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Holomorphic isometries of $\mathbb{B}^m$ into bounded symmetric domains arising from linear sections of minimal embeddings of their compact duals

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Abstract We study general properties of images of holomorphic isometric embeddings of complex unit balls $\mathbb{B}^m$ into irreducible bounded symmetric domains Ω of rank at least 2. In particular, we show that such holomorphic isometries with the minimal normalizing constant arise from linear sections Λ of the compact dual X_c of Ω. The question naturally arises as to which linear sections $Z = \Lambda \cap \Omega$ are actually images of holomorphic isometries of complex unit balls. We study the latter question in the case of bounded symmetric domains Ω of type IV, alias Lie balls, i.e., bounded symmetric domains dual to hyperquadrics. We completely classify images of all holomorphic isometric embeddings of complex unit balls into such bounded symmetric domains Ω. Especially we show that there exist holomorphic isometric embeddings of complex unit balls of codimension 1 incongruent to the examples constructed by Mok [Mok16] from varieties of minimal rational tangents, and that moreover any holomorphic isometric embedding $f : \mathbb{B}^m \to \Omega$ extends to a holomorphic isometric embedding $f : \mathbb{B}^{n-1} \to \Omega$, $\dim \Omega = n$. The case of Lie balls is particularly relevant because holomorphic isometric embeddings of complex unit balls of sufficiently large dimensions into an irreducible bounded symmetric domain other than a type-IV domain are expected to be more rigid.

Keywords Bergman metrics · holomorphic isometric embeddings · bounded symmetric domains · Borel embedding · complex unit balls

Mathematics Subject Classification (2000) 32M15 · 53C55 · 53C42

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1 Introduction

The study of holomorphic isometries between Kähler manifolds endowed with real-analytic Kähler metrics originated with the works of Bochner and Calabi. Especially, Calabi [Ca53] established results on the existence, uniqueness and analytic continuation of germs of holomorphic isometries into Fubini-Study spaces of finite or countably infinite dimension. Embedding any bounded domain $U \Subset \mathbb{C}^n$ into the countably infinite-dimensional Fubini-Study space $(\mathbb{P}^\infty, ds^2_{FS})$ of constant holomorphic sectional curvature $+2$ by means of an orthonormal basis of the Hilbert space $H^2(U)$ of square-integrable holomorphic functions on U, it follows from [Ca53] that any germ of holomorphic isometry $f : (U_1, \lambda ds^2_{U_1}; x_1) \to (U_2, ds^2_{U_2}; x_2)$ extends to a proper holomorphic isometric embedding $F : (U_1, \lambda ds^2_{U_1}; x_1) \hookrightarrow (U_2, ds^2_{U_2}; x_2)$ provided that the Bergman metrics on the bounded domains U_1 and U_2 are complete. (Here ds^2_U stands for the Bergman metric on U and $\lambda > 0$ is a real constant.) Motivated by questions on holomorphic isometries on bounded symmetric domains in Clozel-Ullmo [CU03] arising from the study of commutants of Hecke correspondences, the second author studied in [Mok12] the general question of algebraic extension of germs $f : (U_1, \lambda ds^2_{U_1}; x_1) \to (U_2, ds^2_{U_2}; x_2)$, and proved that $\mathrm{Graph}(f)$ extends as an affine-algebraic variety provided that the Bergman kernel $K_U(z, w)$ is a rational function in $(z, \overline{w})$ for $U = U_1, U_2$.

Restricting to the study of bounded symmetric domains there arises naturally the question of existence and classification of such maps. In view of Hermitian metric rigidity when $\mathrm{rank}(U_1) \geq 2$, the essential question is to classify holomorphic isometric embeddings up to normalizing constants from a complex unit ball $\mathbb{B}^m$ to a bounded symmetric domain Ω. In [Mok12] examples of nonstandard holomorphic isometries from the Poincaré disk into polydisks and into certain Siegel upper half-planes were constructed, and the first examples of nonstandard holomorphic isometries from $\mathbb{B}^m$, $m \geq 2$, were constructed in [Mok16]. They are holomorphic isometries $F : (\mathbb{B}^{p+1}, g_{\mathbb{B}^{p+1}}) \hookrightarrow (\Omega, g_\Omega)$, where Ω is an arbitrary irreducible bounded symmetric domain of rank ≥ 2, the positive integer $p = p(X_c)$ is defined by $c_1(X_c) = (p+2)\delta$ for the compact dual manifold X_c of Ω and the positive generator δ of $H^2(X_c, \mathbb{Z}) \cong \mathbb{Z}$, and $g_{\mathbb{B}^m}$ resp. g_Ω denotes the normalized Kähler-Einstein metric on $\mathbb{B}^m$, $m \geq 1$, resp. Ω with respect to which minimal disks are of constant Gaussian curvature -2. These examples arise from varieties of minimal rational tangents on X_c and they are at the same time *bona fide* holomorphic isometries $F : (\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}) \hookrightarrow (\Omega, ds^2_\Omega)$ with respect to the Bergman metric.

