Geometric structures and substructures on uniruled projective manifolds

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Uniruled projective manifolds play an important role in algebraic geometry. By the seminal work of Mori [Mr79], rational curves always exist on a projective manifold whenever the canonical line bundle fails to be numerically effective, and by Miyaoka-Mori [MM86] any Fano manifold is uniruled. While much knowledge is gained from Mori theory in the case of higher Picard numbers, the structure of uniruled projective manifolds of Picard number 1 is hard to grasp from a purely algebro-geometric perspective. In a series of works on uniruled projective manifolds starting with Hwang-Mok [HM98], Jun-Muk Hwang and the author have developed the basics of a geometric theory of uniruled projective manifolds arising from the study of varieties of minimal rational tangents (VMRTs), i.e., the collection at a general point of the variety of tangents to minimal rational curves passing through the point. The theory was from its onset a cross-over between algebraic geometry and differential geometry. While we dealt with classical problems in algebraic geometry and axiomatics were derived from basics in the deformation theory of rational curves, the heart of our perspective was differential-geometric in nature, revolving around tautological foliations, G-structures, differential systems, etc. and dealing with various issues relating to connections, curvature, integrability, etc., while techniques from several complex variables on analytic continuation were brought in to allow for a passage from transcendental objects defined on open sets in the Euclidean topology to algebraic objects in the Zariski topology.

Given any uniruled projective manifold X, fixing a polarization and minimizing degrees of free rational curves we obtain a minimal rational component $\mathcal{K}$. Basic to $(X, \mathcal{K})$ is the double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, where $\rho : \mathcal{U} \to \mathcal{K}$ is the universal family whose fibers are (unparametrized) minimal rational curves, and $\mu : \mathcal{U} \to X$ is the evaluation map. At a general point $x \in X$ a point on the fiber $\mathcal{U}_x$ corresponds to a minimal rational curve with a marking at x, and the VMRT $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ is the image of $\mathcal{U}_x$ under the tangent map. The double fibration and the VMRT structure $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T(X)$, endowed with a tautological foliation, set the stage on which the basics of our geometric theory have been developed.

In this article, by a geometric structure we mean a VMRT structure $\pi : \mathcal{C}(X) \to X$ or its restriction to a connected open set U on X, and by a geometric substructure we mean a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) \subset \mathbb{P}T(S)$, on a complex submanifold S of some open subset of X, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, which among other things is assumed to dominate S. For a VMRT structure $\pi : \mathcal{C}(X) \to X$, of principal importance here is the tautological foliation on $\mathcal{C}(X)$ transported from the fibration $\rho : \mathcal{U} \to \mathcal{K}$ by means of the tangent map, and solutions to various questions concerning the tautological foliation have strong implications leading to rigidity phenomena or characterization results on uniruled projective manifolds. As to sub-VMRT structures, a basic question is whether the tautological foliation on $\mathcal{C}(X)$ is tangent to $\mathcal{C}(S)$, and an affirmative answer to the question leads to rational saturation for germs of submanifolds inheriting certain types of sub-VMRT structures, and characterization of various classes of special uniruled projective subvarieties.

There is a wide scope of phenomena and problems concerning geometric structures and substructures in complex geometry, and those on uniruled projective manifolds arising from the consideration of minimal rational curves in particular, and the current

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article is an exposition on selected aspects of such phenomena and problems arising from VMRTs. Concerning geometric structures we will be exclusively concerned with those arising from or directly related to known uniruled projective manifolds, especially rational homogeneous spaces of Picard number 1, leaving aside the topic of general VMRT structures, for which the reader is referred to two expository articles of Hwang [Hw12] [Hw15] (and references therein) on VMRTs from the perspective of Cartanian geometry. For geometric substructures our focus will be on sub-VMRT structures on rational homogeneous spaces of Picard number 1 modeled on certain admissible pairs (X_0, X) of such manifolds, while results will also be formulated for sub-VMRT structures on uniruled projective manifolds in general satisfying various notions of nondegeneracy related to the second fundamental form (cf. Mok [Mk08a], Hong-Mok [HoM10] [HoM13], Hong-Park [HoP11], Hwang [Hw14b], Mok-Zhang [MZ15], Zhang [Zh14]). An overview on the topic of germs of complex submanifolds on uniruled projective manifolds will be given in the article.

The current article is written with the aim of highlighting certain aspects in an area of research arising from the study of geometric structures modeled on VMRTs. As a number of surveys and expository articles are available at different stages on various aspects in the development of the subject (Hwang-Mok [HM99a], Hwang [Hw01], Kebekus-Sola Conde [KS06], Mok [Mk08b], Hwang [Hw12] [Hw15]), for the parts of the article where adequate exposition already exists, we are contented with recalling fundamental elements and results in the theory which are essential for the understanding of more recent development and with providing examples occasionally for the purpose of illustration. The presentation will be more systematic in the last section on sub-VMRT structures since the latter topic is relatively new. At the end of the section, we will discuss various perspectives concerning sub-VMRT structures, and indicate how the subject, *a priori* arising from the study of uniruled projective manifolds and their subvarieties, has intimate links with other areas of mathematics including several complex variables, local differential geometry and Kähler geometry. Already on this topic there is the prospect of exciting cross-fertilization of ideas and methodology, and the subject will thrive with further investigation on problems intrinsic to the study of VMRTs and also with applications to be explored on these and other related areas of mathematics.

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§1 Minimal rational curves on uniruled projective manifolds

(1.1) *Minimal rational components and the universal family* For a projective variety $W \subset \mathbb{P}^N$ we denote by Chow(W) the Chow space of all cycles C on W, and by $[C] \in$ Chow(W) the member corresponding to the cycle C. Each irreducible component of Chow(W) is projective. For two projective varieties Y and Z we denote by Hom(Y, Z) the set of all morphisms from Y to Z. Through the use of Hilbert schemes, Hom(Y, Z) is endowed the structure of a complex space such that each of its irreducible components is projective (cf. Kollár [Ko96, Chapter 1]).

By a rational curve on a projective X manifold we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$, which will be denoted by $[f]$ when regarded as an element of

$\mathrm{Hom}(\mathbb{P}^1, X)$. A rational curve $[f]$ is said to be free if and only if the vector bundle f^*T_X on $\mathbb{P}^1$ is semipositive, i.e., isomorphic to a direct sum of holomorphic line bundles $\mathcal{O}(a_k)$ of degree $a_k \geq 0$. The basic objects of our study are the uniruled projective manifolds, i.e., projective manifolds that are "filled up" by rational curves. Equivalently a projective manifold X is uniruled if and only if there exists a free rational curve on X. A smooth hypersurface $X \subset \mathbb{P}^n$ of degree $\leq n-1$ is uniruled by projective lines, and those of degree n are uniruled by rational curves of degree 2. By Mori-Miyaoka [MM86] any Fano manifold is uniruled. For the basics on rational curves in algebraic geometry we refer the reader to Kollár [Ko96].

Let X be a uniruled projective manifold. Fixing an ample line bundle L on X, let $f_0 : \mathbb{P}^1 \to X$ be a free rational curve realizing the minimum of $\deg(h^*L)$ among all *free* rational curves $[h] \in \mathrm{Hom}(\mathbb{P}^1, X)$. Let $\check{\mathcal{H}} \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be an irreducible component containing $[f_0]$ and $\mathcal{H} \subset \check{\mathcal{H}}$ be the subset consisting of free rational curves. $\check{\mathcal{H}}$ is quasi-projective and $\mathcal{H} \subset \check{\mathcal{H}}$ is a dense Zariski open subset. Since each member $[f] \in \mathcal{H}$ is a free rational curve, there is no obstruction in deforming $f : \mathbb{P}^1 \to X$, and, passing to normalization if necessary, $\mathcal{H}$ will be endowed the structure of a quasi-projective manifold. Any member $f : \mathbb{P}^1 \to X$ of $\mathcal{H}$ must be generically injective (i.e., f must be birational onto its image) by the freeness of f and by the minimality of $\deg(f^*L)$ among free rational curves. Thus, $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$ by the assignment $(\gamma, [f]) \mapsto [f \circ \gamma]$ for $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ and $[f] \in \mathcal{H}$. Since $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$, the quotient $\mathcal{K} := \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$ is a complex manifold. There is a canonical morphism $\alpha : \mathcal{H} \to \mathrm{Chow}(X)$ defined by $\alpha([f]) = [f(\mathbb{P}^1)]$ mapping $\mathcal{K}$ onto a Zariski open subset $\mathcal{Q}$ of some irreducible subvariety $\mathcal{Z}$ of $\mathrm{Chow}(X)$. The mapping α is invariant under the action of $\mathrm{Aut}(\mathbb{P}^1)$ and it descends to a bijective holomorphic map $\nu : \mathcal{K} \to \mathcal{Q}$. Hence, ν is a normalization, and $\mathcal{K}$ is a quasi-projective manifold. We call $\mathcal{K}$ a minimal rational component on X. There is a smallest subvariety $B \subset X$ such that every member of $\mathcal{K}$ passing through any point $x \in X - B$ is a free rational curve. We call $B \subset X$ the bad locus of $(X, \mathcal{K})$.

On a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component we have a universal $\mathbb{P}^1-$bundle $\rho : \mathcal{U} \to \mathcal{K}$ called the universal family of $\mathcal{K}$, where $\mathcal{U} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1; 0)$, and $\rho : \mathcal{U} \to \mathcal{K}$ is the canonical projection which realizes $\mathcal{U}$ as the total space of a holomorphic fiber bundle with fibers isomorphic to $\mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1; 0) \cong \mathbb{P}^1$. We have canonically the evaluation map $\mu : \mathcal{U} \to X$, and we write $\mathcal{U}_x := \mu^{-1}(x)$. From the Bend-and-Break Lemma of Mori [Mr79] it follows that a general member κ of a minimal rational component $\mathcal{K}$ corresponds to a standard rational curve, i.e., κ is the equivalence class modulo the action of $\mathrm{Aut}(\mathbb{P}^1)$ of some $[f] \in \mathrm{Hom}(\mathbb{P}^1, X)$ such that $f^*T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$, $1 + p + q = n := \dim(X)$. Note that any standard rational curve $f : \mathbb{P}^1 \to X$ is immersive and generically injective. In the sequel, to avoid clumsy language the term 'minimal rational curve' will sometimes also be used to describe the image of a minimal rational curve belonging to $\mathcal{H}$ under the canonical map $\beta : \mathcal{H} \to \mathcal{K}$.

(1.2) *Varieties of minimal rational tangents and the tautological foliation* Let $(X, \mathcal{K})$ be a uniruled projective manifold X equipped with a minimal rational component. Denote by $\rho : \mathcal{U} \to \mathcal{K}$ the universal family over $\mathcal{K}$, and by $\mu : \mathcal{U} \to X$ the accompanying evaluation map. By definition $\rho : \mathcal{U} \to \mathcal{K}$, as a holomorphic fiber bundle with fibers isomorphic to $\mathbb{P}^1$, is equipped with a tautological foliation whose leaves are the fibers of ρ. Let $x \in X$ be a general point and u be a point on $\mathcal{U}_x$, corresponding to a minimal rational curve with a marking at x. Let $f : \mathbb{P}^1 \to X$ be a parametrization of $\rho(u) \in \mathcal{K}$, $\widehat{f} : \mathbb{P}^1 \to \mathcal{U}$ its tautological lifting so that $\widehat{f}(0) = u$ (hence $f(0) = x$). If f is an immersion at 0 we define $\tau_x(u) = [df(T_0(\mathbb{P}^1))] \in \mathbb{P}T_0(X)$. For a general point $x \in X$ this defines the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathbb{P}T_x(X)$, which is a holomorphic immersion at a general point of $\mathcal{U}_x$ corresponding to a standard rational curve with a marking at x, and we denote by $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ the strict transform of τ_x, so that $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ is *a priori* a generically finite dominant rational map, and $\pi : \mathcal{C}(X) \to X$ is equipped at general points with a multi-foliation $\mathcal{F}$ transported from the tautological foliation on $\mathcal{U}$ by means of the tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$.

Standard rational curves play a special role with regard to the tangent map. For convenience of notation we will state the following result for embedded standard rational curves ℓ. The general case, in which standard rational curves are known only to be immersed, can be stated with a slight modification. For an embedded minimal rational curve ℓ we have $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, and we denote by $P_\ell = \mathcal{O}(2) \oplus \mathcal{O}(1)^p$ the positive part of $T(X)|_\ell$. For the normal bundle $N_{\ell|X}$ for $\ell \subset X$ we have $N_{\ell|X} \cong \mathcal{O}(1)^p \oplus \mathcal{O}^q$. From the deformation theory of rational curves we have (cf. Mok [Mk08b, (2.4), Lemma 2]).

Lemma 1.2.1. *At a general point $x \in X$, and a point $u \in \mathcal{U}_x$ corresponding to a standard rational curve ℓ with a marking at x, the tangent map τ_x is a holomorphic immersion at u. Assuming that $\tau_x(u) := [\alpha]$ is a smooth point of the* VMRT $\mathcal{C}_x(X)$, *we have $d\tau_x(u) : T_u(\mathcal{U}_x) \xrightarrow{\cong} T_{[\alpha]}(\mathcal{C}_x(X)) \subset T_{[\alpha]}(\mathbb{P}T_x(X)) \cong T_x(X)/\mathbb{C}\alpha$. More precisely, assuming for convenience that $\ell \subset X$ is embedded, we have $T_u(\mathcal{U}_x) = \Gamma(\ell, N_{\ell|X} \otimes \mathfrak{m}_x)$, where $\mathfrak{m}_x$ is the maximal ideal sheaf at x on ℓ, and $T_{[\alpha]}(\mathcal{C}_x(X)) = P_\alpha/\mathbb{C}\alpha$, where $P_\alpha = P_{\ell,x}$ is the fiber at x of the positive part $P_\ell \subset T(X)|_\ell$, and for $\nu \in T_u(\mathcal{U}_x) = \Gamma(\ell, N_{\ell|X} \otimes \mathfrak{m}_x)$, we have $d\tau_x(u)(\nu) = \partial_\alpha \nu + \mathbb{C}\alpha \in P_\alpha/\mathbb{C}\alpha \cong T_{[\alpha]}(\mathcal{C}_x(X))$.*

We note that since $\nu(x) = 0$, the partial derivative $\partial_\alpha \nu$ is well-defined. Moreover, while the isomorphism $T_{[\alpha]}(\mathbb{P}T_x(X)) \cong T_x(X)/\mathbb{C}\alpha$ depends on the choice of $\alpha \in T_x(X)$ representing $[T_x(\ell)] \in \mathbb{P}T_x(X)$, there is a canonical isomorphism $T_{[\alpha]}(\mathbb{P}T_x(X)) \otimes L_{[\alpha]} \cong T_x(X)/\mathbb{C}\alpha$, where L denotes the tautological line bundle over $\mathbb{P}T_x(X)$, hence the formula for $d\tau_x(u)(\nu) \in T_{[\alpha]}(\mathcal{C}_x(X)) \subset T_{[\alpha]}(\mathbb{P}T_x(X))$ is independent of the choice of $\alpha \in T_x(\ell)$.

By Hwang-Mok [HM99a] [HM01], at a general point $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ is birational (cf. (2.1)), and it is a morphism by Kebekus [Ke02]. Finally, Hwang-Mok [HM04b] proved that the tangent map is a birational finite morphism, hence $\tau_x : \mathcal{U}_x \to \mathcal{C}_x$ is the normalization. There is a smallest subvariety $B' \supset B$ of X such that every member of $\mathcal{K}$ passing through any point $x \in X - B'$ is a free rational curve immersed at the marked point x and $\tau_x : \mathcal{U}_x \dashrightarrow X$ is a birational morphism. We call $B' \subset X$ the enhanced bad locus of $(X, \mathcal{K})$. In some cases, e.g., in the case of a projective submanifold $X \subset \mathbb{P}^n$ uniruled by projective lines it is easily seen (from the positivity of $T(X)|_\ell \subset T(\mathbb{P}^n)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-1}$ in the latter case) that at a general point $x \in X$ every projective line ℓ passing through x is a standard rational curve, and it has been thought for some time that this may actually be the case in general. Recently, Casagrande-Druel [CD12] found examples of uniruled projective manifolds $(X, \mathcal{K})$ equipped with minimal rational components in a more generalized sense (in the sense that the general VMRT $\mathcal{C}_x(X)$ is projective) on which the VMRT at a general point is actually singular, and Hwang-Kim [HK13b] has now obtained examples where $\mathcal{K}$ is a *bona fide* minimal rational component in the sense of (1.1). This means that for general results in the differential-geometric study of VMRT structures one has to deal with singularities which are smoothed out by normalization.

By the VMRT structure on $(X, \mathcal{K})$ we will mean the fibered space of VMRTs $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T(X)$. In the sequel we will speak of the VMRT structure $\pi : \mathcal{C}(X) \to X$ on X, being understood that we are talking about a subvariety $\mathcal{C}(X) \subset \mathbb{P}T(X)$ which projects onto a Zariski open subset of X. By the birationality of the tangent map we can now speak of the tautological foliation on a VMRT structure $\pi : \mathcal{C}(X) \to X$, and the extent to which the latter foliation is determined by the fibered space of VMRTs will play an important role in the rest of the article.

(1.3) *The affine and projective second fundamental forms* The second fundamental form in affine or projective geometry will be essential in the geometric study of VMRTs. For generalities let V be a finite-dimensional complex vector space and denote by $\nu : V - \{0\} \to \mathbb{P}(V)$ the canonical projection onto the projective space $\mathbb{P}(V)$. For any subset $E \subset \mathbb{P}(V)$ we denote by $\widetilde{E} := \nu^{-1}(E)$ the affinization of E. In terms of the Euclidean flat connection on V, for a complex submanifold S on some open subset of V we have the second fundamental form $\sigma := \sigma_{S|V}$. If $A \subset \mathbb{P}(V)$ is a subvariety and $\eta \in \widetilde{A}$ is a smooth point we have the second fundamental form $\sigma_\eta := \sigma_{\widetilde{A}|V,\eta}$, $\sigma_\eta : S^2T_\eta(\widetilde{A}) \to V/T_\eta(A) := N_{\widetilde{A}|V,\eta}$. We have always $\sigma_\eta(\eta,\xi) = 0$ for any $\xi \in T_\alpha(\widetilde{A})$. Thus, considered as a vector-valued symmetric bilinear form, the kernel of σ_η always contains $\mathbb{C}\eta$. Passing to quotients we have the projective second fundamental form, denoted by $\sigma_{[\eta]} : S^2T_{[\eta]}(A) \to T_{[\eta]}(\mathbb{P}(V))/T_{[\eta]}(A) := N_{A|\mathbb{P}(V),[\eta]}$, which is equivalently defined by the canonical projective connection on $\mathbb{P}(V)$. (We will use the same notation σ for both the Euclidean and the projective second fundamental forms. The subscript, either $\eta \in \widetilde{\mathcal{A}}$ or $[\eta] \in \mathcal{A}$ will indicate which is meant.) Here $T_{[\eta]}(\mathbb{P}(V)) \cong V/\mathbb{C}\eta, T_{[\eta]}(A) \cong T_\eta(\widetilde{A})/\mathbb{C}\eta$, and the two normal spaces $N_{\widetilde{A}|V,\eta} \cong N_{A|\mathbb{P}(V),[\eta]}$ are naturally identified. The projective second fundamental form $\sigma_{[\eta]}$ is the differential at η of the Gauss map, hence the Gauss map is generically injective on $\mathcal{A}$ if and only if $\mathrm{Ker}\,\sigma_{[\eta]} = 0$ for a general point $[\eta]$ of each irreducible component of $\mathcal{A}$. We note that from projective geometry, the Gauss map on a nonlinear projective submanifold $A \subset \mathbb{P}(V)$ is always generically injective.

§2 Analytic continuation along minimal rational curves

(2.1) *Equidimensional Cartan-Fubini extension* Let S be an irreducible Hermitian symmetric space. Denoting by $\mathcal{O}(1)$ the positive generator of the Picard group $\mathrm{Pic}(S) \cong \mathbb{Z}$, S admits an embedding $\theta : S \hookrightarrow \mathbb{P}(\Gamma(S,\mathcal{O}(1))^*)$ and as such S is uniruled by projective lines. When S is of rank ≥ 2, it is endowed with a particular type of G-structure. To explain this we start by recalling the notion of G-structures and flat G-structures. Let n be a positive integer, V be an n-dimensional complex vector space, and M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $\mathrm{GL}(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$.

Definition 2.1.1. *Let* $\mathrm{G} \subset \mathrm{GL}(V)$ *be any complex Lie subgroup. A holomorphic* G-*structure is a* G-*principal subbundle* $\mathcal{G}(M) \subset \mathcal{F}(M)$. *An element of* $\mathcal{G}_x(M)$ *will be called a* G-*frame at* x. *For* $\mathrm{G} \subsetneq \mathrm{GL}(V)$ *we say that* $\mathcal{G}(M)$ *defines a holomorphic reduction of the tangent bundle to* G. *We say that a* G-*structure* $\mathcal{G}(M)$ *on* M *is flat if and only if there exists an atlas of charts* $\{\varphi_\alpha : U_\alpha \to V\}$ *such that the restriction* $\mathcal{G}(U_\alpha)$ *of* $\mathcal{G}(M)$ *to* U_α *is the product* $\mathrm{G} \times U_\alpha \subset \mathrm{GL}(V) \times U_\alpha$ *in terms of Euclidean coordinates on* U_α *given by the chart* $\varphi_\alpha : U_\alpha \to V$.

