On a class of stochastic partial differential equations

Jian Song

Abstract

This paper concerns the stochastic partial differential equation with multiplicative noise $\frac{\partial u}{\partial t} = \mathcal{L}u + u\dot{W}$, where $\mathcal{L}$ is the generator of a symmetric Lévy process X, $\dot{W}$ is a Gaussian noise and $u\dot{W}$ is understood both in the senses of Stratonovich and Skorohod. The Feynman-Kac type of representations for the solutions and the moments of the solutions are obtained, and the Hölder continuity of the solutions is also studied. As a byproduct, when $\gamma(x)$ is a nonnegative and nonngetive-definite function, a sufficient and necessary condition for $\int_0^t \int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds$ to be exponentially integrable is obtained.

1 Introduction

In [39], Walsh developed the theory of stochastic integrals with respect to martingale measures and used it to study the stochastic partial differential equations (SPDEs) driven by space-time Gaussian white noise. Dalang in his seminal paper [17] extended the definition of Wash's stochastic integral and applied it to solve SPDEs with Gaussian noise white in time and homogeneously colored in space (white-colored noise). Recently, the theories on SPDEs with white-colored noise have been extensively developed, and one can refer to, for instance, [13, 15, 16, 32, 37] and the references therein. For the SPDEs with white-colored noise, the methods used in the above-mentioned literature relies on the martingale structure of the noise, and hence cannot be applied to the case when the noise is colored in time. On the other hand, SPDEs driven by a Gaussian noise which is colored in time and (possibly) colored in space have attracted more and more attention.

In the present article, we consider the following SPDE in $\mathbb{R}^d$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{L}u + u\dot{W}, & t \geq 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

In the above equation, $\mathcal{L}$ is the generator of a Lévy process $\{X_t, t \geq 0\}$, $u_0(x)$ is a continuous and bounded function, and the noise $\dot{W}$ is a (generalized) Gaussian random field independent of X with the covariance function given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = |t-s|^{-\beta_0}\gamma(x-y), \tag{1.2}$$

where $\beta_0 \in (0,1)$ and γ is a nonnegative and nonnegative-definite (generalized) function. The product $u\dot{W}$ in (1.1) is understood either in the *Stratonovich* sense or in the *Skorohod* sense. Throughout the paper, we assume that X is a symmetric Lévy process with characteristic exponent $\Psi(\xi)$, i.e., $\mathbb{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$. Note that the symmetry implies that $\Psi(\xi)$ is a real-valued nonnegative function. Furthermore, we assume that X has transition functions denoted by $q_t(x)$, which also entails that $\lim_{|\xi|\to\infty}\Psi(\xi) = \infty$ by Riemann-Lebesgue lemma.

When $\mathcal{L} = \frac{1}{2}\Delta$ where Δ is the Laplacian operator, and $\dot{W}$ is colored in time and white in space, Hu and Nualart [28] investigated the conditions to obtain a unique mild solution for (1.1) in the Skorohod sense, and obtained the Feynman-Kac formula for the moments of the solution. When $\mathcal{L} = \frac{1}{2}\Delta$, and $\dot{W}$ is a fractional white noise with Hurst parameters $H_0 \in (\frac{1}{2}, 1)$ in time and $(H_1, \dots, H_d) \in (\frac{1}{2}, 1)^d$ in space, i.e., $\beta_0 = 2 - 2H_0$ and $\gamma(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$, Hu et al. [30] obtained a Feynman-Kac formula for a weak solution under the condition $2H_0 + \sum_{i=1}^d H_i > d+1$ for the SPDE in the Stratonovich sense. This result was extended to the case $\mathcal{L} = -(-\Delta)^{\alpha/2}$ in Chen et al. [11]. A recent paper [27] by Hu et al. studied (1.1) in both senses when $\mathcal{L} = \frac{1}{2}\Delta$ and $\dot{W}$ is a general Gaussian noise using the techniques of Malliavin calculus and Fourier analysis, obtained the Feynman-Kac formulas for the solutions and the moments of the solutions, and investigated Hölder continuity of the Feynman-Kac functional and the intermittency of the solutions.

There has been fruitful literature on (1.1) in the sense of Skorohod, especially when $\dot{W}$ is white in time. For instance, when $\mathcal{L} = \frac{1}{2}\Delta$, (1.1) is the well-known *parabolic Anderson model* ([1]) and has been extensively investigated in, for example, [6, 7, 8, 10, 35]. Foondun and Khoshnevisan [20, 21] studied the general nonlinear SPDEs. When $\dot{W}$ is colored both in time and in space, $\mathcal{L}$ is a fractional Laplacian, the intermittency property of (1.1) was investigated in Balan and Conus [4, 5].

The main purpose of the current paper is to study (1.1) in both senses of Stratonovich and Skorohod under the assumptions Hypothesis (I) in Section 3 and Hypothesis (II) in Section 5.1 respectively. Under Hypothesis (I), we will obtain Feynman-Kac type of representations for a mild solution to (1.1) in the Stratonovich sense and for the moments of the solution (Theorem 4.6 and Theorem 4.7). Under Hypothesis (II), we will show that the mild solution to (1.1) in the Skorohod sense exists uniquely, and obtain the Feynman-Kac formula for the moments of the solution (Theorem 5.3 and Theorem 5.5). Furthermore, under stronger conditions, we can get Hölder continuity of the solutions in both senses (Theorem 4.11 and Theorem 5.9). As a byproduct, we show that Hypothesis (I) is a sufficient and necessary condition for the Hamiltonian $\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds$ to be exponentially integrable (Proposition 3.2 and Theorem 3.3).

There are two key ingredients to prove the main result Theorem 4.6 for the Stratonovich case. One is to obtain the exponential integrability of $\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds$. When X is a Brownian motion, Le Gall's moment method ([34]) was applied in [30] to get the exponential integrability, and when X is a symmetric α-stable process, the techniques from large deviation were employed in [11, 12]. However, in the current paper, we cannot apply directly either of the two approaches due to the lacks of the self-similarity of the Lévy process X and the homogeneity of the spatial kernel function $\gamma(x)$. Instead, to get the desired exponential integrability, we estimate the moments of $\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds$ directly using Fourier analysis inspired by [27] and the techniques for the computation of moments used in [29]. The other key ingredient is to justify that the Feynman-Kac representation (4.10) is a mild solution to (1.1) in the sense of Definition 4.5. To this goal, we will apply the Malliavin calculus and follow the "standard" approach used in [28, 30, 11, 27].

We get the existence of the solution to (1.1) in the Stratonovich sense by finding its Feynman-Kac representation directly, while in this article we do not address its uniqueness which will be our future work. A possible "probabilistic" treatment that was used in [3] is to express the Duhamel solution as a sum of multiple Stratonovich integrals, and then investigate its relationship (the Hu-Meyer formula [31]) with the Wiener chaos expansion. Another approach is to consider (1.1) pathwisely as a "deterministic" equation. Hu et al. [27] obtained the existence and uniqueness of (1.1) in the Stratonovich sense when $\mathcal{L}=\frac{1}{2}\Delta$ and $\dot{W}$ is a general Gaussian noise, by linking it to a general pathwise equation for which the authors obtained the existence and uniqueness in the framework of weighted Besov spaces. For general SPDEs, one can refer to [9, 19, 23, 24] for the rough path treatment. Recently, Deya [18] applied Hairer's regularity structures theory ([25]) to investigate a nonlinear heat equation driven by a space-time fractional noise.

For (1.1) in the Skorohod sense, we obtain the existence and uniqueness result by studying the chaos expansion of the solution as has been done in [28, 5, 27]. We apply the approximation method initiated in [28] to get the Feynman-Kac type of representation for the moments of the solution. One possibly can also obtain the representation by directly computing the expectations of the products of Wiener chaoses as in [14].

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries on the Gaussian noise and Malliavin calculus. In Section 3, we provide a sufficient and necessary condition for the Hamiltonian $\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds$ to be exponentially integrable. In Section 4, the Feynman-Kac formula for a mild solution to (1.1) in the Stratonovich sense is obtained, the Feynman-Kac formula for the moments of the solution is provided, and the Hölder continuity of the solution is studied. Finally, in Section 5, we obtain the existence and uniqueness of the mild solution in the Skorohod sense under some condition, find the Feynman-Kac formula for the moments, and investigate the Hölder continuity of the solution.

2 Preliminaries

In this section, we introduce the stochastic integral with respect to the noise $\dot{W}$ and recall some material from Malliavin calculus which will be used.

Let $C_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ be the space of smooth functions on $\mathbb{R}_+\times\mathbb{R}^d$ with compact supports, and the Hilbert space $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)$ endowed with the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{H}}=\int_{\mathbb{R}_+^2}\int_{\mathbb{R}^{2d}}\varphi(s,x)\psi(t,y)|t-s|^{-\beta_0}\gamma(x-y)dsdtdxdy, \tag{2.1}$$

where $\beta_0 \in (0,1)$ and γ is a nonnegative and nonnegative-definite function. In a complete probability space $(\Omega, \mathcal{F}, P)$, we define an isonormal Gaussian process (see, e.g., [36, Definition 1.1.1]) $W = \{W(h), h \in \mathcal{H}\}$ with the covariance function given by

$$\mathbb{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}. \tag{2.2}$$

In this paper, we will also use the following stochastic integral to denote $W(\varphi)$,

$$W(\varphi) := \int_0^\infty \int_{\mathbb{R}^d} \varphi(s,x) W(ds, dx).$$

Denote $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions and let $\mathcal{S}'(\mathbb{R}^d)$ denote its dual space of tempered distributions. Let $\widehat{\varphi}$ or $\mathcal{F}\varphi$ be the Fourier transform of $\varphi \in \mathcal{S}'(\mathbb{R}^d)$:

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x} \varphi(x) dx.$$

By the Bochner-Schwartz theorem [22, Theorem 3], the *spectral measure* μ of the process W defined by

$$\int_{\mathbb{R}^d} \gamma(x)\varphi(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\mu(d\xi), \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}^d) \tag{2.3}$$

exists and is tempered (meaning that there exists $p \geq 1$ such that $\int_{\mathbb{R}^d} (1+|\xi|^2)^{-p} \mu(d\xi) < \infty$). The inner product in (2.1) now can be represented by:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^d} \widehat{\varphi}(s,\xi) \overline{\widehat{\psi}(t,\xi)} |t-s|^{-\beta_0} \mu(d\xi) ds dt, \quad \forall\, \varphi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{2.4}$$

where the Fourier transform is with respect to the space variable only, and $\overline{z}$ is the complex conjugate of z.

Throughout the paper, we assume that the covariance function $\gamma(x)$ possesses the following properties .

(1) $\gamma(x)$ is locally integrable.

(2) The Fourier transform $\widehat{\gamma}(\xi)$ is a nonnegative measurable function, and hence $\mu(d\xi) = \widehat{\gamma}(\xi)d\xi$ is absolutely continuous with respect to the Lebesgue measure.

(3) $\gamma(x) : \mathbb{R}^d \to [0, \infty]$ is a continuous function, where $[0,\infty]$ is the usual one-point compactification of $[0, \infty)$.

(4) $\gamma(x) < \infty$ if and only if $x \neq 0$ **OR** $\widehat{\gamma} \in L^\infty(\mathbb{R}^d)$ and $\gamma(x) < \infty$ when $x \neq 0$.

The function $\gamma(x)$ with the above four properties covers a number of kernels such as the Riesz kernel $|x|^{-\beta}$ with $\beta \in (0, d)$, the Cauchy Kernel $\prod_{j=1}^d (x_j^2 + c)^{-1}$, the Poisson kernel

$(|x|^2+c)^{-(d+1)/2}$, and the Ornstein-Uhlenbeck kernel $e^{-c|x|^\alpha}$ with $\alpha \in (0,2]$, for some constant $c \in (0,\infty)$.

Properties (1) and (2) make the spatial kernel $\gamma(x)$ a function of *positive type* ([33, Definition 5.1]). Therefore by [33, Lemma 5.6], for any two Borel probability measures $\nu_1(dx)$ and $\nu_2(dx)$, the following identity holds,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \gamma(x-y)\nu_1(dx)\nu_2(dy) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\gamma}(\xi)\mathcal{F}\nu_1(\xi)\overline{\mathcal{F}\nu_2(\xi)}d\xi, \tag{2.5}$$

where $\mathcal{F}\nu_i(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\nu_i(dx)$ is the Fourier transform of ν_i for $i=1,2$. The above formula, as in [5, Appendix] for instance, can be generalized to $\nu_i(dx) = f_i(x)dx$ with f_i belonging to the space $L^1_{\mathbb{C}}(\mathbb{R}^d)$ of integrable complex-valued functions for $i=1,2$, with $\nu_2(dy)$ on the left-hand side being replaced by its complex conjugate.

If we let $\nu_2(dx)$ be the Dirac delta measure $\delta_0(x)dx$, then we actually have

$$\int_{\mathbb{R}^d} \gamma(x)\nu_1(dx) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\gamma}(\xi)\mathcal{F}\nu_1(\xi)d\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}\nu_1(\xi)\mu(d\xi), \tag{2.6}$$

where $\nu_1(dx)$ is any Borel probability measure or the measure of the form $\nu_1(dx) = f_1(x)dx$ with $f_1 \in L^1_{\mathbb{C}}(\mathbb{R}^d)$. This allows us to have the following lemma.

Lemma 2.1. *For a d-dimensional random variable Y, we have*

$$\mathbb{E}[\gamma(Y)] = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathbb{E}\left[e^{-i\xi\cdot Y}\right]\mu(d\xi).$$

Especially, for any $a \in \mathbb{R}^d$, we have

$$\mathbb{E}[\gamma(X_t+a)] = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathbb{E}\left[e^{-i\xi\cdot(X_t+a)}\right]\mu(d\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot a}e^{-t\Psi(\xi)}\mu(d\xi).$$

Now we briefly recall some useful knowledge in Malliavin calculus. The reader is referred to [36] for more details. Let D be the Malliavin derivative, which is an operator mapping from the Sobolev space $\mathbb{D}^{1,2} \subset L^2(\Omega)$ endowed with the norm $\|F\|_{1,2} = \sqrt{\mathbb{E}[F^2] + \mathbb{E}[\|DF\|^2_{\mathcal{H}}]}$ to $L^2(\Omega;\mathcal{H})$. The divergence operator δ is defined as the the dual operator of D by the duality $\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]$ for all $F \in \mathbb{D}^{1,2}$ and $u \in L^2(\Omega;\mathcal{H})$ in the domain of δ. Note that when $u \in \mathcal{H}, \delta(u) = W(u)$, and that the operator δ is also called the *Skorohod integral* since it coincides with the Skorohod integral in the case of Brownian motion. When $F \in \mathbb{D}^{1,2}$ and $h \in \mathcal{H}$, we have

$$\delta(Fh) = F \diamond \delta(h), \tag{2.7}$$

where $\diamond$ means the wick product. For u in the domain of δ, we also denote $\delta(u)$ by $\int_0^\infty \int_{\mathbb{R}^d} u(s,y)W^\diamond(ds,dy)$ in this article. The following two formulas will be used in the proofs.

$$FW(h) = \delta(Fh) + \langle DF, h\rangle_{\mathcal{H}}, \tag{2.8}$$

for all $F \in \mathbb{D}^{1,2}$ and $h \in \mathcal{H}$.

$$\mathbb{E}\left[FW(h)W(g)\right] = \mathbb{E}\left[\langle D^2F, h\otimes g\rangle_{\mathcal{H}^{\otimes 2}}\right] + \mathbb{E}[F]\langle h, g\rangle_{\mathcal{H}}, \tag{2.9}$$

for all $F \in \mathbb{D}^{2,2}, h \in \mathcal{H}, g \in \mathcal{H}$.

The Wiener chaos expansion has been used in, e.g., [28, 4], to deal with (1.1) in the Skorohod sense. Here we recall some basic facts. Let F be a square integrable random variable measurable with respect to the σ-algebra generated by W. Then F has the chaos expansion

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} F_n,$$

where F_n belongs to the n-th Wiener chaos space $\mathbb{H}_n$. Moreover, $F_n = I_n(f_n)$ for some $f_n \in \mathcal{H}^{\otimes n}$, and the expansion is unique if we require that all f_n's are symmetric in its n variables. Here $I_n : \mathcal{H}^{\otimes n} \to \mathbb{H}_n$ is the multiple Wiener integral. We have the following isometry

$$\mathbb{E}[|I_n(f_n)|^2] = n!\|\widetilde{f}_n\|^2_{\mathcal{H}^{\otimes n}}, \tag{2.10}$$

where $\widetilde{f}_n$ is the symmetrization of f_n.

3 On the exponential integrability

In this section, we will show that Hypothesis (I) below is a sufficient and necessary condition such that for all $\lambda, t > 0$

$$\mathbb{E}\left[\exp\left(\lambda\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds\right)\right] < \infty.$$

Hypothesis (I)**.** *The spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{1}{1+(\Psi(\xi))^{1-\beta_0}}\mu(d\xi) < \infty.$$

Remark 3.1. *When $\mathcal{L} = -(-\Delta)^{\alpha/2}$ for $\alpha \in (0,2]$ and $\gamma(x)$ is of one of the forms $\prod_{j=1}^d |x_j|^{\beta_j}$, $|x|^{-\beta}$ and $\delta_0(x)$, Hypothesis (I) then coincides with the conditions in [29, 11].*

First, we prove that Hypothesis (I) is a necessary condition.

Proposition 3.2.

$$\mathbb{E}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds < \infty, \quad \textit{for all } t > 0$$

if and only if μ satisfies Hypothesis (I).

Proof. By Lemma 2.6,

$$\int_0^t \int_0^t |r-s|^{-\beta_0} \mathbb{E}\left[\gamma(X_r - X_s)\right] drds = \frac{1}{(2\pi)^d} \int_0^t \int_0^t |r-s|^{-\beta_0} \int_{\mathbb{R}^d} e^{-|r-s|\Psi(\xi)} \mu(d\xi) drds,$$

and the result follows from Fubini's theorem and Lemma 3.7. ∎

The following theorem is the main result in this section.