We consider here primarily holomorphic isometries $F : (\mathbb{B}^m, g_{\mathbb{B}^m}) \hookrightarrow (\Omega, g_\Omega)$ where Ω is irreducible, leaving aside the general case to future works. Essential to the study of such maps is the duality between Ω and X_c in the sense both of algebraic geometry and differential geometry. We prove first of all a principal result that the image $S := F(\mathbb{B}^m)$ is an irreducible component of the intersection of Ω with a linear section of $X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big)$ projectively embedded via the minimal embedding. In the special case of Lie balls, i.e., type-IV domains D^{IV}_n, $n \geq 3$, we give a complete classification of such maps (which yields at the same time the full classification with arbitrary normalizing constants). The case of type-IV domains is especially relevant to uniqueness questions arising from [Mok16]. In fact, when $m = p(\Omega) + 1$, possible isomorphism classes of tangent spaces to S have been identified in Mok-Yang [MY16] by an application of duality and the Gauss equation, leading already to uniqueness theorems for holomorphic isometries of $\mathbb{B}^m$ into Ω for some series of classical irreducible bounded symmetric domains of rank ≥ 2 and for the two exceptional bounded symmetric domains. The latter approach fails completely *precisely* in the case of type-IV domains and our result in the current article shows in particular that there exist in this case nonstandard holomorphic isometries which are incongruent to the examples in [Mok16].

After circulating and posting a first version of the article, it has recently been brought to our attention that there are preprints of Xiao-Yuan [XY] and of Upmeier-Wang-Zhang [UWZ] studying holomorphic isometries of the complex unit ball into bounded symmetric domains. Results of both articles overlap with the second half of the current article. Especially, both articles gave explicit parametrizations of all holomorphic isometric embeddings of complex unit balls of codimension 1 into D^{IV}_n, whereas we give a full classification of images of holomorphic isometric embeddings of complex unit balls of any dimension. While functional equations are made use of in [XY], the article [UWZ] is based on entirely different methods, viz., the theory of Jordan algebras and the study of norm-preserving linear operators on Hilbert spaces arising from holomorphic isometries.

2 General properties of holomorphic isometric embeddings of the complex unit ball into irreducible bounded symmetric domains

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain in its Harish-Chandra realization and X_c be the compact dual Hermitian symmetric space of Ω. Recall that $\mathbb{B}^n \subset \mathbb{C}^n$ is the complex unit ball with respect to the standard complex Euclidean metric on $\mathbb{C}^n$. Throughout this article, given an irreducible bounded symmetric domain D, we denote by g_D the canonical Kähler-Einstein metric on D normalized so that minimal disks are of constant Gaussian curvature -2. Denote by $\mathbf{HI}_\lambda(\mathbb{B}^n, \Omega)$ the space of holomorphic isometries $(\mathbb{B}^n, \lambda g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ for a positive real constant $\lambda > 0$. We will show that λ is an integer satisfying $1 \leq \lambda \leq \mathrm{rank}(\Omega)$. Note that our notation $\mathbf{HI}_\lambda(\mathbb{B}^n, \Omega)$ is in general different from the notation in [Mok11, p. 261], since the background metrics in [Mok11, loc. cit.] are the Bergman metrics, which on both $\mathbb{B}^n$ and Ω are canonical Kähler-Einstein metrics of constant Ricci curvature -1, while we make different choices for the canonical Kähler-Einstein metrics $g_{\mathbb{B}^n}$ and g_Ω in the current article. Note also that in this section for notational convenience the complex unit ball is taken to be of dimension n.

2.1 Preliminaries

Let (X_0, g_0) be the Hermitian symmetric manifold of the noncompact type underlying a bounded symmetric domain Ω. Let G_0 be the identity component of the group of holomorphic isometries of X_0, $K \subset G_0$ be a maximal compact subgroup, and write $\beta : X_0 = G_0/K \hookrightarrow X_c = G_c/K \cong G^{\mathbb{C}}/P$ for the Borel embedding, where (X_c, g_c) is the compact dual of (X_0, g_0), G_c is the group of holomorphic isometries of (X_c, g_c), $K \subset G_c$ is a maximal proper subgroup (being isomorphic to and identified with $K \subset G_0$), $G^{\mathbb{C}}$ stands for the complexification of both G_0 and G_c, and $P \subset G^{\mathbb{C}}$ is the isotropy (parabolic) subgroup at a base point of X_c. Writing $\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+$ for the Harish-Chandra decomposition in standard notation of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}$, there is a biholomorphism $\xi : \mathfrak{m}^+ \cong \mathbb{C}^N \hookrightarrow X_c \equiv G^{\mathbb{C}}/P$ onto a dense open subset of X_c containing the Hermitian symmetric space $X_0 \equiv G_0/K$, and the bounded symmetric domain will from now on be identified with $\xi^{-1}(X_0)$, giving the Harish-Chandra realization $\Omega \Subset \mathbb{C}^N$ (cf. [Mok89, p. 94] or Wolf [Wo72, pp. 278-281]).

Suppose that the bounded symmetric domain Ω is irreducible. Then, we let $f : (D, g_\Omega) \to (\Omega, g_\Omega)$ be a holomorphic isometric embedding, where D is an irreducible bounded symmetric domain. We will look for general properties of the image $f(D)$ in $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ via the minimal embedding $\iota : X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big)$ defined by the positive generator $\mathcal{O}(1)$ of the Picard group $\mathrm{Pic}(X_c) \cong \mathbb{Z}$ of X_c. In the case where D is of rank ≥ 2, Clozel-Ullmo [CU03] noted that it already follows from the proof of Hermitian metric rigidity of Mok [Mok87] that f is necessarily totally geodesic. Therefore, we will focus on the case where $D = \mathbb{B}^n$.