As a first example we consider the hyperquadric $Q^n \subset \mathbb{P}^{n+1}$ defined as the zero of a nondegenerate homogeneous quadratic polynomial. The projective second fundamental form σ of $Q^n \subset \mathbb{P}^{n+1}$ defines a section in $\Gamma(Q^n, S^2T^*_{Q^n} \otimes \mathcal{O}(2))$, $\mathcal{O}(2)$ being isomorphic to the normal bundle $N_{Q^n|\mathbb{P}^{n+1}}$ of $Q^n \subset \mathbb{P}^{n+1}$. The twisted symmetric bilinear form σ is everywhere nondegenerate, thereby equipping small open sets of Q^n with holomorphic metrics $\sum g_{\alpha\beta}(z)dz^\alpha \otimes dz^\beta$ in terms of local coordinates, unique up to multiplication by nowhere zero holomorphic functions. This gives a holomorphic conformal structure on Q^n. Here we have a reduction of the frame bundle to the complex conformal group $\mathrm{CO}(n;\mathbb{C}) = \mathbb{C}^*\cdot\mathrm{O}(n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$ being the complex orthogonal group with respect to a nondegenerate complex symmetric bilinear form.

Another example is the Grassmannian $G(p,q)$ of p-planes in a complex vector space $W_0 \cong \mathbb{C}^{p+q}$, where we have a tautological vector bundle F on $G(p,q)$ given by $F_x = E \subset W_0$ for $x = [E] \in G(p,q)$. Writing $V = W/F$, where $W = W_0 \times G(p,q)$ is a trivial vector bundle on $G(p,q)$, we have a canonical isomorphism $T_{G(p,q)} \cong U \otimes V$, $U = F^*$, yielding a Grassmann structure on $G(p,q)$. U and V are called the (semipositive) universal bundles on $G(p,q)$. Here, for a pq-dimensional manifold M on which the holomorphic tangent bundle $T(M) \cong A \otimes B$ where A resp. B is a holomorphic vector bundle of rank p resp. q, representing tangent vectors on X as matrices through the tensor product decomposition, we have a reduction of the frame bundle from $\mathrm{GL}(pq,\mathbb{C})$ to the subgroup $H \subset \mathrm{GL}(pq,\mathbb{C})$ which is the image of $\mathrm{GL}(p,\mathbb{C})\times$ $\mathrm{GL}(q,\mathbb{C})$ in $\mathrm{GL}(pq,\mathbb{C})$ under the homomorphism Φ given by $\Phi(C,D)(X) = CXD^t$. We refer the reader to Manin [Ma97] for Grassmann structures appearing in gauge field theory.

For generalities on Hermitian symmetric spaces we refer the reader to Wolf [Wo72]. Any irreducible Hermitian symmetric space S of the compact type and of rank ≥ 2 carries a canonical S-structure, which is a G-structure for some complex reductive linear subgroup $\mathrm{G} \subsetneq GL(T_0(S))$, as follows. Write $S = G/P$ as a complex homogeneous space, where G is a connected complex simple Lie group, and $P \subset G$ is a maximal parabolic subgroup. Let $P = L \cdot U$ be the Levi decomposition of P, where $U \subset P$ is the unipotent radical and $L \subset P$ is a Levi factor. Equip S with a canonical Kähler-Einstein metric g and write $S = G_c/K$, where G_c is the identity component of the isometry group of (S,g) and $K \subset G_c$ is the isotropy subgroup at a reference point $0 \in S$. Identifying $0 \in S$ with $eP \in G/P \cong S$, in the Levi decomposition $P = L \cdot U$, L can be identified with $K^{\mathbb{C}} \subset \mathrm{GL}(T_0(S))$ by means of the Lie group homomorphism $\Phi : P \to \mathrm{GL}(T_0(S))$ given by $\Phi(\gamma) := d\gamma(0) \in \mathrm{GL}(T_0(S))$, $\Phi|_L : L \xrightarrow{\cong} K^{\mathbb{C}}$. On the other hand, $U = \exp(\mathfrak{m}^-) := M^-$, where $\mathfrak{m}^-$ is the Lie algebra of holomorphic vector fields on S vanishing to the order ≥ 2 at 0, thus $d\gamma(0)$ is the identity map on $T_0(S)$ whenever $\gamma \in M^-$. In other words, $U = M^- = \mathrm{Ker}(\Phi)$. Let η be a nonzero highest weight vector of the isotropy representation of $K^{\mathbb{C}}$ on $T_0(S)$. Since M^- acts trivially on $T_0(S)$, the G-orbit of $[\eta] \in \mathbb{P}T_0(S)$ gives a homogeneous holomorphic fiber subbundle $\mathcal{W} \subset \mathbb{P}T(S)$ whose fiber $\mathcal{W}_0$ over 0 is the $K^{\mathbb{C}}$-orbit of $[\eta]$, i.e., the highest weight orbit. Writing $V = T_0(S)$ and considering at $x \in S$ the set of all linear isomorphisms $\varphi : T_0(S) \xrightarrow{\cong} T_x(S)$ such that $\varphi(\widetilde{\mathcal{W}_0}) = \widetilde{\mathcal{W}}_x$, where $\widetilde{\mathcal{W}_0}$ consists of all nonzero highest weight vectors at 0, etc., we have a reduction of the frame bundle on S from $\mathcal{F}(S)$ to some $\mathcal{G}(S) \subsetneq \mathcal{F}(S)$ defining a G-structure with $\mathrm{G} = K^{\mathbb{C}}$. This canonical $K^{\mathbb{C}}$-structure is also called the canonical S-structure.

Flatness of the canonical $K^{\mathbb{C}}$-structure is not obvious. That this is the case is seen from the Harish-Chandra decomposition. The integrable almost complex structure on S is defined by $ad(j)$ of a certain element j in the 1-dimensional center $\mathfrak{z}$ of the Lie algebra $\mathfrak{k}$ of K. Writing $\mathfrak{g}$ for the Lie algebra of G we have a decomposition $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, where $\mathfrak{k}^{\mathbb{C}}$ is the Lie algebra of $K^{\mathbb{C}} \cong L$, and $\mathfrak{m}^+$ resp. $\mathfrak{m}^-$ is the eigenspace of $ad(j)$ corresponding to the eigenvalue i resp. $-i$. Writing $M^+ := \exp(\mathfrak{m}^+)$, the mapping $M^+ \times K^{\mathbb{C}} \times M^- \mapsto G$ given by $(a,b,c) \mapsto abc$ is injective, leading to the identification of a Zariski open subset W of S with the vector space $\mathfrak{m}^+$ through the mapping $m^+ \mapsto \exp(m^+)P$, yielding Harish-Chandra coordinates $(z_1, \cdots, z_n)$, $n = \dim(S)$. The abelian Lie subalgebra $\mathfrak{m}^+ \subset \mathfrak{g}$ is the Lie algebra of constant vector fields in the coordinates $(z_1, \cdots, z_n)$. The invariance of $\mathcal{W}$ under the vector group M^+ of Euclidean translations shows that $\mathcal{W}|_W = \mathcal{W}_0 \times W$, i.e., the $K^{\mathbb{C}}$-structure on S is flat.

The Harish-Chandra coordinates link immediately to the structure of minimal rational curves on S. A highest weight vector $\eta \in \widetilde{\mathcal{W}}_x$ yields readily a copy of $\mathfrak{sl}(2,\mathbb{C})$ which in standard notations is of the form $\mathbb{C}e_\rho \oplus \mathbb{C}[e_\rho, e_{-\rho}] \oplus \mathbb{C}e_{-\rho}$ in terms of root vectors $e_\rho \in \mathfrak{m}^+$, $e_{-\rho} \in \mathfrak{m}^-$ with respect to suitably chosen Cartan subalgebras, $[e_\rho, e_{-\rho}] \in \mathfrak{k}^{\mathbb{C}}$. Exponentiating one gets a copy of $\mathbb{P}SL(2,\mathbb{C})$, and the orbit of x under the latter group exhausts all the rational curves of degree 1 passing through W as x runs over W and $[\eta]$ runs over $\mathcal{W}_x$. Thus, intersections of minimal rational curves with a Harish-Chandra coordinate chart are given by affine lines ℓ such that $\mathbb{P}T_x(\ell) \in \mathcal{W}_x$ for $x \in \ell$. Moreover, $\mathcal{W}$ is nothing other than $\mathcal{C}(S) \subset \mathbb{P}T(S)$, the VMRT structure on S (cf. (1.2)). Examples of VMRTs $\mathcal{C}_0(S)$ are given in the case of hyperquadrics Q^n, $n \geq 3$, by $\mathcal{C}_0(Q^n) = Q^{n-2} \subset \mathbb{P}T_0(Q^n) \cong \mathbb{P}^{n-1}$, $\widetilde{\mathcal{C}}_0(Q^n) \cup \{0\}$ being the null-cone of the holomorphic conformal structure, and in the case of the Grassmannian $G(p,q)$; $p,q \geq 1$; by $\mathcal{C}_0(G(p,q)) = \varsigma(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}) \subset \mathbb{P}(\mathbb{C}^p \otimes \mathbb{C}^q)$, ς being the Segre embedding given by $\varsigma([u],[v]) = [u \otimes v]$, with the image being projectivizations of decomposable tensors.

The use of Harish-Chandra coordinates allows us to give a differential-geometric and complex-analytic proof (cf. Mok [Mk99]) of the following classical result of Ochiai on S-structures.

Theorem 2.1.1. (Ochiai [Oc70]) *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 equipped with the canonical S-structure. Let $U \subset S$ be a connected open subset and $f : U \xrightarrow{\cong} V \subset S$ be a biholomorphic map preserving the canonical S-structure. Then, there exists $F \in \mathrm{Aut}(S)$ such that $F|_U \equiv f$. As a consequence, any simply-connected compact complex manifold X admitting a flat S-structure must necessarily be biholomorphic to S.*

We refer the reader to [Mk99] and to Mok [Mk08b, (4.2), especially Lemma 4] for detailed discussions on Ochiai's Theorem from a geometric perspective. Denoting by $\mathcal{K}$ the minimal rational component of projective lines on S with the accompanying VMRT structure $\mathcal{C}(S) \subset \mathbb{P}T(S)$, the key issue is to show that f sends a connected open subset of a minimal rational curve onto an open subset of a minimal rational curve, i.e., writing $f_\sharp := [df]$, $\mathcal{F}_S$ for the tautological foliation on $\mathcal{S}$, $\mathcal{C}(U) := \mathcal{C}(S) \cap \mathbb{P}T(U)$, etc., we have to show that $f_{\sharp *}(\mathcal{F}_S|_{\mathcal{C}(U)}) = \mathcal{F}_S|_{\mathcal{C}(V)}$. Granting this, taking U to be a Euclidean ball in Harish-Chandra coordinates so that $\ell \cap U$ is either empty or connected for $[\ell] \in \mathcal{K}$, and by $\mathcal{O} \subset \mathcal{K}$ the subset consisting of all $[\ell] \in \mathcal{K}$ such that $\ell \cap U \neq \emptyset$, $f : U \to S$ induces a holomorphic map $f^\sharp : \mathcal{O} \to \mathcal{K}$. Then, by the method of Mok-Tsai [MT92], Hartogs extension holds true for $\mathcal{O}$, and we conclude that $f^\sharp$ extends *meromorphically* to $\Phi : \mathcal{K} \dashrightarrow \mathcal{K}$. We extend f analytically to F beyond U by defining $F(x)$ to be the intersection of the lines $\Phi([\ell])$, as ℓ ranges over minimal rational curves passing through x. Arguing also with f^{-1} we get a birational extension of f to $F : S \dashrightarrow S$ which transforms minimal rational curves to minimal rational curves, and that is enough to imply that in fact $F \in \mathrm{Aut}(S)$, cf. [Mk99, (2.4)]. We have

Proof that $f_{\sharp *}(\mathcal{F}_S|_{\mathcal{C}(U)}) = \mathcal{F}_S|_{\mathcal{C}(V)}$. We use Harish-Chandra coordinates. Restricted to such a Euclidean chart W, $\widetilde{\mathcal{C}}(S)|_W = \widetilde{\mathcal{C}}_x \times W$ for any $x \in W$, where $\widetilde{\mathcal{C}}_x := \widetilde{\mathcal{C}}_x(S)$. To prove $f_{\sharp *}(\mathcal{F}_S|_{\mathcal{C}(U)}) = \mathcal{F}_S|_{\mathcal{C}(V)}$ it suffices to show $d^2 f(\alpha,\alpha) \in \mathbb{C}df(\alpha)$ for $\alpha \in \widetilde{\mathcal{C}}_x$, $x \in U$. We may assume $df(x) = \mathrm{id}_{T_x(S)}$. For $\beta \in \widetilde{\mathcal{C}}_x$, we have $d^2 f(\alpha,\beta) = \partial_\alpha(df(\widetilde{\beta}))$, where $\widetilde{\beta}$ stands for the constant vector field on U such that $\widetilde{\beta}(x) = \beta$. Since $\widetilde{\mathcal{C}}|_U = \widetilde{\mathcal{C}}_x \times U$, $\partial_\alpha(df(\widetilde{\beta}))$ is the tangent at β to some curve on $\widetilde{\mathcal{C}}_x$, hence $d^2 f(\alpha,\beta) \in P_\beta = T_\beta(\widetilde{\mathcal{C}}_x)$, and by symmetry $d^2 f(\alpha,\beta) \in P_\alpha \cap P_\beta$. To show $d^2 f(\alpha,\alpha) \in \mathbb{C}\alpha$ note that for the second fundamental form σ of $\widetilde{\mathcal{C}}_x \subset T_x(S)$, $\mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$, and it remains to show $d^2 f(\alpha,\alpha) \in \mathrm{Ker}(\sigma_\alpha)$. Fix $\alpha \in \widetilde{\mathcal{C}}_x$ and let $\beta = \alpha(t), \alpha(0) = \alpha$, vary holomorphically on $\widetilde{\mathcal{C}}_x$ in the complex parameter t. Writing $\xi = \frac{d}{dt}\big|_{t=0}\alpha(t) \in P_\alpha$, from $d^2 f(\alpha,\alpha(t)) \in P_\alpha$ it follows that $d^2 f(\alpha,\xi) = \frac{d}{dt}\big|_{t=0} d^2 f(\alpha,\alpha(t)) \in P_\alpha$. On the other hand, $\frac{d}{dt}\big|_{t=0} d^2 f(\alpha(t),\alpha(t)) = 2d^2 f(\alpha,\xi)$. Interpreting $d^2 f(\beta,\beta) \in P_\beta$ as a vector field on $\widetilde{\mathcal{C}}_x$, we have $\nabla_\xi(d^2 f(\beta,\beta)) \in P_\alpha$ for the Euclidean flat connection ∇ on $T_x(S)$, hence $\sigma_\alpha(\xi, d^2 f(\alpha,\alpha)) = 0$. Varying

$\xi \in P_\alpha$, we conclude that $d^2 f(\alpha,\alpha) \in \mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$, as desired. $\square$

A Harish-Chandra coordinate chart flattens the VMRT structure on S on the chart. Although the existence of such coordinates in the Hermitian symmetric case is a very special feature among uniruled projective manifolds, in (2.2) we will explain how the same argument applies in general on a uniruled projective manifold $(X,\mathcal{K})$ endowed with a minimal rational component. The gist of the matter is that, for the computation at a general point $x \in X$ and at a smooth point $[\alpha] \in \mathcal{C}_x(X)$, denoting by ℓ the standard minimal rational curve passing through x such that $T_x(\ell) = \mathbb{C}\alpha$, where that ℓ is assumed embedded for convenience, what one needs is simply a choice of holomorphic coordinates on a neighborhood U of x such that the positive part $P_\ell = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \subset \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q = T(X)|_\ell$ is a constant vector subbundle of $T(X)|_\ell$ on $U \cap \ell$ in terms of the standard trivialization of $T(X)|_\ell$ induced by the holomorphic coordinates, and such choices of holomorphic coordinates exist in abundance.

For a uniruled projective manifold $(X,\mathcal{K})$ equipped with a minimal rational component, we introduced in Hwang-Mok [HM99a] differential systems on the VMRT structure $\mathcal{C}(X) \subset \mathbb{P}T(X)$ and on $\mathcal{K}$, and in [HM01] we gave a proof of a general form of Cartan-Fubini extension using such differential systems. The machinery introduced was used at the same time to prove birationality of the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ at a general point $x \in X$ under the assumption that the Gauss map is generically injective, a result which was later on improved to yield that τ_x is a birational morphism, i.e., τ_x is the normalization map (cf. (1.1)). Restricting to Cartan-Fubini extension, we have

Theorem 2.1.2. (Hwang-Mok [HM01]) *Let $(Z,\mathcal{H})$ and $(X,\mathcal{K})$ be two Fano manifolds of Picard number 1 equipped with minimal rational components. Assume that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$ and that furthermore the Gauss map is an immersion at a general point of each irreducible component of $\mathcal{C}_z(Z)$. Let $f : U \to V$ be a biholomorphic map from a connected open subset $U \subset Z$ onto an open subset $V \subset X$. If $f_\sharp = [df]$ sends each irreducible component of $\mathcal{C}(Z)|_U$ to an irreducible component of $\mathcal{C}(X)|_V$ biholomorphically, then f extends to a biholomorphic map $F : Z \to X$.*

We refer the reader to expositions on the differential systems in [HM99a], [Mk08b] and [Hw12]. Here we will just describe briefly such distributions and their links to analytic continuation. Let $x \in X$ be a general point, and $\ell \subset X$ be a standard minimal rational curve passing through x, assumed embedded for convenience. Let $u \in \mathcal{U}_x$ be the point corresponding to the minimal rational curve ℓ marked at x. Since ℓ is standard, there exists a neighborhood $\mathcal{O}$ of x in $\mathcal{U}$ such that the tangent map τ is a biholomorphism of $\mathcal{O}$ onto a complex submanifold $\mathcal{S}$ of some open subset of $\mathbb{P}T(X)$. Holomorphic distributions can now be defined on $\mathcal{S}$, as follows. The canonical projectio $\varpi := \pi|_\mathcal{S} : \mathcal{S} \to X$ is a submersion, and the kernels of $d\varpi$ defines an integrable distribution $\mathcal{J} \subset T(\mathcal{S})$. In what follows on a coordinate chart $U \subset X$, $0 \in U$ a reference point, we consider the standard trivializations $T(U) \cong U \times T_0(U)$, $T(T(U)) \cong T(U) \times T(T_0(U)) \cong (U \times T_0(U)) \times (T_0(U) \times T_0(U))$, thus at $(x,\xi) \in T_x(U)$, $\eta \in T_0(U)$ is simultaneously used to describe two different vectors, viz., as coordinates for a tangent vector at x and as coordinates for a vector at $(x,\xi) \in T_x(U)$ tangent to $T_x(U)$. To avoid confusion we will write η for the first meaning, and η' for the second, thus η' is a "vertical" tangent vector. Writing $T_x(\ell) = \mathbb{C}\alpha$, and denoting the fibers $\varpi^{-1}(x)$ by $\mathcal{S}_x$, we have $T_{[\alpha]}(\mathcal{S}_x) = P'_\alpha/\mathbb{C}\alpha'$. Define now $\mathcal{P} \subset T(S)$ by $\mathcal{P}_{[\alpha]} = d\varpi^{-1}(P_\alpha)$. Then, $\mathcal{P} \subset T(\mathcal{S})$ is a holomorphic distribution of rank $2p+1$, $p := \dim(\mathcal{C}_x(X))$, $\mathcal{P} \supset \mathcal{J}$.