Theorem 3.3. *Let the measure μ satisfy Hypothesis (I), then for all $t, \lambda > 0$,*

$$\mathbb{E}\left[\exp\left(\lambda \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dsdr\right)\right] < \infty.$$

Remark 3.4. *The above theorem, together with Proposition 3.2, actually declares the equivalence between the integrability and the exponential integrability of $\int_0^t \int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr$. This surprising result is mainly a consequence of the Markovian property of the Lévy process X. A result in the same flavor for $\int_0^t f(B_s)ds$ where B is a standard Brownian motion and f is a positive measurable function has been discovered by Khas'minskii [26] (see, e.g., [38, Lemma 2.1]).*

Proof. Note that $\int_0^t \int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr = 2\int_0^t \int_0^r |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr$, and equivalently we will study the exponential integrability of $\int_0^t \int_0^r |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr$. Inspired by the method in the proof of [29, Theorem 1], we estimate the n-th moments as follows.

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \int_0^r |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr\right)^n = \int_{[0<s<r<t]^n} \mathbb{E}\left(\prod_{j=1}^n |r_j - s_j|^{-\beta_0}\gamma(X_{r_j} - X_{s_j})\right) dsdr \\
=& n! \int_{[0<s<r<t]^n \cap [0<r_1<r_2\cdots<r_n<t]} \mathbb{E}\left(\prod_{j=1}^n |r_j - s_j|^{-\beta_0}\gamma(X_{r_j} - X_{s_j})\right) dsdr \\
\le& n! \int_{[0<s<r<t]^n \cap [0<r_1<r_2\cdots<r_n<t]} \prod_{j=1}^n |r_j - \eta_j|^{-\beta_0} \mathbb{E}\left[\gamma(X_{r_j} - X_{\eta_j})\right] dsdr.
\end{aligned}$$

The last inequality, where η_j is the point in the set $\{r_{j-1}, s_j, s_{j+1}, \dots, s_n\}$ which is closest to r_j from the left, holds since $\mathbb{E}\left[\gamma(X_{r_j} - X_{s_j})\right] = \mathbb{E}\left[\gamma(X_{r_j} - X_{\eta_j} + X_{\eta_j} - X_{s_j})\right] \le \mathbb{E}\left[\gamma(X_{r_j} - X_{\eta_j})\right]$ by the independent increment property of X and Lemma 3.9. Note that $dsdr$ actually means $ds_1 \dots ds_n dr_1 \dots dr_n$ in the above last three integrals. Throughout the article, we will take this kind of abuse of the notation for simpler exposition.

Fix the points $r_1 < \cdots < r_n$, we can decompose the set $[0 < s < r < t]^n \cap [0 < r_1 < r_2 \cdots < r_n < t]$ into $(2n-1)!!$ disjoint subsets depending on which interval the s_i's are placed in. More precisely, s_1 must be in $(0, r_1)$, while s_2 could be in $(0, s_1)$, (s_1, r_1) or (r_1, r_2). Similarly, there are $(2j-1)$ choices to place s_j. Over each subset, we denote the integral by

$$I_\sigma := \int_{[0<z_1<\cdots<z_{2n}<t]} \prod_{j=1}^{n} |z_{\sigma(j)} - z_{\sigma(j)-1}|^{-\beta_0} \mathbb{E}\left[\gamma(X_{z_{\sigma(j)}} - X_{z_{\sigma(j)-1}})\right] dz,$$

where $\sigma_{(1)} < \cdots < \sigma_{(n)}$ are n distinct elements in the set $\{2, 3, \ldots, 2n\}$. Hence

$$\mathbb{E}\left[\left(\int_0^t \int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr\right)^n\right] \le n!\times\left[\text{sum of the } (2n-1)!! \text{ terms of } I_\sigma' s\right]. \quad (3.1)$$

Next, for fixed n, we will provide a uniform upper bound for all $I_\sigma' s$. Noting that $X_{z_{\sigma(j)}} - X_{z_{\sigma(j)-1}} \overset{d}{=} X_{z_{\sigma(j)} - z_{\sigma(j)-1}}$ and letting $y_j = z_j - z_{j-1}$, we have

$$\begin{aligned} I_\sigma &= \int_{[0<y_1+y_2+\cdots+y_{2n}<t,\, 0<y_1,\cdots,y_{2n}<t]} \prod_{j=1}^{n} |y_{\sigma(j)}|^{-\beta_0} \mathbb{E}[\gamma(X_{y_{\sigma(j)}})]dy \\ &\le \frac{t^n}{n!} \int_{[0<y_1+y_2+\cdots+y_n<t,\, 0<y_1,\cdots,y_n<t]} \prod_{j=1}^{n} |y_i|^{-\beta_0} \mathbb{E}[\gamma(X_{y_j})]dy \\ &= \frac{t^n}{n!} \int_{[0<z_1<\cdots<z_n<t]} \prod_{j=1}^{n} |z_j - z_{j-1}|^{-\beta_0} \mathbb{E}[\gamma(X_{z_j - z_{j-1}})]dz \\ &= \frac{t^n}{n!} \int_{[0<z_1<\cdots<z_n<t]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} |z_j - z_{j-1}|^{-\beta_0} e^{-(z_j - z_{j-1})\Psi(\xi_j)} \mu(d\xi)dz. \end{aligned} \quad (3.2)$$

Note that

$$\begin{aligned} &\int_{[0<z_1<\cdots<z_n<t]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} |z_j - z_{j-1}|^{-\beta_0} e^{-(z_j - z_{j-1})\Psi(\xi_j)} \mu(d\xi)dz \\ = &\int_{\Omega_t^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} s_j^{-\beta_0} e^{-s_j\Psi(\xi_j)} \mu(d\xi)ds, \end{aligned} \quad (3.3)$$

where

$$\Omega_t^n = \left\{(s_1, \ldots, s_n) \in [0,\infty)^n : \sum_{j=1}^{n} s_j \le t\right\}. \quad (3.4)$$

For fixed large N, denote

$$\varepsilon_N = \int_{[|\xi|\ge N]} \frac{1}{(\Psi(\xi))^{1-\beta_0}} \mu(d\xi), \quad \text{and} \quad m_N = \mu([|\xi| \le N]). \quad (3.5)$$

Thus, by (3.1), (3.2), (3.3) and Proposition 3.5, we have

$$\begin{aligned} &\mathbb{E}\left[\left(\lambda \int_0^t \int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)dsdr\right)^n\right] \\ \le &(2n-1)!!\lambda^n t^n \sum_{k=0}^{n} \binom{n}{k} \frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^k}{\Gamma(k(1-\beta_0)+1)} m_N^k \left[A_0\varepsilon_N\right]^{n-k}. \end{aligned} \quad (3.6)$$

Now, for fixed t and λ, we can choose N sufficiently large such that $4A_0\lambda t\varepsilon_N < 1$. Consequently,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\lambda\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)dsdr\right)\right]\\
\leq&\sum_{n=0}^{\infty}\lambda^n t^n\frac{(2n-1)!!}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^k}{\Gamma(k(1-\beta_0)+1)}m_N^k\left[A_0\varepsilon_N\right]^{n-k}\\
=&\sum_{k=0}^{\infty}\frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^k}{\Gamma(k(1-\beta_0)+1)}m_N^k\left[A_0\varepsilon_N\right]^{-k}\sum_{n=k}^{\infty}\lambda^n t^n\frac{(2n-1)!!}{n!}\binom{n}{k}\left[A_0\varepsilon_N\right]^{n}\\
\leq&\sum_{k=0}^{\infty}\frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^k}{\Gamma(k(1-\beta_0)+1)}m_N^k\left[A_0\varepsilon_N\right]^{-k}\sum_{n=k}^{\infty}\left[4\lambda A_0 t\varepsilon_N\right]^{n}\\
=&\frac{1}{1-4\lambda A_0 t\varepsilon_N}\sum_{k=0}^{\infty}\frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^k}{\Gamma(k(1-\beta_0)+1)}\left(4\lambda t m_N\right)^k<\infty,
\end{aligned}$$

where in the second inequality we used the estimate $\frac{(2n-1)!!}{n!}\binom{n}{k}\leq 2^n\cdot 2^n=4^n$. The proof is concluded. ■

The following proposition, which plays a key role in this article, is a generalized version of Lemma 3.3 in [27].

Proposition 3.5. *For $\beta_0\in[0,1)$, assume*

$$\int_{\mathbb{R}^d}\frac{1}{1+(\Psi(\xi))^{1-\beta_0}}\mu(d\xi)<\infty.$$

Then there exists a positive constant A_0 depending on β_0 only such that for all $N>0$,

$$\int_{\Omega_t^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}\mu(d\xi)dr\leq\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^k}{\Gamma(k(1-\beta_0)+1)}m_N^k\left[A_0\varepsilon_N\right]^{n-k},$$

where ε_N and m_N are given by (3.5), and Ω_t^n is given by (3.4).

Proof. The proof essentially follows the approach used in the proof of [27, Lemma 3.3].

First note that the assumption implies that $\lim_{N\to\infty}\varepsilon_N=0$, and since $\mu(d\xi)$ is a tempered measure, then $m_N<\infty$ for all $N>0$. For a subset S of $\{1,2,\ldots,n\}$, we denote its complement by S^c, i.e., $S^c := \{1,2,\dots,n\}\backslash S$.

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}}\int_{\Omega_t^n}\prod_{j=1}^n r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}dr\mu(d\xi)\\
&=\int_{\mathbb{R}^{nd}}\int_{\Omega_t^n}\prod_{j=1}^n r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}\left[I_{[|\xi_j|\le N]}+I_{[|\xi_j|>N]}\right]dr\mu(d\xi)\\
&=\sum_{S\subset\{1,2,\dots,n\}}\int_{\mathbb{R}^d}\int_{\Omega_t^n}\prod_{l\in S} r_l^{-\beta_0}e^{-r_l\Psi(\xi_l)}I_{[|\xi_l|\le N]}\prod_{j\in S^c} r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}I_{[|\xi_j|>N]}dr\mu(d\xi)\\
&\le\sum_{S\subset\{1,2,\dots,n\}}\int_{\mathbb{R}^d}\int_{\Omega_t^n}\prod_{l\in S} r_l^{-\beta_0}I_{[|\xi_l|\le N]}\prod_{j\in S^c} r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}I_{[|\xi_j|>N]}dr\mu(d\xi).
\end{aligned}$$

Note that $\Omega_t^n\subset\Omega_t^S\times\Omega_t^{S^c}$, where $\Omega_t^I=\{(r_i, i\in I): r_i\ge 0, \sum_{j\in I} r_i\le t\}$ for any $I\subset\{1,2,\dots,n\}$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}}\int_{\Omega_t^n}\prod_{j=1}^n r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}dr\mu(d\xi)\\
&\le\sum_{S\subset\{1,2,\dots,n\}}\int_{\mathbb{R}^{nd}}\int_{\Omega_t^S\times\Omega_t^{S^c}}\prod_{l\in S} r_l^{-\beta_0}I_{[|\xi_l|\le N]}\prod_{j\in S^c} r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}I_{[|\xi_j|>N]}dr\mu(d\xi).
\end{aligned}$$

By Lemma 3.11, we have

$$\int_{\Omega_t^S}\prod_{l\in S} r_l^{-\beta_0}dr=\frac{\left(\Gamma(1-\beta_0)t^{1-\beta_0}\right)^{|S|}}{\Gamma(|S|(1-\beta_0)+1)}.$$

On the other hand, there exists $A_0>0$ depending on β_0 only such that

$$\begin{aligned}
&\int_{\Omega_t^{S^c}}\prod_{j\in S^c} r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}dr\le\int_{[0,t]^{|S^c|}}\prod_{j\in S^c} r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}dr\\
&\le\prod_{j\in S^c}\int_0^t r^{-\beta_0}e^{-r\Psi(\xi_j)}dr\le\prod_{j\in S^c}A_0(\Psi(\xi_j))^{-1+\beta_0},
\end{aligned}$$

where the last equality holds since $\int_0^t r^{-\beta_0}e^{-ar}dr=a^{-1+\beta_0}\int_0^{at}s^{-\beta_0}e^{-s}ds\le a^{-1+\beta_0}\int_0^\infty s^{-\beta_0}e^{-s}ds$.

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{R}^{nd}}\int_{\Omega_t^n}\prod_{j=1}^n r_j^{-\beta_0}e^{-r_j\Psi(\xi_j)}dr\mu(d\xi)\\
&\le\sum_{S\subset\{1,2,\dots,n\}}\int_{\mathbb{R}^{nd}}\frac{\big(\Gamma(1-\beta_0)t^{1-\beta_0}\big)^{|S|}}{\Gamma(|S|(1-\beta_0)+1)}\prod_{l\in S}I_{[|\xi_l|\le N]}\prod_{j\in S^c}A_0(\Psi(\xi_j))^{-1+\beta_0}I_{[|\xi_j|>N]}\mu(d\xi)\\
&=\sum_{S\subset\{1,2,\dots,n\}}\frac{\big(\Gamma(1-\beta_0)t^{1-\beta_0}\big)^{|S|}}{\Gamma(|S|(1-\beta_0)+1)}A_0^{|S^c|}m_N^{|S|}\varepsilon_N^{|S^c|}\\
&=\sum_{k=0}^n\binom{n}{k}\frac{\big(\Gamma(1-\beta_0)t^{1-\beta_0}\big)^{k}}{\Gamma(k(1-\beta_0)+1)}A_0^{n-k}m_N^k\varepsilon_N^{n-k},
\end{aligned}
$$

and the proof is concluded. ∎

Remark 3.6. *If we assume the following stronger condition,*

$$\int_{\mathbb{R}^d}\frac{1}{1+(\Psi(\xi))^{1-\beta_0-\varepsilon_0}}\mu(d\xi)<\infty$$

for some $\varepsilon_0\in(0,1-\beta_0)$, we may prove that for all $\lambda,t>0$

$$\mathbb{E}\left[\lambda\exp\left(\left|\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-X_s)dsdr\right|^p\right)\right]<\infty,\quad \text{when } p<\frac{1}{1-\varepsilon_0},\tag{3.7}$$

without involving Proposition 3.5.

Now we estimate the integral over $\mathbb{R}^{nd}$ in the last term of (3.2) first. By (3.2) and Lemma 3.10, there exists $C>0$ depending only on $1-\beta_0-\varepsilon_0$ and $\mu(d\xi)$, such that

$$I_\sigma\le C^n\frac{t^n}{n!}\int_{[0<z_1<z_2\cdots<z_n<t]}\prod_{j=1}^n|z_j-z_{j-1}|^{-\beta_0}\prod_{j=1}^n[1+(z_j-z_{j-1})^{-1+\beta_0+\epsilon}]dz.$$

Denote $\tau=(\tau_1,\dots,\tau_n)$ and $|\tau|=\sum_{j=1}^n\tau_j$. Then

$$\prod_{j=1}^n[1+(z_j-z_{j-1})^{-1+\beta_0+\epsilon_0}]=\sum_{\tau\in\{0,1\}^n}\prod_{j=1}^n(z_j-z_{j-1})^{\tau_j(-1+\beta_0+\varepsilon_0)}=\sum_{\tau\in\{0,1\}^n}J_\tau=\sum_{m=0}^n\sum_{|\tau|=m}J_\tau.$$

When $|\tau|=m$ and $t\ge1$, by Lemma 3.11, we have

$$\int_{[0<z_1<z_2\cdots<z_n<t]}\prod_{j=1}^n|z_j-z_{j-1}|^{-\beta_0}J_\tau dz\le\frac{C^nt^{m\varepsilon_0+(n-m)(1-\beta_0)}}{\Gamma(m\varepsilon_0+(n-m)(1-\beta_0)+1)}\le\frac{C^nt^{n(1-\beta_0)}}{\Gamma(n\varepsilon_0+1)},$$

noting that $\varepsilon_0<1-\beta_0$.

Note that there are $\binom{n}{m}$ J_τ's for $|\tau| = m$, and hence

$$I_\sigma \le C^n \frac{t^{n(2-\beta_0)}}{n!} \sum_{m=0}^{n} \binom{n}{m} \frac{1}{\Gamma(n\varepsilon_0 + 1)} \le C^n \frac{t^{n(2-\beta_0)}}{n!} (n+1) 2^n \frac{1}{(n\varepsilon_0/3)^{n\varepsilon_0}}, \tag{3.8}$$

where in the last step we use the properties $\binom{n}{m} \le 2^n$ and $\Gamma(x+1) \ge (x/3)^x$.

Combining (3.1) and (3.8), we have, for all $\lambda > 0$ and $t > 0$,

$$\mathbb{E}\left[\left(\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) ds dr\right)^n\right] \le \left(C t^{2-\beta_0}\right)^n (n!)^{1-\varepsilon_0},$$

where $C > 0$ depends on β_0, ε_0 and $\mu(d\xi)$, and then (3.7) follows.

Lemma 3.7. *There exist positive constants C_1 and C_2 depending on β_0 only such that*

$$\frac{1}{1 + x^{1-\beta_0}} \int_0^t s^{-\beta_0} e^{-s} ds \le \int_0^t s^{-\beta_0} e^{-sx} ds \le \frac{1}{1 + x^{1-\beta_0}} (C_1 + C_2 t^{1-\beta_0}),\ \forall\, x > 0.$$

Similarly, there exist positive constants D_1 and D_2 depending on β_0 only such that

$$\frac{2}{1 + x^{1-\beta_0}} \int_0^t \int_0^s r^{-\beta_0} e^{-r} dr ds \le \int_0^t \int_0^t |r-s|^{-\beta_0} e^{-|r-s|x} dr ds \le \frac{2}{1 + x^{1-\beta_0}} (D_1 t + D_2 t^{2-\beta_0}),\ \forall\, x > 0.$$

Proof. An change of variable implies that

$$\int_0^t s^{-\beta_0} e^{-sx} ds = x^{\beta_0 - 1} \int_0^{tx} r^{-\beta_0} e^{-r} dr.$$

The first inequality is a consequence of the following observation. When $x \ge 1$,

$$x^{\beta_0-1} \int_0^t r^{-\beta_0} e^{-r} dr \le x^{\beta_0-1} \int_0^{tx} r^{-\beta_0} e^{-r} dr \le x^{\beta_0-1} \int_0^\infty r^{-\beta_0} e^{-r} dr,$$

and when $0 < x < 1$,

$$\int_0^t s^{-\beta_0} e^{-s} ds \le \int_0^t s^{-\beta_0} e^{-sx} ds \le \int_0^t s^{-\beta_0} ds.$$

The second estimate follows from the first one and the following equality

$$\int_0^t \int_0^t |r-s|^{-\beta_0} e^{-x|r-s|} ds dr = 2 \int_0^t \int_0^r (r-s)^{-\beta_0} e^{-x(r-s)} ds dr = 2 \int_0^t \int_0^r s^{-\beta_0} e^{-xs} ds dr. \qquad \blacksquare$$

Remark 3.8. *Using similar approach in the above proof, we can show that the two inequalities hold for*

$$\sup_{a \in \mathbb{R}} \int_0^t s^{-\beta_0} e^{-|s+a|x} ds \quad \text{and} \quad \sup_{a \in \mathbb{R}} \int_0^t \int_0^t |r-s|^{-\beta_0} e^{-|r-s+a|x} dr ds$$

as well. It suffices to show that the upper bounds hold. We prove the first one as an illustration. When $0 < x < 1$,

$$\int_0^t s^{-\beta_0} e^{-|s+a|x} ds \le \int_0^t s^{-\beta_0} ds;$$

when $x \ge 1$,

$$\int_0^t s^{-\beta_0} e^{-|s+a|x} ds \le x^{\beta_0 - 1} \int_0^\infty s^{-\beta_0} e^{-|s+ax|} ds \le C x^{\beta_0 - 1}$$

where

$$C = \sup_{a \in \mathbb{R}} \int_0^\infty s^{-\beta_0} e^{-|s+ax|} ds \le \int_0^1 s^{-\beta_0} ds + \sup_{a\in\mathbb{R}} \int_1^\infty e^{-|s+ax|} ds \le \int_0^1 s^{-\beta_0} ds + \int_{-\infty}^\infty e^{-|s|} ds < \infty,$$

and the upper bound is obtained.

