6.5 in [8]) via another method (cf. Lemma 6.4 in [8]).

Proof. We shall use notations mentioned in Proposition 4.3. If $p \geq 2$ is odd, then since $s|p$, we have $s = \frac{p}{|\pi^{-1}(a_i)|} \geq 3$ for each i by Proposition 4.3. Therefore, $b_i = p - |\pi^{-1}(a_i)| \geq p - \frac{p}{3} = \frac{2p}{3}$ and thus

$$2p - 2 = \sum_{i=1}^{m} b_i \geq m \cdot \frac{2p}{3} \implies m \leq 3 \cdot \frac{p-1}{p} < 3 \implies m \leq 2.$$

On the other hand, we have $m \geq 2$, so we have $m = 2$. The rest follows from arguments in the proof of Theorem 6.5 in [8]. □

If $m = 2$ and $p \geq 2$ is an integer, then $(v_1, v_2) = (p, p)$ and $(b_1, b_2) = (p-1, p-1)$. Now, suppose that $m = 3$ and $p \geq 4$ is even, then there are three distinct branch points a_1, a_2, a_3 with branching order b_1, b_2, b_3 respectively. Moreover $v_i|\pi^{-1}(a_i)| = p$ and $b_i = p\left(1 - \frac{1}{v_i}\right)$ for $i = 1, 2, 3$. Now, we determine all possible cases of (v_1, v_2, v_3) as in [10], p. 30-31. Note that $2 \leq v_1, v_2, v_3 \leq p$. Without loss of generality, assume that $v_1 \geq v_2 \geq v_3$. From the Riemann-Hurwitz formula, we have

$$-2 = p\left(1 - \frac{1}{v_1} - \frac{1}{v_2} - \frac{1}{v_3}\right).$$

Then

$$-2 = p\left(1 - \frac{1}{v_1} - \frac{1}{v_2} - \frac{1}{v_3}\right) \geq p\left(1 - \frac{3}{v_3}\right).$$

Hence $1 - \frac{3}{v_3} < 0$ and thus $v_3 < 3$, but then $v_3 \geq 2$ so that $v_3 = 2$. Now,

$$2 = p\left(\frac{1}{v_1} + \frac{1}{v_2} - \frac{1}{2}\right) \leq p\left(\frac{2}{v_2} - \frac{1}{2}\right).$$

Then, $\frac{2}{v_2} - \frac{1}{2} > 0$ so that $v_2 < 4$, i.e. $v_2 \leq 3$. If $v_2 = 2$, then $p = 2v_1$. Thus, $m = 3$, $(v_1, v_2, v_3) = \left(\frac{p}{2}, 2, 2\right)$. If $v_2 = 3$, then $2 = p\left(\frac{1}{v_1} - \frac{1}{6}\right)$. Thus $\frac{1}{v_1} - \frac{1}{6} > 0 \implies 6 > v_1 \implies 5 \geq v_1$. Now, $(v_1, v_2, v_3) = (v_1, 3, 2)$ with $5 \geq v_1 \geq 3$. If $v_1 = 3$, then $2 = p\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{p}{6} \implies p = 12$. If $v_1 = 4$, then $2 = p\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{p}{12} \implies p = 24$. If $v_1 = 5$, then $2 = p\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{p}{30} \implies p = 60$.
Thus, we have determined all possibilities of (v_1, v_2, v_3) in case $m = 3$ and p is even as follows: In case $m = 3$, we have

(v_1, v_2, v_3)	(b_1, b_2, b_3)	degree of π
$(\frac{p}{2}, 2, 2)$	$(p-2, \frac{p}{2}, \frac{p}{2})$	p
$(3, 3, 2)$	$(8, 8, 6)$	12
$(4, 3, 2)$	$(18, 16, 12)$	24
$(5, 3, 2)$	$(48, 40, 30)$	60

TABLE 1. All possible cases when $m = 3$

Proposition 4.5. (Global Rigidity of the $2q$-th Root Embedding)
Suppose that $p = 2q$ for some integer $q \geq 2$. Let $f = (f^1, \ldots, f^{2q}) : \Delta \to \Delta^{2q}$ be a holomorphic isometric embedding with isometric constant $k = 1$, sheeting number $n = p = 2q$. Let $\pi : V \to \mathbb{P}^1$ be the $2q$-sheeted branched covering, where $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^{2q}$ is the irreducible projective-algebraic subvariety which extends the graph of f. Then the number of distinct branch points of π is exactly 2. In particular, f is precisely the $2q$-th root embedding up to reparametrizations.