The Borel embedding $\Omega \subset X_c$ identifies the irreducible bounded symmetric domain Ω as an open subset of its dual Hermitian symmetric space of the compact type X_c, which is a Fano manifold of Picard number 1, and the minimal canonical embedding $\iota : X_c \hookrightarrow \mathbb{P}^{N'}$ identifies X_c as a projective submanifold uniruled by projective lines. For uniruled projective manifolds X Hwang-Mok [HM99] introduced the notion of the variety of minimal rational tangents (VMRT) at a general point $x \in X$. In the special case of a projective submanifold $X \subset \mathbb{P}^m$ uniruled by projective lines, as is the case of $X = \iota(X_c) \subset \mathbb{P}^{N'}$, the VMRT $\mathscr{C}_x(X) \subset \mathbb{P}(T_x(X))$ consists of all $[\alpha] \in \mathbb{P}(T_x(X))$ such that, defining $\ell(\alpha)$ to be the unique line on $\mathbb{P}^m$ passing through x and satisfying $T_x(\ell(\alpha)) = \mathbb{C}\alpha$, we have $\ell(\alpha) \subset X$. Moreover, for a general point $[\alpha] \in \mathscr{C}_x(X)$, we have $T(X)|_{\ell(\alpha)} = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for nonnegative integers p, q depending only on X, and we have $\dim \mathscr{C}_x(X) = p$. For $\Omega \subset X_c$ and for a point $x \in \Omega$ we will speak of the VMRT $\mathscr{C}_x(\Omega)$ at $x \in \Omega$ to mean $\mathscr{C}_x(X_c)$. The subvariety $\mathscr{C}_x(\Omega) \subset \mathbb{P}(T_x(\Omega))$ is equivalently the collection of projectivizations of nonzero vectors $\alpha \in T_x(\Omega)$ tangent to minimal disks on Ω passing through the point x.

From now on we denote also by X_c the image of X_c in $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ via the minimal embedding, i.e., we identify X_c with $\iota(X_c)$. From [DL08,

p. 2341], we may assume that the minimal embedding $\iota : X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c,\mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ maps $\mathbb{C}^N$ biholomorphically onto the set $X_c \setminus \{[\xi_0,\ldots,\xi_{N'}]\in\mathbb{P}^{N'} : \xi_0=0\} = X_c\cap U_0 \subset \mathbb{P}^{N'}$, where $U_0 = \left\{[\xi_0,\ldots,\xi_{N'}]\in\mathbb{P}^{N'} : \xi_0\neq 0\right\}$. Let $\omega_{\mathbb{P}^{N'}}$ be the Kähler form on $\mathbb{P}^{N'}$ which corresponds to the normalized Fubini-Study metric $g_{\mathbb{P}^{N'}}$ on $\mathbb{P}^{N'}$ such that $(\mathbb{P}^{N'}, g_{\mathbb{P}^{N'}})$ is of constant holomorphic sectional curvature $+2$. Then, the pullback of the Kähler form $\omega_{\mathbb{P}^{N'}}$ to $\mathbb{C}^{N'+1}\setminus\{\mathbf{0}\}$ is given by $\omega_{\mathbb{P}^{N'}} = \sqrt{-1}\partial\overline{\partial}\log\left(\sum_{j=0}^{N'}|\xi_j|^2\right)$ for $(\xi_0,\ldots,\xi_{N'})\in\mathbb{C}^{N'+1}\setminus\{\mathbf{0}\}$ so that the induced Kähler form ω_{X_c} on X_c is given by the restriction of $\omega_{\mathbb{P}^{N'}}$ to X_c. Therefore, in terms of the Harish-Chandra coordinates $z=(z_1,\ldots,z_N)\in\mathbb{C}^N$, we may assume that ι is written as $\iota(z) = [\sigma_0(z),\ldots,\sigma_{N'}(z)]$, $\sigma_k\in\Gamma(X_c,\mathcal{O}(1))$ for $0\le k\le N'$, such that $\sigma_l(\mathbf{0})=0$ for $1\le l\le N'$ and σ_0 is non-vanishing on $\mathbb{C}^N$. Here, the key point is the existence of a holomorphic section in $\Gamma(X_c,\mathcal{O}(1))$ which does not vanish on the Harish-Chandra coordinate chart $\mathbb{C}^N\supset\Omega$. In particular, replacing σ_l by $G_l := \frac{\sigma_l}{\sigma_0}$ for $1\le l\le N'$, the restriction of the Kähler form ω_{X_c} to $\mathbb{C}^N$ is given by $\omega_{X_c}\big|_{\mathbb{C}^N} = \sqrt{-1}\partial\overline{\partial}\log\left(1+\sum_{l=1}^{N'}|G_l(z)|^2\right)$. On the other hand, the Bergman kernel of Ω can be written as $K_\Omega(z,w) = \frac{1}{(C_\Omega h_\Omega(z,w))^{m_\Omega}}$ (cf. [FK90, p.77]), where $h_\Omega(z,w)$ is a polynomial in $(z,\overline{w})$ satisfying $h_\Omega(z,\mathbf{0})=1$, $h_\Omega(w,z)=\overline{h_\Omega(z,w)}$, and m_Ω is some positive integer depending on Ω. Actually, $C_\Omega^{m_\Omega} = \mathrm{Vol}(\Omega)$ is the Euclidean volume of Ω, $h_\Omega(z,w) = N(z,\overline{w})$ is the generic norm (cf. [Lo77]) and $c_1(X_c) = m_\Omega\cdot\delta$ so that $m_\Omega = p(\Omega)+2$, where $p(\Omega) := p(X_c) = \dim\mathscr{C}_o(X_c)$ is the complex dimension of the VMRT $\mathscr{C}_o(X_c)\subseteq\mathbb{P}(T_o(X_c))$ of X_c at some base point $o\in X_c$ (cf. [Mok16]). More precisely, $K_\Omega(z,w) = \frac{1}{\mathrm{Vol}(\Omega)}h_\Omega(z,w)^{-(p(\Omega)+2)}$. Then, we have the Kähler forms

$$\omega_{g_\Omega} = -\sqrt{-1}\partial\overline{\partial}\log h_\Omega(z,z),\quad \omega_{X_c}\big|_{\mathbb{C}^N} = \sqrt{-1}\partial\overline{\partial}\log h_\Omega(z,-z)$$