Lemma 3.9. $\mathbb{E}[\gamma(X_t + a)] \le \mathbb{E}[\gamma(X_t)]$, *for all* $a \in \mathbb{R}^d$.

Proof. By Lemma 2.1,

$$\mathbb{E}[\gamma(X_t + a)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot a} e^{-t\Psi(\xi)} \mu(d\xi) \le \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} \mu(d\xi) = \mathbb{E}[\gamma(X_t)]. \quad \blacksquare$$

Lemma 3.10. *Suppose*

$$\int_{\mathbb{R}^d} \frac{1}{1 + (\Psi(\xi))^\alpha} \mu(d\xi) < \infty,$$

for some $\alpha > 0$, *then there exists a constant* $C > 0$ *depending on* $\mu(d\xi)$ *and* α *only, such that*

$$\int_{\mathbb{R}^d} e^{-x\Psi(\xi)} \mu(d\xi) \le C(1 + x^{-\alpha}),\ \forall x > 0.$$

Proof. Since $\lim_{|\xi|\to\infty} \Psi(\xi) = \infty$, we can choose $M > 0$ such that $\Psi(\xi) > 1$ when $|\xi| > M$. Clearly

$$\int_{\mathbb{R}^d} e^{-x\Psi(\xi)} \mu(d\xi) = \int_{[|\xi| \le M|]} e^{-x\Psi(\xi)} \mu(d\xi) + \int_{[|\xi| > M]} e^{-x\Psi(\xi)} \mu(d\xi).$$

The first integral on the right-hand side is bounded by $\mu([|\xi| \le M])$ which is finite. For the second integral, note that $y^\alpha e^{-y}$ is uniformly bounded for all $y \ge 0$, and hence there exists a constant C depending on α only such that

$$\int_{[|\xi|>M]} e^{-x\Psi(\xi)} \mu(d\xi) \le C \int_{[|\xi|>M]} x^{-\alpha} (\Psi(\xi))^{-\alpha} \mu(d\xi) \le x^{-\alpha} \int_{[|\xi|>M]} \frac{2C}{1 + (\Psi(\xi))^\alpha} \mu(d\xi). \quad \blacksquare$$

Lemma 3.11. *Suppose* $\alpha_i \in (-1, 1), i = 1, \dots, n$ *and let* $\alpha = \alpha_1 + \cdots + \alpha_n$. *Then*

$$\int_{[0 < r_1 < \cdots < r_n < t]} \prod_{i=1}^n (r_i - r_{i-1})^{\alpha_i}\ dr_1 \dots dr_n = \frac{\prod_{i=1}^n \Gamma(\alpha_i + 1) t^{\alpha+n}}{\Gamma(\alpha + n + 1)},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ *is the Gamma function.*

Proof. The result follows from a direct computation of the iterated integral with respect to $r_n, r_{n-1}, \ldots, r_1$ orderly. The properties $\Gamma(x+1) = x\Gamma(x)$ and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ are used in the computation, where $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt$ for $x, y > 0$ is the beta function. ∎

4 Stratonovich equation

In the this section, we will use the approximation method ([28, 30, 11, 27]) to study (1.1) in the Stratonovich sense.

4.1 Definition of $\int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y)W(dr, dy)$

Denote $g_\delta(t) := \frac{1}{\delta} I_{[0,\delta]}(t)$ for $t \geq 0$ and $p_\varepsilon(x) = \frac{1}{\varepsilon^d} p(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^d$, where $p(x) \in C_0^\infty(\mathbb{R}^d)$ is a symmetric probability density function and its Fourier transform $\widehat{p}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$. We also have that for all $\xi \in \mathbb{R}^d$, $\lim_{\varepsilon \to 0+} \widehat{p}_\varepsilon(\xi) = 1$.

Let

$$\Phi_{t,x}^{\varepsilon,\delta}(r, y) := \int_0^t g_\delta(t - s - r)p_\varepsilon(X_s^x - y)ds \cdot I_{[0,t]}(r). \tag{4.1}$$

Formal computations suggest that

$$\lim_{\varepsilon,\delta \downarrow 0} \int_0^t \int_{\mathbb{R}^d} \Phi_{t,x}^{\varepsilon,\delta}(r, y)W(dr, dy) = \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y)W(dr, dy),$$

where $\delta_0(x)$ is the Dirac delta function. This formal derivation is validated by the following theorem.

Theorem 4.1. *Let the measure μ satisfy Hypothesis (I), then $W(\Phi_{t,x}^{\varepsilon,\delta})$ is well-defined a.s. and forms a Cauchy sequence in L^2 when $(\varepsilon, \delta) \to 0$ with the limit denoted by*

$$W(\delta_0(X_{t-\cdot}^x - \cdot)I_{[0,t]}(\cdot)) = \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y)W(dr, dy).$$

Furthermore, $W(\delta_0(X_{t-\cdot}^x - \cdot)I_{[0,t]}(\cdot))$ is Gaussian distributed conditional on X with variance

$$\mathrm{Var}\left[W(\delta_0(X_{t-\cdot}^x - \cdot)I_{[0,t]}(\cdot))|X\right] = \int_0^t \int_0^t |r - s|^{-\beta_0}\gamma(X_r - X_s)dsdr. \tag{4.2}$$

Proof. Let $\varepsilon_i, \delta_i, i = 1, 2$ be positive numbers, then by (2.1)

$$\begin{aligned}\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} = &\int_{[0,t]^4} \int_{\mathbb{R}^{2d}} p_{\varepsilon_1}(X_{s_1}^x - y_1)p_{\varepsilon_2}(X_{s_2}^x - y_2)\gamma(y_1 - y_2)\\ &g_{\delta_1}(t - s_1 - r_1)g_{\delta_2}(t - s_2 - r_2)|r_1 - r_2|^{-\beta_0}dy_1dy_2dr_1dr_2ds_1ds_2.\end{aligned} \tag{4.3}$$

Hence

$$\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} \geq 0.$$

By [30, Lemma A.3], there exists a positive constant C depending on β_0 only, such that

$$\int_{[0,t]^2} g_{\delta_1}(t-s_1-r_1) g_{\delta_2}(t-s_2-r_2) |r_1-r_2|^{-\beta_0} dr_1 dr_2 \leq C|s_1-s_2|^{-\beta_0}. \tag{4.4}$$

Therefore,

$$\begin{aligned}
\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} &\leq C \int_{[0,t]^2} \int_{\mathbb{R}^{2d}} p_{\varepsilon_1}(X_{s_1}^x - y_1) p_{\varepsilon_2}(X_{s_2}^x - y_2) |s_1-s_2|^{-\beta_0} dy_1 dy_2 ds_1 ds_2 \\
&= \frac{C}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \mathcal{F}\left(p_{\varepsilon_1}(X_{s_1}^x - \cdot)\right)(\xi) \overline{\mathcal{F}\left(p_{\varepsilon_2}(X_{s_2}^x - \cdot)\right)(\xi)} |s_1-s_2|^{-\beta_0} \mu(d\xi) ds_1 ds_2 \\
&= \frac{C}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi) \widehat{p}_{\varepsilon_2}(\xi) \exp\big(-i\xi\cdot(X_{s_1}-X_{s_2})\big) |s_1-s_2|^{-\beta_0} \mu(d\xi) ds_1 ds_2 \\
&\leq C(\varepsilon_1,\varepsilon_2) \int_{[0,t]^2} |s_1-s_2|^{-\beta_0} ds_1 ds_2 < \infty.
\end{aligned} \tag{4.5}$$

The second equality above holds because $\mathcal{F}(\phi(\cdot - a))(\xi) = \exp(-ia\cdot\xi)\widehat{\phi}(\xi)$ and that we can apply the Parseval's identity since $\int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi)\widehat{p}_{\varepsilon_2}(\xi)\mu(d\xi) \leq \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi)\|p_\varepsilon\|_1 \mu(d\xi) = \int_{\mathbb{R}^d} p_{\varepsilon_1}(x)\gamma(x)dx < \infty$. Hence, for $\varepsilon, \delta > 0$, $\Phi_{t,x}^{\varepsilon,\delta} \in \mathcal{H}$ a.s. and $W(\Phi_{t,x}^{\varepsilon,\delta})$ is well-defined a.s..

Now we show that $W(\Phi_{t,x}^{\varepsilon,\delta})$ forms a Cauchy sequence in L^2 when $(\varepsilon,\delta) \to 0$, for which it suffices to show that $\mathbb{E}[\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}}]$ converges as (ε_1,δ_1) and (ε_2,δ_2) tend to zero. By the formula (2.4) for the inner product using Fourier transforms,

$$\begin{aligned}
\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} =& \frac{1}{(2\pi)^d} \int_{[0,t]^4} \int_{\mathbb{R}^d} \mathcal{F}\left(p_{\varepsilon_1}(X_{s_1}^x - \cdot)\right)(\xi) \overline{\mathcal{F}\left(p_{\varepsilon_2}(X_{s_2}^x - \cdot)\right)(\xi)} \\
& \qquad g_{\delta_1}(t-s_1-r_1) g_{\delta_2}(t-s_2-r_2) |r_1-r_2|^{-\beta_0} \mu(d\xi) dr_1 dr_2 ds_1 ds_2 \\
=& \frac{1}{(2\pi)^d} \int_{[0,t]^4} \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi) \widehat{p}_{\varepsilon_2}(\xi) \exp\big(-i\xi\cdot(X_{s_1}-X_{s_2})\big) \\
& \qquad g_{\delta_1}(t-s_1-r_1) g_{\delta_2}(t-s_2-r_2) |r_1-r_2|^{-\beta_0} \mu(d\xi) dr_1 dr_2 ds_1 ds_2.
\end{aligned}$$

By Fubini's theorem and thanks to (4.4) and Proposition 3.2, we can apply the dominated convergence theorem and get that

$$\begin{aligned}
\mathbb{E}[\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}}] \longrightarrow & \frac{1}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \mathbb{E} \exp\left(-i\xi\cdot(X_{s_1}-X_{s_2})\right) |s_1-s_2|^{-\beta_0} \mu(d\xi) ds_1 ds_2 \\
=& \int_{[0,t]^2} |s_1-s_2|^{-\beta_0} \mathbb{E}\gamma(X_{s_2}-X_{s_1}) ds_1 ds_2
\end{aligned} \tag{4.6}$$

as (ε_1,δ_1) and (ε_2,δ_2) go to zero.

Finally, conditional on X, $W(\Phi_{t,x}^{\varepsilon,\delta})$ is Gaussian and hence the limit (in probability) $W(\delta_0(X_{t-\cdot}^x - \cdot))$ is also Gaussian. To show the formula (4.2) for conditional variance, it suffices to show that

$$\langle \Phi_{t,x}^{\varepsilon,\delta}, \Phi_{t,x}^{\varepsilon,\delta} \rangle_{\mathcal{H}} \longrightarrow \int_{[0,t]^2} |s_1 - s_2|^{-\beta_0} \gamma(X_{s_2} - X_{s_1}) ds_1 ds_2 \tag{4.7}$$

in $L^1(\Omega)$ as $(\varepsilon, \delta) \to 0$. Noting that, by Lemma 4.2, the inside integral in (4.3)

$$\int_{[0,t]^2} \int_{\mathbb{R}^{2d}} p_\varepsilon(X_{s_1}^x - y_1) p_\varepsilon(X_{s_2}^x - y_2) \gamma(y_1 - y_2)$$
$$g_\delta(t - s_1 - r_1) g_\delta(t - s_2 - r_2) |r_1 - r_2|^{-\beta_0} dy_1 dy_2 dr_1 dr_2$$

converges to $|s_1 - s_2|^{-\beta_0} \gamma(X_{s_1} - X_{s_2})$ a.s. as (ε, δ) goes to zero, because of (4.6) we can apply Scheffé's lemma to get that the convergence is also in $L^1(\Omega \times [0,t]^2, P \times m)$ where m is the Lebesgue measure on $[0,t]^2$. Consequently it follows that the convergence (4.7) holds in $L^1(\Omega)$. ■

Lemma 4.2. *When $a - b \neq 0$,*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} p_\varepsilon(a - y_1) p_\varepsilon(b - y_2) \gamma(y_1 - y_2) dy_1 dy_2 = \gamma(a - b).$$

Proof. The change of variables $x_1 = y_1 - y_2, x_2 = y_2$ implies that $\int_{\mathbb{R}^{2d}} p_\varepsilon(a - y_1) p_\varepsilon(b - y_2) \gamma(y_1 - y_2) dy_1 dy_2 = \int_{\mathbb{R}^{2d}} p_\varepsilon(a - x_1 - x_2) p_\varepsilon(b - x_2) \gamma(x_1) dx_1 dx_2 = \int_{\mathbb{R}^d} (p_\varepsilon * p_\varepsilon)(a - b - x_1) \gamma(x_1) dx_1 = \int_{\mathbb{R}^d} \frac{1}{\varepsilon}(p * p)(\frac{a-b-x}{\varepsilon}) \gamma(x) dx$. Since the convolution $p * p$ is also a smooth probability density function with compact support, it suffices to prove the following result. ■

Lemma 4.3. *Let $f_\varepsilon(x) = \frac{1}{\varepsilon^d} f(\frac{x}{\varepsilon})$, where $f \in C_0^\infty(\mathbb{R}^d)$ is a symmetric probability density function. Then we have*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} f_\varepsilon(a - x) \gamma(x) dx = \gamma(a), \ \forall a \neq 0.$$

Proof. Suppose that the support of the function f is inside $[-M, M]$. Let the positive number ε be sufficiently small such that $\gamma(x)$ is continuous on $[a - M\varepsilon, a + M\varepsilon]$. By the mean value theorem, we have

$$\int_{\mathbb{R}^d} f_\varepsilon(a - x) \gamma(x) dx = \int_{[a - M\varepsilon, a + M\varepsilon]} f_\varepsilon(a - x) \gamma(x) dx$$
$$= \gamma(a_\varepsilon) \int_{[a - M\varepsilon, a + M\varepsilon]} f_\varepsilon(a - x) dx = \gamma(a_\varepsilon),$$

where $a_\varepsilon \in [a - M\varepsilon, a + M\varepsilon]$. The result follows if we let ε go to zero. ■

4.2 Feynman-Kac formula

For positive numbers ε and δ, define

$$\dot{W}^{\epsilon,\delta}(t,x) := \int_0^t \int_{\mathbb{R}^d} g_\delta(t-s)p_\epsilon(x-y)W(ds,dy) = W(\phi_{t,x}^{\varepsilon,\delta}), \tag{4.8}$$

where

$$\phi_{t,x}^{\varepsilon,\delta}(s,y) = g_\delta(t-s)p_\epsilon(x-y)\cdot I_{[0,t]}(s).$$

Then $\dot{W}^{\epsilon,\delta}(t,x)$ exists in the classical sense and it is an approximation of $\dot{W}(t,x)$. Taking advantage of $\dot{W}^{\epsilon,\delta}(t,x)$, we can define the integral $\int_0^T \int_{\mathbb{R}^d} v(t,x)W(dt,dx)$ in the Stratonovich sense as follows.

Definition 4.4. *Suppose that $v = \{v(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a random field satisfying*

$$\int_0^T \int_{\mathbb{R}^d} |v(t,x)|dxdt < \infty, \quad a.s.,$$

and that the limit in probability $\lim\limits_{\epsilon,\delta\downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t,x)\dot{W}^{\epsilon,\delta}(t,x)dxdt$ exists. The we denote the limit by

$$\int_0^T \int_{\mathbb{R}^d} v(t,x)W(dt,dx) := \lim_{\epsilon,\delta\downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t,x)\dot{W}^{\epsilon,\delta}(t,x)dxdt.$$

and call it Stratonovich integral.

Let $\mathcal{F}_t$ be the σ-algebra generated by $\{W(s,x), 0 \leq s \leq t, x \in \mathbb{R}^d\}$, and we say that a random field $\{F(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is adapted if $\{F(t,x), t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ for all $x \in \mathbb{R}^d$. Denote the convolution between the function q_t and f by $Q_t f$, i.e.,

$$Q_t f(x) := \int_{\mathbb{R}^d} q_t(x-y)f(y)dy.$$

A mild solution to (1.1) in the Stratonovich sense is defined as follows.

Definition 4.5. *An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to (1.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for all $t \geq 0$ and $x \in \mathbb{R}^d$ the following integral equation holds*

$$u(t,x) = Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)u(s,y)W(ds,dy), \tag{4.9}$$

where the stochastic integral is in the Stratonovich sense of Definition 4.4.

The following theorem is the main result in this section.

Theorem 4.6. *Let the measure μ satisfy Hypothesis (I). Then*

$$u(t,x) = \mathbb{E}^X \left[u_0(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y) W(dr, dy) \right) \right] \tag{4.10}$$

is well-defined and it is a mild solution to (1.1) in the Stratonovich sense.