Lemma 4.6. *Under the same assumptions in Proposition 4.5, and suppose that π has 3 distinct branch points $a_1, a_2, a_3 \in \partial\Delta$. Then, there is a component function f^j of f such that $\widetilde{f^j}(\overline{\Delta}) \subset \Delta$, where $\widetilde{f} = (\widetilde{f^1}, \ldots, \widetilde{f^{2q}}) : \overline{\Delta} \to \overline{\Delta^{2q}}$ is the continuous mapping such that $\widetilde{f}|_\Delta = f$.*

Proof. Let the ramification index of π at a_i be v_i for $i = 1, 2, 3$, then all possible (v_1, v_2, v_3) are listed in table 1.
We can write $a_j = e^{\theta_j}$ for $j = 1, 2, 3$ and assume that $0 \leq \theta_1 < \theta_2 < \theta_3 < 2\pi$ without loss of generality. Let $A_{3,1} = \{e^{i\theta} \in \partial\Delta \mid \theta \in (\theta_3, \theta_1 + 2\pi)\}$, $A_{1,2} = \{e^{i\theta} \in \partial\Delta \mid \theta \in (\theta_1, \theta_2)\}$ and $A_{2,3} = \{e^{i\theta} \in \partial\Delta \mid \theta \in (\theta_2, \theta_3)\}$. Then, by the properness of the holomorphic isometric embedding f (from [6]), Lemma 3.1 and Lemma 3.2, we can suppose that $\widetilde{f^1}(A_{3,1}) \subset \partial\Delta$ and $\widetilde{f^\mu}(A_{3,1}) \not\subset \partial\Delta$ for $2 \leq \mu \leq 2q$; $\widetilde{f^{2q}}(A_{1,2}) \subset \partial\Delta$ and $\widetilde{f^\mu}(A_{1,2}) \not\subset \partial\Delta$ for $1 \leq \mu \leq 2q-1$; $\widetilde{f^2}(A_{2,3}) \subset \partial\Delta$ and $\widetilde{f^\mu}(A_{2,3}) \not\subset \partial\Delta$ for $\mu \neq 2$.
For all cases listed in table 1, we have $v_3 = 2$. In order to be consistent to above settings, by the continuity of the map $\widetilde{f}$, we would have $|\widetilde{f^1}(a_3)| = |\widetilde{f^2}(a_3)| = 1$, $|\widetilde{f^\mu}(a_3)| < 1$ for $3 \leq \mu \leq 2q$ by Lemma 4.2, $|\widetilde{f^2}(a_2)| = |\widetilde{f^{2q}}(a_2)| = 1$ and $|\widetilde{f^1}(a_1)| = |\widetilde{f^{2q}}(a_1)| = 1$. Now, we assume that contrary that

$$\nexists\, j \in \{1, \ldots, 2q\} \text{ such that } \widetilde{f^j}(\overline{\Delta}) \subset \Delta. \tag{4.5}$$

Then, for $3 \leq \mu \leq 2q-1$, we should have $|\widetilde{f^\mu}(a_2)| = 1$ or $|\widetilde{f^\mu}(a_1)| = 1$.
In any cases listed in table 1, the number of elements in the set

$$I_1 := \{\mu \in \mathbb{Z} \mid 3 \leq \mu \leq 2q-1,\ |\widetilde{f^\mu}(a_2)| = 1 \text{ or } |\widetilde{f^\mu}(a_1)| = 1\}$$

is at most $2q-4$ because we already have $|\widetilde{f^2}(a_2)| = |\widetilde{f^{2q}}(a_2)| = 1$, $|\widetilde{f^1}(a_1)| = |\widetilde{f^{2q}}(a_1)| = 1$ and $v_1, v_2 \leq q = \frac{p}{2}$. In case $q = 2$ (i.e. $p = 2q = 4$), the above statements would imply $I_1 = \varnothing$. Note that $|\widetilde{f^\mu}(a_3)| < 1$ for $3 \leq \mu \leq 2q-1$, by the assumption 4.5, the set I_1 must have precisely $2q-3$ elements. This leads to a contradiction. Hence, we conclude that $\exists\, j \in \{1, \ldots, 2q\}$ such that $\widetilde{f^j}(\overline{\Delta}) \subset \Delta$. □