On the other hand, writing $\mathcal{K}_{\rm st} \subset \mathcal{K}$ for the Zariski open subset consisting of standard minimal rational curves, we have on $\mathcal{K}_{\rm st}$ a holomorphic distribution $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$, defined as follows. For $[\ell] \in \mathcal{K}$, $T_{[\ell]}(\mathcal{K}) = H^0(\ell, N_{\ell|X})$, where $N_{\ell|X}$ stands for the normal bundle of ℓ in X. When $[\ell] \in \mathcal{K}_{\rm st}$, we have $N_{\ell|X} \cong \mathcal{O}(1)^p \oplus \mathcal{O}^q$, noting that $Q = \mathcal{O}(1)^p$

is the (strictly) positive part of the normal bundle $N_{\ell|X}$. $Q \subset N_{\ell|X}$ is characterized by the fact that $Q \otimes \mathcal{O}(-1)$ is spanned by $\Gamma(\ell, N_{\ell|X} \otimes \mathcal{O}(-1)) \cong \mathbb{C}^{2p}$, hence intrinsically defined. The assignment $[\ell] \mapsto \Gamma(\ell, \mathcal{O}(1)^p)$ defines a distribution $\mathcal{D}$ on $\mathcal{K}_{\rm st}$ of rank $2p$. We have

Proposition 2.1.1. (Hwang-Mok [HM01]) *Denoting by $\gamma : \mathcal{S} \to \mathcal{K}$ the canonical projection, we have $\mathcal{P} = d\gamma^{-1}(\mathcal{D})$. As a consequence $[\mathcal{F}, \mathcal{P}] \subset \mathcal{P}$, i.e., $\mathcal{F}$ lies on the Cauchy characteristic of the distribution $\mathcal{P}$. Moreover, assuming that for a general point $x \in X$, the projective second fundamental form σ on $\mathcal{C}_x$ is nondegenerate at a general smooth point $[\alpha] \in \mathcal{C}_x$. Then, $\mathcal{F} \subset P$ is exactly the Cauchy characteristic of $\mathcal{P}$.*

For the proof of the proposition we refer the reader to Hwang-Mok [HM99a, Corollary 3.1.5] and to Mok [Mk08b, (5.1), Proposition 5]. It suffices here to make a couple of remarks. First, given any holomorphic distribution $W \subset T(M)$ on a complex manifold, there is a holomorphic bundle homomorphism $\theta : \Lambda^2 W \to T(M)/W$ such that for any $\xi, \eta \in \Gamma(M, W)$ and for $x \in M$, we have $[\xi, \eta](x) \bmod W = \theta(\xi, \eta)(x)$. We call θ the Frobenius form of $W \subset T(M)$. Denote now by φ the Frobenius form of $\mathcal{P}|_{\mathcal{S}} \subset T(\mathcal{S})$ and by ψ the Frobenius form of $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$. From the fact that $\gamma : \mathcal{S} \to \mathcal{K}_{\rm st}$ is a holomorphic submersion and from $\mathcal{P}|_{\mathcal{S}} = d\gamma^{-1}(\mathcal{D})$, for $\xi \in \mathcal{S}$ and $u, v \in \mathcal{S}_\xi$ it follows readily that $\psi(d\gamma(u), d\gamma(v)) = \beta(\varphi_\xi(u, v))$ where the bundle isomorphism $\beta : T(\mathcal{S})/\mathcal{P}|_{\mathcal{S}} \xrightarrow{\cong} \gamma^*(T(\mathcal{K}_{\rm st})/\mathcal{D})$ is naturally induced by $d\gamma$. From $\mathcal{F} = (d\gamma)^{-1}(0)$ it now follows readily that $[\mathcal{F}, \mathcal{P}] \subset \mathcal{P}$. The Frobenius form φ and equivalently the Frobenius form ψ can furthermore be computed in terms of the second fundamental forms σ of $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ for a general point $x \in X$, and the last statement of Proposition 2.1.1 follows from the computation.

Proposition 2.1.1 yields immediately that the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x$ is a birational map for a general point. Moreover, the Cartan-Fubini Extension Principle holds true. In fact, given $f : U \xrightarrow{\cong} V$ such that $f_\sharp(\mathcal{C}(X)|_U) = \mathcal{C}(Z)|_V$ as in the hypothesis of Theorem 2.1.2, denoting $\mathcal{P}$ by $\mathcal{P}(X)$ and the analogous distribution on $\mathcal{C}(Z)$ by $\mathcal{P}(Z)$, obviously $f_{\sharp *}(\mathcal{F}_X)$ lies on the Cauchy characteristic of $\mathcal{P}(Z)$, and from the characterization of the Cauchy characteristic as in Proposition 2.1.1 it follows that the local VMRT-preserving map $f : U \xrightarrow{\cong} V$ actually preserves the tautological foliation. That the foliation-preserving property implies the extendibility of f to a biholomorphism $F : X \xrightarrow{\cong} Z$ was established in [HM01] by a combination of techniques of analytic continuation in several complex variables and the deformation theory of rational curves.

On top of being intrinsic, the approach in [HM01] introduced into the subject differential systems on VMRT structures $\mathcal{C}(X) \subset \mathbb{P}T(X)$ and on minimal rational components $\mathcal{K}$. So far the distributions $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$ have not been much studied. Especially, when a uniruled projective manifold X carries some extra geometric structure, e.g., a contact structure, the distribution $\mathcal{D} \subset T(\mathcal{K}_{\rm st})$ can be further enriched leading to an enhanced differential system on $\mathcal{K}_{\rm st}$, and it is tempting to believe that in certain cases this could lead to uniqueness or rigidity results concerning $\mathcal{K}$. The case where X carries a contact structure is especially interesting in view of the long-standing conjecture that a Fano contact manifold of Picard number 1 is homogeneous.

For applications of Cartan-Fubini extension in the equidimensional case we refer the reader to [HM01] and [HM04b]. Here we only note that in [HM04b] we obtained a new solution to the Lazarsfeld Problem, viz., proving that for $S := G/P$ a rational homogeneous space of Picard number 1 other than the projective space, any finite surjective holomorphic map $f : S \to X$ onto a projective manifold X must necessarily be a biholomorphism. The proof there was based on Cartan-Fubini extension applied to VMRTs of the uniruled projective manifold X. The original proof in [HM99b] was an application of our geometric theory of VMRTs at an early stage of its development relying heavily on Lie theory, especially on results concerning G-structures of Ochiai [Oc70] (Theorem 2.1.1 here) in the symmetric cases and those concerning differential systems on G/P of Yamaguchi [Ya93] in the non-symmetric cases. In [HM04b] the geometric theory on VMRTs was more self-contained, and we succeeded in entirely removing the detailed knowledge about G/P from the solution of Lazarsfeld's Problem.

(2.2) *Cartan-Fubini extension in the non-equidimensional case* Generalizing the arguments for the differential-geometric proof of Ochiai's Theorem, Hong-Mok [HoM10] established the following non-equidimensional Cartan-Fubini extension theorem.

Theorem 2.2.1. (Hong-Mok [HoM10], Theorem 1.1) *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds equipped with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$. Let $f : U \to X$ be a holomorphic embedding defined on a connected open subset $U \subset Z$. If f respects varieties of minimal rational tangents and is nondegenerate with respect to $(\mathcal{H}, \mathcal{K})$, then f extends to a rational map $F : Z \dashrightarrow X$.*

Here we say that f respects VMRTs if and only if $df(\widetilde{\mathcal{C}}_z(Z)) = \widetilde{\mathcal{C}}_{f(z)}(X) \cap df(T_z(Z))$, i.e., $f_\sharp(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(X) \cap f_\sharp(\mathbb{P}T_z(Z))$, where $f_\sharp$ is the projectivization of df. The holomorphic embedding $f : U \to X$ is said to be nondegenerate with respect to $(\mathcal{K}, \mathcal{H})$ if (a) its image $f(U)$ is not contained in the bad locus of $(X, \mathcal{K})$, and (b) at a general point $z \in U$ and a general smooth point $\alpha \in \widetilde{\mathcal{C}}_z(Z)$, $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$ such that the second fundamental form $\sigma(\eta, \xi)$ of $\widetilde{\mathcal{C}}_{f(z)}(X) \subset T_{f(z)}(X)$ at $df(\alpha)$, when restricted in ξ to the vector subspace $T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))) \subset T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X))$ and regarded as a family of linear maps on $T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X))$ in η, has common kernel $\mathbb{C}df(\alpha)$. Thus,

$$(\dagger) \quad \left\{\eta \in T_{df(\alpha)}(\widetilde{\mathcal{C}}_{f(z)}(X)) : \sigma(\eta, \xi) = 0 \text{ for any } \xi \in T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z)))\right\} = \mathbb{C}df(\alpha)\,.$$

Alternatively (†) means that on $\mathcal{C}_{f(z)}(X)$, considering the projective second fundamental form $\sigma_{f_\sharp([\alpha])}$ of $f_\sharp(\mathcal{C}_z(Z)) \subset \mathcal{C}_{f_\sharp([\alpha])}(X)$ at $f_\sharp([\alpha])$, the common kernel of $\sigma(\,\cdot\,, \xi), \xi \in T_{f_\sharp([\alpha])}(\mathcal{C}_{f(z)}(X))$ reduces to 0. Using the arguments of analytical continuation along minimal rational curves as developed in Hwang-Mok [HM01], the key issue that we settled in Hong-Mok [HoM10] was to prove that f maps a germ of minimal rational curve on $(Z, \mathcal{H})$ into a germ of minimal rational curve on $(X, \mathcal{K})$. Equivalently we showed that the image $f_{\sharp*}(\mathcal{F}_Z)$ on $f_\sharp(\mathcal{C}(Z)|_U) \subset \mathcal{C}(X)$ agrees with the restriction of $\mathcal{F}_X$ to $f_\sharp(\mathcal{C}(Z)|_U)$ as holomorphic line subbundles of $T(\mathcal{C}(X))|_{f_\sharp(\mathcal{C}(Z)|_U)}$. We prove Theorem 2.2.1 along the line of a proof of Ochiai's Theorem (Theorem 2.1.1) using the Euclidean flat connection and Harish-Chandra coordinates as explained, showing that, in the case where Z and X are Hermitian symmetric and using Harish-Chandra coordinates, for the Hessian $d^2 f(\xi, \eta)$ we have $d^2 f(\alpha, \alpha) \in \mathbb{C}df(\alpha)$ when evaluated at a point $z \in Z$ and at a vector $\alpha \in \widetilde{\mathcal{C}}_z(Z)$ under the nondegeneracy condition as stated.

When Z and X are irreducible Hermitian symmetric spaces of the compact type and of rank ≥ 2 the proof is the same as in Mok [Mk99]. In general, one makes use of special coordinate systems, as follows. Let $z \in Z$ be a general point and $\alpha \in \widetilde{\mathcal{C}}_z(Z)$ be a smooth point such that $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$. Let $\ell \subset Z$ be the minimal rational curve with a marking at z, and assume that z is a smooth point of ℓ for convenience. Let $z' \in \ell$ be a smooth point close to z. Write $T_{z'}(\ell) = \mathbb{C}\alpha'$. Let $\mathcal{D} \subset \mathcal{C}_{z'}(Z)$ be a smooth neighborhood of $[\alpha']$ on $\mathcal{C}_{z'}(Z)$, and $\mathcal{O}$ be a neighborhood of 0 in $\mathbb{C}^p$ such that the assignment $t = (t_1, \cdots, t_p) \mapsto [\alpha'_t] \in \mathcal{D}$, $\alpha'_0 = \alpha'$, defines a biholomorphism from $\mathcal{O}$ onto $\mathcal{D}$. Consider now the family of minimal rational curves parametrized by $\mathcal{O}$ given by a holomorphic map $\Phi : \mathbb{P}^1 \times \mathcal{O} \to Z$ such that $\Phi(0, t) = z'$ for $t \in \mathcal{O}$, $\Phi(s, 0) \in \ell'_0 := \ell$ and such that, for $t \in \mathcal{O}$, $\varphi_t(s) := \Phi(s, t)$ parametrizes the minimal rational curve ℓ'_t

passing through z' such that $T_{z'}(\ell'_t) = \mathbb{C}\alpha'_t$, $\alpha'_t \in \mathcal{D}$. We may assume that $z = \varphi_0(s_0)$ for some $s_0 \in \Delta$, $\Phi|_{\Delta^* \times \mathcal{O}}$ is an embedding and that $\varphi_t|_\Delta$ is an embedding for $t \in \mathcal{O}$.

Consider a holomorphic coordinate chart on a neighborhood U of z' in Z, $z \in U$, in which the minimal rational curves near $\ell'_0 = \ell$ passing through z' are represented on the chart as open subsets of lines through the origin. For each general point $w \in Z$, let $\mathcal{V}_w$ be the union of minimal rational curves passing through w. Thus, $\Sigma := \Phi(\Delta \times \mathcal{O}) \subset \mathcal{V}_{z'}$. Writing the parametrization as $\Phi(s,t) = s(\psi_1(t), \cdots, \psi_n(t)) = s\psi(t)$ in terms of the chosen Euclidean coordinates, observe that Σ is smooth along $\varphi_0(\Delta^*)$ and that for $w \in \varphi_0(\Delta^*)$ we have $T_w(\Sigma) = \mathrm{Span}\big\{(\psi(0), \frac{\partial\psi}{\partial t_1}(0), \cdots, \frac{\partial\psi}{\partial t_p}(0)\big\}$, which is independent of s. We call this the tangential constancy of Σ along the minimal rational curve ℓ. From the basics in the deformation theory of rational curves this implies that VMRTs are tangentially constant (in an obvious sense) along ℓ. The latter applies to all minimal rational curves at the same time when there is a coordinate system in which all minimal rational curves are represented by affine lines, which is in particular the case for Harish-Chandra coordinates in the Hermitian symmetric case, and that was the reason underlying the differential-geometric proof of Ochiai's Theorem.

In general for the computation at $z \in \ell \subset Z$, we have to resort to the special coordinates arising from some nearby point z' lying on ℓ, as described in the above. Compared to Riemannian geometry, the latter may be taken as an analogue of normal geodesic coordinates at z' in which minimal rational curves passing through z' appear as radial lines. Since other minimal rational curves intersecting with the chart need not be represented as affine lines, an elementary approximation argument was needed to carry through the proof, as was done in Hong-Mok [HoM10, Lemma 2.7].

Recently Hwang [Hw14b] has a generalized formulation of non-equidimensional Cartan-Fubini extension, as follows.

Theorem 2.2.2. (Hwang [Hw14b], Theorem 1.3) *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is positive-dimensional at a general point $z \in Z$. Let $f : U \to X$ be a holomorphic embedding defined on a connected open subset $U \subset Z$. Suppose $f_\sharp(\mathcal{C}(X)|_U) \subset \mathcal{C}(Z)$ and*

$$\big\{\eta \in T_{f_\sharp([\alpha])}(\mathcal{C}_{f(z)}(X)) : \sigma(\eta, \xi) = 0 \text{ for any } \xi \in T_{f_\sharp([\alpha])}(f_\sharp(\mathcal{C}_z(Z)))\big\} = 0\ ,$$

then f extends to a rational map $F : Z \to X$.

Hwang [Hw14b] made use of differential systems and Lie brackets of holomorphic vector fields more in the spirit of the proof in Hwang-Mok [Hw01] and does not require the use of adapted coordinates, and the proof is therefore more intrinsic, although the original proof in Hong-Mok [HoM10] also applies to give the same statement. One motivation for the more generalized formulation is that even when $\dim(Z) = \dim(X)$, Theorem 2.2.2 exceeds Cartan-Fubini extension in Theorem 2.1.2. The context applies, by the method of Hwang-Kim [HK13a] to equidimensional maps given by suitable double covers branched over Fano manifolds of Picard number 1 of large index.

§3 Characterization and recognition of homogeneous VMRT structures

(3.1) *Uniruled projective manifolds equipped with reductive holomorphic* G-*structures*
Just as the flat Euclidean space (as a germ) is characterized among Riemannian manifolds by the vanishing of the curvature tensor, flat G-structures are characterized by the vanishing of certain structure functions (Guillemin [Gu65]). For $k \geq 1$, a G-structure $\mathcal{G}(X) \subset \mathcal{F}(X)$ is k-flat at x if and only if there exists a germ of biholomorphism $f : (X; x) \to (V; 0)$ such that $f_*\mathcal{G}$ is tangent to the flat G-structure $\mathcal{G}' = \mathrm{G} \times V$ along $\mathcal{G}'_0$ to the order $\geq k$. When a given G-structure on X is k-flat at every point, there

is a naturally defined structure function c^k which measures the obstruction to $(k+1)$-flatness, which is a holomorphic 2-form taking values in some quotient bundles of tensor bundles of the form $T(X) \otimes S^k T^*(X)$. G acts on these quotient bundles. In the event that $\mathrm{G} \subsetneq \mathrm{GL}(V)$ is *reductive*, by identifying the latter quotient bundles with G-invariant vector subbundles of $T(X) \otimes S^k T^*(X)$, the structure functions concerned correspond to holomorphic sections θ_k of $\mathrm{Hom}\big(\Lambda^2 T(X), T(X) \otimes S^k T^*(X)\big)$. In this case, to prove flatness it suffices to check the vanishing of a finite number of θ_k. Concerning uniruled projective manifolds endowed with reductive G-structures we have the following result of Hwang-Mok [HM97].

Theorem 3.1.1. (Hwang-Mok [HM97]) *Let X be a uniruled projective manifold admitting an irreducible reductive G-structure, $\mathrm{G} \subsetneq \mathrm{GL}(V)$. Then, X is biholomorphic to an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2.*

We refer the reader to Hwang-Mok [HM99a] and Mok [Mk08b, (4.3)] for discussions on G-structures on uniruled projective manifolds surrounding the above theorem, and will be contented here with some remarks on the proof of the theorem. When a G-structure $\mathcal{G} \subset \mathcal{F}(X)$ is defined we have an associated homogeneous holomorphic fiber subbundle $\mathcal{W} \subset \mathbb{P}T(X)$, where the fibers $\mathcal{W}_x \subset \mathbb{P}T_x(X)$ are highest weight orbits. The first step of the proof consists of showing that $\mathcal{W}$ agrees with the VMRT structure $\mathcal{C}(X) \subset \mathbb{P}T(X)$, and the proof is based on Grothendieck's classification of G-principal bundles on $\mathbb{P}^1$ (Grothendieck [Gro57]). The identification $\mathcal{C}(X) = \mathcal{W}$ implies that every minimal rational curve is standard, and that $\mathcal{C}_x(X)$ agrees with the VMRT of an irreducible Hermitian symmetric space S of the compact type of rank $r \geq 2$, i.e., the G-structure is an S-structure. After that it remains to check the vanishing of structure functions interpreted as elements $\theta_k \in \Gamma\big(X, \mathrm{Hom}(\Lambda^2 T(X), T(X) \otimes S^k T^*(X)\big)$. When θ_k is restricted to elements of the form $\alpha \wedge \xi$, where $\alpha \in \widetilde{\mathcal{C}}_x(X)$ and $\xi \in P_\alpha$, then $\theta_k(\alpha, \xi) = 0$ follows by restricting to standard minimal rational curves ℓ and checking degrees of summands in Grothendieck decomposition basing on $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, $T_x(\ell) = \mathbb{C}\alpha$ (assuming ℓ to be embedded), noting that for $\xi \in P_\alpha$, $\alpha \wedge \xi \in \Lambda^2 T(X)|_\ell$ belongs to a direct summand of degree 3, while all direct summands of $(T(X) \otimes S^k T^*(X))|_\ell$ are of degree ≤ 2. The flatness of the G-structure results from the fact that $\{\alpha \wedge P_\alpha : \alpha \in \widetilde{\mathcal{C}}_x(X)\}$ spans $\Lambda^2 T_x(X)$, cf. Hwang-Mok [HM98, (5.1), Proposition 14].

(3.2) *Recognizing rational homogeneous spaces of Picard number 1 from the VMRT at a general point* For a rational homogeneous space $S = G/P$ of Picard number 1, denoting by $\mathcal{O}(1)$ the positive generator of $\mathrm{Pic}(S) \cong \mathbb{Z}$ we identify $S \subset \mathbb{P}(\Gamma(S, \mathcal{O}(1))^*)$ as a projective submanifold via the minimal embedding by $\mathcal{O}(1)$, and equip S with the minimal rational component on S consisting of projective lines on S. We are interested in characterizing a given uniruled projective manifold in terms of its VMRTs as projective submanifolds. Especially we have the following Recognition Problem for rational homogeneous spaces of Picard number 1 formulated in Mok [Mk08c].

Definition 3.2.1. *Let $S = G/P$ be a rational homogeneous space of Picard number 1, and $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ be the VMRT of S at $0 = eP \in S$. For any uniruled projective manifold X of Picard number 1 equipped with a minimal rational component $\mathcal{K}$, we denote by $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ the VMRT of $(X, \mathcal{K})$ at a general point $x \in X$. We say that the Recognition Problem for S is solved in the affirmative if any uniruled projective manifold X of Picard number 1 must necessarily be biholomorphic to S whenever $\big(\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)\big)$ is projectively equivalent to $\big(\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)\big)$.*

Cho, Miyaoka and Shepherd-Barron [CMS02] proved the characterization of the projective space $\mathbb{P}^n$ among uniruled projective manifolds by the fact that for a minimal rational curve ℓ we have $K_X^{-1} \cdot \ell = n + 1$, i.e., equipping X with some minimal rational component, the assumption $\mathcal{C}_x(X) = \mathbb{P}T_x(X)$ at a general point $x \in X$ implies that X is biholomorphic to $\mathbb{P}^n$. The purpose of the Recognition Problem was to deal with the characterization of $S = G/P$ different from a projective space, i.e., where $\mathcal{C}_0(S) \subsetneq \mathbb{P}T_0(S)$ at $0 = eP$. The following theorem gives our current state of knowledge on the Recognition Problem for rational homogeneous spaces of Picard number 1.