Proof. Consider the following approximation of (1.1)

$$\begin{cases} u^{\varepsilon,\delta}(t,x) = \mathcal{L} u^{\varepsilon,\delta}(t,x) + u^{\varepsilon,\delta}(t,x) \dot{W}^{\varepsilon,\delta}(t,x), \\ u^{\varepsilon,\delta}(0,x) = u_0(x). \end{cases} \tag{4.11}$$

By the classical Feynman-Kac formula,

$$u^{\varepsilon,\delta}(t,x) = \mathbb{E}^X \left[u_0(X_t^x) \exp\left(\int_0^t \dot{W}^{\varepsilon,\delta}(r, X_{t-r}^x) dr \right) \right] = \mathbb{E}^X \left[u_0(X_t^x) \exp\left(W(\Phi_{t,x}^{\varepsilon,\delta}) \right) \right]$$

where $\Phi_{t,x}^{\varepsilon,\delta}$ is defined in (4.1) and the last equality follows from the stochastic Fubini's theorem, is a mild solution to (4.11), i.e.,

$$u^{\varepsilon,\delta}(t,x) = Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y) u^{\varepsilon,\delta}(s,y) \dot{W}^{\varepsilon,\delta}(s,y) ds dy. \tag{4.12}$$

To prove the result, it suffices to show that as (ε, δ) tends to zero, both sides of (4.12) converge respectively in probability to those of (4.9) with $u(t,x)$ given in (4.10). We split the proof in two steps for easier interpretation.

Step 1. First, we show that $u^{\varepsilon,\delta}(t,x) \to u(t,x)$ in L^p for all $p > 1$. By Theorem 4.1, as $(\varepsilon,\delta) \to 0$, $W(\Phi_{t,x}^{\varepsilon,\delta})$ converges to $W(\delta_0(X_{t-\cdot}^x - \cdot) I_{[0,t]}(\cdot))$ in probability, and hence it suffices to show that

$$\sup_{\varepsilon,\delta>0} \sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbb{E}[|u^{\varepsilon,\delta}(t,x)|^p] < \infty.$$

Note that $W(\Phi_{t,x}^{\varepsilon,\delta})$ is Gaussian conditional on X, and hence

$$\mathbb{E}\left[\exp\left(p W(\Phi_{t,x}^{\varepsilon,\delta}) \right) \right] = \mathbb{E}\left[\exp\left(\frac{p^2}{2} \|\Phi_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2 \right) \right].$$

By (2.4) and (4.4), in a similar way of proving (4.5), we can show that there exists a positive constant C depending on β_0 only such that

$$\|\Phi_{t,x}^{\varepsilon,\delta}\|_{\mathcal{H}}^2 \le C \int_{[0,t]^2} \int_{\mathbb{R}^d} (\widehat{p}_\varepsilon(\xi))^2 \exp(-i\xi \cdot (X_r - X_s)) |r-s|^{-\beta_0} \mu(d\xi) dr ds.$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[\|\Phi^{\varepsilon,\delta}_{t,x}\|^{2n}_{\mathcal{H}}] \le & C^n \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n (\widehat{p}_\varepsilon(\xi_j))^2\, \mathbb{E} \exp(-i \sum_{j=1}^n \xi_j \cdot (X_{r_j} - X_{s_j})) \\
& \qquad \prod_{j=1}^n |r_j - s_j|^{-\beta_0} \prod_{j=1}^n \mu(d\xi_j) dr_j ds_j \\
\le & C^n \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \mathbb{E} \exp(-i \sum_{j=1}^n \xi_j \cdot (X_{r_j} - X_{s_j})) \prod_{j=1}^n |r_j - s_j|^{-\beta_0} \prod_{j=1}^n \mu(d\xi_j) dr_j ds_j \\
= & \mathbb{E}\left[\left(C \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \right)^n \right].
\end{aligned}
$$

The second inequality above holds because $\sup_{\xi \in \mathbb{R}^d} \widehat{p}_\varepsilon(\xi) \le 1$ and $\mathbb{E}\exp(-i\sum_{j=1}^n \xi_j \cdot (X_{r_j} - X_{s_j}))$ is a positive real number. Thus there is constant $C > 0$ depending on β_0 only such that

$$
\sup_{\varepsilon,\delta>0} \sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbb{E}\left[\exp\left(\frac{p^2}{2} \|\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}} \right) \right] \le \mathbb{E}\left[\exp\left(C \frac{p^2}{2} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \right) \right],
$$

where the term on the right-hand side is finite by Theorem 3.3.

Step 2. Now by Definition 4.4, it suffices to show that

$$
I^{\varepsilon,\delta} := \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)(u^{\varepsilon,\delta}(s,y) - u(s,y)) \dot{W}^{\varepsilon,\delta}(s,y) ds dy
$$

converges in L^2 to zero. Denoting $v^{\varepsilon,\delta}_{s,y} = u^{\varepsilon,\delta}(s,y) - u(s,y)$ and noting that $\dot{W}^{\varepsilon,\delta}(s,y) = W(\phi^{\varepsilon,\delta}_{s,y})$ we have

$$
\mathbb{E}[(I^{\varepsilon,\delta})^2] = \int_{[0,t]^2} \int_{\mathbb{R}^{2d}} q_{t-s_1}(x-y_1) q_{t-s_2}(x-y_2) \mathbb{E}\left[v^{\varepsilon,\delta}_{s_1,y_1} v^{\varepsilon,\delta}_{s_2,y_2} W(\phi^{\varepsilon,\delta}_{s_1,y_1}) W(\phi^{\varepsilon,\delta}_{s_2,y_2}) \right] dy_1 dy_2 ds_1 ds_2.
$$

Use the following notations $V^{\varepsilon,\delta}_{t,x}(X) = \int_0^t \dot{W}^{\varepsilon,\delta}(r, X^x_{t-r}) dr = W(\Phi^{\varepsilon,\delta}_{t,x}(X))$, $V_{t,x}(X) = \int_0^t \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - y) W(dr,dy) = W(\delta_0(X^x_{t-\cdot} - \cdot) I_{[0,t]}(\cdot))$, and

$$
A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2) = \prod_{j=1}^2 u_0(X^j_{s_j} + y_j) \left[\exp\left(V^{\varepsilon,\delta}_{s_j,y_j}(X^j) \right) - \exp\left(V_{s_j,y_j}(X^j) \right) \right],
$$

where X^1 and X^2 are two independent copies of X. Then

$$
\mathbb{E}\left[v^{\varepsilon,\delta}_{s_1,y_1} v^{\varepsilon,\delta}_{s_2,y_2} W(\phi^{\varepsilon,\delta}_{s_1,y_1}) W(\phi^{\varepsilon,\delta}_{s_2,y_2}) \right] = \mathbb{E}^{X^1,X^2}\left[\mathbb{E}^W\left[A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2) W(\phi^{\varepsilon,\delta}_{s_1,y_1}) W(\phi^{\varepsilon,\delta}_{s_2,y_2}) \right] \right].
$$

By the integration by parts formula (2.9),

$$
\begin{aligned}
& \mathbb{E}^W\left[A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2) W(\phi^{\varepsilon,\delta}_{s_1,y_1}) W(\phi^{\varepsilon,\delta}_{s_2,y_2}) \right] \\
= & \mathbb{E}^W\left[\langle D^2 A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2), \phi^{\varepsilon,\delta}_{s_1,y_1} \otimes \phi^{\varepsilon,\delta}_{s_2,y_2} \rangle_{\mathcal{H}^{\otimes 2}} \right] + \mathbb{E}^W[A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)] \langle \phi^{\varepsilon,\delta}_{s_1,y_1}, \phi^{\varepsilon,\delta}_{s_2,y_2} \rangle_{\mathcal{H}},
\end{aligned}
$$

and hence we have

$$\mathbb{E}\left[v^{\varepsilon,\delta}_{s_1,y_1}v^{\varepsilon,\delta}_{s_2,y_2}W(\phi^{\varepsilon,\delta}_{s_1,y_1})W(\phi^{\varepsilon,\delta}_{s_2,y_2})\right]=\mathbb{E}[A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)B^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)]+\mathbb{E}[v^{\varepsilon,\delta}_{s_1,y_1}v^{\varepsilon,\delta}_{s_2,y_2}]\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\rangle_{\mathcal{H}},$$

where

$$\begin{aligned}&B^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)\\&=\sum_{j,k=1}^{2}\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\Phi^{\varepsilon,\delta}_{s_j,y_j}(X^j)-\delta(X^j_{s_j-\cdot}+y_j-\cdot)I_{[0,s_j]}(\cdot)\rangle_{\mathcal{H}}\langle\phi^{\varepsilon,\delta}_{s_2,y_2},\Phi^{\varepsilon,\delta}_{s_k,y_k}(X^k)-\delta(X^k_{s_k-\cdot}+y_k-\cdot)I_{[0,s_k]}(\cdot)\rangle_{\mathcal{H}}.\end{aligned}$$

Therefore,

$$\mathbb{E}[(I^{\varepsilon,\delta})^2]=J^{\varepsilon,\delta}_1+J^{\varepsilon,\delta}_2,$$

with the notations

$$\begin{aligned}J^{\varepsilon,\delta}_1&=\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}q_{t-s_1}(x-y_1)q_{t-s_2}(x-y_2)\mathbb{E}[A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)B^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)]dy_1dy_2ds_1ds_2\\&\le\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}q_{t-s_1}(x-y_1)q_{t-s_2}(x-y_2)\\&\qquad\left(\mathbb{E}[(A^{\varepsilon,\delta}(s_1,y_1,s_2,y_2))^2]\right)^{1/2}\left(\mathbb{E}[(B^{\varepsilon,\delta}(s_1,y_1,s_2,y_2))^2]\right)^{1/2}dy_1dy_2ds_1ds_2\end{aligned}$$

and

$$\begin{aligned}J^{\varepsilon,\delta}_2&=\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}q_{t-s_1}(x-y_1)q_{t-s_2}(x-y_2)\mathbb{E}[v^{\varepsilon,\delta}_{s_1,y_1}v^{\varepsilon,\delta}_{s_2,y_2}]\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\rangle_{\mathcal{H}}dy_1dy_2ds_1ds_2\\&\le\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}q_{t-s_1}(x-y_1)q_{t-s_2}(x-y_2)\left(\mathbb{E}[(v^{\varepsilon,\delta}_{s_1,y_1})^2]\right)^{1/2}\left(\mathbb{E}[(v^{\varepsilon,\delta}_{s_2,y_2})^2]\right)^{1/2}\langle\phi^{\varepsilon,\delta}_{s_1,y_1},\phi^{\varepsilon,\delta}_{s_2,y_2}\rangle_{\mathcal{H}}dy_1dy_2ds_1ds_2.\end{aligned}$$

Now the problem is reduced to show that both $J^{\varepsilon,\delta}_1$ and $J^{\varepsilon,\delta}_2$ converge to zero as $(\varepsilon,\delta)\to 0$. By the result in **Step 1**, we have

$$\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}[(v^{\varepsilon,\delta}_{s,y})^2]=0,$$

and similar arguments imply that

$$\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}[(A^{\varepsilon,\delta}_{s,y})^2]=0,$$

for all $(s,y)\in[0,T]\times\mathbb{R}^d$. Also note that both $\sup_{\varepsilon,\delta>0}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\mathbb{E}[(v^{\varepsilon,\delta}_{s,y})^2]$ and $\sup_{\varepsilon,\delta>0}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\mathbb{E}[(A^{\varepsilon,\delta}_{s,y})^2]$ are finite.

First we can prove $\lim_{\varepsilon,\delta\downarrow 0}J^{\varepsilon,\delta}_1=0$ by the dominated convergence theorem, noting that Lemma 4.8 implies

$$\sup_{\varepsilon,\delta>0}\sup_{(s_1,y_1)\in[0,T]\times\mathbb{R}^d}\sup_{(s_2,y_2)\in[0,T]\times\mathbb{R}^d}\mathbb{E}[(B^{\varepsilon,\delta}(s_1,y_1,s_2,y_2))^2]<\infty.$$

Now we show $\lim_{\varepsilon,\delta\downarrow 0} J_2^{\varepsilon,\delta} = 0$. By (2.4) and (4.4), we have

$$\langle \phi^{\varepsilon,\delta}_{s_1,y_1}, \phi^{\varepsilon,\delta}_{s_2,y_2}\rangle_{\mathcal{H}} \le C|s_1 - s_2|^{-\beta_0} \int_{\mathbb{R}^d} \exp\left(-i\xi\cdot(y_1-y_2)\right) \left(\widehat{p}_\varepsilon(\xi)\right)^2 \mu(d\xi),$$

therefore,

$$J_2^{\varepsilon,\delta} \le C \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} q_{t-s_1}(x-y_1) q_{t-s_2}(x-y_2) K^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)|s_1-s_2|^{-\beta_0} dy_1 dy_2 ds_1 ds_2$$

where

$$\begin{aligned} K^{\varepsilon,\delta}(s_1,y_1,s_2,y_2) :=& \left(\mathbb{E}[(v^{\varepsilon,\delta}_{s_1,y_1})^2]\right)^{1/2} \left(\mathbb{E}[(v^{\varepsilon,\delta}_{s_2,y_2})^2]\right)^{1/2} \int_{\mathbb{R}^d p_\varepsilon(\xi)} \exp(-i\xi\cdot(y_1-y_2)) \left(\widehat{p}_\varepsilon(\xi)\right)^2 \mu(d\xi) \\ \le& C \int_{\mathbb{R}^d} \exp(-i\xi\cdot(y_1-y_2)) \left(\widehat{p}_\varepsilon(\xi)\right)^2 \mu(d\xi). \end{aligned}$$

Denote

$$L^\varepsilon_{s_1,s_2} := \int_{\mathbb{R}^d} \exp(-i\xi\cdot(y_1-y_2)) \left(\widehat{p}_\varepsilon(\xi)\right)^2 \mu(d\xi).$$

Hence

$$K^{\varepsilon,\delta}(s_1,y_1,s_2,y_2) \le C L^\varepsilon_{s_1,s_2}.$$

For the integral of $L^\varepsilon_{s_1,s_2}$, we have

$$\begin{aligned} &\int_0^t \int_0^t \int_{\mathbb{R}^{2d}} q_{t-s_1}(x-y_1) q_{t-s_2}(x-y_2) L^\varepsilon_{s_1,s_2} |s_1-s_2|^{-\beta_0} dy_1 dy_2 ds_1 ds_2 \\ &= \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} q_{t-s_1}(x-y_1) q_{t-s_2}(x-y_2) \exp(-i\xi\cdot(y_1-y_2)) \\ &\qquad\qquad \left(\widehat{p}_\varepsilon(\xi)\right)^2 |s_1-s_2|^{-\beta_0} \mu(d\xi) dy_1 dy_2 ds_1 ds_2 \\ &= \int_0^t \int_0^t \int_{\mathbb{R}^d} \exp\left(-(t-s_1)\Psi(\xi)\right) \exp\left(-(t-s_2)\Psi(\xi)\right) \left(\widehat{p}_\varepsilon(\xi)\right)^2 |s_1-s_2|^{-\beta_0} \mu(d\xi) ds_1 ds_2 \\ &\xrightarrow{\varepsilon\to 0} \int_0^t \int_0^t \int_{\mathbb{R}^d} \exp\left(-(t-s_1)\Psi(\xi)\right) \exp\left(-(t-s_2)\Psi(\xi)\right) |s_1-s_2|^{-\beta_0} \mu(d\xi) ds_1 ds_2 \\ &= \int_0^t \int_0^t \int_{\mathbb{R}^{2d}} q_{t-s_1}(x-y_1) q_{t-s_2}(x-y_2) \gamma(y_1-y_2) |s_1-s_2|^{-\beta_0} dy_1 dy_2 ds_1 ds_2 w, \end{aligned} \tag{4.13}$$

where the convergence follows from the dominated convergence theorem, the last equality follows from the formula (2.5), and the last term is finite by Lemma 3.7.

We have shown that $K^{\varepsilon,\delta}(s_1,y_1,s_2,y_2)$ which converges to zero almost everywhere, is bounded by the sequence $L^\varepsilon_{s_1,s_2}$ which converges to $\gamma(y_1-y_2)$, and thanks to (4.13), we can apply the generalized dominated convergence theorem to get that $\lim_{\varepsilon,\delta\downarrow 0} J_2^{\varepsilon,\delta} = 0$. ■

Using Theorem 4.6, by direct computation we can get the following Feynman-Kac type of representation for the moments of the solution to (1.1).

Theorem 4.7. *Let μ satisfy Hypothesis (I), then the solution given by (4.10) has finite moments of all orders. Furthermore, for any positive integer p,*

$$\mathbb{E}[u(t,x)^p] = \mathbb{E}\left[\prod_{j=1}^{p} u_0(X_t^j + x) \exp\left(\frac{1}{2}\sum_{j,k=1}^{p}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^k)drds\right)\right], \quad (4.14)$$

where $X_1, \ldots, X_p$ are p independent copies of X.