Proof of Proposition 4.5. Suppose that π has m distinct branch points. By proposition 4.3, we have $2 \leq m \leq 3$. Suppose that $m = 3$, then π has precisely three distinct branch points $a_1, a_2, a_3 \in \partial\Delta$. Let the ramification index of π at a_i be v_i for $i = 1, 2, 3$, then (v_1, v_2, v_3) is determined by table 1.
By Lemma 4.6, there is a component function f^j of f such that $\widetilde{f^j}(\overline{\Delta}) \subset \Delta$. Choose any continuous path $\gamma : [0, 1] \to \mathbb{P}^1$ joining $0 \in \mathbb{C} \subset \mathbb{P}^1$ to a point $z_0 \in \mathbb{P}^1 \smallsetminus \{a_1, a_2, a_3, 0\}$, then $\gamma(0) = 0$ and $\gamma(1) = z_0$. If $z_0 = \infty \in \mathbb{P}^1$, we assume that $\gamma(t) \in \mathbb{C}\ \ \forall\, t \in [0, 1)$. If $z_0 \neq \infty$, we assume that $\gamma(t) \in \mathbb{C}\ \ \forall\, t \in [0, 1]$. If $|f^j(z_0)| \geq 1$, then since γ is continuous, and f^j is continuous along the path γ by doing analytic continuation along γ, $\exists\, t_0 \in (0, 1)$ such that $|f^j(\gamma(t_0))| = 1$, but then from the functional equation

$$\prod_{\mu=1}^{2q} \left(1 - |f^\mu(z)|^2\right) = 1 - |z|^2,$$

we have $|\gamma(t_0)| = 1$ because $\gamma(t_0) \in \mathbb{C}$. But this contradicts to the assumption that$f^j(\overline{\Delta}) \subset \Delta$. If $z_0 = \infty$ and $|f^j(\gamma(t))| \to 1$ as $t \to 1$, then $\exists\, l \in \{1, \ldots, p\}$ such that $f^j_l(\infty) = \lim_{t \to \infty} f^j(\gamma(t))$. But then $\exists$ a rational function $R^j : \mathbb{P}^1 \to \mathbb{P}^1$ such that $z = R^j(f^j_l(z))$ for $1 \leq l \leq p$. This implies that R^j would map some element in $\partial\Delta$ to $\infty \in \mathbb{P}^1$, this contradicts to the fact that $R^j(\partial\Delta) \subset \partial\Delta$ in [8].
Hence, whenever f^j is extended complex-analytically along a continuous path $\gamma : [0, 1] \to \mathbb{P}^1 \smallsetminus \{a_1, a_2, a_3\}$ joining 0 to a point in $\mathbb{P}^1 \smallsetminus \{a_1, a_2, a_3, 0\}$, we have $|f^j(\gamma(t))| < 1\ \ \forall\, t \in [0, 1]$.
Now, we can construct a branched holomorphic covering map $S \to \mathbb{P}^1$ which branches over a_1, a_2, a_3 for some Riemann surface S, which is indeed the graph of the multivalued holomorphic function

extending the graph of f^j. But then from the above arguments, the image of any branch of f^j lies completely inside the unit disk Δ. The multivalued holomorphic function on $\mathbb{C}$, which extends f^j, can be realized as a non-constant holomorphic function $\widehat{f^j} : S \to \mathbb{C}$ defined on the Riemann surface S (since f^j is non-constant), but then image of $\widehat{f^j}$ would lie inside the union of all images of different branches of f^j, which is known to be lying completely inside Δ, i.e. $\widehat{f^j}(S) \subset \Delta$. However, by Maximum Principle (Corollary 2.6 in [2]), there does not exist a non-constant bounded holomorphic function $S \to \mathbb{C}$ on S, this leads to a contradiction. Hence the number of distinct branch points of π cannot be 3, i.e. $m \neq 3$. Thus $m = 2$ and the rest follows from arguments in the proof of Theorem 6.5 in [8]. □

Proof of Theorem 1.1. The case $p = 2$ follows from [8] already. If $p \geq 3$ is odd, the theorem follows from the corollary 4.4 (also follows from [8]). If $p \geq 4$ is even, the theorem follows from Proposition 4.5. □

Remark. We have proven that for any integer $p \geq 2$,

$$\mathbf{HI}_1(\Delta, \Delta^p; p) = \{\varphi \circ F_p \circ \psi \mid \varphi \in \mathrm{Aut}(\Delta^p),\ \psi \in \mathrm{Aut}(\Delta)\},$$

where $F_p : \Delta \to \Delta^p$ is the p-th root embedding.

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