Theorem 3.2.1. (Mok [Mk08c], Hong-Hwang [HH08]) *Let G be a simple complex Lie group, $P \subset G$ be a maximal parabolic subgroup corresponding to a long simple root, and $S := G/P$ be the corresponding rational homogeneous space of Picard number 1. Then, the Recognition Problem for S is solved in the affirmative.*

We refer the reader to Mok [Mk08b, (6.3)] for an exposition revolving around the Recognition Problem and the proof of Theorem 3.2.1. Here in its place we will explain the principle underlying our approach and give some highlights on how the principle applies in the proof of the theorem. The first geometric link between VMRT structures and differential geometry was the author's proof of the Generalized Frankel Conjecture (Mok [Mk88]) in Kähler geometry which characterizes compact Kähler manifolds of semipositive holomorphic bisectional curvature. In particular, if X is a Fano manifold of Picard number 1 admitting a Kähler metric g of semipositive holomorphic bisectional curvature, then it must be biholomorphic to an irreducible Hermitian symmetric space S of the compact type. Regarding X as a uniruled projective manifold, it was proven that the VMRT structure $\mathcal{C}(X) \subset \mathbb{P}T(X)$ on X is invariant under holonomy of a metric g_t, $t > 0$, obtained from g by the Kähler-Ricci flow. (This was the core result of [Mk88] even though the term VMRT had not been introduced at that point.) In the absence of a Kähler metric with special properties, it was a challenge to introduce some notion of parallel transport that makes sense in the general setting of a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$ and hence with the associated VMRT structure $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T(X)$.

Our study of the Recognition Problem actually went back to the work Mok [Mk02] concerning the conjecture of Campana-Peternell [CP91] on compact complex manifolds with nef tangent bundle (cf. (3.3)), where we devised a method for reconstructing a Fano manifold X of Picard number 1 with nef tangent bundle from its VMRT $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ in the very special case where $\dim(\mathcal{C}_x(X)) = 1$. In the general case where the VMRT of $(X, \mathcal{K})$ at a general point $x \in X$ is congruent to that of the model, i.e., $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$, $S = G/P$, *a priori* there is a subvariety $E \subset X$ over which $\mathcal{C}_y(X)$ is not known to be congruent to the model. The key of the affirmative solution in the Hermitian symmetric case is a removable singularity theorem in codimension 1, viz., the assertion that for each irreducible component H of E which is a hypersurface of X, a general point y on H is a removable singularity for the VMRT structure. Note that in the Hermitian symmetric case the orbit of $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ is parametrized by an affine-algebraic variety $\mathcal{M} \subset \mathbb{C}^N$. In a neighborhood U of y in X, the VMRT structure can be described by a holomorphic map $\varphi : U - E \to \mathcal{M} \subset \mathbb{C}^N$, and saying that the S-structure has a removable singularity at y is the same as saying that the vector-valued holomorphic map has a removable singularity. Once we have proven such a result in codimension 1, it follows by Hartogs extension that the S-structure extends holomorphically to X, and we conclude that $X \cong S$ by Theorem 3.1.1, as desired.

To prove the removable singularity theorem for VMRT structures in the Hermitian symmetric case we introduce a method of parallel transport of VMRTs. Note that every curve on X must intersect a hypersurface since X is of Picard number 1. Starting with a standard rational curve $\ell \not\subset E$ which is assumed embedded and considering the lifting $\widehat{\ell} \subset \mathbb{P}T(X)|_\ell$, we transport the second fundamental form $\sigma_{[\alpha]}$ of $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ at $[T_x(\ell] := [\alpha]$ over $x \in \ell \cap (X - E)$ to the second fundamental form $\sigma_{[\beta]}$ of $\mathcal{C}_y(X) \subset \mathbb{P}T_y(X)$ at $[T_y(\ell)] := [\beta]$ over $y \in \ell \cap H$. Observing that the second fundamental form σ

along $\widehat{\ell}$ gives a holomorphic section of a holomorphic vector bundle V (of rank $:= r$) which is *holomorphically trivial* ([Mk08c, (6.1), Proposition 6]), which results from the fact that ℓ is a standard rational curve, we have a *parallel transport* in the sense that $V|_{\widehat{\ell}} \cong \mathbb{C}^r \times \widehat{\ell}$, and that an element $\xi \in V_{[\alpha]}$ is transported to $\gamma([\beta]) \in V_{[\alpha]}$ for the unique holomorphic section $\gamma \in \Gamma(\widehat{\ell}, V)$ such that $\gamma([\alpha]) = \xi$. This already yields a removable singularity theorem for VMRT structures in the case of the hyperquadric, since the smooth hyperquadric $Q^{n-2} \cong \mathcal{C}_0(Q^n) \subset \mathbb{P}T_0(Q^n) \cong \mathbb{P}^{n-1}$ cannot be deformed to a singular hyperquadric unless its second fundamental form at a general point also degenerates. In the general case of S-structures more work is required, e.g., in the case where the VMRT $\mathcal{C}_0(S)$ is itself an irreducible Hermitian symmetric space of rank ≥ 2, parallel transport of the second fundamental form along $\widehat{\ell}$ implies a parallel transport of the VMRT structure of $\mathcal{C}_x(X)$, in a neighborhood of $[\alpha]$ to a neighborhood of $[\beta]$ on $\mathcal{C}_y(X)$. Here $\mathcal{C}_x(X)$ is regarded itself as a uniruled projective manifold equipped with the minimal rational component $\mathcal{H}$ consisting of projective lines in $\mathbb{P}T_x(X)$ lying on $\mathcal{C}_x(X)$, and the parallel transport of the VMRT structure of $(\mathcal{C}_x(X), \mathcal{H})$ is shown to be enough to force a removable singularity theorem of S-structures in codimension 1.

In [Mk08c] the argument was carried through also in the contact case, where on top of parallel transport of the second fundamental form we also introduced parallel transport of the third fundamental form, and resort to the work of Hong [Ho00] on the characterization of contact homogeneous spaces which replaces Theorem 3.1.1. When the VMRT of X at a general point is congruent to $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ for a Fano homogeneous contact manifold S of Picard number 1 other than an odd-dimensional projective space, the linear span of VMRTs on X defines a meromorphic distribution D of co-rank 1 on X, and parallel transport of the third fundamental form by the same principle as explained in the last paragraph is made possible by the splitting type $D|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^r \oplus \mathcal{O}^r \oplus \mathcal{O}(-1)$. The argument was generalized in the other long-root cases by Hwang-Hong [HH08], in which analogues of the results of [Ho00] were obtained to solve the Recognition Problem in the affirmative for the remaining long-root cases.

Remarks

(a) In [Mk88], in the event that $\mathcal{C}_x(X) = \mathbb{P}T_x(X)$ and the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x$ is not a biholomorphism at a general point, we proved that there exists a hypersurface $\mathcal{H} \subset \mathbb{P}T(X)$ which is invariant under holonomy of (X, g_t), $t > 0$ sufficiently small.

(b) In Mok [Mk08b, (6.3), Conjecture 6], $S = G/P$ should have been "a Fano homogeneous space of Picard number 1" (instead of "a Fano homogeneous contact manifold of Picard number 1"). The Recognition Problem was expected to be always solved in the affirmative for $S = G/P$ of Picard number 1.

(3.3) *Rationale, applications and generalizations of the Recognition Problem* The original motivation of the Recognition Problem was an attempt to tackle the Campana-Peternell Conjecture in [CP91], which may be regarded as a generalization of the Hartshorne Conjecture, or as an algebro-geometric analogue of the Generalized Frankel Conjecture in Kähler geometry. It concerns projective manifolds X with nef tangent bundle. Especially, assuming that X is Fano and of Picard number 1, according to the Campana-Peternell Conjecture X is expected to be biregular to a rational homogeneous space, i.e., $X \cong G/P$ where G is a simple complex Lie group, and $P \subset G$ is a maximal parabolic subgroup. In Mok [Mk02] we took the perspective of reconstructing X from its VMRTs, and solved the problem in the very special case where X admits a minimal rational component $\mathcal{K}$ for which VMRTs are 1-dimensional, and where in addition $b_4(X) = 1$. The latter topological condition was removed by Hwang [Hw07], leading to

Theorem 3.3.1. (Mok [Mk02], Hwang [Hw07]) *Let X be a Fano manifold of Picard number 1 with nef tangent bundle. Suppose X is equipped with a minimal rational* *component for which the variety of minimal rational tangents at a general point $x \in X$ is 1-dimensional. Then, X is biholomorphic to the projective plane $\mathbb{P}^2$, the 3-dimensional hyperquadric Q^3, or the 5-dimensional Fano contact homogeneous space $K(G_2)$ of type G_2. In particular, X is a rational homogeneous space.*

Here we note that a weaker condition than the nefness of the tangent bundle was used in the proof of the theorem, viz., only the nefness of the restriction of the tangent bundle $T(X)$ to rational curves was used. The latter implies that deformation of rational curves is unobstructed, and hence the minimal rational component $\mathcal{K}$ is a projective manifold, and we have the universal family $\rho : \mathcal{U} \to \mathcal{K}$ accompanied by the evaluation map $\mu : \mathcal{U} \to X$ which is also a holomorphic submersion. When VMRTs are 1-dimensional, under the nefness assumption one shows easily that the fibers $\mathcal{U}_x$ of $\mu : \mathcal{U} \to X$ are smooth rational curves. It was proven in [Mk02] that for the $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ the direct image of the relative tangent bundle T_ρ gives a rank-3 bundle which is stable, yielding by an application of the Bogomolov inequality that $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is of degree $d \leq 4$ under the additional assumption $b_4(X) = 1$, and for the equality case we resorted to the existence result of Uhlenbeck-Yau [UY86] on Hermitian-Einstein metrics on stable holomorphic vector bundles over projective manifolds to arrive at a contradiction, leaving behind the options $d = 1, 2, 3$. These options do exist as is given in the statement of Theorem 3.3.1. The Recognition Theorem now enters, allowing us to recover the hyperquadric Q^3 from the VMRT in case $d = 2$, and to recover the 5-dimensional Fano contact homogeneous space $K(G_2)$ in case $d = 3$. They served as prototypes for the Recognition Problem for the Hermitian symmetric case and the Fano contact case as solved in the affirmative in Mok [Mk08c].

While Theorem 3.3.1 concerns a very special case of the Campana-Peternell Conjecture, it is worth noting that there is no assumption on the dimension of X itself. It is tempting to think that for VMRTs which are of dimension 2 one could identify the possible VMRTs for X of Picard number 1 and of nef tangent bundle, and recover X through the Recognition Problem. It appears for the time being conceptually difficult to devise a strategy for a solution of the Campana-Peternell Conjecture for Fano manifolds of Picard number 1 along the lines of thought in Theorem 3.3.1 since in the short-root case VMRTs are only almost homogeneous. Since a key element in the proof of Theorem 3.3.1 was a bound on the degree of the tangent map it would be meaningful to try to get an *a priori* bound of $\dim(X)$ in terms of dimensions of VMRTs under the nefness assumption on the tangent bundle.

Another reason for introducing the Recognition Problem was to give a conceptually unified proof of rigidity under Kähler deformation of rational homogeneous spaces $S = G/P$ of Picard number 1. The rigidity problem was taken up in Hwang-Mok [HM98], Hwang [Hw97], and Hwang-Mok [HM02] [HM04a] [HM05], according to an underlying classification scheme for $X = G/P$ in terms of complexity into the Hermitian symmetric case [HM98], the Fano contact case [Hw97], the remaining long-root cases [HM02], and the short-root cases [HM04a] [HM05]. As it turned out, the case of the 7-dimensional Fano homogeneous contact manifold $\mathbb{F}^5$, i.e., the Chow component of minimal rational curves on the hyperquadric Q^5, was missed in Hwang [Hw97] and again in Hwang-Mok [HM02], and it was later found by Pasquier-Perrin [PP10] that $\mathbb{F}^5$ admits a deformation to a G_2-horospherical variety X^5. The corrected statement about the rigidity problem under Kähler deformation of rational homogeneous spaces $S = G/P$ of Picard number 1 is given by

Theorem 3.3.2. *Let $S = G/P$ be a rational homogeneous space of Picard number 1 other than the 7-dimensional Fano homogeneous contact manifold $\mathbb{F}^5$. Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C}, |t| < 1\}$ be a regular family of projective manifolds such that the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$. Then, X_0 is also biholomorphic to S.*

There was a general scheme of proof adopted in the series of articles mentioned on the rigidity of $S = G/P$ under Kähler deformation. Note that X_0 is a uniruled projective manifold equipped with the minimal rational component $\mathcal{K}_0$ whose general point corresponds a free rational curve of degree 1 with respect to the positive generator $\mathcal{O}(1)$ of $\mathrm{Pic}(X_0) \cong \mathbb{Z}$. The general scheme consists first of all of a proof that at a general point x_0 of the central fiber X_0, the VMRT $\mathcal{C}_{x_0}(X_0) \subset \mathbb{P}T_{x_0}(X_0)$ is projectively equivalent to the VMRT $\mathcal{C}_{x_t}(X_t) \subset \mathbb{P}T_{x_t}(X_t)$ at any point $x_t \in X_t$ for $t \neq 0$, i.e., projectively equivalent to the VMRT $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ of the model manifold $S = G/P$ at the reference point $0 = eP$. We may call this the invariance of VMRTs (at a general point) under Kähler deformation. Shrinking Δ around 0 and rescaling the variable t if necessary, the comparison of VMRTs on different fibers was done by choosing $x_t = \sigma(t)$ for a holomorphic section $\sigma : \Delta \to \mathcal{X}$ where $\sigma(0)$ avoids the bad set B of $(X_0, \mathcal{K}_0)$. The VMRTs $\mathcal{C}_{\sigma(t)}(X_t)$, $t \in \Delta$, are images of the tangent map $\tau_{\sigma(t)} : \mathcal{U}_{\sigma(t)}(X_t) \dashrightarrow \mathcal{C}_{\sigma(t)}(X_t)$, where the set $\{\mathcal{U}_{\sigma(t)}(X_t) : t \in \Delta\}$ constitutes a regular family of projective manifolds. With some oversimplification, in most long-root cases the latter fact allows us to deduce $\mathcal{U}_{\sigma(0)} \cong \mathcal{U}_0(S)$ by an inductive argument. For the inductive argument we recall that the tangent map $\tau_0 : \mathcal{U}_0(S) \xrightarrow{\cong} \mathcal{C}_0(S)$ is a biholomorphism and note that in the long-root case $\mathcal{U}_0(S) \cong \mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ is a Hermitian symmetric space of the compact type of rank ≤ 3. An inductive argument applies in the long-root case when $\mathcal{C}_0(S)$ is irreducible but there are cases where $\mathcal{C}_0(S)$ is reducible, as for example the Grassmannian $G(p, q)$ of rank ≥ 2 where the VMRT is given by the Segre embedding $\varsigma : \mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \to \mathbb{P}^{pq-1}$. In the latter case invariance of VMRTs under Kähler deformation was established in [HM98] by cohomological considerations in the deformation of projective subspaces of X_t associated to factors of $\mathcal{C}_0(G(p, q)) \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$. After the invariance of VMRTs under deformation has been established, properly speaking the geometric theory of VMRTs enters in the study of the tangent map. In [HM98] it was proven in a general setting for a uniruled projective manifold X of Picard number 1 that the distribution $\mathcal{W}$ spanned by VMRTs is not integrable unless $\mathcal{W} = T(X)$, while sufficient projective-geometric conditions were given for the integrability of $\mathcal{W}$ forcing the tangent map τ_{x_0} to be an isomorphism at a general point $x_0 \in X_0$ in [HM98]. In the other long root cases the VMRTs of the model manifold are linearly degenerate, and we made use of the more general result that $\mathcal{W}$ must be bracket generating for a uniruled projective manifold X of Picard number 1 to reach the same conclusion in [HM02]. Here we say that a distribution $\mathcal{W}$ on X is bracket generating to mean that the tangent subsheaf generated by $\mathcal{W}$ from taking successive Lie brackets is the tangent sheaf. The same reasoning was applied to the short-root cases in [HM04a] and [HM05]. In the cases of [HM05], the VMRTs of the model spaces are linearly nondegenerate.

For the long-root case to conclude $X_0 \cong S$ one makes use of results from differential systems, viz., Ochiai's theorem [Oc70] in the Hermitian symmetric case and Yamaguchi's result on differential systems [Ya93] for long-root cases other than the symmetric and contact cases. (In the contact case [Hw97] uses a simpler argument.) There is one short-root case of type F_4 in which the VMRT is linearly degenerate which was treated in [HM04a] along the line of [HM02], while the most difficult case of the rigidity problem was the short-root cases of the symplectic Grassmannian $S_{k,\ell}$ (cf. below) and the remaining F_4 short-root case, where the VMRTs are linearly nondegenerate. In the latter cases it was after establishing invariance of VMRTs that the real difficulty emerges, viz., the key issue was how one can recover S from its VMRTs. It was tempting to give a unified argument on rigidity under Kähler deformation as a consequence of (a) invariance of VMRTs under deformation and (b) an affirmative solution of the Recognition Problem. Such a unified scheme of proof would apply to the central fiber X_0 as a separate uniruled projective manifold equipped with the minimal rational component $\mathcal{K}$ without using the fact that it is the central fiber of a family such that the other fibers are biholomorphic to the model manifold $S = G/P$.

While the affirmative solutions of the Recognition Problem in the long-root cases give a unified explanation of the phenomenon of deformation rigidity (with one exception) in those cases, the same problem for the short-root cases remains unresolved. An important feature for the primary examples of symplectic Grassmannians $X = S_{k,\ell}$ is the existence of local differential-geometric invariants which cannot possibly be captured by the VMRT at a general point. To explain this we describe the symplectic Grassmannian. Consider a complex vector space W of dimension 2ℓ equipped with a symplectic form ω. Let k be an integer, $1 < k < \ell$, and consider the subset $S_{k,\ell} \subset \mathrm{Gr}(k, W)$ of k-planes in W isotropic with respect to ω. Let $x = [E] \in S_{k,\ell}$ be an arbitrary point. Suppose $A^{(k-1)}$ resp. $B^{(k+1)}$ are vector subspaces of W of dimension $k-1$ resp. $k+1$ such that $A^{(k-1)} \subset E \subset B^{(k+1)}$. Suppose furthermore that $B^{(k+1)}$ is isotropic with respect to ω. Let $\Gamma \subset \mathrm{Gr}(k, W)$ be the rational curve consisting of all k-planes F such that $A^{(k-1)} \subset F \subset B^{(k+1)}$. Then, $\omega|_F \equiv 0$ for every $[F] \in \Gamma$, hence $\Gamma \subset S_{k,\ell}$. For a minimal rational curve Γ as described we have $T_x(\Gamma) = \mathbb{C}\lambda$, where $\lambda \in \mathrm{Hom}(E, W/E) = T_x(S_{k,\ell})$ such that $\lambda|_{A^{(k-1)}} \equiv 0$ and $\mathrm{Im}(\lambda) = B^{(k+1)}/E$. The set of all $[T_x(\Gamma)] \in \mathbb{P}T_x(S_{k,\ell})$ thus described is given by $\mathcal{S}_x := \big\{[a \otimes b] : 0 \neq a \in E^*, 0 \neq b := \beta + E \in W/E, \omega|_{E+\mathbb{C}\beta}\big\} \equiv 0$. Thus, writing $Q_x = E^{\perp}/E$ we have $\mathcal{S}_x = \zeta(\mathbb{P}(E^*) \times \mathbb{P}(Q_x))$, where $\zeta : \mathbb{P}(E^*) \times \mathbb{P}(Q_x) \to \mathbb{P}(E^* \otimes Q_x)$ is the Segre embedding. The assignment $E \to E^* \otimes Q_x$ defines a holomorphic distribution D on $S_{k,\ell}$ which is invariant under the symplectic group $\mathrm{Sp}(W, \omega) = \mathrm{Aut}(S_{k,\ell})$.

The minimal rational curves $\Gamma \subset S_{k,\ell}$ described in the above are special. In the definition of a minimal rational curve containing the k-plane $[E]$, in place of requiring ω to be isotropic on $B^{(k+1)}$ it suffices to have $B^{(k+1)} = E + \mathbb{C}\beta$, where $0 \neq \beta \in A^{\perp} - E$, $A = A^{(k-1)}$. In fact, writing $E = A^{(k-1)} + \mathbb{C}e$ and assuming $\beta \in W - E$, the condition that $E := A^{(k-1)} + \mathbb{C}\gamma$ is isotropic with respect to ω for any $\gamma \in \mathbb{C}e + \mathbb{C}\beta$ is equivalent to the requirement that $\beta \in A^{\perp}$. Thus, the VMRT $\mathcal{C}_x(S_{k,\ell})$ at x is given by $\mathcal{C}_x(S_{k,\ell}) = \big\{[a \otimes b] : 0 \neq a \in E^*, 0 \neq b := \beta + E \in W/E, \omega(\beta, \alpha) = 0$ for any α such that $a(\alpha) = 0\big\}$, and the locus of tangents of the 'special' minimal rational curves at x is given by $\mathcal{S}_x(S_{k,\ell}) = \mathcal{C}_x(S_{k,\ell}) \cap \mathbb{P}D_x$. The distribution $D \subsetneq T(S_{k,\ell})$ is not integrable. Observing that (W, ω) induces a symplectic form ϖ_x on $Q_x = E^{\perp}/E$, $\dim(Q_x) = 2(\ell - k)$, the Frobenius form $\varphi_x : \Lambda^2 D_x \to T_x(S_{k,\ell})/D$ is determined in a precise way by ϖ_x (cf. Hwang-Mok [HM05, Proposition 5.3.1]), so that for $0 \neq a, a' \in E^*$; $b, b' \in Q_x$, $\varphi_x(a \otimes b, a' \otimes b') \neq 0$ if and only if $\varpi_x(b, b') \neq 0$. The symplectic form ϖ on Q_x cannot be recovered from the VMRTs alone, in fact there exists a uniruled projective manifold Z of Picard number $\neq 1$ with isotrivial VMRTs $\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$ projectively equivalent to $\mathcal{C}_x(S_{k,\ell}) \subset \mathbb{P}T_x(S_{k,\ell})$ such that the analogous distribution D, is *integrable* (cf. Hwang [Hw12]). Here $D_z \subset \mathbb{P}T_z(Z)$ is retrieved from $\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)$ as a projective subvariety, e.g., as the locus where the second fundamental form is degenerate.