Lemma 4.8. *Let the measure μ satisfy Hypothesis (I). Then, for any $n \in \mathbb{N}$,*

$$\sup_{\varepsilon,\delta>0}\sup_{\varepsilon',\delta'>0}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\sup_{(r,z)\in[0,T]\times\mathbb{R}^d}\mathbb{E}\left[\left(\langle\phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon',\delta'}(X)\rangle_{\mathcal{H}}\right)^n\right] < \infty,$$

and

$$\sup_{\varepsilon,\delta>0}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\sup_{(r,z)\in[0,T]\times\mathbb{R}^d}\mathbb{E}\left[\left(\langle\phi_{s,y}^{\varepsilon,\delta}, \delta(X_{r-\cdot}^z - \cdot)I_{[0,r]}(\cdot)\rangle_{\mathcal{H}}\right)^n\right] < \infty.$$

Proof. First of all, $\langle\phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon',\delta'}(X)\rangle_{\mathcal{H}}$ is a nonnegative real number by (2.1), and by (2.4)

$$\langle\phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon',\delta'}(X)\rangle_{\mathcal{H}} = \int_0^r\int_0^s\int_0^r\int_{\mathbb{R}^d}\widehat{p}_\varepsilon(\xi)\widehat{p}_{\varepsilon'}(\xi)\exp(-i\xi\cdot(X_\tau^z - y))g_{\delta'}(r-\mu-\tau)$$
$$g_\delta(s-\nu)|\mu-\nu|^{-\beta_0}\mu(\xi)d\tau d\mu d\nu.$$

Therefore, denoting $D = [0,r]\times[0,s]\times[0,r]$, as in the first step of the proof for Theorem 4.6, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle\phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon',\delta'}(X)\rangle_{\mathcal{H}}\right)^n\right]\\
=&\int_{D^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n} g_{\delta'}(r-\mu_j-\tau_j)g_\delta(s-\nu_j)|\mu_j-\nu_j|^{-\beta_0}\\
&\qquad\prod_{j=1}^{n}\widehat{p}_\varepsilon(\xi_j)\widehat{p}_{\varepsilon'}(\xi_j)\mathbb{E}\left[\exp\left(-i\sum_{j=1}^{n}\xi_j\cdot(X_{\tau_j}^z - y)\right)\right]\mu(d\xi)d\tau d\mu d\nu\\
\le& C^n\int_{[0,r]^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}|r-s-\tau_j|^{-\beta_0}\exp\left(-\frac{\varepsilon+\varepsilon'}{2}\sum_{j=1}^{n}|\xi_j|^2\right)\left|\mathbb{E}\left[\exp\left(-i\sum_{j=1}^{n}\xi_j\cdot(X_{\tau_j}^z - y)\right)\right]\right|\mu(d\xi)d\tau\\
\le& C^n\int_{[0,r]^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}|r-s-\tau_j|^{-\beta_0}\mathbb{E}\left[\exp\left(-i\sum_{j=1}^{n}\xi_j\cdot X_{\tau_j}\right)\right]\mu(d\xi)d\tau.
\end{aligned}$$

Thus, we have, denoting $\eta_j = \xi_j + \xi_{j+1} + \cdots + \xi_n$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\langle \phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon',\delta'}(X)\rangle_{\mathcal{H}}\right)^n\right] \\
\leq & C^n n! \int_{[0<\tau_1<\cdots<\tau_n<r]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |r-s-\tau_j|^{-\beta_0} \mathbb{E}\left[\exp\left(-i\sum_{j=1}^n \xi_j \cdot X_{\tau_j}\right)\right] \mu(d\xi) d\tau \\
= & C^n n! \int_{[0<\tau_1<\cdots<\tau_n<r]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |r-s-\tau_j|^{-\beta_0} \mathbb{E}\left[\exp\left(-i\sum_{j=1}^n \eta_j \cdot (X_{\tau_j} - X_{\tau_{j-1}})\right)\right] \mu(d\xi) d\tau \text{ (let } \tau_0 = 0) \\
= & C^n n! \int_{[0<\tau_1<\cdots<\tau_n<r]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |r-s-\tau_j|^{-\beta_0} \exp\left(-\sum_{j=1}^n (\tau_j - \tau_{j-1})\Psi(\eta_j)\right) \mu(d\xi) d\tau \\
\leq & C^n n! \int_{[0<\tau_1<\cdots<\tau_n<r]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |\tau_j + (s-r)|^{-\beta_0} \exp\left(-\sum_{j=1}^n (\tau_j - \tau_{j-1})\Psi(\xi_j)\right) \mu(d\xi) d\tau \text{ (by Lemma 4.9)} \\
=: & C^n n! U_n(r,s)
\end{aligned}
$$

When $s - r \geq 0$, for all $0 < r \leq s < T$,

$$
\begin{aligned}
U_n(r,s) &\leq \int_{[0<\tau_1<\cdots<\tau_n<r]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |\tau_j|^{-\beta_0} \exp\left(-\sum_{j=1}^n (\tau_j - \tau_{j-1})\Psi(\xi_j)\right) \mu(d\xi) d\tau \\
&\leq \int_{[0<\tau_1<\cdots<\tau_n<T]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |\tau_j - \tau_{j-1}|^{-\beta_0} \exp\left(-\sum_{j=1}^n (\tau_j - \tau_{j-1})\Psi(\xi_j)\right) \mu(d\xi) d\tau.
\end{aligned}
$$

By Proposition 3.5, $U_n(r,s)$ is uniformly bounded by a finite number depending on (T, n, β_0) and the measure μ only.

When $r - s > 0$, the set $[0 < \tau_1 < \cdots < \tau_n < r]$ is the union of $A_k's$ for $k = 0, 1, 2, \ldots, n$

where $A_k = [0 = \tau_0 < \tau_1 < \cdots < \tau_k < r - s < \tau_{k+1} < \cdots < \tau_n < r]$. On each A_k, we have

$$\begin{aligned}
&\int_{A_k}\int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} |r-s-\tau_j|^{-\beta_0} \exp\left(-\sum_{j=1}^{n}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\tau \\
&= \int_{A_k}\int_{\mathbb{R}^{nd}} \prod_{j=1}^{k}(r-s-\tau_j)^{-\beta_0} \exp\left(-\sum_{j=1}^{k}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right) \\
&\qquad\qquad (\tau_{k+1}-(r-s))^{-\beta_0}\exp\left(-(\tau_{k+1}-(r-s)+(r-s)-\tau_k)\Psi(\xi_j)\right) \\
&\qquad\qquad \prod_{j=k+2}^{n}(\tau_j-(r-s))^{-\beta_0}\exp\left(-\sum_{j=k+2}^{n}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\tau \\
&\le \int_{A_k}\int_{\mathbb{R}^{nd}} \prod_{j=1}^{k}(r-s-\tau_j)^{-\beta_0} \exp\left(-\sum_{j=1}^{k}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right) \\
&\qquad\qquad (\tau_{k+1}-(r-s))^{-\beta_0}\exp\left(-(\tau_{k+1}-(r-s))\Psi(\xi_{k+1})\right) \\
&\qquad\qquad \prod_{j=k+2}^{n}(\tau_j-(r-s))^{-\beta_0}\exp\left(-\sum_{j=k+2}^{n}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\tau \\
&= \int_{[0<\tau_1<\cdots<\tau_k<r-s]}\int_{\mathbb{R}^{kd}} \prod_{j=1}^{k}(r-s-\tau_j)^{-\beta_0}\exp\left(-\sum_{j=1}^{k}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\tau \\
&\quad \times \int_{[r-s<\tau_{k+1}<\cdots<\tau_n<r]}\int_{\mathbb{R}^{(n-k)d}} \prod_{j=k+1}^{n}(\tau_j-(r-s))^{-\beta_0}\exp\left(-(\tau_{k+1}-(r-s))\Psi(\xi_{k+1})\right) \\
&\qquad\qquad\qquad \exp\left(-\sum_{j=k+2}^{n}(\tau_j-\tau_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\tau \\
&=: M_1(s,r)\times M_2(s,r).
\end{aligned}$$

By Lemma 3.10, we have

$$\begin{aligned}
\sup_{0<s<r<T} M_1(s,r) &\le C^k \sup_{0<s<r<T}\int_{[0<\tau_1<\cdots<\tau_k<r-s]}\prod_{j=1}^{k}(r-s-\tau_j)^{-\beta_0}(1+(\tau_j-\tau_{j-1})^{-1+\beta_0})d\tau \\
&< \infty.
\end{aligned}$$

For $M_2(s,r)$, let $\theta_j = \tau_j - (r-s), j = k+1, \ldots, n$, and assume $\theta_k = 0$, then for all $0 < s < r < T$,

$$\begin{aligned}
M_2(s,r) &= \int_{[0<\theta_{k+1}<\cdots<\theta_n<s]}\int_{\mathbb{R}^{(n-k)d}} \prod_{j=k+1}^{n}\theta_j^{-\beta_0}\exp\left(-\sum_{j=k+1}^{n}(\theta_j-\theta_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\theta \\
&\le \int_{[0<\theta_{k+1}<\cdots<\theta_n<T]}\int_{\mathbb{R}^{(n-k)d}} \prod_{j=k+1}^{n}(\theta_j-\theta_{j-1})^{-\beta_0}\exp\left(-\sum_{j=k+1}^{n}(\theta_j-\theta_{j-1})\Psi(\xi_j)\right)\mu(d\xi)d\theta,
\end{aligned}$$

and the last integral is bounded by a finite number depending on $(n-k, T, \beta_0)$ and μ by Proposition 3.5.

Thus we have shown that when $r - s > 0, \sup_{0<s<r<T} U_n(r, s) < \infty$, and the first inequality is obtained. Finally, since $\langle \phi^{\varepsilon,\delta}_{s,y}, \Phi^{\varepsilon',\delta'}_{r,z}(X)\rangle_{\mathcal{H}}$ converges to $\langle \phi^{\varepsilon,\delta}_{s,y}, \delta(X^z_{r-\cdot} - \cdot)I_{[0,r]}(\cdot)\rangle_{\mathcal{H}}$ in probability as $(\varepsilon', \delta') \to 0$, the second the inequality follows from the first one and Fatou's lemma. ∎

Lemma 4.9. *For any $t > 0$ and $a \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \exp(-t\Psi(\xi + a))\mu(d\xi) \le \int_{\mathbb{R}^d} \exp(-t\Psi(\xi))\mu(d\xi).$$

Proof. By the formula (2.6), we have

$$\begin{aligned}
&\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp(-t\Psi(\xi + a))\mu(d\xi) = \int_{\mathbb{R}^d} q_t(x)e^{-ia\cdot x}\gamma(x)dx \\
&\le \int_{\mathbb{R}^d} q_t(x)\gamma(x)dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp(-t\Psi(\xi))\mu(d\xi).
\end{aligned}$$

∎

4.3 Hölder continuity

Hypothesis (S1)**.** *The spectral measure μ satisfies that for all $z \in \mathbb{R}^d$, there exist $\alpha_1 \in (0, 1]$ and $C > 0$ such that*

$$\int_0^T \int_0^T \int_{\mathbb{R}^d} |r - s|^{-\beta_0} e^{-|r-s|\Psi(\xi)} \left(1 - e^{-i\xi\cdot z}\right) \mu(d\xi)drds \le C|z|^{2\alpha_1}.$$

Hypothesis (T1)**.** *The spectral measure μ satisfies that for all a in a bounded subset of $\mathbb{R}$, there exist $\alpha_2 \in (0, 1]$ and $C > 0$ such that*

$$\int_0^T \int_0^T \int_{\mathbb{R}^d} |r - s|^{-\beta_0}\bigg|\exp\Big(-|r - s|\Psi(\xi)\Big) - \exp\Big(-|r - s + a|\Psi(\xi)\Big)\bigg|\mu(d\xi)drds \le C|a|^{\alpha_2}.$$

Remark 4.10. *A sufficient condition for Hypothesis (S1) to hold is the following*

$$\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha_1}}{1 + (\Psi(\xi))^{1-\beta_0}}\mu(d\xi) < \infty \tag{4.15}$$

due to Lemma 3.7 and the fact that $1 - \cos x \le |x|^{2\alpha_1}$. Note that $\alpha_1 < 1 - \beta_0$ is a necessary condition for (4.15) to hold. This is because $\mu(A) < \infty$ for any bounded set $A \subset \mathbb{R}^d$, $\mu(\mathbb{R}^d) = \gamma(0) = \infty$, $\lim_{\xi\to\infty} \Psi(\xi) = \infty$ and $\limsup_{\|\xi\|\to\infty} \frac{\Psi(\xi)}{\|\xi\|^2} < \infty$.

Similarly, a sufficient condition for Hypothesis (T1) to be true is that

$$\int_{\mathbb{R}^d} \frac{(\Psi(\xi))^{\alpha_2}}{1 + (\Psi(\xi))^{1-\beta_0}}\mu(d\xi) < \infty \tag{4.16}$$

because of Remark 3.8 and the fact that $|e^{-x} - e^{-y}| \le (e^{-x} + e^{-y})|x - y|^{\alpha}$ for $x, y \ge 0$ and $\alpha \in (0, 1]$. Indeed for $a > 0, e^a - 1 \le (e^a + 1)(a \wedge 1)$, and hence $e^a - 1 \le (e^a + 1)a^{\alpha}$ for $\alpha \in (0, 1]$. One necessary condition for (4.16) to hold is $\alpha_2 < 1 - \beta_0$.

Theorem 4.11. *Let $u_0(x) \equiv 1$. If the measure μ satisfies Hypothesis (S1), then the solution $u(t,x)$ given by the Feynman-Kac formula (4.10) has a version that is θ_1-Hölder continuous in x on any compact set of $[0,\infty) \times \mathbb{R}^d$, with $\theta_1 < \alpha_1$; Similarly, if μ satisfies Hypothesis (T1), the solution $u(t,x)$ has a version that is θ_2-Hölder continuous in t on any compact set of $[0,\infty) \times \mathbb{R}^d$, with $\theta_2 < \alpha_2/2$.*

Proof. Recall that $V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(X^x_{t-s} - y) W(ds,dy)$. Noting that $|e^a - e^b| \le (e^a + e^b)|a-b|$, we have for any $p > 0$

$$\mathbb{E}^W[\left|\mathbb{E}^X\left[\exp(V_{t,x}) - \exp(V_{s,y})\right]\right|^p] \le C\mathbb{E}^W\left[\left(\mathbb{E}^X\left[\exp(2V_{t,x}) + \exp(2V_{s,y})\right]\right)^{p/2}\left(\mathbb{E}^X\left[|V_{t,x} - V_{s,y}|^2\right]\right)^{p/2}\right]$$
$$\le C\mathbb{E}\left[\exp(pV_{t,x}) + \exp(pV_{s,y})\right]\left(\mathbb{E}^W\left[\left(\mathbb{E}^X\left[|V_{t,x} - V_{s,y}|^2\right]\right)^p\right]\right)^{1/2}.$$

By Theorem 3.3, $\mathbb{E}\left[\exp(pV_{t,x}) + \exp(pV_{t,y})\right] < \infty$. On the other hand,

$$\left(\mathbb{E}^W\left[\left(\mathbb{E}^X\left[|V_{t,x} - V_{s,y}|^2\right]\right)^p\right]\right)^{1/2} \le \left(\mathbb{E}^X\left(\mathbb{E}^W[|V_{t,x} - V_{s,y}|^{2p}]\right)^{1/p}\right)^{p/2} \le C_p\left(\mathbb{E}^X\mathbb{E}^W[|V_{t,x} - V_{s,y}|^2]\right)^{p/2},$$

where the first inequality follows from Minkowski's inequality and the second one holds because of the equivalence between the L^p-norm and L^2-norm of Gaussian random variables. For the spatial estimate, by Hypothesis (S1),

$$\mathbb{E}^X\mathbb{E}^W[|V_{t,x} - V_{t,y}|^2] = 2\int_0^t \int_0^t |r-s|^{-\beta_0}\mathbb{E}^X\left[\gamma(X_r - X_s) - \gamma(X_r - X_s + x - y))\right] drds$$
$$= 2\int_0^t \int_0^t \int_{\mathbb{R}^d} |r-s|^{-\beta_0} e^{-|r-s|\Psi(\xi)}\left(1 - e^{-i\xi\cdot(x-y)}\right)\mu(d\xi)drds \le C|x-y|^{2\alpha_1}.$$

Therefore

$$\mathbb{E}^W[\left|\mathbb{E}^X\left[\exp(V_{t,x}) - \exp(V_{t,y})\right]\right|^p] \le C_p|x-y|^{\alpha_1 p},$$

and the Hölder continuity of $u(t,x)$ in space follows from Komogorov's continuity criterion.

Now assume that $0 \le s < t \le T$, then

$$\begin{aligned}
&\mathbb{E}[(V_{t,x} - V_{s,x})^2] \\
=&\mathbb{E}\left[\left(\int_0^s \int_{\mathbb{R}^d}\left(\delta_0(X^x_{t-r} - z) - \delta_0(X^x_{s-r} - z)\right)W(dr,dz) + \int_s^t \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - z)W(dr,dx)\right)^2\right] \\
\le& 2(A+B),
\end{aligned}$$

where

$$A = \mathbb{E}\left[\left(\int_0^s \int_{\mathbb{R}^d}\left(\delta_0(X^x_{t-r} - z) - \delta_0(X^x_{s-r} - z)\right)W(dr,dz)\right)^2\right],$$

and

$$B = \mathbb{E}\left[\left(\int_s^t \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - z)W(dr,dx)\right)^2\right].$$

For the first term A, by Hypothesis (T1), we have

$$
\begin{aligned}
A &= \mathbb{E}\left[\int_0^s\int_0^s |s_1-s_2|^{-\beta_0}\big[\gamma(X_{t-s_1}-X_{t-s_2})+\gamma(X_{s-s_1}-X_{s-s_2})-2\gamma(X_{t-s_1}-X_{s-s_2})\big]ds_1ds_2\right] \\
&\le 2\int_0^s\int_0^s\int_{\mathbb{R}6d}|s_1-s_2|^{-\beta_0}\left|\exp\Big(-|s_1-s_2|\Psi(\xi)\Big)-\exp\Big(-|t-s-s_1+s_2|\Psi(\xi)\Big)\right|\mu(d\xi)ds_1ds_2 \\
&\le C|t-s|^{\alpha_2}.
\end{aligned}
$$

For the term B, we have

$$
\begin{aligned}
B &= \int_s^t\int_s^t |s_1-s_2|^{-\beta_0}\mathbb{E}\gamma(X_{s_1}-X_{s_1})ds_1ds_2 = \int_0^{t-s}\int_0^{t-s}|s_1-s_2|^{-\beta_0}\mathbb{E}\gamma(X_{s_1}-X_{s_1})ds_1ds_2 \\
&= \int_{\mathbb{R}^d}\int_0^{t-s}\int_0^{t-s}|s_1-s_2|^{-\beta_0}\exp\left(-|s_1-s_2|\Psi(\xi)\right)ds_1ds_2\mu(d\xi).
\end{aligned}
$$

By Lemma 3.7, we have that for $(t-s)$ in a bounded domain, there exists a constant C such that

$$
\int_0^{t-s}\int_0^{t-s}|s_1-s_2|^{-\beta_0}\exp\left(-|s_1-s_2|\Psi(\xi)\right)ds_1ds_2 \le C(t-s)\frac{1}{1+(\Psi(\xi))^{1-\beta_0}}.
$$

Hence $B\le C(t-s)$, and

$$
\mathbb{E}^W\left[\left|\mathbb{E}^X\left[\exp(V_{t,x})-\exp(V_{s,x})\right]\right|^p\right] \le C\left(\mathbb{E}[(V_{t,x}-V_{s,x})^2]\right)^{p/2} \le C(A+B)^{p/2}\le C(t-s)^{p\alpha_2/2}.
$$

The Hölder continuity in time is obtained by Kolmogorov's criterion. ∎

5 Skorohod equation

In this section, we consider (1.1) in the Skorohod sense, i.e., we consider the following SPDE,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \mathcal{L}u + u\diamond \dot{W}, & t\ge 0, x\in\mathbb{R}^d \\
u(0,x)=u_0(x), & x\in\mathbb{R}^d,
\end{cases}
\tag{5.1}
$$

where the symbol $\diamond$ means the wick product.