(cf. [LM11, p. 1061]) so that the minimal disks (resp. minimal rational curves) are of constant Gaussian curvature -2 (resp. $+2$) with respect to the induced Kähler metric. In other words,

$$\sqrt{-1}\partial\overline{\partial}\log h_\Omega(z,-z) = \sqrt{-1}\partial\overline{\partial}\log\left(1+\sum_{l=1}^{N'}|G_l(z)|^2\right).$$

Lemma 1 *Let $\Omega\Subset\mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. Then, there are Harish-Chandra coordinates $z=(z_1,\ldots,z_N)$ on $\Omega\Subset\mathbb{C}^N$ such that*

$$h_\Omega(z,z) = 1-\sum_{j=1}^N z_j\overline{z_j} + \sum_{l=1}^{N''}(-1)^{\chi_l}\hat{G}_l(z)\overline{\hat{G}_l(z)},$$

where $N''\ge 1$ is some integer, $\hat{G}_l(z)$ is a homogeneous polynomial in z of degree $\deg\hat{G}_l\ge 2$ and $\chi_l\in\{0,1\}$ for $1\le l\le N''$. As a consequence, we have *the polarization*

$$h_\Omega(z,\xi) = 1 - \sum_{j=1}^{N} z_j\overline{\xi_j} + \sum_{l=1}^{N''}(-1)^{\chi_l}\hat{G}_l(z)\overline{\hat{G}_l(\xi)}.$$

Proof Since $\Omega \Subset \mathbb{C}^N$ is a bounded complete circular domain, we can write $h_\Omega(z,\xi) = 1 + \sum_{1\le|I|=|J|\le N_\Omega} b_{I\overline{J}} z^I \overline{\xi}^J$ for some positive integer N_Ω and some $b_{I\overline{J}} \in \mathbb{C}$. Note that the Harish-Chandra coordinates $(z_1,\ldots,z_N)$ can be regarded as complex geodesic coordinates of Ω at $\mathbf{0}$ [Mok89, p. 88]. Hence, up to rescaling of the coordinate system $(z_1,\ldots,z_N)$ we may suppose that the Harish-Chandra realization $\Omega \Subset \mathbb{C}^N$ is chosen so that

$$-m_\Omega \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log h_\Omega(z,z)\Big|_{z=\mathbf{0}} = \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K_\Omega(z,z)\Big|_{z=\mathbf{0}} = m_\Omega \delta_{ij}$$

and thus $b_{i\overline{j}} = -\delta_{ij}$. Note that $\overline{h_\Omega(z,\xi)} = h_\Omega(\xi,z)$, so it follows from the identity

$$2\mathrm{Re}(b_{I\overline{J}} z^I \overline{z}^J) = b_{I\overline{J}} z^I \overline{z}^J + \overline{b_{I\overline{J}}} z^J \overline{z}^I = \left|b_{I\overline{J}} z^I + z^J\right|^2 - \left|b_{I\overline{J}} z^I\right|^2 - \left|z^J\right|^2$$

for multi-indices I, J ($|I| = |J| \ge 2$) that

$$h_\Omega(z,z) = 1 - \sum_{j=1}^{N} z_j\overline{z_j} + \sum_{l=1}^{N''}(-1)^{\chi_l}\hat{G}_l(z)\overline{\hat{G}_l(z)},$$

where $N'' \ge 1$ is an integer, $\hat{G}_l(z)$ is a homogeneous polynomial in z of degree ≥ 2 and $\chi_l \in \{0,1\}$, $1 \le l \le N''$. Here the existence of the homogeneous polynomials $\hat{G}_l(z)$ in z of degree ≥ 2 is due to the fact that $\mathbb{B}^N$ and Ω could not be holomorphically isometric to each other. Obviously, we have analogously the formula for the polarization $h_\Omega(z,\xi)$ of $h_\Omega(z,z)$ as stated, yielding the lemma. □

If we restrict $\omega_{X_c}|_{\mathbb{C}^N}$ to Ω, then from the previous observations, $-\log h_\Omega(z,-z)$ and $-\log(1+\sum_{l=1}^{N'}|G_l(z)|^2)$ differ by the real part of some holomorphic function on Ω. It follows from partial differentiation with respect to $z_1,\ldots,z_N$ and $h_\Omega(z,\mathbf{0}) = 1 = 1 + \sum_{l=1}^{N'} G_l(z)\overline{G_l(\mathbf{0})}$ that $1 + \sum_{l=1}^{N'}|G_l(z)|^2 = h_\Omega(z,-z)$ on Ω and thus on the whole $\mathbb{C}^N$. Then, we have the polarized equation $1 + \sum_{l=1}^{N'} G_l(z)\overline{G_l(w)} = h_\Omega(z,-w)$ so that

$$\omega_{g_\Omega} = -\sqrt{-1}\partial\overline{\partial}\log\left(1 + \sum_{l=1}^{N'} G_l(z)\overline{G_l(-z)}\right).$$