It remains to recognize $X \cong S_{k,\ell}$ for a uniruled projective manifold X of Picard number 1 such that the VMRT $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ is projectively equivalent to $\mathcal{C}_x(S_{k,\ell}) \subset \mathbb{P}T_x(S_{k,\ell})$. While one of the original motivations of the Recognition Problem is to prove rigidity under Kähler deformation in a conceptually uniform manner this has so far not been possible for the short-root cases. Reasoning in the opposite direction, the method of proof of rigidity under Kähler deformation in Hwang-Mok [HM05] may give a hint to the solution of the Recognition Problem for $S_{k,\ell}$, which is conceptually an important problem in its own right. In Hwang-Mok [HM05] we obtained a foliation on the central fiber in the deformation problem by means of the Frobenius form $\varphi : \Lambda^2 D \to T(X)/D$ associated to the specific VMRT structure. Recall that in [HM05] we considered a regular family $\pi : \mathcal{X} \to \Delta$ of Kähler manifolds with $X_t := \pi^{-1}(t) \cong S_{k,\ell}$ for $t \neq 0$, and, using estimates of vanishing orders and dimensions of linear spaces of vector fields we studied also the possibility that the Frobenius form is *a priori* degenerate on $D \subset X$. In this case we showed that X_0 is the image of some Grassmann bundle over a symplectic Grassmannian under a birational morphism, and the existence of such a model was finally shown to contradict the smoothness of X_0 unless the rank of ϖ_x is maximal at a general point, in which case we showed $X_0 \cong S_{k,\ell}$, In the case of the Recognition Problem for the symplectic Grassmannian one has to first get local models for isotrivial VMRTs modeled on $\mathcal{C}_x(S_{k,\ell}) \subset \mathbb{P}T_x(S_{k,\ell})$ depending on the rank of some skew-symmetric bilinear form ϖ_x on a $2(\ell - k)$-dimensional vector space Q_x at a general point $x \in X$, where Q_x can be retrieved from the VMRT and ϖ_x is determined by the Frobenius form φ_x. The construction and parametrization of local models are by themselves a challenging problem requiring new ideas on the local study of differential systems arising from these specific VMRTs.

Regarding the exceptional case of $S = \mathbb{F}^5$ for deformation rigidity of rational homogeneous spaces of Picard number 1 as stated in Theorem 3.3.1, Hwang [Hw14a] has established the following result giving two alternatives for the central fiber.

Theorem 3.3.3. (Hwang [Hw14a]) *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of projective manifolds over the unit disk Δ, and denote by $X_t := \pi^{-1}(t)$ the fiber over $t \in \Delta$. Suppose X_t is biholomorphic to $\mathbb{F}^5$ for each $t \in \Delta - \{0\}$, then the central fiber is biholomorphic to either $\mathbb{F}^5$ or to the G_2-horospherical variety X^5 in Pasquier-Perrin* [PP10].

Paradoxically, the failure of rigidity under Kähler deformation in the exceptional case of $S \cong \mathbb{F}^5$, coupled with Hwang's result above, lends credence to an important general principle in the geometric theory of VMRTs, viz., that in the case of a uniruled projective manifold X of Picard number 1, the underlying complex structure should be recognized by its VMRT at a general point. For the question on Kähler deformation considered, this means that rigidity fails if and only if invariance of VMRTs under deformation breaks down. In [Hw14], Theorem 3.3.3 was proven precisely by identifying two alternatives for a general VMRT on the central fiber and by recovering the complex structure of X_0 via an affirmative solution to the associated Recognition Problem.

For $S = \mathbb{F}_5 = G/P$ with a reference point $0 = eP \in S$, denoting by $D \subsetneq T(S)$ the unique G-invariant proper holomorphic distribution on S, to be called the minimal distribution, the VMRT $\mathcal{C}_0(S) \subset \mathbb{P}D_0(S)$ is projectively equivalent to the image of $\eta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$, where, writing π_i; $i = 1, 2$; for the canonical projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto its i-th factor, η is the embedding given by $\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(2)$. Moreover, it can be proven that for the regular family $\pi : \mathcal{X} \to \Delta$, letting $\mathcal{D}^\flat$ be the distribution on $\mathcal{X}|_{\Delta^*}$ such that $\mathcal{D}^\flat|_{X_t} \subset T(X_t)$ agrees with the minimal distribution on X_t for $t \neq 0$, $\mathcal{D}^\flat$ extends across the central fiber to give a meromorphic distribution $\mathcal{D}$ on $\mathcal{X}$. In the proof of Theorem 3.3.3, taking a holomorphic section $\sigma : \Delta(\epsilon) \to \mathcal{X}$ for some $\epsilon > 0$ such that $\sigma(0)$ is a general point of X_0, and considering the regular family $\tau : \mathcal{E} \to \Delta(\epsilon)$ of projective submanifolds of $E_t := \sigma^*\mathcal{C}_{\sigma(t)}(X_t) \subset \sigma^*\mathbb{P}D_{\sigma(t)} \cong \mathbb{P}^5$, $\big(E_0 \subset \mathbb{P}^5\big)$ is proven to be projectively equivalent to either $\big(\eta(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^5\big)$ or to $\big(\Sigma \subset \mathbb{P}^5\big)$, where $\Sigma := \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-3))$ is a Hirzebruch surface embedded into $\mathbb{P}^5$ by some ample line bundle. In the latter case the VMRT as a projective submanifold at a general point of X_0 is projectively equivalent to the VMRT of the 7-dimensional G_2-horospherical variety X^5 in the notation of Pasquier-Perrin [PP10]. Hwang then solved the Recognition Problem for X^5 by resorting to the obstruction theory for the construction of an appropriate connection on the central fiber X_0. This involves G-structures for a certain non-reductive linear group and the proof in Hwang [Hw14a] is a *tour de force*.

§4 Germs of complex submanifolds of uniruled projective manifolds

(4.1) *An overview* Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, and $\pi : \mathcal{C}(X) \to X$, $\mathcal{C}(X) \subset \mathbb{P}T_X$, be the associated VMRT structure. We are interested in studying germs of complex submanifolds of X in relation to the VMRT structure. Since very little on the topic has been discussed in earlier surveys, we will be more systematic here with the exposition. In Mok [Mk08a] we examined the question of characterizing Grassmannians $G(p', q') \subset G(p, q)$ of rank $r = \min(p', q') \geq 2$ realized as complex submanifolds by means of standard embeddings. The fundamental analytic tool was the non-equidimensional Cartan-Fubini extension developed in full generality in Hong-Mok [HoM10] as stated in Theorem 2.2.1, which was applied in [HoM10] to yield characterization theorems on standard embeddings for pairs of rational homogeneous spaces (X_0, X) of Picard number 1, $X_0 \subset X$, where X is defined by a Dynkin diagram marked at a long simple root, and X_0 is nonlinear and obtained from a marked sub-diagram. This result, and generalizations by Hong-Park [HoP11] to the short-root case and to the case of maximal linear subspaces, will be the focus in (4.2), where the geometric idea of parallel transport of VMRTs along minimal rational curves will be explained. In (4.3) we explain an application in Hong-Mok [HoM13] of the methods of (4.2) to homological rigidity of smooth Schubert cycles with a few identifiable exceptions, where the rigidity statement was reduced to the question whether local deformations of a smooth Schubert cycle $Z \subset X$ must be translates $\gamma(Z)$ of Z, $\gamma \in \mathrm{Aut}(X)$. The latter question was answered in the affirmative in Hong-Mok [HoM13] in most cases by using the rigidity of VMRTs as projective submanifolds under local deformation of Z coupled with the argument of parallel transport of VMRTs. We explain the complex-analytic argument of [HoM13] which deduces parallel transport of VMRTs along minimal rational curves in the case of homogeneous Schubert cycles from the compactness of the moduli of the homogeneous submanifold $\mathcal{C}_0(Z) \subset \mathcal{C}_0(X)$. In (4.4) we introduce the notion of sub-VMRT structures given by $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, for complex submanifolds S and discuss the rigidity result of Mok-Zhang [MZ15] for sub-VMRT structures which strengthens those in (4.2) from [HoM10] and [HoP11]. In (4.5) we formulate a rigidity result for sub-VMRT structures on general uniruled projective manifolds $(X, \mathcal{K})$, and formulate the Recognition Problem for the characterization of classes of uniruled projective subvarieties on $(X, \mathcal{K})$ in terms of sub-VMRT structures. We consider in (4.6) examples of sub-VMRT structures related to Hermitian symmetric spaces, and discusses analytic continuation of sub-VMRT structures for special classes of uniruled projective subvarieties.

It is worth noting that [Mk08a] originated from a method of proof of Tsai's Theorem [Ts93] concerning proper holomorphic maps between bounded symmetric domains of rank ≥ 2 in the equal rank case, and as such the study of sub-VMRT structures is at the same time a topic in local differential geometry in a purely transcendental setting. In (4.7) we explore various links of VMRT sub-structures to algebraic geometry, several complex variables, Kähler geometry and the geometry of submanifolds in Riemannian geometry, and describe some sources of examples, including those from holomorphic isometries of Mok [Mk15] and from the classification of sub-VMRT structures in the Hermitian symmetric case of Zhang [Zh14].

(4.2) *Germs of VMRT-respecting holomorphic embeddings modeled on certain pairs of rational homogeneous spaces of sub-diagram type and a rigidity phenomenon* Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be uniruled projective manifolds equipped with minimal rational components with positive-dimensional VMRTs. When Z is of Picard number 1, given a VMRT-respecting holomorphic embedding f of a connected open subset $U \subset Z$ into X satisfying some genericity condition and a nondegeneracy condition for the pair $\big(f_\sharp(\mathcal{C}_z(Z)) \subset \mathcal{C}_{f(z)}(X)\big)$ expressed in terms of second fundamental forms, non-equidimensional Cartan-Fubini extension (Theorem 2.2.1) gives an extension of $f(U)$ to a projective subvariety $Y \subset X$ such that $\dim(Y) = \dim(U)$. In the case of irreducible Hermitian symmetric spaces S of rank ≥ 2, for which Ochiai's Theorem on S-structures serves as a prototype for equidimensional Cartan-Fubini extension, and more generally in the case of rational homogeneous spaces $X = G/P$ of Picard number 1, one expects to be able to say more about the projective subvariety $Y \subset X$. This was first undertaken in Mok [Mk08a] in the special case of Grassmannians of rank ≥ 2. For a pair (X_0, X) of rational homogeneous spaces of Picard number 1 obtained from marked Dynkin diagrams, the works of Hong-Mok [HoM10] and Hong-Park [HoP11] yielded the following characterization theorem. Here we have a holomorphic equivariant embedding $\Phi : X_0 = G_0/P_0 \hookrightarrow G/P = X$ and a mapping $F : X_0 \to X$ is said to be a standard embedding if and only if $F = \gamma \circ \Phi$ for some $\gamma \in \mathrm{Aut}(X)$. We have the following result due to [HoM10] in the long-root case and [HoP11] in the short-root case.

Theorem 4.2.1. (Hong-Mok [HoM10, Theorem 1.2], Hong-Park [HoP11, Theorem 1.2]) *Let $X_0 = G_0/P_0$ and $X = G/P$ be rational homogeneous spaces associated to simple roots determined by marked Dynkin diagrams $(\mathcal{D}(G_0), \gamma_0), (\mathcal{D}(G), \gamma)$ respectively. Suppose $\mathcal{D}(G_0)$ is obtained from a sub-diagram of $\mathcal{D}(G)$ with γ_0 being identified with γ. If X_0 is nonlinear and $f : U \to X$ is a holomorphic embedding from a connected open subset $U \subset X_0$ into X which respects VMRTs at a general point $x \in U$, then f is the restriction to U of a standard embedding of X_0 into X.*

For finite-dimensional complex vector spaces $E \cong E'$ and subvarieties $A \subset \mathbb{P}E$, $A' \subset \mathbb{P}E'$, we say that $\big(A \subset \mathbb{P}E\big)$ is projectively equivalent to $\big(A' \subset \mathbb{P}E'\big)$ if and only if there exists a projective linear isomorphism $\Psi : \mathbb{P}E \xrightarrow{\cong} \mathbb{P}E'$ such that $\Psi(A) = A'$. To compare to Theorem 2.2.1, in what follows we will write Z for X_0. Write $S := f(U)$, which is a complex submanifold of some open subset of X. Obviously for $z \in U$, $\big(\mathcal{C}_z(Z) \subset \mathbb{P}T_z(Z)\big)$ is projectively equivalent to $\big(f_\sharp(\mathcal{C}_z(Z)) \subset f_\sharp(\mathbb{P}T_z(Z))\big)$, i.e., $\big(\mathcal{C}_{f(z)}(S) \subset \mathbb{P}T_{f(z)}(S)\big)$. Here and henceforth we write $\mathcal{C}_{f(z)}(S)$ for $\mathcal{C}_{f(z)}(X) \cap \mathbb{P}T_{f(z)}(S)$, which is the same as $f_\sharp(\mathcal{C}_z(Z))$ by the hypothesis that f respects VMRTs. (We will also write $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$.) Write $\Lambda_z = [df(z)] : \mathbb{P}T_z(Z) \xrightarrow{\cong} \mathbb{P}T_{f(z)}(S)$ for the projective linear isomorphism inducing the projective equivalence. In the long-root case, by the proof of Hong-Mok [HoM10, Proposition 3.4], denoting by $D(Z) \subset T(Z)$ the holomorphic distribution spanned by VMRTs, $\Lambda_z|_{\mathbb{P}D_z(Z)}$ can be extended to $\Phi_z = [d\gamma(z)] : \mathbb{P}T_z(X) \xrightarrow{\cong} \mathbb{P}T_{f(z)}(X)$ for some $\gamma \in \mathrm{Aut}(X)$ such that $\gamma(z) = f(z)$. Thus, given the germ of VMRT-respecting holomorphic map $f : U \to X$, $Z' := \gamma(Z)$ gives a rational homogeneous submanifold $Z' \subset X$ such that $\mathcal{C}_{f(0)}(Z') = \mathcal{C}_{f(0)}(S)$ and the proof of Theorem 4.2.1 consists of fitting S into the model Z'. (We remark that in the statement of [HoM10, Proposition 3.4], in place of $V = T_x(X)$ and $W := T_x(Z)$ one could have replaced V by the linear span $D_x(X)$ of $\widetilde{\mathcal{C}}_x(X)$ and W by the linear span $D_x(Z)$ of $\widetilde{\mathcal{C}}_x(Z)$. The arguments for the proof of [HoM10, Proposition 3.4] actually yield the following stronger statement, which we will use in what follows. If $\mathcal{B}' = \mathcal{C} \cap \mathbb{P}W'$ is another linear section such that $\big(\mathcal{B}' \subset \mathbb{P}W'\big)$ is projectively equivalent to $\big(\mathcal{B} \subset \mathbb{P}W\big)$, then there is $h \in P$ such that $\mathcal{B}' = h\mathcal{B}$. In other words, in place of $h \in \mathrm{Aut}(\mathcal{C}_x(X))$ the proof actually gives $h \in P$. The latter fact was used in [HoM10] in the process of fitting S into a model $Z' = \gamma(Z)$.)

For the proof of Theorem 4.2.1 the strategy was to compare the germ of manifold S at some base point with the model complex submanifold $Z' = \gamma(Z) \subset X$. In the ensuing discussion, replacing f by $\gamma^{-1} \circ f$ we will assume without loss of generality that $Z' = Z$, so that $\mathcal{C}_0(S) = \mathcal{C}_0(Z)$. Starting with the base point $0 = f(0) \in S$ and considering the union $\mathcal{V}_1$ of minimal rational curves on Z passing through 0, preservation of the tautological foliation, i.e., $f_{\sharp *}(\mathcal{F}_Z) = \mathcal{F}_X|_{\mathcal{C}(S)}$, as in the proof of non-equidimensional Cartan-Fubini extension (Theorem 2.2.1) implies that the germ $(S; 0)$ contains $(\mathcal{V}_1; 0)$. By the repeated adjunction of minimal rational curves, we have $0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_m = Z$. In order to prove that S is an open subset of Z it suffices to prove inductively that the germ $(S;0)$ contains $(\mathcal{V}_k;0)$ for each $k \geq 1$, $\mathcal{V}_0 = \{0\}$. For $k \geq 1$ the statement $(S;0) \supset (\mathcal{V}_k;0)$ is the same as saying that $\mathcal{C}_{x_{k-1}}(S) = \mathcal{C}_{x_{k-1}}(Z)$ for any $x_{k-1} \in \mathcal{V}_{k-1} \cap S$ sufficiently close to 0. The inductive argument was done in [Mk08a] and [HoM10] by means of parallel transport of VMRTs along minimal rational curves, as follows. Assume $(S;0) \supset (\mathcal{V}_k;0)$. If for a general point $x \in S \cap \mathcal{V}_k$, $x = f(z)$, we have $f_\sharp(\mathcal{C}_z(Z)) = \mathcal{C}_{f(z)}(Z)$, then $(S;x)$ contains the germ $(\mathcal{V}_1(x);0)$, where $\mathcal{V}_1(x)$ is the union of all rational curves on Z passing through $x = f(z) \in S \cap \mathcal{V}_k$. At a point $x = x_k \in S \cap (\mathcal{V}_k - \mathcal{V}_{k-1})$ we have a line ℓ joining some point $x' = x_{k-1} \in S \cap \mathcal{V}_{k-1}$ to x, and we know that $S = f(U)$ and Z share a nonempty connected open subset of the rational curve ℓ, which contains both x' and x. From $\mathcal{C}_{x'}(S) = \mathcal{C}_{x'}(Z)$ it follows that $\mathcal{C}_x(S)$ and $\mathcal{C}_x(Z)$ are tangent to each other at x. The argument of parallel transport of VMRTs along minimal rational curves consists of the following statement established in the long-root case by Hong-Mok [HoM10, Proposition 3.6] using the theory of Lie groups and representation theory.

Proposition 4.2.1. *Let (Z,X) be a pair of rational homogeneous spaces of Picard number 1 marked at a simple root, $X = G/P$, $0 = eP$. Suppose $\gamma \in P$ and $\mathcal{C}_0(\gamma(Z))$ and $\mathcal{C}_0(Z)$ are tangent to each other at a common smooth point $[\alpha] \in \mathcal{C}_0(\gamma(Z)) \cap \mathcal{C}_0(Z)$, then $\mathcal{C}_0(\gamma(Z)) = \mathcal{C}_0(Z)$.*

For a proof of Proposition 4.2.1 which works in both the long-root and the short root cases we defer to (4.3). Returning to Theorem 4.2.1 for the short-root case, by Hong-Park [HoP11, Proposition 2.2] the existence of $\varphi \in P$ such that $Z' := \varphi(Z)$ is tangent to $S := f(U)$ at $f(0) = 0$ holds true in the short-root case excepting in the case of certain pairs (Z,X) where X is a symplectic Grassmannian and Z is a Grassmannian. For the short-root case other than the exceptional pairs (Z,X) Hong-Park proceeded along the line of Hong-Mok [HoM10], but for the verification of the nondegeneracy condition they resorted to the explicit description of VMRTs as projective submanifolds as given in Hwang-Mok [HM04] and [HM05].