5.1 Existence and uniqueness of the mild solution

In this subsection, we will obtain the existence and uniqueness of the mild solution to (5.1) under the following assumption.

Hypothesis (II). *The spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{1}{1+\Psi(\xi)}\mu(d\xi) < \infty.$$

Remark 5.1. *When $\mathcal{L} = -(-\Delta)^{\alpha/2}$ for $\alpha \in (0,2]$ and $\gamma(x)$ is of one of the forms $\prod_{j=1}^d |x_j|^{\beta_j}$, $|x|^{-\beta}$ and $\delta_0(x)$, Hypothesis (II) is equivalent to $\beta < \alpha$, where $\beta = \sum_{j=1}^d \beta_j$ for the first case and $\beta = 1$ for the third one. It is also a necessary condition for (5.1) to have a unique mild solution ([5]).*

Definition 5.2. *An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to (5.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for all $t \geq 0$ and $x \in \mathbb{R}^d$, $\mathbb{E}[u^2(t,x)] < \infty$, and the following integral equation holds*

$$u(t,x) = Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)u(s,y)W^{\diamond}(ds,dy), \tag{5.2}$$

where the stochastic integral is in the Skorohod sense.

Suppose that $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a solution to (5.2), then for fixed (t,x), the square integrable random variable $u(t,x)$ can be expressed uniquely as the Wiener chaos expansion,

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{5.3}$$

where $f_n(\cdot,t,x)$ is symmetric in $\mathcal{H}^{\otimes n}$. On the other hand, if we apply (5.2) repeatedly, as in [28, 29], we can find an explicit representations for f_n with $n \geq 1$

$$f_n(s_1,x_1,\ldots,s_n,x_n,t,x) = \frac{1}{n!} q_{t-s_{\sigma(n)}}(x-x_{\sigma(n)}) \cdots q_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}) Q_{s_{\sigma(1)}} u_0(x_{\sigma(1)}).$$

Here σ denotes the permutation of $\{1,2,\ldots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Note that $f_0(t,x) = Q_t u_0(x)$.

Therefore, to obtain the existence and uniqueness of the solution to (5.2), it suffices to prove

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} < \infty, \quad \forall\, (t,x) \in [0,T] \times \mathbb{R}^d. \tag{5.4}$$

Theorem 5.3. *Let the measure μ satisfy Hypothesis (II). Then (5.4) holds, and consequently, $u(t,x)$ given by (5.3) is the unique mild solution to (5.1).*

Proof. Without loss of generality, we assume that $u_0(x) \equiv 1$. Now we have

$$\begin{aligned} &n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\ =&n! \int_{[0,t]^{2n}} \int_{\mathbb{R}^{2nd}} h_n(s,y,t,x) h_n(r,z,t,x) \prod_{j=1}^n |s_j - r_j|^{-\beta_0} \prod_{j=1}^n \gamma(y_j - z_j) dydzdsdr, \end{aligned}$$

where

$$h_n(s_1,\dots,s_n,x_1,\dots,x_n,t,x)=\frac{1}{n!}q_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})\cdots q_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}). \tag{5.5}$$

Then by (2.1) and (2.5),

$$\begin{aligned}
&n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\\
\le& n!\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(s,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(r,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|s_j-r_j|^{-\beta_0}dsdr,\\
\le& n!\int_{[0,t]^{2n}}A_{t,x}(s)A_{t,x}(r)\prod_{j=1}^n|s_j-r_j|^{-\beta_0}dsdr\\
\le& n!\int_{[0,t]^{2n}}A^2_{t,x}(s)\prod_{j=1}^n|s_j-r_j|^{-\beta_0}dsdr,\ \text{(using } 2ab\le a^2+b^2 \text{ and the symmetry of the integral)}
\end{aligned}$$

where

$$A_{t,x}(s)=\left(\int_{\mathbb{R}^{nd}}|\mathcal{F}h_n(s,\cdot,t,x)(\xi)|^2\mu(d\xi)\right)^{1/2}, \tag{5.6}$$

with

$$\mathcal{F}h_n(s,\cdot,t,x)(\xi)=\frac{1}{n!}e^{-ix\cdot(\xi_1+\cdots+\xi_n)}\prod_{j=1}^n\exp\Big[-[s_{\sigma(j+1)}-s_{\sigma(j)}]\Psi(\xi_{\sigma(1)}+\cdots+\xi_{\sigma(j)})\Big], \tag{5.7}$$

where we use the convention $s_{\sigma(n+1)}=t$.

Note that $\int_0^t\int_0^t f(s)|s-r|^{-\beta_0}dsdr\le 2\int_0^t r^{-\beta_0}dr\int_0^t|f(s)|ds$ and let $D_t=2\int_0^t r^{-\beta_0}dr$. Therefore,

$$\begin{aligned}
&n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\le D_t^n n!\int_{[0,t]^n}A^2_{t,x}(s)ds\\
=&D_t^n n!\int_{[0,t]^n}\int_{\mathbb{R}^{nd}}|\mathcal{F}h_n(s,\cdot,t,x)(\xi)|^2\mu(d\xi)ds\\
=&D_t^n\frac{1}{n!}\int_{[0,t]^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\Big[-2[s_{\sigma(j+1)}-s_{\sigma(j)}]\Psi(\xi_{\sigma(1)}+\cdots+\xi_{\sigma(j)})\Big]\mu(d\xi)ds\\
\le&D_t^n\frac{1}{n!}\int_{[0,t]^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\Big[-2[s_{\sigma(j+1)}-s_{\sigma(j)}]\Psi(\xi_{\sigma(j)})\Big]\mu(d\xi)ds\ \ \text{(by Lemma 4.9)}\\
=&D_t^n\int_{[0<s_1<\cdots<s_n<t]}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n\exp\Big[-2[s_{j+1}-s_j]\Psi(\xi_j)\Big]\mu(d\xi)ds.
\end{aligned}$$

Similar as in the proof of Theorem 3.3, we can apply Proposition 3.5 with $\beta_0=0$ for the last integral and then get the following estimate

$$n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\le D_t^n\sum_{k=0}^n\binom{n}{k}\frac{t^k}{k!}m_N^k\left[A_0\varepsilon_N\right]^{n-k},$$

where ε_N and m_N are given in (3.5) with $\beta_0 = 0$. Hence, if we choose N sufficiently large such that $2D_t A_0 \varepsilon_N < 1$, then we have

$$\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \le \sum_{n=0}^{\infty} D_t^n \sum_{k=0}^{n} \binom{n}{k} \frac{t^k}{k!} m_N^k \left[A_0 \varepsilon_N\right]^{n-k}$$
$$\le \sum_{k=0}^{\infty} \frac{t^k}{k!} m_N^k \sum_{n=k}^{\infty} D_t^n 2^n \left[A_0 \varepsilon_N\right]^{n-k} = \frac{1}{1 - 2D_t A_0 \varepsilon_N} \sum_{k=0}^{\infty} \frac{t^k}{k!} m_N^k D_t^k 2^k < \infty. \qquad \blacksquare$$

Remark 5.4. *Let $\eta(x)$ be a locally integrable function, then as in [27], the result of the above theorem still holds if the temporal kernel $|r-s|^{-\beta_0}$ is replaced by $\eta(r-s)$.*

The following theorem provides the Feynman-Kac type of representations for the solution and the moments of the solution when the spectral measure μ satisfies the stronger condition Hypothesis (I). The proof is similar to the one in [30] and we omit it here.

Theorem 5.5. *If we assume that μ satisfy Hypothesis (I), then*

$$u(t,x) = \mathbb{E}^X \left[u_0(X_t^x) \exp\left(\int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y) W(dr, dy) - \frac{1}{2} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) drds \right) \right] \tag{5.8}$$

is the unique mild solution to (5.1) in the Skorohod sense. Consequently, for any positive integer p, we have

$$\mathbb{E}[u(t,x)^p] = \mathbb{E}\left[\prod_{j=1}^{p} u_0(X_t^j + x) \exp\left(\sum_{1 \le j < k \le p} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r^j - X_s^k) drds \right) \right], \tag{5.9}$$

where $X_1, \ldots, X_p$ are p independent copies of X.

5.2 Feynman-Kac formula for the moments of the solution

When the measure μ satisfies Hypothesis (II) but not Hypothesis (I), the representation (5.8) may be invalid since $\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) drds$ might be infinite a.s. (see [30] for the case that X is a d-dimensional Brownian motion and $\gamma(x) = \prod_{j=1}^d |x_j|^{-\beta_j}, \beta_j \in (0,1), j = 1, \ldots, d$). However, the Feynman-Kac formula (5.9) for the moments still holds as stated in the following theorem.

Theorem 5.6. *Let the measure μ satisfy Hypothesis (II), then the Feynman-Kac formula (5.9) for the moments of the mild solution to (5.1) holds.*

Proof. We will adopt the approximation method used in [28, Section 5] to prove the result. The proof is split into three steps for easier reading.

Step 1. Consider the approximation of (5.1),

$$\begin{cases} u^{\varepsilon,\delta}(t,x) = \mathcal{L}u^{\varepsilon,\delta}(t,x) + u^{\varepsilon,\delta}(t,x) \diamond \dot{W}^{\varepsilon,\delta}(t,x), \\ u^{\varepsilon,\delta}(0,x) = u_0(x). \end{cases} \tag{5.10}$$

Recall that $\dot{W}^{\varepsilon,\delta}(t,x)$ is defined in (4.8). If $u^{\varepsilon,\delta}(t,x) \in \mathbb{D}^{1,2}$, then by (2.7)

$$u^{\varepsilon,\delta}(t,x) \diamond \dot{W}^{\varepsilon,\delta}(t,x) = \int_0^t \int_{\mathbb{R}^d} g_\delta(t-s) p_\varepsilon(x-y) u^{\varepsilon,\delta}(t,x) W^\diamond(ds,dy).$$

Therefore, the mild solution to (5.10) is, as defined in [28], an adapted random field $\{u^{\varepsilon,\delta}(t,x), t \geq 0, x \in \mathbb{R}^d\}$ which is square integrable for all fixed (t,x) and satisfies the following integral equation,

$$\begin{aligned} u^{\varepsilon,\delta}(t,x) &= Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y) u^{\varepsilon,\delta}(s,y) \diamond \dot{W}^{\varepsilon,\delta}(s,y) ds dy \\ &= Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y) g_\delta(s-r) p_\varepsilon(y-z) u^{\varepsilon,\delta}(s,y) ds dy \right) W^\diamond(dr,dz). \end{aligned}$$

Denote

$$Z^{\varepsilon,\delta}_{t,x}(r,z) = \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y) g_\delta(s-r) p_\varepsilon(y-z) u^{\varepsilon,\delta}(s,y) ds dy.$$

Thus to show that an adapted and square integrable process $\{u^{\varepsilon,\delta}(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to (5.10), it is equivalent to show $u^{\varepsilon,\delta}(t,x) = Q_t u_0(x) + \delta(Z^{\varepsilon,\delta}_{t,x})$. Therefore by the definition of the divergence operator δ, it is equivalent to show that for any $F \in \mathbb{D}^{1,2}$ with mean zero,

$$\mathbb{E}[F u^{\varepsilon,\delta}(t,x)] = \mathbb{E}[\langle Z^{\varepsilon,\delta}_{t,x}, DF \rangle_{\mathcal{H}}]. \tag{5.11}$$

Let

$$u^{\varepsilon,\delta}(t,x) = \mathbb{E}^X \left[u_0(X^x_t) \exp\left(W(\Phi^{\varepsilon,\delta}_{t,x}) - \frac{1}{2} \|\Phi^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}} \right) \right], \tag{5.12}$$

where $\Phi^{\varepsilon,\delta}_{t,x}$ is given by (4.1). Using a similar argument based on the technique of S-transform as in the proof of [28, Proposition 5.2], we can show that $u^{\varepsilon,\delta}(t,x)$ given by (5.12) satisfies (5.11), and hence it is a mild solution to (5.10).

Step 2. In this step, we will show that

$$\lim_{\varepsilon,\delta \downarrow 0} \mathbb{E}\left[\left(u^{\varepsilon,\delta}(t,x) \right)^p \right] = \mathbb{E}\left[\prod_{j=1}^p u_0(X^j_t + x) \exp\left(\sum_{1 \leq j < k \leq p} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X^j_r - X^k_s) dr ds \right) \right]. \tag{5.13}$$

Without loss of the generality, we assume $u_0(x) \equiv 1$ from now on. Denote

$$\Phi^{\varepsilon,\delta,j}_{t,x}(r,y) := \int_0^t g_\delta(t-s-r) p_\varepsilon(X^j_s + x - y) ds \cdot I_{[0,t]}(r), \;\; j = 1, \ldots, p.$$

The p-moment of $u^{\varepsilon,\delta}(t,x)$ is

$$\begin{aligned}
\mathbb{E}\left[\left(u^{\varepsilon,\delta}(t,x)\right)^p\right] &= \mathbb{E}^W\mathbb{E}^X\prod_{j=1}^{p}\exp\left(W(\Phi_{t,x}^{\varepsilon,\delta,j}) - \frac{1}{2}\|\Phi_{t,x}^{\varepsilon,\delta,j}\|_{\mathcal{H}}^2\right)\\
&= \mathbb{E}^X\exp\left(\frac{1}{2}\|\sum_{j=1}^{p}\Phi_{t,x}^{\varepsilon,\delta,j}\|_{\mathcal{H}}^2 - \frac{1}{2}\sum_{j=1}^{p}\|\Phi_{t,x}^{\varepsilon,\delta,j}\|_{\mathcal{H}}^2\right) = \mathbb{E}^X\exp\left(\sum_{1\le i<j\le p}\langle\Phi_{t,x}^{\varepsilon,\delta,j},\Phi_{t,x}^{\varepsilon,\delta,k}\rangle_{\mathcal{H}}\right).
\end{aligned}$$

As in the proof of Theorem 4.1, we can show that

$$\begin{aligned}
\langle\Phi_{t,x}^{\varepsilon,\delta,j},\Phi_{t,x}^{\varepsilon,\delta,k}\rangle_{\mathcal{H}} =& \frac{1}{(2\pi)^d}\int_{[0,t]^4}\int_{\mathbb{R}^d}(\widehat{p}_\varepsilon(\xi))^2\exp\left(-i\xi\cdot(X_{s_1}^j - X_{s_2}^k)\right)\\
&g_\delta(t-s_1-r_1)g_\delta(t-s_2-r_2)|r_1-r_2|^{-\beta_0}\mu(d\xi)dr_1dr_2ds_1ds_2,
\end{aligned}$$

and that $\langle\Phi_{t,x}^{\varepsilon,\delta,j},\Phi_{t,x}^{\varepsilon,\delta,k}\rangle_{\mathcal{H}}$ converges to $\int_{[0,t]^2}|s_1-s_2|^{-\beta_0}\gamma(X_{s_1}^j - X_{s_2}^k)ds_1ds_2$ in L^1 as (ε,δ) tends to zero. Now to prove the equality (5.13), it suffices to show that for any $\lambda>0$,

$$\sup_{\varepsilon,\delta>0}\mathbb{E}\left[\exp\left(\lambda\langle\Phi_{t,x}^{\varepsilon,\delta,j},\Phi_{t,x}^{\varepsilon,\delta,k}\rangle_{\mathcal{H}}\right)\right]<\infty. \tag{5.14}$$

By (2.4) and (4.4), there exists a positive constant C depending on β_0 only such that

$$\langle\Phi_{t,x}^{\varepsilon,\delta,j},\Phi_{t,x}^{\varepsilon,\delta,k}\rangle_{\mathcal{H}} \le C\int_{[0,t]^2}\int_{\mathbb{R}^d}(\widehat{p}_\varepsilon(\xi))^2\exp(-i\xi\cdot(X_r^j - X_s^k))|r-s|^{-\beta_0}\mu(d\xi)drds.$$

Hence to obtain (5.14), it is sufficient to prove that for any $\lambda>0$,

$$\sup_{\varepsilon>0}\mathbb{E}\left[\exp\left(\lambda\int_{[0,t]^2}\int_{\mathbb{R}^d}(\widehat{p}_\varepsilon(\xi))^2\exp(-i\xi\cdot(X_r-\widetilde{X}_s))|r-s|^{-\beta_0}\mu(d\xi)drds\right)\right]<\infty, \tag{5.15}$$

where $\widetilde{X}$ is an independent copy of X. For the n-th moment of the exponent, similar to the proof of Theorem 4.6, we have that for any $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{[0,t]^2}\int_{\mathbb{R}^d}(\widehat{p}_\varepsilon(\xi))^2\exp(-i\xi\cdot(X_r-\widetilde{X}_s))|r-s|^{-\beta_0}\mu(d\xi)drds\right)^n\right]\\
=&\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}(\widehat{p}_\varepsilon(\xi_j))^2E\exp(-i\sum_{j=1}^{n}\xi_j\cdot(X_{r_j}-\widetilde{X}_{s_j}))\prod_{j=1}^{n}|r_j-s_j|^{-\beta_0}\mu(d\xi)drds\\
\le&\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\mathbb{E}\exp(-i\sum_{j=1}^{n}\xi_j\cdot(X_{r_j}-\widetilde{X}_{s_j}))\prod_{j=1}^{n}|r_j-s_j|^{-\beta_0}\mu(d\xi)drds\\
=&\mathbb{E}\left[\left(\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-\widetilde{X}_s)drds\right)^n\right].
\end{aligned}$$

Now to prove (5.14), it is sufficient to prove that for any $\lambda>0$,

$$\mathbb{E}\left[\exp\left(\lambda\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-\widetilde{X}_s)drds\right)\right]<\infty. \tag{5.16}$$

Note that

$$\mathbb{E}\left[\left(\lambda\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-\widetilde{X}_s)drds\right)^n\right] = \lambda^n\int_{[0,t]^{2n}}\prod_{j=1}^n |r_j-s_j|^{-\beta_0}\mathbb{E}\left[\prod_{j=1}^n\gamma(X_{r_j}-\widetilde{X}_{s_j})\right]drds$$

$$= \lambda^n(n!)^2\int_{[0,t]^{2n}}\int_{\mathbb{R}^{2nd}} h_n(s,y,t,0)h_n(r,z,t,0)\prod_{j=1}^n|s_j-r_j|^{-\beta_0}\prod_{j=1}^n\gamma(y_j-z_j)dydzdrds,$$

where h_n is given by (5.5), and the last equality is obtained by using the independent increment property of X. Then (5.16) can be obtained as in the proof of Theorem 5.3.