Remark 1 Recall that $h_\Omega(z,\xi) = N(z,\overline{\xi})$ is indeed the generic norm and Loos [Lo77] wrote down the formula of $N(z,\overline{\xi})$ explicitly for each irreducible bounded symmetric domain Ω of rank r by using Jordan triple systems. Then, one can deduce from sections 4 and 7 in [Lo77] that $G_l(z)$ can be chosen to be homogeneous polynomials of z and of degree $\deg(G_l)$ such that $2 \le \deg(G_l) \le r$ for $N+1 \le l \le N'$ and $G_j(z) = z_j$ for $1 \le j \le N$ by using Jordan triple systems, i.e.,

$$h_\Omega(z,\xi) = 1 + \sum_{l=1}^{N'} (-1)^{\deg(G_l)} G_l(z)\overline{G_l(\xi)}.$$

Nevertheless, it is not necessary to use the above explicit form and properties of generic norms in the current article.

The following proposition shows that any mapping in $\mathbf{HI}_r(\mathbb{B}^n, \Omega)$ is totally geodesic whenever $r = \operatorname{rank}(\Omega)$.

Proposition 1 *Let $F : (\mathbb{B}^n, r g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ be a holomorphic isometric embedding, where $\Omega \Subset \mathbb{C}^N$ is an irreducible bounded symmetric domain of rank r and $n \ge 1$ is an integer. Then, F is totally geodesic.*

Proof Write $S := F(\mathbb{B}^n)$. Then, for any $y \in S$ and $\alpha \in T_y^{1,0}(S)$ with $\|\alpha\|_{g_\Omega}^2 = 1$, we have $-\frac{2}{r} = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(S, g_\Omega|_S) \le R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(\Omega, g_\Omega) \le -\frac{2}{r}$ so that $R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(S, g_\Omega|_S) = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(\Omega, g_\Omega) = -\frac{2}{r}$. Denoting by σ the $(1,0)$-part of the second fundamental form of S in (Ω, g_Ω), it follows from the Gauss equation that $\sigma(\alpha,\alpha) = 0$ for any $\alpha \in T_y^{1,0}(S)$ and $y \in S$. Then, for any $\alpha, \beta \in T_y^{1,0}(S)$ and $y \in S$, we have $0 = \sigma(\alpha+\beta, \alpha+\beta) = 2\sigma(\alpha,\beta)$ so that $\sigma \equiv 0$. Hence, F is totally geodesic. □

2.2 System of functional equations induced from holomorphic isometries between bounded symmetric domains

Let $D \Subset \mathbb{C}^n$, $\Omega \Subset \mathbb{C}^N$ be irreducible bounded symmetric domains in their Harish-Chandra realizations such that $\operatorname{rank}(\Omega) \ge 2$. Recall that for a bounded domain U we denote by ds_U^2 the Bergman metric on U. Let $f : (D, \lambda ds_D^2; \mathbf{0}) \to (\Omega, ds_\Omega^2; \mathbf{0})$ be a germ of holomorphic isometry. We can write the Bergman kernels as $K_\Omega(z,\xi) = C_\Omega^{-m_\Omega} h_\Omega(z,\xi)^{-m_\Omega}$, $K_D(w,\zeta) = C_D^{-m_D} h_D(w,\zeta)^{-m_D}$ of Ω, D respectively such that $h_\Omega(z,\xi)$ (resp. $h_D(w,\zeta)$) is a polynomial in $(z,\overline{\xi})$ (resp. $(w,\overline{\zeta})$) satisfying $h_\Omega(\mathbf{0},\mathbf{0}) = 1$ (resp. $h_D(\mathbf{0},\mathbf{0}) = 1$), where m_Ω, m_D are some positive integers depending on Ω, D respectively. From Proposition 1.1.2. in [Mok12], we consider the system of functional equations $K_\Omega(z, f(\zeta)) = A \cdot K_D(w,\zeta)^\lambda$ for $\zeta \in B^n(\mathbf{0},\varepsilon) \Subset D$, where $A := K_\Omega(\mathbf{0},\mathbf{0}) K_D(\mathbf{0},\mathbf{0})^{-\lambda}$ and $\varepsilon > 0$ is some real number. Then, Mok [Mok12] defined

$$V_\zeta^0 := \left\{(w,z) \in D \times \Omega : K_\Omega(z, f(\zeta)) = A \cdot K_D(w,\zeta)^\lambda\right\} \tag{1}$$

and $V^0 := \bigcap_{\zeta \in B^n(\mathbf{0},\varepsilon)} V_\zeta^0$. (Here, V_ζ^0 (resp. V^0) is a notation different from that being used in [Mok12].) Note that λ is a positive rational number (cf. [Mok12, proof of Theorem 1.3.1, p. 1634]). Write $\lambda' := \frac{\lambda m_D}{m_\Omega}$, which is a positive integer (cf. [Mok12, p. 1635]). For each irreducible component Γ of V_ζ^0, by taking fractional powers of both sides of the defining equation of V_ζ^0 in Eq. (1), one gets

$$h_\Omega(z, f(\zeta)) = c_\Gamma \cdot h_D(w,\zeta)^{\lambda'},$$

where c_Γ is some non-zero complex number depending on Γ. For each $\zeta \in B^n(\mathbf{0},\varepsilon)$, we only consider the variety V_ζ, which is the union of all irreducible components of V_ζ^0 containing the point $(\mathbf{0},\mathbf{0}) \in D \times \Omega$. Since $h_\Omega(\mathbf{0},\mathbf{0}) = 1$ and $h_D(\mathbf{0},\mathbf{0}) = 1$, we see that $(\mathbf{0},\mathbf{0}) \in \Gamma$, i.e., $\Gamma \subset V_\zeta$, if and only if $c_\Gamma = 1$. In other words, we have