To describe the exceptional pairs (Z,X) recall that for $X = S_{k,\ell} \subset \mathrm{Gr}(k,W)$, there is an invariant distribution $D \subset T(X)$ given by $D = U \otimes Q$, where U and Q are homogeneous vector bundles over X with fibers $U_x = E^*$ and $Q_x = E^\perp/E$ at $x = [E] \in X$. Recall that $\mathcal{C}_x(S_{k,\ell}) \cap \mathbb{P}D_x(S_{k,\ell}) = \mathcal{C}_x(S_{k,\ell}) \cap \mathbb{P}(U_x \otimes Q_x) = \varsigma(\mathbb{P}U_x \times \mathbb{P}Q_x)$, where $\varsigma : \mathbb{P}U_x \times \mathbb{P}Q_x \to \mathbb{P}(U_x \otimes Q_x)$ denotes the Segre embedding given by $\varsigma([u],[q]) = [u \otimes q]$. Fix now isotropic subspaces $0 \neq F_1 \subset F_2$ and consider $Z \subset X$ consisting of all (isotropic) k-planes E such that $F_1 \subset E \subset F_2$. Then, $Z \subset X = S_{k,\ell} \subset \mathrm{Gr}(k,W)$ is a Grassmannian, and $T(Z) = U' \otimes Q'$, where $U'_x = (E/F_1)^* \subset U_x$, and $Q'_x = F_2/E \subset E^\perp/E = Q_x$. We have $\mathrm{rank}(U') = k - \dim F_1 := a$, $\mathrm{rank}(Q') = \dim F_2 - k := b$. Then, writing $x = [E] \in X$, for any a-plane $A_x \subset U_x$ and any b-plane $B_x \subset Q_x$, $\big(\varsigma(\mathbb{P}A_x \times \mathbb{P}B_x) \subset \mathbb{P}(A_x \otimes B_x)\big)$ is projectively equivalent to $\big(\varsigma(\mathbb{P}U'_x \times \mathbb{P}Q'_x) \subset \mathbb{P}(U_x \otimes Q_x)\big)$. Recall that (W,ω) induces a symplectic form ϖ_x on $Q_x = E^\perp/E$ (cf. (3.3)). By definition ϖ_x vanishes on $Q'_x = F_2/E$. If we choose now $B_x \subset Q_x$ to be such that $\varpi_x|_{B_x} \not\equiv 0$, then any projective linear isomorphism $\Lambda_x : \mathbb{P}(A_x \otimes B_x) \xrightarrow{\cong} \mathbb{P}(U'_x \otimes Q'_x)$ such that $\Lambda(\varsigma(\mathbb{P}A_x \times \mathbb{P}B_x)) = \varsigma(\mathbb{P}U'_x \times \mathbb{P}Q'_x)$ cannot possibly extend to an element of the parabolic subgroup P_x of G at x since ϖ_x is invariant under P_x.

For the solution of the special case (Z,X) above for the symplectic Grassmannian $X = S_{k,\ell}$ it is sufficient to embed X into the Grassmannian $\mathrm{Gr}(k,2\ell)$ and solve the problem for $(Z, \mathrm{Gr}(k,2\ell))$. The end result is that $S = F(U)$ is still an open subset of a sub-Grassmannian Z' in $\mathrm{Gr}(k,2\ell)$ which lies on $X = S_{k,\ell}$. We observe that the proof yields the following. If we start with $f : (Z;0) \to (S_{k,\ell};0)$ such that f respects VMRTs, and suppose $f_\sharp(\mathcal{C}_0(Z)) = \zeta(\mathbb{P}A_x \times \mathbb{P}B_x)$, then we have necessarily $\varpi_x|_{B_x} \equiv 0$.

Hong-Park [HoP11] considered in addition the cases involving maximal linear subspaces $Z = X_0$, as follows.

Theorem 4.2.2. (Hong-Park [HoP11, Theorem 1.3]) *Let $X = G/P$ be a rational homogeneous space associated to a simple root and let $Z \subset X$ be a maximal linear subspace. Let $f : U \to X$ be a holomorphic embedding from a connected open subset $U \subset Z$ into X such that $\mathbb{P}(df(T_z(U))) \subset \mathcal{C}_{f(z)}(X)$ for any $z \in U$. If there is a maximal linear space $Z_{\max}$ of X of dimension* $\dim(U)$ *which is tangent to $f(U)$ at some point $x_0 = f(z_0), z_0 \in U$, then $f(U)$ is contained in $Z_{\max}$, excepting when $(Z_{\max}; X)$ is given by* (a) *X is associated to $(B_\ell, \alpha_i), 1 \le i \le \ell - 1$, and $Z_{\max}$ is $\mathbb{P}^{\ell-i}$;* (b) *X is associated to (C_ℓ, α_ℓ) and $Z_{\max}$ is $\mathbb{P}^1$; or* (c) *X is associated to (F_4, α_1) and $Z_{\max}$ is $\mathbb{P}^2$.*

The maximal linear subspaces $\Pi \subset \mathcal{C}_0(X)$ divide into a finite number of isomorphism types under the action of the parabolic subgroup P on $\mathcal{C}_0(X)$ (cf. Landsberg-Manivel [LM03]). The assumption that there is a maximal linear space $Z_{\max}$ of X of dimension $\dim(U)$ which is tangent to $S := f(U)$ at some point $x_0 := f(z_0) \in S$ implies that $\mathbb{P}T_{x_0}(S) = \mathbb{P}T_{x_0}(Z_{\max})$, which forces $\mathbb{P}T_x(S) \subset \mathcal{C}_x(X)$ to be a maximal linear subspace of the same type for any $x = f(z) \in S$. Here the question is whether the tautological foliation $\mathcal{F}_X$ on $\mathbb{P}T(X)$, regarded as a holomorphic line subbundle, is tangent to $\mathbb{P}T(S)$ and hence restricts to $\mathcal{C}(X) \cap \mathbb{P}T(S) = \mathcal{C}(S)$. An affirmative answer to the latter question implies that S is an open set on a projective linear subspace obtained by adjoining projective lines at a single base point. Obviously $f_{\sharp *}(\mathcal{F}_Z)$ need not agree with $\mathcal{F}_X|_S$. In fact, trivially any immersion between projective spaces respects VMRTs.

Suppose $\alpha \in df(T_{z_0}(U))$. To show that $\mathcal{F}_X|_S$ defines a foliation on S, in place of requiring the nondegeneracy condition $\operatorname{Ker} \sigma_\alpha(\,\cdot\,, df(T_{z_0}(Z))) = \mathbb{C}\alpha$ it suffices to have $\operatorname{Ker} \sigma_\alpha(\,\cdot\,, df(T_{z_0}(Z))) \subset df(T_{z_0}(Z))$, which was checked to be the case for $Z = X_0$ being a maximal linear subspace, in which case $\operatorname{Ker} \sigma_\alpha(\,\cdot\,, df(T_{z_0}(Z))) = df(T_{z_0}(Z))$ with the exceptions as stated in the theorem. For the exceptional cases (a)-(c), counter-examples had been constructed by Choe-Hong [CH04].

We note that Theorem 4.2.2, while formulated in terms of a holomorphic embedding $f : U \to X$, concerns only the germ of complex submanifold $(S; x_0)$, where $S = f(U)$. Here f plays only an auxiliary role. In fact, any germ of biholomorphism $h : (U; z_0) \to (S; x_0)$ satisfies the condition $\mathbb{P}(dh(T_z(U))) \subset \mathcal{C}_{h(z)}(X)$ whenever h is defined at $z \in U$. As such Theorem 4.2.2 may be regarded as a result on geometric substructures.

In the long-root and nonlinear case of admissible pairs (Z, X), in which case $\mathcal{C}_0(Z) \subset \mathbb{P}T_0(Z)$ is homogeneous and nonlinear, given a germ of VMRT-respecting holomorphic immersion $f : (Z; z_0) \to (X; x_0)$, the condition $\operatorname{Ker} \sigma_\alpha\big(\,\cdot\,, T_\alpha(\widetilde{\mathcal{C}}_{x_0}(Z))\big) \subset P_\alpha \cap df(T_{z_0}(Z))$, where $P_\alpha = T_\alpha(\widetilde{\mathcal{C}}_{x_0}(X))$, forces $\operatorname{Ker} \sigma_\alpha\big(\,\cdot\,, T_\alpha(\widetilde{\mathcal{C}}_0(Z))\big) = \mathbb{C}\alpha$ since the projective second fundamental form on the nonlinear homogeneous projective submanifold $\mathcal{C}_{z_0}(Z) \subset \mathbb{P}T_{z_0}(Z)$ itself is everywhere nondegenerate. The same implication holds true in the short-root case provided that $[\alpha] \in \mathcal{C}_{x_0}(X) \cap \mathbb{P}(df(T_{z_0}(Z))$ is a general point of both $\mathcal{C}_{x_0}(X)$ and $f_\sharp(\mathcal{C}_{z_0}(Z))$. Thus, in the nonlinear case of admissible pairs (Z, X) one can examine the question of rational saturation for a germ of complex submanifold endowed with a certain type of geometric substructure (e.g. a sub-Grassmann structure) without the assumption of the existence of an underlying holomorphic map. We will show that this is indeed possible, and we will introduce a variation of the notion of nondegeneracy for that purpose. This will be taken up in (4.4) and (4.5).

(4.3) *Characterization of smooth Schubert varieties in rational homogeneous spaces of Picard number 1* Characterization of standard embeddings between certain pairs of rational homogeneous spaces (X_0, X) of Picard number 1 was achieved by means of non-equidimensional Cartan-Fubini extension and parallel transport of VMRTs along minimal rational curves. In Hong-Mok [HoM13] this approach was further adopted to deal with a problem of homological rigidity of smooth Schubert cycles on rational homogeneous spaces. Recall that a Schubert cycle $Z \subset X$ on a rational homogeneous space $X = G/P$ is one for which the G-orbit of $G\cdot[Z]$ in $\mathrm{Chow}(X)$ is projective. Suppose $S \subset X$ is a cycle homologous to the Schubert cycle Z, the homological rigidity problem is to ask whether S is necessarily equivalent to Z under the action of $\mathrm{Aut}(X)$. We note that by the extremality of the homology class of a Schubert cycle among homology classes of effective cycles, S is reduced and irreducible.

We reformulate the homological rigidity problem so as to relate it to the geometric theory of uniruled projective manifolds modeled on VMRTs, as follows. To start with, considering the irreducible component $\mathcal{Q}$ of $\mathrm{Chow}(X)$ containing the point $[S]$, there always exists a closed G-orbit in $\mathcal{Q}$, and as such it contains a point corresponding to some Schubert cycle, and thus $\mathcal{Q}$ must contain $[Z]$ itself by the uniqueness modulo G-action of Schubert cycles representing the same homology class. The homological rigidity problem would be solved in the affirmative if we established the local rigidity of Z. In Hong-Mok [HoM13] we dealt with the case of smooth Schubert cycles. When $X = G/P$ is a rational homogeneous space defined by a Dynkin diagram $\mathcal{D}(G)$ marked at a simple root, and $Z \subset X$ is defined by a marked sub-diagram, then $Z \subset X$ is a Schubert cycle (cf. [HoM13, §2, Example 1]). In [HoM13] we proved

Theorem 4.3.1. (Hong-Mok [HoM13, Theorem 1.1]) *Let $X = G/P$ be a rational homogeneous space associated to a simple root and let $X_0 = G_0/P_0$ be a homogeneous submanifold associated to a sub-diagram $\mathcal{D}(G_0)$ of the marked Dynkin diagram $\mathcal{D}(G)$ of X. Then, any subvariety of X having the same homology class as X_0 is induced by the action of $\mathrm{Aut}_0(X)$, excepting when (X_0, X) is given by*
(a) $X = (C_n, \{\alpha_k\})$, $\Lambda = \{\alpha_{k-1}, \alpha_b\}$, $2 \le k < b \le n$;
(b) $X = (F_4, \{\alpha_3\})$, $\Lambda = \{\alpha_1, \alpha_4\}$ *or* $\{\alpha_2, \alpha_4\}$;
(c) $X = (F_4, \{\alpha_4\})$, $\Lambda = \{\alpha_2\}$ *or* $\{\alpha_3\}$,
where Λ denotes the set of simple roots in $\mathcal{D}(G)\backslash\mathcal{D}(G_0)$ which are adjacent to the sub-diagram $\mathcal{D}(G_0)$.

When $X = G/P$ is defined by the marked Dynkin diagram $(\mathcal{D}(G), \gamma)$, where γ is a long simple root, it was established in [HoM13, Proposition 3.7] that any smooth Schubert cycle Z on X is a rational homogeneous submanifold corresponding to a marked sub-diagram of $(\mathcal{D}(G), \gamma)$, hence Theorem 4.3.1 in the long-root case exhausts all smooth Schubert cycles up to G-action. When γ is a short root, this need not be the case, and we refer the reader to [HoM13, Theorem 1.2] for results on the homological rigidity problem for the symplectic Grassmannian pertaining to smooth Schubert cycles which are not rational homogeneous submanifolds.

Parallel transport of VMRTs along a minimal rational curve was used in an essential way in [HoM13] for the proof of Theorem 4.3.1. Using a complex-analytic argument, we established in [HoM13, Proposition 3.3] a proof of Proposition 4.2.1 applicable to the case of Schubert cycles $Z \subset X = G/P$ corresponding to marked sub-diagrams as in Theorem 4.3.1, as follows. (We say that (Z, X) is of sub-diagram type.)

Proof of Proposition 4.2.1. Consider the point $[Z]$ in $\mathrm{Chow}(X)$ corresponding to the reduced cycle Z. Since $Z \subset X$ is a Schubert cycle, the G-orbit of $[Z]$ in $\mathrm{Chow}(X)$ is projective. From this and the fact that $Z \subset X$ is a rational homogeneous submanifold one deduces that the P-orbit of $[Z]$ in $\mathrm{Chow}(X)$ is also projective. Given this, the failure of Proposition 4.2.1 would imply the existence of a holomorphic family $\varpi : \mathcal{Q} \to \Gamma$ of projective submanifolds $Q_t \subset \mathcal{C}_0(X)$, $[Q_t] \in \mathcal{Q}$, parametrized by a *projective* curve Γ, such that all members Q_t contain $[\alpha]$ and such that they all share the *same tangent space* $V = T_{[\alpha]}(\mathcal{C}_0(Z))$. Consider the holomorphic section σ of $\varpi : \mathcal{M} \to \Gamma$ corresponding to the common base point $[\alpha] \in Q_t$ for all $t \in \Gamma$. The assumption that $T_{[\alpha]}(Q_t) = V$ for all $t \in \Gamma$ would imply that the normal bundle N of $\sigma(\Gamma)$ in $\mathcal{Q}$ is holomorphically trivial, which would contradict the negativity of the normal bundle resulting from the existence of a canonical map $\chi : \mathcal{Q} \to \mathcal{C}_0(X)$ collapsing $\sigma(\Gamma)$ to the single point $[\alpha]$ (cf. Grauert [Gra62]). This proves Proposition 4.2.1 by argument by contradiction. $\square$

(4.4) *Sub-VMRT structures arising from admissible pairs of rational homogeneous spaces of Picard number 1 and a rigidity phenomenon* Consider the Grassmann manifold $G(p,q)$ of rank $r = \min(p,q) \geq 2$. We have $T(G(p,q)) = U \otimes V$ where U (resp. V) is a semipositive universal bundle of rank p (resp. q). Let $W \subset G(p,q)$ be an open subset and $S \subset W$ be a complex submanifold such that $T(S) = A \otimes B$, where $A \subset U|_S$ (resp. $B \subset V|_S$) is a holomorphic vector subbundle of rank p' (resp. q') such that $r' = \min(p',q') \geq 2$. Thus, by assumption S inherits a $G(p',q')$-structure, and the question we posed was whether S is an open subset of a projective submanifold $Z \subset G(p,q)$ such that Z is the image of $G(p',q')$ under a standard embedding. When the natural Grassmann structure (of rank $r' \geq 2$) on S is flat, then, given any $x_0 \in S$, there exists an open neighborhood $\mathcal{O}$ of x_0 on S, an open neighborhood U of $0 \in G(p',q')$, and a biholomorphic mapping $f : U \xrightarrow{\cong} \mathcal{O}$ such that f preserves $G(p',q')$-structures, equivalently $f_\sharp(\mathcal{C}(G(p',q'))|_U) = \mathcal{C}(S)$ where $\mathcal{C}(S) := \mathcal{C}(G(p,q)) \cap \mathbb{P}T(S)$. This is the situation dealt with in Mok [Mk08a] where it was proven that S is indeed an open subset of a sub-Grassmannian $Z \subset G(p,q)$ which is the image of $G(p',q')$ under a standard embedding. The question here is whether the flatness assumption is superfluous.

One can contrast Grassmann structures with holomorphic conformal structures. Let $n \geq 4$ and $(S;0)$ be an m-dimensional germ of complex submanifold on the hyperquadric Q^n, $3 \leq m < n$. We say that $(S;0)$ inherits a holomorphic conformal structure if and only if the restriction of the standard holomorphic conformal structure on Q^n to $(S;0)$ is nondegenerate. Here examples abound, indeed a *generic* germ of complex submanifold $(S;0)$ inherits such a structure. On a small coordinate neighborhood W of 0 the holomorphic conformal structure is given by the equivalence class (up to conformal factors) of a holomorphic metric g, i.e., a nondegenerate holomorphic covariant symmetric 2-tensor $g = \sum g_{\alpha\beta}(z)dz^\alpha \otimes dz^\beta$ in local coordinates, and the latter restricts to a holomorphic conformal structure on $(S;0)$ if and only if g is nondegenerate on $T_0(S)$. The case of the pair $(G(p',q');G(p,q))$ is very different. In the latter case, denoting by W a coordinate neighborhood of $0 \in G(p,q)$, it is *a priori* very special that $(S;0) \subset (W;0)$ inherits a $G(p',q')$-structure. Indeed, for a general $p'q'$-dimensional linear subspace $\Pi \subset T_0(G(p,q))$, the intersection $\mathcal{C}_0(G(p,q)) \cap \mathbb{P}(\Pi)$, if nonempty, is of codimension $pq - p'q'$ in $\mathcal{C}_0(G(p,q)) \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, of dimension $p + q - 2$. Thus the expected dimension of intersection is strictly less than that of a $G(p',q')$-structure, i.e., $p' + q' - 2$, as soon as $(p',q') \neq (p,q)$. (For an example of low dimension, consider a *generic* 4-dimensional complex submanifold $S \subset W \subset G(2,3)$. The intersection $\mathcal{C}_0(G(p,q)) \cap \mathbb{P}T_0(S)$ is expected to be of codimension 2 in $\mathcal{C}_0(G(2,3)) \cong \mathbb{P}^1 \times \mathbb{P}^2$, i.e., a curve, while it is a surface $\mathcal{C}_0(S) \cong \mathcal{C}_0(G(2,2)) \cong \mathbb{P}^1 \times \mathbb{P}^1$ when S inherits a $G(2,2)$-structure.) In view of the excessive intersection of VMRTs with projectivized tangent spaces it is perceivable that rigidity already follows from excessive intersection of VMRTs with projectivized tangent spaces and from the specific forms of the intersections, and that the flatness assumption is unnecessary.

In the case of Grassmann structures or other G-structures modeled on irreducible Hermitian symmetric spaces of rank ≥ 2, by Guillemin [Gu65] we could resort to proving the vanishing of a finite number of obstruction tensors to demonstrate flatness, but the method of G-structures is ill-adapted even for rational homogeneous spaces. In its place we examined the tautological foliation on VMRT structures and raised the question whether the restriction to $\mathbb{P}T(S)$ already defines a foliation. In the special and simpler case of linear model submanifolds such a question was formulated and solved by Hong-Park [HoP11] (Theorem 4.2.2 in the above). For the general formulation of rigidity phenomena on complex submanifolds we introduce the notion of admissible pairs of rational homogeneous spaces of Picard number 1 (cf. Mok-Zhang [MZ15, Definition 1.1] and the ensuing discussion). Recall that for a holomorphic immersion $\nu : M \to N$ between complex manifolds we write $\nu_\sharp$ for the projectivization of the differential $d\nu : T(M) \to T(N)$ of the map.

Definition 4.4.1. *Let X_0 and X be rational homogeneous spaces of Picard number 1, and $i : X_0 \hookrightarrow X$ be a holomorphic embedding equivariant with respect to a homomorphism of complex Lie groups $\Phi : \mathrm{Aut}_0(X_0) \to \mathrm{Aut}_0(X)$. We say that $(X_0, X; i)$ is an admissible pair (of rational homogeneous spaces of Picard number 1) if and only if* (a) *i induces an isomorphism $i_* : H_2(X_0, \mathbb{Z}) \xrightarrow{\cong} H_2(X, \mathbb{Z})$, and* (b) *denoting by $\mathcal{O}(1)$ the positive generator of* $\mathrm{Pic}(X)$ *and by $\rho : X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(1))^*) := \mathbb{P}^N$ the first canonical projective embedding of X, $\rho \circ i : X_0 \hookrightarrow \mathbb{P}^N$ embeds X_0 as a linear section of $\rho(X)$.*

It follows immediately from (b) that $i : X_0 \to X$ respects VMRTs, i.e. $i_\sharp(\mathcal{C}_x(X_0)) = \mathcal{C}_{i(x)}(X) \cap i_\sharp(\mathbb{P}T_x(X_0))$ for every point $x \in X_i$. Next we introduce the notion of a sub-VMRT structure modeled on an admissible pair (X_0, X) of rational homogeneous spaces (cf. Mok-Zhang [MZ15, Definition 1.2]).