Step 3. As in the proof of Theorem 4.6, we can show that $\sup_{\varepsilon,\delta>0}\sup_{t\in[0,T],x\in\mathbb{R}^d}\mathbb{E}[|u^{\varepsilon,\delta}(t,x)|^p] < \infty$, $u^{\varepsilon,\delta}(t,x)$ converges to a limit denoted by $u(t,x)$ in L^p for any $p>0$ as (ε,δ) goes to zero, and moreover, $u(t,x)$ satisfies the formula (5.9). Therefore, by the uniqueness of the mild solution to (5.1), to conclude the proof, we only need to show that $u(t,x)$ is a mild solution to (5.1), i.e.,

$$\mathbb{E}[Fu(t,x)] = \mathbb{E}[\langle Z_{t,x}, DF\rangle_{\mathcal{H}}], \tag{5.17}$$

for any $F\in\mathbb{D}^{1,2}$ with $\mathbb{E}[F]=0$, where $Z_{t,x}(r,z)=q_{t-r}(x-z)u(r,z)$.

In a way similar to the proof of Theorem 4.1, we can prove that $\lim_{\varepsilon,\delta\downarrow 0}\mathbb{E}[\|Z^{\varepsilon,\delta}_{t,x}-Z_{t,x}\|^2_{\mathcal{H}}]=0$. Then we can show the equality (5.17) by letting (ε,δ) in (5.11) go to zero, noting that $F\in\mathbb{D}^{1,2}$ and $\lim_{\varepsilon,\delta\downarrow 0}u^{\varepsilon,\delta}(t,x)=u(t,x)$ in L^2. ∎

Remark 5.7. *In the second step of the proof, actually we proved that under Hypothesis (II), (5.16) holds, i.e., for any* $\lambda>0$

$$\mathbb{E}\left[\exp\left(\lambda\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-\widetilde{X}_s)drds\right)\right]<\infty.$$

5.3 Hölder continuity

Hypothesis (S2)**.** *The spectral measure* μ *satisfies that for all* $a\in\mathbb{R}^d$, *there exist* $\alpha_1\in(0,1]$ *and* $C>0$ *such that*

$$\sup_{z\in\mathbb{R}^d}\int_0^T\int_{\mathbb{R}^d} e^{-s\Psi(\xi+z)}\left(1-e^{-i(\xi+z)\cdot a}\right)\mu(d\xi)ds \le C|a|^{2\alpha_1}.$$

Hypothesis (T2)**.** *The spectral measure* μ *satisfies, for some* $\alpha_2\in(0,1)$,

$$\int_{\mathbb{R}^d}\frac{(\Psi(\xi))^{\alpha_2}}{1+\Psi(\xi)}\mu(d\xi)<\infty.$$

Remark 5.8. *Similar to the Stratonovich case, we have the following sufficient condition for Hypothesis (S2) to hold:*

$$\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\xi+z|^{2\alpha_1}}{1+\Psi(\xi+z)}\mu(d\xi)<\infty. \tag{5.18}$$

Furthermore, if $\eta(\xi) := \Psi(\xi)/|\xi|^{2\alpha_1}$ is a Lévy characteristic exponent (which is equivalent to say that $-\eta(\xi)$ is continuous, conditionally positive definite and $\eta(0) = 0$, see, e.g., [2, Theorem 1.2.17]; a special case in which $\eta(\xi)$ is the characteristic exponent of a symmetric stable process is that $\Psi(\xi) = |\xi|^\alpha$ with $\alpha > 2\alpha_1$), and the condition (5.20) below holds, then condition (5.18) is equivalent to

$$\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha_1}}{1+\Psi(\xi)}\mu(d\xi) < \infty. \tag{5.19}$$

Clearly (5.18) implies (5.19). Now we show that the inverse is true, if we assume that $\eta(\xi)$ is the characteristic exponent of a certain Lévy process $\{Y_t, t \geq 0\}$ and that the following condition holds

$$\sup_{z\in\mathbb{R}^d} \mu([|\xi+z| \leq M]) < \infty, \tag{5.20}$$

where M is a positive number such that $\eta(\xi) \geq 1$ for all $|\xi| \geq M$.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\xi+z|^{2\alpha_1}}{1+\Psi(\xi+z)}\mu(d\xi) = \int_{[|\xi+z|\leq M]} \frac{|\xi+z|^{2\alpha_1}}{1+\Psi(\xi+z)}\mu(d\xi) + \int_{[|\xi+z|>M]} \frac{|\xi+z|^{2\alpha_1}}{1+\Psi(\xi+z)}\mu(d\xi)\\
\leq& M^{2\alpha_1} \sup_{z\in\mathbb{R}^d} \mu([|\xi+z| \leq M]) + 2\int_{\mathbb{R}^d} \frac{1}{1+\eta(\xi+z)}\mu(d\xi)\\
=& C + 2\int_{\mathbb{R}^d}\int_0^\infty e^{-t}\mathbb{E}\left[e^{i(\xi+z)\cdot Y_t}\right] dt\mu(d\xi) = C + 2\mathbb{E}\left[\int_{\mathbb{R}^d}\int_0^\infty e^{-t}e^{i(\xi+z)\cdot Y_t} dt\mu(d\xi)\right]\\
=& C + 2\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} e^{i\xi\cdot Y_t}\mu(d\xi)\ e^{-t}e^{iz\cdot Y_t}dt\right] = C + 2\mathbb{E}\left[\int_0^\infty \gamma(Y_t)e^{-t}e^{iz\cdot Y_t}dt\right]\\
\leq& C + 2\mathbb{E}\left[\int_0^\infty \gamma(Y_t)e^{-t}dt\right] = C + 2\int_{\mathbb{R}^d}\frac{1}{1+\eta(\xi)}\mu(d\xi) \leq D + 2\int_{\mathbb{R}^d}\frac{|\xi|^{2\alpha_1}}{1+\Psi(\xi)}\mu(d\xi),
\end{aligned}$$

where D is another constant that may be different from C.

On the other hand, Hypothesis (T2) actually implies and hence is equivalent to the condition

$$\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\Psi(\xi+z))^{\alpha_2}}{1+\Psi(\xi+z)}\mu(d\xi) < \infty. \tag{5.21}$$

Note that for all $z \in \mathbb{R}^d$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{(\Psi(\xi+z))^{\alpha_2}}{1+\Psi(\xi+z)}\mu(d\xi) \leq \int_{\mathbb{R}^d}\left(\frac{1}{1+\Psi(\xi+z)}\right)^{1-\alpha_2}\mu(d\xi)\\
=& \int_{\mathbb{R}^d} \frac{1}{\Gamma(1-\alpha_2)}\int_0^\infty t^{-\alpha_2}e^{-[1+\Psi(\xi+z)]t}dt\ \mu(d\xi)\\
=& \int_{\mathbb{R}^d} \frac{1}{\Gamma(1-\alpha_2)}\int_0^\infty t^{-\alpha_2}e^{-t}\mathbb{E}[e^{i(\xi+z)\cdot Y_t)}]dt\ \mu(d\xi)\\
=& \frac{1}{\Gamma(1-\alpha_2)}\mathbb{E}\left[\int_0^\infty t^{-\alpha_2}e^{-t}e^{iz\cdot Y_t}\int_{\mathbb{R}^d} e^{i\xi\cdot Y_t}\mu(d\xi)\ dt\right]\\
\leq& \frac{1}{\Gamma(1-\alpha_2)}\mathbb{E}\left[\int_0^\infty t^{-\alpha_2}e^{-t}\int_{\mathbb{R}^d} e^{i\xi\cdot Y_t}\mu(d\xi)\ dt\right] = \int_{\mathbb{R}^d}\left(\frac{1}{1+\Psi(\xi)}\right)^{1-\alpha_2}\mu(d\xi),
\end{aligned}$$

where the first equality follows from the formula $c^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-ct}dt$ for $c > 0$ and $\alpha \in (0,1)$. Finally Hypothesis (T2) implies (5.21) because of the following equivalence

$$\int_{\mathbb{R}^d} \frac{(\Psi(\xi))^{\alpha_2}}{1+\Psi(\xi)}\mu(d\xi) < \infty \Longleftrightarrow \int_{\mathbb{R}^d}\left(\frac{1}{1+\Psi(\xi)}\right)^{1-\alpha_2}\mu(d\xi) < \infty$$

which is due to the facts $\lim_{|\xi|\to\infty}\Psi(\xi) = \infty$ and $\mu(A) < \infty$ for bounded $A \in \mathcal{B}(\mathbb{R}^d)$,

Theorem 5.9. *Let $u_0(x) \equiv 1$ and $u(t,x)$ be the unique mild solution to (5.1). If μ satisfies Hypothesis (S2), then $u(t,x)$ has a version that is θ_1-Hölder continuous in x with $\theta_1 < \alpha_1$ on any compact set of $[0,\infty)\times\mathbb{R}^d$; Similarly, if μ satisfies Hypothesis (T2), the solution $u(t,x)$ has a version that is θ_2-Hölder continuous in t with $\theta_2 < [\alpha_2 \wedge (1-\beta_0)]/2$ on any compact set of $[0,\infty)\times\mathbb{R}^d$.*

Proof. Let $u(t,x) = 1 + \sum_{n=1}^\infty I_n(h_n(\cdot,t,x))$ and $u(s,y) = 1+\sum_{n=1}^\infty I_n(h_n(\cdot,s,y))$, where h_n is given by (5.5). Then for $p>2$,

$$\begin{aligned}\|u(t,x)-u(s,y)\|_{L^p} &\le \sum_{n=1}^\infty \|I_n(h_n(\cdot,t,x)) - I_n(h_n(\cdot,s,y))\|_{L^p}\\ &\le \sum_{n=1}^\infty (p-1)^{n/2}\|I_n(h_n(\cdot,t,x)) - I_n(h_n(\cdot,s,y))\|_{L^2}\\ &= \sum_{n=1}^\infty (p-1)^{n/2}\sqrt{n!}\|h_n(\cdot,t,x)-h_n(\cdot,s,y)\|_{\mathcal{H}^{\otimes n}},\end{aligned} \tag{5.22}$$

where the last inequality holds due to the equivalence of L^p norms for $p>1$ on any Wiener chaos space $\mathbb{H}_n$ ([36, Theorem 1.4.1]), and the last equality follows from (2.10).

Step 1. First, we study the spatial continuity. Suppose that $s=t$, similar as in the proof of Theorem 5.3, we have

$$\begin{aligned}&n!\|h_n(\cdot,t,x)-h_n(\cdot,t,y)\|^2_{\mathcal{H}^{\otimes n}}\\ =&n!\left(\|h_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} + \|h_n(\cdot,t,y)\|^2_{\mathcal{H}^{\otimes n}} - 2\langle h_n(\cdot,t,x), h_n(\cdot,s,y)\rangle_{\mathcal{H}^{\otimes n}}\right)\\ =&\frac{2}{n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\left[1-e^{-i(x-y)\cdot(\xi_1+\cdots+\xi_n)}\right]\prod_{j=1}^n \exp\Big[-[r_{\sigma(j+1)}-r_{\sigma(j)}]\Psi(\xi_{\sigma(1)}+\cdots+\xi_{\sigma(j)})\Big]\\ &\quad\prod_{j=1}^n\exp\Big[-[s_{\eta(j+1)}-s_{\eta(j)}]\Psi(\xi_{\eta(1)}+\cdots+\xi_{\eta(j)})\Big]\mu(d\xi)\prod_{j=1}^n|r_j-s_j|^{-\beta_0}drds\end{aligned}$$

where σ and η are permutations of the set $\{1,2,\dots,n\}$ such that $r_{\sigma(1)} < r_{\sigma(2)} < \cdots < r_{\sigma(n)}$ and $s_{\eta(1)} < s_{\eta(2)} < \cdots < s_{\eta(n)}$. Denote

$$A^2(r) = \int_{\mathbb{R}^{nd}}\left[1-e^{-i(x-y)\cdot(\xi_1+\cdots+\xi_n)}\right]\prod_{j=1}^n\exp\Big[-2[r_{\sigma(j+1)}-r_{\sigma(j)}]\Psi(\xi_{\sigma(1)}+\cdots+\xi_{\sigma(j)})\Big]\mu(d\xi).$$

Recall the notations $D_t = 2\int_0^t s^{-\beta_0} ds$ and $\Omega_t^n = \Big\{(s_1, \dots, s_n) \in [0,\infty)^n : \sum_{j=1}^n s_j \le t\Big\}$. We have

$$
\begin{aligned}
&n!\|h_n(\cdot,t,x) - h_n(\cdot,t,y)\|^2_{\mathcal{H}^{\otimes n}} \le \frac{2}{n!}\int_{[0,t]^{2n}} A^2(r)\prod_{j=1}^n |s_j - r_j|^{-\beta_0} ds dr \le \frac{2}{n!} D_t^n \int_{[0,t]^n} A^2(r) dr \\
=&2D_t^n \int_{[0<r_1<r_2<\cdots<r_n<t]} \int_{\mathbb{R}^{nd}} \left[1 - e^{-i(x-y)\cdot(\xi_1+\cdots+\xi_n)}\right] \prod_{j=1}^n \exp\Big[-2[r_{j+1} - r_j]\Psi(\xi_1 + \cdots + \xi_j)\Big]\mu(d\xi)dr \\
=&2D_t^n \int_{\Omega_t^n}\int_{\mathbb{R}^{nd}} \left[1 - e^{-i(x-y)\cdot(\xi_1+\cdots+\xi_n)}\right] \prod_{j=1}^n \exp\Big[-2s_j\Psi(\xi_1 + \cdots + \xi_j)\Big]\mu(d\xi)ds \\
\le&2D_t^n \sup_{z\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \left[1 - e^{-i(x-y)\cdot(z+\xi_n)}\right] \exp[-2s_n\Psi(z+\xi_n)]\mu(d\xi_n)ds_n \\
&\qquad\times \int_{\Omega_t^{n-1}} \int_{\mathbb{R}^{(n-1)d}} \prod_{j=1}^{n-1} \exp\left[-2s_j\Psi(\xi_1+\cdots+\xi_j)\right]\mu(d\xi_1)\dots\mu(d\xi_{n-1})ds_1\dots ds_{n-1} \\
\le&CD_t^n|x-y|^{2\alpha_1} \int_{\Omega_t^{n-1}} \int_{\mathbb{R}^{(n-1)d}} \prod_{j=1}^{n-1} \exp\left[-2s_j\Psi(\xi_j)\right]\mu(d\xi)ds. \quad \text{(By Hypothesis (S2))}
\end{aligned}
$$

Applying Lemma (3.5), we have

$$
\sqrt{n!}\|h_n(\cdot,t,x) - h_n(\cdot,t,y)\|_{\mathcal{H}^{\otimes n}} \le |x-y|^{\alpha_1} C D_t^{n/2} \sum_{k=0}^{n-1} \sqrt{\binom{n-1}{k}\frac{t^k}{k!} m_N^k \left[A_0\varepsilon_N\right]^{n-1-k}}.
$$

As in the proof of Theorem 5.3, we can choose N large enough, such that

$$
\sum_{n=1}^\infty D_t^{n/2} \sum_{k=0}^n \sqrt{\binom{n}{k}\frac{t^k}{k!} m_N^k \left[A_0\varepsilon_N\right]^{n-k}} < \infty,
$$

and hence there exists a constant C such that

$$
\|u(t,x) - u(t,y)\|_{L^p} \le C|x-y|^{\alpha_1},
$$

which implies the spatial Hölder continuity of $u(t,x)$.

Step 2. Now we consider the Hölder continuity in time, assuming that $0 \le s < t \le T$

and $x=y$. Then for the estimation on the n-th chaos space, we have

$$
\begin{aligned}
&n!\|h_n(\cdot,t,x)-h_n(\cdot,s,x)\|^2_{\mathcal{H}^{\otimes n}}\\
=&n!\left(\|h_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}+\|h_n(\cdot,s,x)\|^2_{\mathcal{H}^{\otimes n}}-2\langle h_n(\cdot,t,x),h_n(\cdot,s,x)\rangle_{\mathcal{H}^{\otimes n}}\right)\\
=&n!\Bigg[\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu\\
&\quad+\int_{[0,s]^{2n}}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,s,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,s,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu\\
&\quad-2\int_{[0,t]^n\times[0,s]^n}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,s,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu\Bigg].
\end{aligned}
$$

Therefore

$$
n!\|h_n(\cdot,t,x)-h_n(\cdot,s,x)\|^2_{\mathcal{H}^{\otimes n}}\le n!(D_n+D_n'), \tag{5.23}
$$

where

$$
\begin{aligned}
D_n=&\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu\\
&-\int_{[0,t]^n\times[0,s]^n}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,s,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu,
\end{aligned}
$$

and

$$
\begin{aligned}
D_n'=&\int_{[0,t]^n\times[0,s]^n}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu\\
&-\int_{[0,s]^{2n}}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,s,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu.
\end{aligned}
$$

We will just estimate D_n, and D_n' will share the same upper bound of D_n.

Clearly, $D_n=A_n+B_n$ where

$$
A_n=\int_{[0,t]^n\times([0,t]^n\setminus[0,s]^n)}\int_{\mathbb{R}^{nd}}\mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu \tag{5.24}
$$

and

$$
\begin{aligned}
B_n=\int_{[0,t]^n\times[0,s]^n}\int_{\mathbb{R}^{nd}}&\left(\overline{\mathcal{F}h_n(v,\cdot,t,x)(\xi)}-\overline{\mathcal{F}h_n(v,\cdot,s,x)(\xi)}\right)\\
&\mathcal{F}h_n(u,\cdot,t,x)(\xi)\mu(d\xi)\prod_{j=1}^n|u_j-v_j|^{-\beta_0}dvdu.
\end{aligned} \tag{5.25}
$$

To get an estimation for the right-hand side of (5.23), we will separate the rest of the proof into three parts for easier reading.