$$V_\zeta = \left\{(w,z) \in D \times \Omega : h_\Omega(z, f(\zeta)) = h_D(w,\zeta)^{\lambda'}\right\}, \tag{2}$$

which is the union of all irreducible components of V_ζ^0 containing the point $(\mathbf{0},\mathbf{0}) \in D \times \Omega$. Moreover, we define $V := \bigcap_{\zeta \in B^n(\mathbf{0},\varepsilon)} V_\zeta$. In order to study all irreducible components of V^0 containing $(\mathbf{0},\mathbf{0})$, it suffices to consider another system of functional equations $h_\Omega(z, f(\zeta)) = h_D(w,\zeta)^{\lambda'}$ for $\zeta \in B^n(\mathbf{0},\varepsilon) \Subset D$.

Letting $F_\zeta(w,z) := h_\Omega(z, f(\zeta)) - h_D(w,\zeta)^{\lambda'}$, it is obvious that $F_\zeta(\mathbf{0},\mathbf{0}) = 0$ for any $\zeta \in B^n(\mathbf{0},\varepsilon)$ and we have $(w,z) \in V$ if and only if $F_\zeta(w,z) = 0$ for all $\zeta \in B^n(\mathbf{0},\varepsilon)$. For each fixed (w,z), the function $\overline{F_\zeta(w,z)}$ is holomorphic in ζ. Thus, after shrinking $B^n(\mathbf{0},\varepsilon)$ if necessary, we can write the Taylor expansion of $\overline{F_\zeta(w,z)}$ as a holomorphic function of ζ around $\mathbf{0} \in B^n(\mathbf{0},\varepsilon)$. Hence, $(w,z) \in V$ if and only if $\frac{\partial^{|I|}}{\partial \overline{\zeta}^I} F_\zeta(w,z)\big|_{\zeta=\mathbf{0}} = 0$ for any multi-index I satisfying $|I| \geq 1$. In particular, we have

$$V = \left\{(w,z) \in D \times \Omega : \frac{\partial^{|I|}}{\partial \overline{\zeta}^I} F_\zeta(w,z)\bigg|_{\zeta=\mathbf{0}} = 0 \quad \forall\, I,\ |I| \geq 1\right\} \tag{3}$$

by Eq. (2). From [Mok12], the system of functional equations is said to be sufficiently non-degenerate if any irreducible component of V containing $\mathrm{Graph}(f)$ is of dimension $n = \dim \mathrm{Graph}(f)$. Write the positive rational number λ as $\frac{p}{q}$ where p and q are relatively prime positive integers. Let $W^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ be the affine-algebraic subvariety given by

$$W^\sharp := \left\{(w,z) \in \mathbb{C}^n \times \mathbb{C}^N : K_\Omega(z, f(\zeta))^q = A^q \cdot K_D(w,\zeta)^p\right\}. \tag{4}$$

($W^\sharp$ is defined as in the proof of Theorem 1.3.1. in [Mok12], except that the constant A appearing in the definition of $W^\sharp$ there should have been A^q.) Note that for each $\zeta \in B^n(\mathbf{0},\varepsilon)$, $F_\zeta(w,z)$ is defined for any $(w,z) \in \mathbb{C}^n \times \mathbb{C}^N$. Then, the union of all irreducible components of $W^\sharp$ containing $(\mathbf{0},\mathbf{0})$ lies inside the variety W' defined by

$$W' := \left\{(w,z) \in \mathbb{C}^n \times \mathbb{C}^N : \frac{\partial^{|I|}}{\partial \overline{\zeta}^I} F_\zeta(w,z)\bigg|_{\zeta=\mathbf{0}} = 0 \quad \forall\, I,\ |I| \geq 1\right\}. \tag{5}$$

2.3 Properties of the common zero set of a family of extremal functions

Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded symmetric domains in their Harish-Chandra realizations such that Ω is irreducible and of rank ≥ 2. Recall that for a bounded domain U we denote by ds_U^2 the Bergman metric on U. Let $f : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$ be a holomorphic isometric embedding. Assume without loss of generality that $f(\mathbf{0}) = \mathbf{0}$. Suppose that the system of functional equations is not sufficiently non-degenerate. Then, from the proof of Proposition 1.1.2. in [Mok12], there is a complex-analytic one-parameter family $\{f_t\}_{t\in\Delta}$ such that $f_0 = f$ and $K_\Omega(f_t(z), f(w)) = A \cdot K_D(z,w)^\lambda$, where $A := K_\Omega(\mathbf{0},\mathbf{0})K_D(\mathbf{0},\mathbf{0})^{-\lambda}$. Furthermore, under the assumption that $\frac{\partial^k}{\partial t^k} f_t(z)\big|_{t=0} \equiv 0$ for $k < l$, Mok [Mok12] defined $\eta(f(z)) := \frac{\partial^l}{\partial t^l} f_t(z)\big|_{t=0} \not\equiv 0$. Denote by $H^2(\Omega)$ the space of all square-integrable holomorphic functions on Ω. Then, for each $z_0 \in D_\varepsilon := B^n(\mathbf{0},\varepsilon) \Subset D$ such that $\eta(f(z_0)) \neq 0$, Mok [Mok12, Lemma 1.1.2.] showed that an extremal function on Ω is given by