Definition 4.4.2. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, $W \subset X$ be an open subset, and $S \subset W$ be a complex submanifold. Consider the fibered space $\pi : \mathcal{C}(X) \to X$ of varieties of minimal rational tangents on X. For every point $x \in S$ define $\mathcal{C}_x(S) := \mathcal{C}_x(X) \cap \mathbb{P}T_x(S)$ and write $\varpi : \mathcal{C}(S) \to S$ for $\varpi = \pi\big|_{\mathcal{C}(S)}, \varpi^{-1}(x) := \mathcal{C}_x(S)$ for $x \in S$. We say that $S \subset W$ inherits a sub-VMRT structure modeled on (X_0, X) if and only if for every point $x \in S$ there exists a neighborhood U of x on S and a trivialization of the holomorphic projective bundle $\mathbb{P}T(X)|_U$ given by $\Phi : \mathbb{P}T(X)|_U \xrightarrow{\cong} \mathbb{P}T_0(X) \times U$ such that* (1) *$\Phi(\mathcal{C}(X)|_U) = \mathcal{C}_0(X) \times U$ and* (2) *$\Phi(\mathcal{C}(S)|_U) = \mathcal{C}_0(X_0) \times U$.*

The definition that $S \subset W$ inherits a sub-VMRT structure modeled on the admissible pair (X_0, X) can be reformulated as requiring

(†) For any $x \in X$ there exists a projective linear isomorphism $\Lambda_x : \mathbb{P}T_x(X) \xrightarrow{\cong} \mathbb{P}T_0(X)$ such that $\Lambda_x(\mathcal{C}_x(X)) = \mathcal{C}_0(X)$ and $\Lambda_x(\mathcal{C}_x(S)) = \mathcal{C}_0(X_0)$.

With an aim to generalize the characterization theorems of Hong-Mok [HoM10] and Hong-Park [HoP11] on standard embeddings between rational homogeneous spaces of Picard number 1 we introduced in Mok-Zhang [MZ15, Definition 5.1] the notion of rigid pairs (X_0, X), as follows.

Definition 4.4.3. *For an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1, we say that (X_0, X) is rigid if and only if, for every complex submanifold S of some open subset of X inheriting a sub-VMRT structure modeled on (X_0, X), there exists some $\gamma \in \mathrm{Aut}(X)$ such that S is an open subset of $\gamma(X_0)$.*

We are now ready to state one of the main results of Mok-Zhang [MZ15].

Theorem 4.4.1. (Mok-Zhang [MZ15, Main Theorem 1]) *Let (X_0, X) be an admissible pair of sub-diagram type of rational homogeneous spaces of Picard number 1 of sub-diagram type marked at a simple root. Suppose $X_0 \subset X$ is nonlinear. Then, (X_0, X) is rigid.*

Combining with Hong-Park [HoP11] on the maximal linear case, the question on rigidity of sub-VMRT structures modeled on admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1 of sub-diagram type is completely settled (cf. [MZ15, Corollary 1.1]).

Corollary 4.4.1. *An admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1 of sub-diagram type is a rigid pair excepting when $X_0 \subset X$ is a non-maximal linear subspace, or when $X_0 \subset X$ is a maximal linear subspace $Z_{\max}$ given by* (a) *X is associated to (B_ℓ, α_i), $1 \le i \le \ell - 1$, and $Z_{\max}$ is $\mathbb{P}^{\ell-i}$;* (b) *X is associated to (C_ℓ, α_ℓ) and $Z_{\max}$ is $\mathbb{P}^1$; or* (c) *X is associated to (F_4, α_1) and X_0 is $\mathbb{P}^2$.*

For the study of rigidity of admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1, Zhang [Zh14, Main Theorem 2] classified all such pairs in the case where X is an irreducible Hermitian symmetric space of the compact type and $X_0 \subset X$ is nonlinear. (The linear cases for all rational homogeneous spaces X of Picard number 1 have been enumerated in Hong-Park [HoP11].) From the classification Zhang established

Theorem 4.4.2. (Zhang [Zh14, Main Theorem 2]) *An admissible pair (X_0, X) of irreducible Hermitian symmetric space of the compact type is non-rigid whenever (X_0, X) is degenerate for substructures.*

The key issue in the proof of Theorem 4.4.1 is to show that the tautological foliation $\mathcal{F}_X$ on $\mathcal{C}(X)$ is tangent to the total space $\mathcal{C}(S)$ of the sub-VMRT structure. Pick any $x \in S$ and any $[\alpha] \in \mathcal{C}_x(S)$, and denote by ℓ the minimal rational curve passing through x such that $T_x(\ell) = \mathbb{C}\alpha$. From the sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ regarded as a holomorphic fiber bundle over S, there exists a holomorphic vector field θ on some neighborhood U of x on S such that $\theta(x) = \alpha$ and such that $\theta(y) \in \widetilde{\mathcal{C}}_y(S)$ whenever $y \in U$. The integral curve of θ passing through x gives a smooth holomorphic curve γ on S tangent to ℓ at x such that the lifting $\check{\gamma}$ of γ to $\mathbb{P}T(S)$ lies on $\mathcal{C}(S)$. At the point $[\alpha] \in \mathcal{C}_x(S)$ the difference between $T_{[\alpha]}(\check{\gamma})$ and $T_{[\alpha]}(\check{\ell})$, where $\check{\ell}$ denotes the tautological lifting $\check{\ell} \subset \mathcal{C}(X)$ of ℓ, gives a vector $\eta \in T_{[\alpha]}(\mathcal{C}_x(X))$. In view of the flexibility in the choice of γ, the vector η is only well-defined modulo $T_{[\alpha]}(\mathcal{C}_x(S)) \cong (P_\alpha \cap T_x(S))/\mathbb{C}\alpha$. Let $D(X) \subset T(X)$ be the G-invariant distribution spanned at each point $x \in X$ by $\widetilde{\mathcal{C}}_x(X)$. There is a vector-valued symmetric bilinear form $\tau_{[\alpha]} : S^2 T_{[\alpha]}(\mathcal{C}_x(X)) \to T_{[\alpha]}(\mathbb{P}T_x(X))/(T_{[\alpha]}(\mathcal{C}_x(X)) + T_{[\alpha]}(\mathbb{P}(T_x(S) \cap D_x(X))$, given by $\tau_{[\alpha]} = \nu \circ \sigma_{[\alpha]}$, $\nu : T_{[\alpha]}(\mathbb{P}T_x(X))/T_{[\alpha]}(\mathcal{C}_x(X)) \to T_{[\alpha]}(\mathbb{P}T_x(X))/(T_{[\alpha]}(\mathcal{C}_x(X)) + T_{[\alpha]}(\mathbb{P}(T_x(S) \cap D_x(X)))$ being the canonical projection. Here $T_{[\alpha]}(\mathbb{P}T_x(X))/T_{[\alpha]}(\mathcal{C}_x(X))$ is the normal space of the inclusion $\mathcal{C}_x(X) \subset \mathbb{P}T_x(X)$ at $[\alpha] \in \mathcal{C}_x(S) \subset \mathcal{C}_x(X)$ and σ denotes the projective second fundamental form of the said inclusion. For both the second fundamental form σ and the variant τ, we use the same notation when passing to affinizations $\widetilde{\mathcal{C}}_x(X) \subset T_x(X)$. The context will make it clear which is meant.

Obviously $\tau_{[\alpha]}(\xi_1, \xi_2) = 0$ whenever $\xi_1, \xi_2 \in T_{[\alpha]}(\mathcal{C}_x(S))$. For the proof that $\mathcal{F}_X$ is tangent to $\mathcal{C}(S)$ it suffices to show that for $\eta \in T_{[\alpha]}(\mathcal{C}_x(X))$ as defined in the last paragraph we have actually $\eta \in T_{[\alpha]}(\mathcal{C}_x(S))$. In [MZ15] we show that $\tau_{[\alpha]}(\eta, \xi) = 0$ whenever $\xi \in T_{[\alpha]}(\mathcal{C}_x(S))$ and derive the rigidity of the pairs (X_0, X) in Theorem 4.4.1 by checking that $\tau_{[\alpha]}(\eta, \xi) = 0$ for all $\xi \in T_{[\alpha]}(\mathcal{C}_x(S))$ implies that $\eta \in T_{[\alpha]}(\mathcal{C}_x(S))$. We say in this case that $(\mathcal{C}_x(S), \mathcal{C}_x(X))$ is nondegenerate for substructures (cf. Definition 4.5.2 in the next subsection), noting that $T_x(S) \cap D_x(X)$ is the linear span of $\widetilde{\mathcal{C}}_x(S)$. The checking is derived from statements about the second fundamental form σ concerning nondegeneracy of Hong-Mok [HoM10] and the proof of Theorem 4.4.1 is completed by means of parallel transport of VMRTs along minimal rational curves and the standard argument of adjunction of minimal rational curves (cf. (4.2) and (4.3)). In the Hermitian symmetric case the proof that $\tau_{[\alpha]}(\eta, \xi) = 0$ for $\xi \in T_{[\alpha]}(\mathcal{C}_x(S))$ results from a differential-geometric calculation with respect to the flat connection in Harish-Chandra coordinates, and the general case is derived from adapted coordinates in the same setting as in [HoM10] explained in (4.2).

Our arguments apply to uniruled projective manifolds to give a sufficient condition for a germ of complex submanifold to be rationally saturated, making it applicable to study sub-VMRT structures in general. This will be explained in (4.5).

For the formulation of sub-VMRT structures and nondegeneracy for substructures, one has to make use of the distribution on X spanned by VMRTs. In the event that the distribution $D(X) \subset T(X)$ spanned by VMRTs is linearly *degenerate*, the proof that $\mathcal{F}_X$ is tangent to $\mathcal{C}(S)$ relies on the fact that the kernel of the Frobenius form $\varphi : \Lambda^2 D(X) \to T(X)/D(X)$ contains the linear span of $\{\alpha \wedge P_\alpha : \alpha \in \widetilde{\mathcal{C}}_x(X)\}$, where $P_\alpha = T_\alpha(\widetilde{\mathcal{C}}_x(X))$, a basic fact about distributions spanned by VMRTs that was established in Hwang-Mok [HM98, (4.2), Proposition 10].

(4.5) *Criteria for rational saturation and algebraicity of germs of complex submanifolds* Let now $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$. Using a generalization of the argument of Theorem 4.4.1 and the method of analytic continuation by the adjunction of (open subsets of) minimal rational curves of Hwang-Mok [HM01], Mok [Mk08a] and Hong-Mok [HoM11], Mok-Zhang [Mk14, Theorem 1.4] also obtained a general result on the analytic continuation of a germ of complex submanifold S on X when S inherits a certain geometric substructure. For its formulation, we consider a locally closed complex submanifold $S \subset X$. Writing $\mathcal{C}(S) := \mathcal{C}(X)|_S \cap \mathbb{P}T(S)$ and $\varpi := \pi|_{\mathcal{C}(S)} : \mathcal{C}(S) \to S$ we defined in [MZ15, Definition 5.1] the notion of a sub-VMRT structure, as follows.

Definition 4.5.1. *We say that $\varpi := \pi|_{\mathcal{C}(S)} : \mathcal{C}(S) \to S$ is a sub-VMRT structure on $(X, \mathcal{K})$ if and only if* (a) *at every point $x \in S$ every minimal rational curve ℓ on $(X, \mathcal{K})$ passing through x is a free rational curve immersed at x and furthermore the tangent map $\tau_x : \mathcal{U}_x \dashrightarrow \mathcal{C}_x(X)$ at x of $(X, \mathcal{K})$ is a birational morphism;* (b) *the restriction of ϖ to each irreducible component of $\mathcal{C}(S)$ is surjective, and;* (c) *at a general point $x \in S$ and for any irreducible component Γ_x of $\mathcal{C}_x(S)$, we have $\Gamma_x \not\subset \mathrm{Sing}(\mathcal{C}_x(X))$.*

Given a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ of $\pi : \mathcal{C}(X) \to X$, there is some integer $m \geq 1$ such that over a general point $x \in S$, $\mathcal{C}_x(S)$ has exactly m irreducible components and such that ϖ is a submersion at a general point χ_k of each irreducible component $\Gamma_{k,x}$ of $\mathcal{C}_x(S)$. We introduce now the notions of proper pairs of projective subvarieties and nondegeneracy for substructures for such pairs (cf [MZ15, Definitions 5.2 & 5.3]).

Definition 4.5.2. *Let V be a Euclidean space and $\mathcal{A} \subset \mathbb{P}(V)$ be an irreducible subvariety. We say that $(\mathcal{B}, \mathcal{A})$ is a proper pair if and only if $\mathcal{B}$ is a linear section of $\mathcal{A}$, and for each irreducible component Γ of $\mathcal{B}$, $\Gamma \not\subset \mathrm{Sing}(\mathcal{A})$.*

For a uniruled projective manifold X and a locally closed complex submanifold $S \subset X$ inheriting a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ as in Definition 4.5.1, at a general point $x \in S$, $(\mathcal{C}_x(S), \mathcal{C}_x(X))$ is a proper pair of projective subvarieties. We introduce now the notion of nondegeneracy for substructures for $(\mathcal{B}, \mathcal{A}; E)$. Here for convenience we assume that $\mathcal{A}$ is irreducible. When applied to sub-VMRT structures this means that the VMRT $\mathcal{C}_x(X)$ at a general point on the ambient manifold X is assumed irreducible.

Definition 4.5.3. *Let V be a finite-dimensional vector space, $E \subset V$ be a vector subspace and $(\mathcal{B}, \mathcal{A})$ be a proper pair of projective subvarieties in $\mathbb{P}(V)$, $\mathcal{B} := \mathcal{A} \cap \mathbb{P}(E) \subset \mathcal{A} \subset \mathbb{P}(V)$. Assume that $\mathcal{A}$ is irreducible. Let $\xi \in \widetilde{\mathcal{B}}$ be a smooth point of both $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$, and let $\sigma : S^2T_\xi(\widetilde{\mathcal{A}}) \to V/T_\xi(\widetilde{\mathcal{A}})$ be the second fundamental form of $\widetilde{\mathcal{A}}$ in V with respect to the Euclidean flat connection on V. Write $V' \subset V$ for the linear span of $\widetilde{\mathcal{A}}$ and define $E' := E \cap V'$. Let $\nu : V/T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ be the canonical projection and define $\tau : S^2T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ by $\tau := \nu \circ \sigma$. For the proper pair $(\mathcal{B}, \mathcal{A})$,*

$\mathcal{B} = \mathcal{A} \cap \mathbb{P}(E)$, *we say that* $(\mathcal{B}, \mathcal{A}; E)$ *is nondegenerate for substructures if and only if for each irreducible component* Γ *of* $\mathcal{B}$ *and for a general point* $\chi \in \Gamma$, *we have*

$$\left\{\eta \in T_\chi(\widetilde{\mathcal{A}}) : \tau(\eta, \xi) = 0 \text{ for any } \xi \in T_\chi(\widetilde{\mathcal{B}})\right\} = T_\chi(\widetilde{\mathcal{B}}).$$

In case $E' = E \cap V'$ is the same as the linear span of $\widetilde{\mathcal{B}}$ we drop the reference to E, with the understanding that the projection map ν is defined by using the linear span of $\widetilde{\mathcal{B}}$ as E'. In the case of an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1, writing $D(X) \subset T(X)$ for the G-invariant distribution spanned by VMRTs at each point $x \in X$, $D(X) \cap T(X_0)$ is the same as the G_0-invariant distribution on X_0 spanned by VMRTs. (When the Dynkin diagram is marked at a long simple root, $D(X)$ and $D(X_0)$ are the minimal nonzero invariant distributions, but the analogue fails for the short-root case.) In order to adapt the arguments for rational saturation to the general situation of sub-VMRT structures, we need to introduce an auxiliary condition on the intersection $\mathcal{C}(S) = \mathcal{C}(X) \cap \mathbb{P}T(S)$, to be called Condition (T), which is automatically satisfied in the case of admissible pairs (X_0, X). We have

Definition 4.5.4. *Let* $\varpi : \mathcal{C}(S) \to S$, $\mathcal{C}(S) := \mathcal{C}(X) \cap \mathbb{P}T(S)$, *be a sub-VMRT structure on* $S \subset X$ *as in Definition 5.1. For a point* $x \in S$, *and* $[\alpha] \in \mathrm{Reg}(\mathcal{C}_x(S)) \cap \mathrm{Reg}(\mathcal{C}_x(X))$, *we say that* $(\mathcal{C}_x(S), [\alpha])$, *or equivalently* $(\widetilde{\mathcal{C}_x}(S), \alpha)$, *satisfies* Condition (T) *if and only if* $T_\alpha(\widetilde{\mathcal{C}_x}(S)) = T_\alpha(\widetilde{\mathcal{C}_x}(X)) \cap T_x(S)$. *We say that* $\varpi : \mathcal{C}(S) \to S$ *satisfies* Condition (T) *at* x *if and only if* $(\widetilde{\mathcal{C}_x}(S), [\alpha])$ *satisfies* Condition (T) *for a general point* $[\alpha]$ *of each irreducible component of* $\mathrm{Reg}(\mathcal{C}_x(S)) \cap \mathrm{Reg}(\mathcal{C}_x(X))$. *We say that* $\varpi : \mathcal{C}(S) \to S$ *satisfies* Condition (T) *if and only if it satisfies the condition at a general point* $x \in S$.

The argument for proving that $\mathcal{F}_X$ is tangent to $\mathcal{C}(S)$ remains valid for the general set-up of sub-VMRT structures on uniruled projective manifolds $(X, \mathcal{K})$, and the method of analytic continuation by adjoining minimal rational curves remains applicable.We have

Theorem 4.5.1. (Mok-Zhang [MZ15, Theorem 1.4]) *Let* $(X, \mathcal{K})$ *be a uniruled projective manifold* X *equipped with a minimal rational component* $\mathcal{K}$ *with associated VMRT structure given by* $\pi : \mathcal{C}(X) \to X$. *Assume that at a general point* $x \in X$, *the VMRT* $\mathcal{C}_x(X)$ *is irreducible. Write* $B' \subset X$ *for the enhanced bad locus of* $(X, \mathcal{K})$. *Let* $W \subset X - B'$ *be an open set, and* $S \subset W$ *be a complex submanifold such that, writing* $\mathcal{C}(S) := \mathcal{C}(X)|_S \cap \mathbb{P}T(S)$ *and* $\varpi := \pi\big|_{\mathcal{C}(S)}$, $\varpi : \mathcal{C}(S) \to S$ *is a sub-VMRT structure satisfying* Condition (T). *Suppose furthermore that for a general point* x *on* S *and for each of the irreducible components* $\Gamma_{k,x}$ *of* $\mathcal{C}_x(S)$, $1 \le k \le m$, *the inclusion* $\Gamma_{k,x} \subset \mathcal{C}_x(X)$ *at a general smooth point* χ_k *of* $\Gamma_{k,x}$ *is nondegenerate for substructures. Then,* S *is rationally saturated with respect to* $(X, \mathcal{K})$. *In other words,* S *is uniruled by open subsets of minimal rational curves belonging to* $\mathcal{K}$.

When X is of Picard number 1, by a line ℓ on X we mean a rational curve ℓ of degree 1 with respect to the positive generator of $\mathrm{Pic}(X)$. We say that $(X, \mathcal{K})$ is a uniruling by lines to mean that members of $\mathcal{K}$ are lines. We prove for these uniruled projective manifolds a sufficient condition for the *algebraicity* of germs of sub-VMRT structures on them. Recall that a holomorphic distribution $\mathcal{D}$ on a complex manifold M is said to be bracket generating if and only if, defining inductively $\mathcal{D}_1 = \mathcal{D}$, $\mathcal{D}_{k+1} = \mathcal{D}_k + [\mathcal{D}, \mathcal{D}_k]$, we have $\mathcal{D}_m|_U = T(U)$ on a neighborhood U of a general point $x \in M$ for m sufficiently large. By a distribution we will mean a coherent subsheaf of the tangent sheaf.

Theorem 4.5.2. (Mok-Zhang [MZ15, Main Theorem 2]) *In the statement of Theorem 4.5.1 suppose furthermore that* $(X, \mathcal{K})$ *is a projective manifold of Picard number 1 uniruled by lines and that the distribution* $\mathcal{D}$ *on* S *defined at a general point* $x \in X$ *by* $\mathcal{D}_x := \mathrm{Span}(\widetilde{\mathcal{C}_x}(S))$ *is bracket generating. Then, there exists an irreducible subvariety* $Z \subset X$ *such that* $S \subset Z$ *and such that* $\dim(Z) = \dim(S)$.

Thus, the subvariety $Z \subset X$, which is rationally saturated with respect to $\mathcal{K}$, is in particular uniruled by minimal rational curves belonging to $\mathcal{K}$. $Z \subset X$ is thus a uniruled projective subvariety. We say that S admits a projective-algebraic extension. Note that the hypothesis that $\mathcal{D}$ is bracket generating is trivially satisfied when $\mathcal{D}$ is linearly nondegenerate at a general point.

Modulo Theorem 4.5.1, which yields rational saturation for sub-VMRT structures under a condition of nondegeneracy for substructures, for the proof of Theorem 4.5.2 we reconstruct a projective-algebraic extension of S by a process of adjunction of minimal rational curves as in Hwang-Mok [HM98], Mok [Mk08a] and Hong-Mok [HM10]. As opposed to the situation of these articles where the adjunction process is *a priori* algebraic, the major difficulty in the proof of Theorem 4.5.2 lies in showing that, starting from a *transcendental* germ of complex manifold $(S; x_0)$ on $X - B'$ equipped with a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ obtained from taking intersections with tangent subspaces, the process of adjoining minimal rational curves starting with those emanating from x_0 and tangent to S is actually *algebraic*. For this purpose we introduce a method of propagation of sub-VMRT structures along chains of special minimal rational curves and apply methods of extension of holomorphic objects in several complex variables coming from the Hartogs phenomenon (cf. Siu [Si74]), viz., we show that for the inductive process of propagation of the germ $(S; x_0)$ along chains of rational curves, the obstruction in essence lies on subvarieties of codimension ≥ 2 on certain universal families of chains of rational curves. Crucial to this process is a proof of the "Thickening Lemma" which allows us to show that the sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ can be propagated along a general member of certain algebraic families of standard rational curves which are defined inductively.