Step 2(a). In this part, we will estimate A_n given in (5.24). Note that $[0,t]^n = \cup_{k_j\in\{0,1\}} I_{k_1} \times I_{k_2} \times \cdots \times I_{k_n}$ with $I_1 = [0,s]$ and $I_2 = [s,t]$. Hence $[0,t]^n \backslash [0,s]^n$ is the union of $2^n - 1$ disjoint interval products, each of which contains at least one $[s,t]$. Denote $E_{n,j}$ the product of n intervals, all of which are $[0,t]$ except that the j-th interval is $[s,t]$. Therefore, for the term A_n, we have

$$\begin{aligned}A_n \leq & 2^n \sup_{j=1,\dots,n} \int_{[0,t]^n\times E_{n,j}} \int_{\mathbb{R}^{nd}} \mathcal{F}h_n(u,\cdot,t,x)(\xi)\overline{\mathcal{F}h_n(v,\cdot,t,x)(\xi)}\mu(d\xi)\prod_{j=1}^n |u_j - v_j|^{-\beta_0} dvdu \\ \leq & 2^n \sup_{j=1,\dots,n} \int_{[0,t]^n\times E_{n,j}} \left(A_{t,x}^2(u) + A_{t,x}^2(v)\right)\prod_{j=1}^n |u_j - v_j|^{-\beta_0} dvdu \end{aligned} \tag{5.26}$$

with $A_{t,x}(u)$ given in (5.6). Denoting $D_t = 2\int_0^t |s|^{-\beta_0} ds$, for positive function f, we have the following estimates

$$\int_0^t \int_0^t f(u)|u-v|^{-\beta_0} dvdu \leq D_t \int_0^t f(u)du,$$

$$\int_0^t \int_s^t f(u)|u-v|^{-\beta_0} dvdu \leq \frac{2^{\beta_0}}{1-\beta_0}(t-s)^{1-\beta_0} \int_0^t f(u)du,$$

and

$$\int_0^t \int_s^t f(v)|u-v|^{-\beta_0} dvdu \leq D_t \int_s^t f(v)dv.$$

Applying those estimates, we get

$$\begin{aligned}&\int_{[0,t]^n\times E_{n,j}} \left(A_{t,x}^2(u) + A_{t,x}^2(v)\right)\prod_{j=1}^n |u_j - v_j|^{-\beta_0} dvdu \\ \leq & \frac{2^{\beta_0}}{1-\beta_0}(t-s)^{1-\beta_0} D_t^{n-1} \int_{[0,t]^n} A_{t,x}^2(u)du + D_t^n \int_{E_{n,j}} A_{t,x}^2(v)dv. \end{aligned} \tag{5.27}$$

Note that Hypothesis (T2) implies

$$\int_{\mathbb{R}^d} \frac{1}{1+(\Psi(\xi))^{1-\alpha_2}}\mu(d\xi) < \infty, \tag{5.28}$$

and hence there exists $C>0$ depending on the measure μ and α_2 such that for all $x>0$

$$\int_{\mathbb{R}^d} e^{-x\Psi(\xi)}\mu(d\xi) \leq C(1+x^{\alpha_2-1})$$

by Lemma 3.10. On the other hand, by Lemma 3.11, we have

$$\begin{aligned}&\int_{[0<v_1<v_2<\cdots<v_n<t]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n (1+(v_{j+1}-v_j)^{\alpha_2-1})\mu(d\xi)dv \\ \leq & C^n \sum_{\tau\in\{0,1\}^n} \frac{\prod_{j=1}^n \Gamma(\tau_j(\alpha_2-1)+1)}{\Gamma(\sum_{j=1}^n \tau_j(\alpha_2-1)+n+1)} t^{\sum_{j=1}^n \tau_j(\alpha_2-1)+n}. \end{aligned} \tag{5.29}$$

Combining (5.28) and (5.29) and using the approach in Remark 3.6, we have for $t \in [0,T]$ with $T \geq 1$,

$$\int_{[0,t]^n} A_{t,x}^2(u)du \leq \frac{C^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\frac{t^{m(\alpha_2-1)+n}}{\Gamma(m(\alpha_2-1)+n+1)} \leq \frac{(2C)^n}{n!}\frac{T^n}{\Gamma(n\alpha_2+1)}. \tag{5.30}$$

Similarly, for all $j \in \{1,2,\dots,n\}$ and $0 \leq s < t \leq T$ with $T \geq 1$, we have

$$\begin{aligned}
&\int_{E_{n,j}} A_{t,x}^2(v)dv \leq \frac{1}{(n!)^2}\int_{[0,t]^{n-1}\times[s,t]}\int_{\mathbb{R}^{nd}}\prod_{j=1}^{n}\exp\left(-2(v_{\sigma(j+1)}-v_{\sigma(j)})\psi(\xi_{\sigma(j)})\right)\mu(d\xi)dv\\
&\leq \frac{C^n}{(n!)^2}\int_{[0,t]^{n-1}\times[s,t]}\prod_{j=1}^{n}\left(1+(v_{\sigma(j+1)}-v_{\sigma(j)})^{\alpha_2-1}\right)dv\\
&\leq \frac{C^n}{(n!)^2}\left(\int_{[0,t]^n}-\int_{[0,s]^n}\right)\prod_{j=1}^{n}\left(1+(v_{\sigma(j+1)}-v_{\sigma(j)})^{\alpha_2-1}\right)dv\\
&= \frac{C^n}{n!}\left(\int_{[0<v_1<\cdots<v_n<t]}-\int_{[0<v_1<\cdots<v_n<s]^n}\right)\prod_{j=1}^{n}\left(1+(v_{j+1}-v_j)^{\alpha_2-1}\right)dv\\
&= \frac{C^n}{n!}\sum_{\tau\in\{0,1\}^n}\frac{\prod_{j=1}^n\Gamma(\tau_j(\alpha_2-1)+1)}{\Gamma(\sum_{j=1}^n\tau_j(\alpha_2-1)+n+1)}(t^{\sum_{j=1}^n\tau_j(\alpha_2-1)+n}-s^{\sum_{j=1}^n\tau_j(\alpha_2-1)+n})\\
&\leq \frac{1}{n!}\frac{C^n}{\Gamma(n\alpha_2+1)}nT^n(t-s)^{\alpha_2}.
\end{aligned} \tag{5.31}$$

The last inequality holds because $t^{\sum_{j=1}^n\tau_j(\alpha_2-1)+n}-s^{\sum_{j=1}^n\tau_j(\alpha_2-1)+n} \leq nT^n(t-s)^{\alpha_2}$ for all n and τ. Combining the above (5.30) and (5.31) with (5.26) and (5.27), we have

$$A_n \leq \frac{1}{n!}\frac{C^n}{\Gamma(n\alpha_2+1)}\left((t-s)^{1-\beta_0}+(t-s)^{\alpha_2}\right), \tag{5.32}$$

where C depends on the measure μ, T, β_0 and α_2.

Step 2(b). The term B_n given in (5.25) will be estimated in this part.

$$\begin{aligned}
B_n \leq& \frac{1}{(n!)^2}\int_{[0,t]^n\times[0,s]^n}\int_{\mathbb{R}^{nd}}\left|e^{-(t-v_{\sigma(n)})\Psi(\xi_1+\cdots+\xi_n)}-e^{-(s-v_{\sigma(n)})\Psi(\xi_1+\cdots+\xi_n)}\right|\\
&\prod_{j=1}^{n-1}e^{-(v_{\sigma(j+1)}-v_{\sigma(j)})\Psi(\xi_{\sigma(1)}+\cdots+\xi_{\sigma(j)})}\mathcal{F}h_n(u,\dots,t,x)(\xi)\mu(d\xi)\prod_{j=1}^{n}|u_j-v_j|^{-\beta_0}dvdu\\
\leq& 2(t-s)^{\alpha_2}\frac{1}{(n!)^2}\int_{[0,t]^n\times[0,s]^n}\int_{\mathbb{R}^{nd}}(\Psi(\xi_1+\cdots+\xi_n))^{\alpha_2}\prod_{j=1}^{n}e^{-(v_{\sigma(j+1)}-v_{\sigma(j)})\Psi(\xi_{\sigma(1)}+\cdots+\xi_{\sigma(j)})}\\
&\prod_{j=1}^{n}e^{-(u_{\eta(j+1)}-u_{\eta(j)})\Psi(\xi_{\eta(1)}+\cdots+\xi_{\eta(j)})}\mu(d\xi)\prod_{j=1}^{n}|u_j-v_j|^{-\beta_0}dvdu,
\end{aligned}$$

where $v_{n+1} = s, u_{n+1} = t$ and σ and η are permutations such that $0 < v_{\sigma(1)} < \cdots < v_{\sigma(n)} < t$ and $0 < u_{\eta(1)} < \cdots < u_{\eta(n)} < t$, and in the last step we used the inequality $|e^{-x} - e^{-y}| \le |e^{-x} + e^{-y}||x-y|^{\alpha} \le 2|x-y|^{\alpha}$ for $x, y > 0$ and $\alpha \in (0,1]$.

Let

$$A_t^2(u) = \int_{\mathbb{R}^{nd}} (\Psi(\xi_1 + \cdots + \xi_n))^{\alpha_2} \prod_{j=1}^{n} e^{-2(u_{\eta(j+1)} - u_{\eta(j)})\Psi(\xi_{\eta(1)} + \cdots + \xi_{\eta(j)})} \mu(d\xi)$$

and

$$A_s^2(v) = \int_{\mathbb{R}^{nd}} (\Psi(\xi_1 + \cdots + \xi_n))^{\alpha_2} \prod_{j=1}^{n} e^{-2(v_{\sigma(j+1)} - v_{\sigma(j)})\Psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)})} \mu(d\xi).$$

we have

$$\begin{aligned}
&\int_{[0,t]^n} A_t^2(u) du \\
=& n! \int_{[0<u_1<\cdots<u_n<t]} \int_{\mathbb{R}^{nd}} (\Psi(\xi_1 + \cdots + \xi_n))^{\alpha_2} \prod_{j=1}^{n} e^{-2(u_{j+1} - u_j)\Psi(\xi_1 + \cdots + \xi_j)} \mu(d\xi) du \\
\le& n! \sup_{z \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Psi(\xi_n + z))^{\alpha_2} e^{-2(t-u_n)\Psi(\xi_n + z)} \mu(d\xi_n) du_n \\
&\qquad \times \int_{[0<u_1<\cdots<u_{n-1}<t]} \int_{\mathbb{R}^{(n-1)d}} \prod_{j=1}^{n-1} e^{-2(u_{j+1} - u_j)\Psi(\xi_1 + \cdots + \xi_j)} \mu(d\xi) du \\
\le& n! C \int_{[0<u_1<\cdots<u_{n-1}<t]} \int_{\mathbb{R}^{(n-1)d}} \prod_{j=1}^{n-1} e^{-2(u_{j+1} - u_j)\Psi(\xi_j)} \mu(d\xi) du \\
\le& n! \frac{C^{n+1} T^n}{\Gamma(n\alpha_2 + 1)},
\end{aligned}$$

where the last second step follows from Lemma 3.7, Hypothesis (T2), Remark 5.8 and Lemma 4.9, and the last step follows by a similar argument for (5.30). Now we have the estimation for B_n,

$$\begin{aligned}
B_n &\le (t-s)^{\alpha_2} \frac{1}{(n!)^2} \int_{[0,t]^n \times [0,s]^n} (A_t^2(u) + A_s^2(v)) \prod_{j=1}^{n} |u_j - v_j|^{-\beta_0} dv du \\
&\le 2(t-s)^{\alpha_2} \frac{1}{(n!)^2} D_t^n \int_{[0,t]^n} A_t^2(u) du \\
&\le 2(t-s)^{\alpha_2} \frac{1}{n!} D_t^n \frac{C^{n+1} T^n}{\Gamma(n\alpha_2 + 1)}.
\end{aligned} \tag{5.33}$$

Step 2(c). Therefore, combining (5.32) and (5.33), we have that there exists a constant C depending on the measure μ, T, α_2 and β_0 such that

$$\sum_{n=1}^{\infty} (p-1)^{n/2} \sqrt{n!} \sqrt{D_n} = \sum_{n=1}^{\infty} (p-1)^{n/2} \sqrt{n!} \sqrt{A_n + B_n} \le C(t-s)^{[\alpha_2 \wedge (1-\beta_0)]/2}. \tag{5.34}$$

Note that we can get estimation for D_n' analogous to (5.34), by an argument similar as the above for D_n. Finally, by (5.22), (5.23) and (5.34), we have

$$\|u(t,x) - u(s,x)\|_{L^p} \le \sum_{n=1}^{\infty}(p-1)^{n/2}\sqrt{n!}\|h_n(\cdot,t,x) - h_n(\cdot,s,y)\|_{\mathcal{H}^{\otimes n}}$$

$$\le \sum_{n=1}^{\infty}(p-1)^{n/2}\sqrt{n!}\sqrt{D_n + D_n'} \le C(t-s)^{[\alpha_2\wedge(1-\beta_0)]/2}.$$

The Hölder continuity in time now is concluded by the Kolmogorov's criterion. ∎

References

[1] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109(5):1492–1505, 1958.

[2] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[3] G. Bal. Convergence to SPDEs in Stratonovich form. *Comm. Math. Phys.*, 292(2):457–477, 2009.

[4] R. M. Balan and D. Conus. Intermittency for the wave and heat equations driven by fractional noise in time. *http://arxiv.org/pdf/1311.0021v1.pdf*.

[5] R. M. Balan and D. Conus. A note on intermittency for the fractional heat equation. *Statist. Probab. Lett.*, 95:6–14, 2014.

[6] L. Bertini and N. Cancrini. The stochastic heat equation: Feynman-Kac formula and intermittence. *J. Statist. Phys.*, 78(5-6):1377–1401, 1995.

[7] R. A. Carmona and S. A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.*, 108(518):viii +125 pp., 1994.

[8] R. A. Carmona and S. A. Molchanov. Stationary parabolic Anderson model and intermittency. *Probab. Theory Related Fields*, 102(4):433–453, 1995.

[9] M. Caruana, P. K. Friz, and H. Oberhauser. A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(1):27–46, 2011.

[10] L. Chen and R. C. Dalang. Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *http://arxiv.org/pdf/1307.0600v2.pdf*.

[11] X. Chen, Y. Hu, and J. Song. Feynman-kac formula for the fractional heat equations driven by fractional white noises. *http://arxiv.org/pdf/1203.0477.pdf.*

[12] X. Chen, Y. Hu, J. Song, and F. Xing. Exponential asymptotics for time-space hamiltonians. *Ann. Inst. Henri Poincaré., to appear.*

[13] P.-L. Chow. *Stochastic partial differential equations.* Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[14] D. Conus. Moments for the parabolic Anderson model: on a result by Hu and Nualart. *Commun. Stoch. Anal.*, 7(1):125–152, 2013.

[15] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1992.

[16] R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[17] R. C. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.*, 4(6):29 pp. (electronic), 1999.

[18] A. Deya. On a modelled rough heat equation. *http://arxiv.org/abs/1404.5438.*

[19] A. Deya, M. Gubinelli, and S. Tindel. Non-linear rough heat equations. *Probab. Theory Related Fields*, 153(1-2):97–147, 2012.

[20] M. Foondun and D. Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.*, 14(21):548–568, 2009.

[21] M. Foondun and D. Khoshnevisan. On the stochastic heat equation with spatially-colored random forcing. *Trans. Amer. Math. Soc.*, 365(1):409–458, 2013.

[22] I. M. Gel′fand and N. Y. Vilenkin. *Generalized functions. Vol. 4.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964 [1977]. Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein.

[23] M. Gubinelli, A. Lejay, and S. Tindel. Young integrals and SPDEs. *Potential Anal.*, 25(4):307–326, 2006.

[24] M. Gubinelli and S. Tindel. Rough evolution equations. *Ann. Probab.*, 38(1):1–75, 2010.

[25] M. Hairer. Solving the KPZ equation. *Ann. of Math. (2)*, 178(2):559–664, 2013.

[26] R. Z. Has′minskiĭ. On positive solutions of the equation $\mathcal{U} + Vu = 0$. *Theor. Probability Appl.*, 4:309–318, 1959.

[27] Y. Hu, J. Huang, D. Nualart, and S. Tindel. Stochastic heat equations with general multiplicative gaussian noises: Hölder continuity and intermittency. *http://arxiv.org/pdf/1402.2618.pdf.*

[28] Y. Hu and D. Nualart. Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields*, 143(1-2):285–328, 2009.

[29] Y. Hu, D. Nualart, and J. Song. Integral representation of renormalized self-intersection local times. *J. Funct. Anal.*, 255(9):2507–2532, 2008.

[30] Y. Hu, D. Nualart, and J. Song. Feynman-Kac formula for heat equation driven by fractional white noise. *Ann. Probab.*, 39(1):291–326, 2011.

[31] Y. Z. Hu and P.-A. Meyer. Sur les intégrales multiples de Stratonovitch. In *Séminaire de Probabilités, XXII*, volume 1321 of *Lecture Notes in Math.*, pages 72–81. Springer, Berlin, 1988.

[32] D. Khoshnevisan. *Analysis of stochastic partial differential equations*, volume 119 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2014.

[33] D. Khoshnevisan and Y. Xiao. Harmonic analysis of additive Lévy processes. *Probab. Theory Related Fields*, 145(3-4):459–515, 2009.

[34] J.-F. Le Gall. Exponential moments for the renormalized self-intersection local time of planar Brownian motion. In *Séminaire de Probabilités, XXVIII*, volume 1583 of *Lecture Notes in Math.*, pages 172–180. Springer, Berlin, 1994.

[35] S. A. Molchanov. Ideas in the theory of random media. *Acta Appl. Math.*, 22(2-3):139–282, 1991.

[36] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[37] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[38] A.-S. Sznitman. *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[39] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.

Jian Song
Department of Mathematics and Department of Statistics & Actuarial Science
The University of Hong Kong, Hong Kong
`txjsong@hku.hk`