$$h_{\eta(f(z_0))}(\zeta) = \frac{\overline{\partial_{\eta(f(z_0))} K_\Omega(f(z_0),\zeta)} - \overline{(\partial_{\eta(f(z_0))} h_0)} h_0(\zeta)}{\overline{\partial_{\eta(f(z_0))} h_{\eta(f(z_0))}}}, \tag{6}$$

where $h_0 \in H^2(\Omega)$ is chosen so that $|h(f(z_0))|$ attains its maximum value at $h = h_0$ among all $h \in H^2(\Omega)$ of unit L^2-norm (see [Mok89, p. 55]). Here $\partial_{\eta(f(z_0))} h_0$ means $\partial_{\eta(f(z_0))} h_0(f(z_0))$, etc. For the construction of such extremal functions, we refer the reader to [Mok12, pp. 1626-1627]. Recall that $h_\Omega(z,\zeta) = 1 + \sum_{l=1}^{N'} G_l(z)\overline{G_l(-\zeta)}$. Then, we also have $h_\Omega(z,\zeta) = \overline{h_\Omega(\zeta,z)} = 1 + \sum_{l=1}^{N'} \overline{G_l(\zeta)} G_l(-z)$. Define $H_l(z) = G_l(-z)$, we have

$$K_\Omega(z,\zeta) = C_\Omega^{-m_\Omega} \left(1 + \sum_{\mu=1}^{N'} H_\mu(z)\overline{G_\mu(\zeta)}\right)^{-m_\Omega}.$$

Lemma 2 *In the above setting, we have* $\mathrm{Zero}(h_{\eta(f(z_0))}) = \mathrm{Zero}(h'_{\eta(f(z_0))}|_\Omega)$, *where* $h'_{\eta(f(z_0))}$ *is some holomorphic function on* $\mathbb{C}^N$ *which is a* $\mathbb{C}$*-linear combination of* $G_1, \ldots, G_{N'}$*. In particular, for the family of holomorphic functions* h_α, $\alpha \in \mathbf{A}$, *on* Ω *constructed in* [Mok12, Proposition 1.1.2.], *we have* $\mathrm{Zero}(h_\alpha) = \mathrm{Zero}(h'_\alpha|_\Omega)$, *where* h'_α *is a holomorphic function on* $\mathbb{C}^N$ *which is a* $\mathbb{C}$*-linear combination of* $G_1, \ldots, G_{N'}$.

Proof We follow the notation in the proof of Proposition 1.1.2. in [Mok12]. Note that $z_0 \in D$ is chosen so that $\eta(f(z_0)) \neq 0$. Recall that we may write $K_\Omega(z,\zeta) = C_\Omega^{-m_\Omega} h_\Omega(z,\zeta)^{-m_\Omega}$ and $h_\Omega(z,\zeta) = 1 + \sum_{l=1}^{N'} H_l(z)\overline{G_l(\zeta)}$ because Ω is irreducible. We compute

$$\overline{\frac{\partial}{\partial z_j} K_\Omega(z,\zeta)\bigg|_{z=f(z_0)}} = -m_\Omega \frac{K_\Omega(\zeta, f(z_0))}{h_\Omega(\zeta, f(z_0))} \sum_{\mu=1}^{N'} \overline{\frac{\partial H_\mu}{\partial z_j}(f(z_0))} G_\mu(\zeta). \tag{7}$$

Denote by $(\partial_j H_\mu)(f(z_0)) = \frac{\partial H_\mu}{\partial z_j}(f(z_0))$. We write $f_t = (f_t^1, \ldots, f_t^N)$. From [Mok12], we have $h_0(z) = \frac{K_\Omega(z, f(z_0))}{\sqrt{K_\Omega(f(z_0), f(z_0))}}$. Hence, by Eq. (7)

$$\begin{aligned}
&\overline{\partial_{\eta(f(z_0))} h_0(f(z_0))} h_0(\zeta) \\
&= - m_\Omega \sum_{j=1}^N \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\Big|_{t=0}} \frac{\sum_{\mu=1}^{N'} \overline{(\partial_j H_\mu)(f(z_0))} G_\mu(f(z_0))}{h_\Omega(f(z_0), f(z_0))} K_\Omega(\zeta, f(z_0)).
\end{aligned} \tag{8}$$

Therefore, we have

$$\begin{aligned}
&\overline{\partial_{\eta(f(z_0))} K_\Omega(f(z_0), \zeta)} - \overline{(\partial_{\eta(f(z_0))} h_0)} h_0(\zeta) \\
&= - m_\Omega K_\Omega(\zeta, f(z_0)) \\
&\quad \cdot \sum_{j=1}^N \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\Big|_{t=0}} \left(\frac{\sum_{\mu=1}^{N'} \overline{(\partial_j H_\mu)(f(z_0))} G_\mu(\zeta)}{h_\Omega(\zeta, f(z_0))} - \frac{\sum_{\mu=1}^{N'} \overline{(\partial_j H_\mu)(f(z_0))} G_\mu(f(z_0))}{h_\Omega(f(z_0), f(z_0))} \right) \\
&= - m_\Omega \frac{K_\Omega(\zeta, f(z_0))}{h_\Omega(\zeta, f(z_0)) h_\Omega(f(z_0), f(z_0))} \\
&\quad \cdot \sum_{j=1}^N \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\Big|_{t=0}} \left(\sum_{\mu=1}^{N'} \left(A'_{\mu,j}(f(z_0)) - B'_j(f(z_0)) \overline{H_\mu(f(z_0))} \right) G_\mu(\zeta) - B'_j(f(z_0)) \right),
\end{aligned} \tag{9}$$