Proposition 4.5.1. (Mok-Zhang [MZ, Proposition 6.1]) *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component,* $\dim(X) := n$, *and $\varpi : \mathcal{C}(S) \to S$ be a sub-VMRT structure as in Theorem 1.4,* $\dim(S) := s$. *Let $[\alpha] \in \mathcal{C}(S)$ be a smooth point of both $\mathcal{C}(S)$ and $\mathcal{C}(X)$, $\varpi([\alpha]) := x$, and $[\ell] \in \mathcal{K}$ be the minimal rational curve (which is smooth at x) such that $T_x(\ell) = \mathbb{C}\alpha$, and $f : \mathbf{P}_\ell \to \ell$ be the normalization of ℓ, $\mathbf{P}_\ell \cong \mathbb{P}^1$. Suppose $(\mathcal{C}_x(S), [\alpha])$ satisfies* Condition (T) *in Definition 4.5.4. Then, there exists an s-dimensional complex manifold E, $\mathbf{P}_\ell \subset E$, and a holomorphic immersion $F : E \to X$ such that $F|_{\mathbf{P}_\ell} \equiv f$, and such that $F(E)$ contains an open neighborhood of x in S.*

In relation to Theorem 4.5.2 there is the problem of recognizing special classes of uniruled projective subvarieties, which we formulate as

Problem 4.5.1. (The Recognition Problem for sub-VMRT structures) *Let X be a uniruled projective manifold endowed with a minimal rational component $\mathcal{K}$, and Φ be a class of projective subvarieties $Z \subset X$ which are rationally saturated with respect to $(X, \mathcal{K})$. Denote by B' the enhanced bad locus of $(X, \mathcal{K})$. We say that the Recognition Problem for the class $\Phi \subset$* Chow(X) *is solved in the affirmative if one can assign to each $x \in X - B'$ a variety of linear sections $\Psi_x \subset$* Chow$(\mathcal{C}_x(X))$ *in such a way that a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ of $\pi : \mathcal{C}(X) \to X$ admits a projective-algebraic extension to a member Z of Φ if and only if $[\mathcal{C}_s(S)] \in \Psi_s$ for a general point $s \in S$.*

As an example, let X be the Grassmannian $G(p, q)$; $p, q \geq 2$; identified as a submanifold of some $\mathbb{P}^N$ by means of the Plücker embedding. Suppose $2 \leq p' \leq p$, $2 \leq q' \leq q$ and let Φ to be the class of linear sections $Z = G(p, q) \cap \Pi$, $\Pi \subset \mathbb{P}^N$ a projective linear subspace, such that Z is the image of a standard embedding of $G(p', q')$ into $G(p, q)$. Then, by Theorem 4.5.1, the Recognition Problem is solved for Φ by taking Ψ_x at $x \in X$ to consist of linear sections $\varsigma(\mathbb{P}(U_x) \times \mathbb{P}(V_x)) \cap \mathbb{P}(U'_x \otimes V'_x)$ where $T(G(p, q)) \cong U \otimes V$, where ς denotes the Segre embedding, and $U'_x \subset U_x$ (resp. $V'_x \subset V_x$) runs over the set of

p'-dimensional (resp. q'-dimensional) vector subspaces of U_x (resp. V_x). Another example is the Recognition of maximal linear subspaces. The result of Hong-Park [HoP11] (Theorem 4.2.1 here) says that maximal linear subspaces on $X = G/P$ can be recognized only with a few exceptions. By making use of a quantitative version of nondegeneracy for substructures the Recognition Problem for maximal linear subspaces can be solved in the affirmative on certain linear sections of rational homogeneous spaces. As an example we have the following result in the case of linear sections of Grassmannians taken from Mok-Zhang [MZ15, Corollary 9.1].

Proposition 4.5.2. *Consider the Grassmannian $G(p,q)$, $3 \le p \le q$, of rank $p \ge 3$. Let $Z \subset G(p,q)$ be a smooth linear section of codimension $\le p-2$, $\mathcal{H}$ be the space of projective lines on Z, and $E \subset Z$ be the bad locus of $(Z,\mathcal{H})$. Let $(S;x_0)$ be a germ of complex submanifold on $Z-E$ such that $\mathbb{P}T(S) \subset \mathcal{C}(Z)|_S$ and $\mathbb{P}T(S)$ contains a smooth point of $\mathcal{C}(Z)|_S$. Suppose $\mathbb{P}T_x(S) \subset \mathcal{C}_x(Z)$ is a maximal linear subspace for a general point $x \in S$. Then, $S \subset Z$ is a maximal linear subspace.*

Note that for a projective submanifold Z equipped with a uniruling $\mathcal{H}$ by projective lines, the bad locus E and the emhanced bad locus E' of $(Z,\mathcal{H})$ are the same. By the very nature of the notion of sub-VMRT structures, viz., by taking linear sections with tangent subspaces, the Recognition Problem concerns primarily the recognition of a global linear section Z of a projective manifold X uniruled by projective lines from the fact that VMRTs of Z at a general point is a linear section of the VMRT of X with special properties. We may say that this amounts to recognizing certain global linear sections from sub-VMRTs which are special linear sections of VMRTs. In a direction beyond the current article, one could define higher order sub-VMRT structures by considering minimal rational curves which are tangent to the submanifold S to higher orders. There for instance one could raise the problem of recognizing the intersection of a sub-Grassmannians with a number of quadric hypersurfaces in terms of second order sub-VMRT structures.

In (4.6) we will discuss some concrete examples to which Theorem 4.5.1 and Theorem 4.5.2 apply.

(4.6) *Examples of sub-VMRT structures related to irreducible Hermitian symmetric spaces of the compact type* Theorem 4.5.1 gives sufficient conditions for proving that certain sub-VMRT structures are rationally saturated. In the event that the sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ satisfies Condition (T) and it is furthermore linearly nondegenerate for substructures at a general point, it shows that the sub-VMRT structure arises from some uniruled projective subvariety. Here are some examples of sub-VMRT structures to which Theorem 4.5.1 and Theorem 4.5.2 apply.

(a) Let X be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 other than a Lagrangian Grassmannian equipped with the minimal rational component $\mathcal{K}$ of projective lines. Let $[\alpha] \in \mathcal{C}_x(X)$ and consider $\mathcal{S}_{[\alpha]} := \mathcal{C}_x(X) \cap \mathbb{P}(P_\alpha)$, where $P_\alpha = T_\alpha(\widetilde{\mathcal{C}}_x(X))$. Then $\mathcal{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)$ is linearly nondegenerate. Note here that a Lagrangian Grassmannian is equivalently an irreducible symmetric space $G^{\mathrm{III}}(n,n)$, $n \ge 2$, of type III for which the VMRTs are Veronese embeddings $\nu : \mathbb{P}(V) \to \mathbb{P}(S^2V)$ for $V \cong \mathbb{C}^n$ for some $n \ge 2$, given by $\nu([v]) = [v \otimes v]$, in which case the analogue of $\mathcal{S}_{[\alpha]}$ is the single point $[\alpha]$ since the image of the Veronese embedding contains no lines. $\mathcal{S}_{[\alpha]}$ is the cone over a copy of the VMRT of $\mathcal{C}_{[\alpha]}(X)$. It can be proven that the proper pair $(\mathcal{S}_{[\alpha]}, \mathcal{C}_x(X))$ is nondegenerate for substructures excepting in the cases where $X = Q^n$, $n \ge 3$, or when $X = G(2,q), q \ge 2$. Thus, by Theorem 4.5.1, any complex submanifold $S \subset W$ on an open subset $W \subset X$ carrying a sub-VMRT structure $\varpi : \mathcal{C}_x(S) \to S$ with fibers $\big(\mathcal{C}_x(S) \subset \mathcal{C}_x(X)\big)$ projectively equivalent to $\big(\mathcal{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)\big)$ is rationally saturated and in particular uniruled by projective lines. By the linear nondegeneracy of

$\mathcal{S}_{[\alpha]}$ in $\mathbb{P}(P_\alpha)$, Theorem 4.5.2 applies to show that $S \subset W$ admits a projective-algebraic extension. As a model let $x \in X$ and consider the union $\mathcal{V}$ of all projective lines on X passing through x. Then $\mathcal{V} \subset X$ is a projective subvariety inheriting a sub-VMRT structure modeled on $\big(\mathcal{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)\big)$.

We note that in the case where X is the Grassmannian $G(p,q)$ of rank $r = \min(p,q) \geq 2$, $T_{G(p,q)} \cong U \otimes V$, where U resp. V is a universal bundle of rank p resp. q, and $\mathcal{C}_x(X) = \varsigma\big(\mathbb{P}(U_x) \times \mathbb{P}(V_x)\big)$ for the Segre embedding ς, so that, writing $\alpha = u \otimes v$ we have $\mathcal{S}_{[\alpha]} = \mathbb{P}(\mathbb{C}u \otimes V_x) \cup \mathbb{P}(U_x \otimes \mathbb{C}v)$ is the union of two projective subspaces of dimension $p-1$ resp. $q-1$ intersecting at a single point. This gives an example of a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ with 2 irreducible components in each fiber $\mathcal{C}_x(S)$. Both components have to be taken into account at the same time in order to have linear nondegeneracy in $\mathbb{P}T_x(S)$ so that one can apply the last statement in Theorem 4.5.2.

(b) The following are particular cases of examples discussed in Mok-Zhang [MZ15, §9] Embed the Grassmann manifold $G(p,q)$; $p,q \geq 2$; into the projective space by the Plücker embedding $\varphi : G(p,q) \to \mathbb{P}(\Lambda^p\mathbb{C}^{p+q}) := \mathbb{P}^N$ and thus identify $G(p,q)$ as a projective submanifold. Consider a smooth complete intersection $X = G(p,q) \cap (H_1 \cap \cdots \cap H_m)$ of codimension k, where for $1 \leq i \leq m$, $H_i \subset \mathbb{P}^N$ is a smooth hypersurface of degree k_i, $k := k_1 + \cdots + k_m$. Let δ be the restriction to X of the positive generator of $H^2(G(p,q),\mathbb{Z}) \cong \mathbb{Z}$. If $k \leq p+q-1$ then $c_1(X) = (p+q-k)\delta \geq \delta$ and X is Fano. If $k \leq p+q-2$ then $c_1(X) \geq 2\delta$. For the latter range X is uniruled by the minimal rational component $\mathcal{K}$ of projective lines on X and the associated VMRT $\mathcal{C}_x(X)$ of X at a general point of X is the intersection of $\mathcal{C}_x(G(p,q)) = \varsigma(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1})$, where ς stands for the Segre embedding, of codimension k, with k hypersurfaces in $\mathbb{P}T_x(X)$ of degrees $(1,\cdots,k_1;\cdots;1,\cdots,k_m)$, and $\dim(\mathcal{C}_x(X)) = (p-1)+(q-1)-k = (p+q-2)-k \geq 0$. Suppose now $2 \leq p' < p$, $2 \leq q' < q$, and suppose $X_0 := G(p',q') \cap (H_1 \cap \cdots \cap H_m)$ is also smooth. We have $c_1(X_0) = (p'+q'-k)\delta \geq 2\delta$ if and only if $k \leq p'+q'-2$, in which case the pair $\big(\mathcal{C}_x(X_0) \subset \mathcal{C}_x(X)\big)$ consists of projective submanifolds of $\mathbb{P}T(X)$ of the form $\big(\mathcal{C}_0(G(p',q')) \cap \mathcal{J} \subset \mathcal{C}_0(G(p,q)) \cap \mathcal{J}\big)$ for some subvariety $\mathcal{J} \subset \mathbb{P}T_0(X)$ of codimension k at a reference point $0 \in G(p',q') \subset G(p,q)$. Consider now a germ of complex submanifold $(S;x_0)$ on X, and assume that by intersecting with projectivized tangent spaces we have a sub-VMRT structure $\varpi : \mathcal{C}(S) \to S$ satisfying Condition (T), where over a general point $x \in S$, the pair $\big(\mathcal{C}_x(S) \subset \mathcal{C}_x(X)\big)$ is projectively equivalent to $\big(\mathcal{C}_0(G(p',q')) \cap \mathcal{J}_x \subset \mathcal{C}_0(G(p,q)) \cap \mathcal{J}_x\big)$ for some projective subvariety $\mathcal{J}_x \subset \mathbb{P}T_0(X)$ of codimension k, and where $\mathcal{C}_x(S) \subset \mathbb{P}T_x(S)$ is linearly nondegenerate. We show that if $k \leq \min(p'-2, q'-2)$, then the sub-VMRT structure $\varpi : \mathcal{C}(S) \to \mathcal{S}$ satisfies the hypotheses of Theorem 4.5.2 and must hence extend to a projective subvariety $Z \subset X$, $\dim(Z) = \dim(S) = p'q'-k$ which is uniruled by projective lines. This gives examples of germs of complex submanifolds on a uniruled projective manifold with variable VMRTs for which the arguments for proving algebraicity of Theorem 4.5.2 are applicable to prove analytic continuation of S to a projective subvariety. We note also that nondegeneracy for substructures fails in general if we consider S as a germ of complex submanifold on $G(p,q)$ instead of X. In fact, it already fails in general for $S \subset G(p',q')$, $S := G(p',q') \cap H$ (hence *a fortiori* for $S \subset G(p,q)$) when H is a smooth hypersurface of $G(p,q)$).

Complete intersections give examples of complex submanifolds $Y \subset G(p,q)$ uniruled by projective lines for which the isomorphism type of $\mathcal{C}_y(Y) \subset \mathcal{C}_y(G(p,q))$ at a general point can be described, but this precise information is not necessary for the application of Theorem 4.5.2. The same argument in fact applies to any projective submanifold $Y \subset G(p,q)$ uniruled by projective lines under the assumption that $c_1(Y) = (p+q-k)\delta \geq 2\delta$.

(4.7) *Perspectives on geometric substructures* While a first motivation on the study of the geometric theory on uniruled projective manifolds was to tackle problems in algebraic geometry on such manifolds, as the theory was developing, it was clear that our theory carries a strong differential-geometric flavor. It was at least in part developed in a self-contained manner basing on the study of the double fibration arising from the universal family, the tautological foliation and associated differential systems, and the axiomatics of the theory were derived from the deformation theory of rational curves. Varieties of minimal rational tangents appear naturally as the focus of our study, in the context of the VMRT structure $\pi : \mathcal{C}(X) \to X$. While basic results in the early part of the theory, such as those on integrability issues concerning distributions spanned by VMRTs and on Cartan-Fubini extension, have led to solutions of a number of guiding problems, the theory also takes form on its own. It is legitimate to raise questions regarding the VMRTs themselves, such as the Recognition Problem and problems on the classification of isotrivial VMRT structures on uniruled projective manifolds.

One may develop the theory of sub-VMRT structures in analogy to the study of Riemannian submanifolds in Riemannian manifolds, where rationally saturated subvarieties may be taken as weak analogues to geodesic subspaces and, at least in cases of rational homogeneous spaces of Picard number 1, certain cycles such as (possibly singular) Schubert cycles can be taken as strong analogues. Concerning problems in algebraic geometry that may be treated by the study of sub-VMRT substructures, first of all it is natural to extend the characterization of smooth Schubert varieties in rational homogeneous spaces of Picard number 1 to the case of singular Schubert varieties, where one has to examine sub-VMRT structures $\varpi : \mathcal{C}(S) \to S$ with singular sub-VMRTs and to study parallel transport in such a broader context. In view of the application of equidimensional Cartan-Fubini to prove rigidity of finite surjective holomorphic maps (Hwang-Mok [HM01] [HM04]) it is tempting to believe that non-equidimensional Cartan-Fubini can have implications for rigidity of certain non-equidimensional maps between uniruled projective varieties.

Cartan-Fubini extension can be taken as a generalization of Ochiai's Theorem in the context of S-structures (i.e., G-structures arising from irreducible Hermitian symmetric spaces S of the compact type and of rank ≥ 2) from an entirely different angle, viz., from the perspectives of local differential geometry and several complex variables. The proof itself reveals the interaction of these aspects with algebraic geometry, notably with Mori's theory on rational curves. The study of sub-VMRT structures on a uniruled projective manifold in Mok [Mk08a] was first of all motivated by the desire to understand the heart of a rigidity phenomenon in several complex variables, viz., the rigidity of proper holomorphic maps between irreducible bounded symmetric domains of the same rank $r \geq 2$, established by Tsai [Ts93] by considering boundary values on certain product submanifolds, as was done in Mok-Tsai [MT92], and applying methods of Kähler geometry. Regarding a bounded symmetric domain Ω of rank ≥ 2 as an open subset of its dual Hermitian symmetric space S of the compact type by means of the Borel embedding, Ω carries a VMRT structure by restriction. The gist of the arguments of [Mk08a] consists of exploiting boundary values of the map, from which one shows that the mapping respects VMRTs because of properness and because of the decomposition of $\partial\Omega$ [Wo72] into the disjoint union of boundary faces of different ranks, and rigidity of the map results from a non-equidimensional Cartan-Fubini extension as was later developed in full generality by Hong-Mok [HoM10]. For a proper holomorphic map $f : \Omega \to \Omega'$ where $r' = \mathrm{rank}(\Omega') > \mathrm{rank}(\Omega) = r$ the theory has yet to be further developed. In some very special cases they have led to VMRT-respecting holomorphic maps (Tu [Tu02]), but in general to a different form of geometric structures where VMRTs are mapped into vectors tangent to rational curves of degree $\leq r' - r + 1 < r'$, a context which was first discussed by Neretin [Ne99] in the case of classical domains of type I (which are dual to Grassmannians). The study of proper holomorphic mappings will remain a source of motivation for the further study of VMRTs or more general geometric substructures.

Another source of examples with sub-VMRT structures, somewhat surprisingly, is the study of holomorphic isometries of the complex unit ball into an irreducible bounded symmetric domain Ω of rank ≥ 2. Denote by $\Omega \subset S$ the Borel embedding of Ω as an open subset of its dual Hermitian symmetric space S of the compact type. There, the construction in Mok [Mk14] shows that, given a regular boundary point $q \in \mathrm{Reg}(\partial\Omega)$, and denoting by $\mathcal{V}_q$ the union of minimal rational curves passing through q, the intersection $\Sigma := \mathcal{V}_q \cap \Omega$ is the image of a holomorphic isometric embedding of B^{p+1}, $p = \dim(\mathcal{C}_0(S))$. These were the examples which inherit, excepting in the case of Lagrangian Grassmannians, *singular* sub-VMRT structures $\varpi : \mathcal{C}(Z) \to Z$ which are nondegenerate for substructures as explained in (4.5). The dimension $p+1$ is the maximal possible dimension n for a holomorphic isometry $f : (B^n, g) \to (\Omega, h)$, where g resp. h are canonical Kähler-Einstein metrics normalized so that minimal disks are of Gaussian curvature -2, and, questions on uniqueness and rigidity in the case of $n = p$ have led to the study of normal forms of tangent spaces of $T_x(Z)$ and interesting questions on the reconstruction of complex submanifolds from their sub-VMRT structures. Another exciting area where sub-VMRT structures enter is the study of geometric substructures on a quotient $X_\Gamma = \Omega/\Gamma$ of a bounded symmetric domains Ω by a torsion-free discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$, where X_Γ is a quasi-projective manifold inheriting by descent an S-structure, which is equivalently a VMRT structure.

Taking VMRT structures as an area of research in its own right, it is necessary to examine more examples of interesting sub-VMRT structures. In the Hermitian symmetric case Zhang [Zh14] has now completely classified such pairs (X_0, X) and determined those which are nondegenerate for substructures. In addition, beyond the standard example of the holomorphic conformal structure on Q^n, $n \geq 3$, where germs of complex submanifolds with variable Bochner-Weyl curvature tensors abound, for the other admissible pairs in the Hermitian symmetric case where nondegeneracy for substructures fails, Zhang [Zh14] constructed examples of nonstandard complex submanifolds modeled on (X_0, X) (cf. Theorem 4.4.2 in the current article). At the same time, new cases (not of sub-diagram type) of admissible pairs (X_0, X) which are nondegenerate for substructures have been identified. These admissible pairs in the Hermitian symmetric case, said to be of special type, are not Schubert cycles and the argument of parallel transport fails (cf. Proposition 4.2.1 used in the sub-diagram cases). It will be interesting to classify in general admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1. Moreover, there are many interesting uniruled projective subvarieties such as Schubert cycles on $X = G/P$, for which the theory of sub-VMRT structures apply, and they may provide new sources for the study of the Recognition Problem for sub-VMRT structures and for formulating other geometric problems in the theory.

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