where $A'_{\mu,j}(f(z_0)) := h_\Omega(f(z_0), f(z_0)) \overline{(\partial_j H_\mu)(f(z_0))}$ and $B'_j(f(z_0)) := \sum_{\mu=1}^{N'} \overline{(\partial_j H_\mu)(f(z_0))} G_\mu(f(z_0))$. From [Mok12], $dh_0(\eta(f(z_0)) = 0$. On the other hand, we compute $dh_0(\eta(f(z_0))) = \frac{\partial_{\eta(f(z_0))} K_\Omega(z, f(z_0))\big|_{z=f(z_0)}}{\sqrt{K_\Omega(f(z_0), f(z_0))}}$ so that

$$\begin{aligned}
\sum_{j=1}^N \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\Big|_{t=0}} B'_j(f(z_0)) &= - \frac{h_\Omega(f(z_0), f(z_0))}{m_\Omega K_\Omega(f(z_0), f(z_0))} \overline{\partial_{\eta(f(z_0))} K_\Omega(z, f(z_0))\Big|_{z=f(z_0)}} \\
&= 0.
\end{aligned} \tag{10}$$

Hence,

$$\begin{aligned}
&\overline{\partial_{\eta(f(z_0))} h_{\eta(f(z_0))}} \cdot h_{\eta(f(z_0))}(\zeta) \\
&= - m_\Omega \frac{K_\Omega(\zeta, f(z_0))}{h_\Omega(\zeta, f(z_0)) h_\Omega(f(z_0), f(z_0))} \cdot \sum_{\mu=1}^{N'} \sum_{j=1}^N \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\Big|_{t=0}} A'_{\mu,j}(f(z_0)) G_\mu(\zeta).
\end{aligned} \tag{11}$$

Let $w \in D$ be sufficiently close to z_0 so that $f(w)$ is defined. From [Mok12], we have $h_{\eta(f(z_0))}(f(w)) = 0$, hence by Eq. (11) we deduce

$$\sum_{\mu=1}^{N'} \sum_{j=1}^N \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\big|_{t=0}} A'_{\mu,j}(f(z_0)) G_\mu(f(w)) = 0. \tag{12}$$

From properties of Bergman kernels on bounded symmetric domains, note that for an arbitrary but fixed point ξ_0 on Ω, the holomorphic function $s_{\xi_0}(\zeta) := K_\Omega(\zeta, \xi_0)$ on Ω has no zeros. This is the case since Ω is a complete circular domain, hence $K_\Omega(\zeta, \mathbf{0})$ is a constant function, and since $\mathrm{Aut}(\Omega)$ acts transitively on Ω. Thus, beyond Eq. (12) it follows in fact from Eq. (11) that the zero set of $h_{\eta(f(z_0))}$ agrees with that of the holomorphic function $h'_{\eta(f(z_0))}|_\Omega$, where

$$\begin{aligned} h'_{\eta(f(z_0))}(\zeta) &:= \sum_{\mu=1}^{N'} \left(\sum_{j=1}^{N} \overline{\frac{\partial^l f_t^j}{\partial t^l}(z_0)\Big|_{t=0}} A'_{\mu,j}(f(z_0)) \right) G_\mu(\zeta) \\ &= h_\Omega(f(z_0), f(z_0)) \cdot \overline{\partial_{\eta(f(z_0))} h_\Omega(z, \zeta)\Big|_{z=f(z_0)}}, \end{aligned} \tag{13}$$

i.e., $\mathrm{Zero}(h_{\eta(f(z_0))}) = \mathrm{Zero}(h'_{\eta(f(z_0))}|_\Omega)$. Moreover $h'_{\eta(f(z_0))}$ is a $\mathbb{C}$-linear combination of the holomorphic functions $G_1, \ldots, G_{N'}$. For the family h_α, $\alpha \in \mathbf{A}$, constructed in [Mok12, Proposition 1.1.2.], each h_α, $\alpha \in \mathbf{A}$, is of the form $h_{\eta(f(z_0))}$ for some $z_0 \in D$ satisfying $\eta(f(z_0)) \neq \mathbf{0}$, so the result follows from the above computations. □

Proposition 2 *If* $\dim_{(w,f(w))}(V \cap (\{w\} \times \Omega)) \geq 1$ *for a general point* $w \in D$, *then there is a non-trivial family of holomorphic functions* h_α, $\alpha \in \mathbf{A}$, *on* Ω *such that* $f(D) \subset E := \bigcap_{\alpha \in \mathbf{A}} \mathrm{Zero}(h_\alpha)$ *and* $\iota(E) = P \cap \iota(\Omega)$ *for some projective linear subspace* P *in* $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$. *Viewing* h'_α *as a holomorphic function on* $\mathbb{C}^N$ *for each* $\alpha \in \mathbf{A}$, $H := \bigcap_{\alpha \in \mathbf{A}} \mathrm{Zero}(h'_\alpha) \subset \mathbb{C}^N$ *is a complex-analytic subvariety satisfying* $H \cap \Omega = E$ *and* $\iota(H) = P \cap \iota(\mathbb{C}^N)$.

Proof From the proof of Proposition 1.1.2. in [Mok12], we have obtained a non-trivial family of holomorphic functions h_α, $\alpha \in \mathbf{A}$, on Ω such that $f(D) \subset E := \bigcap_{\alpha \in \mathbf{A}} \mathrm{Zero}(h_\alpha)$. By Lemma 2, we have $\mathrm{Zero}(h_\alpha) = \mathrm{Zero}(h'_\alpha|_\Omega)$, where $h'_\alpha|_\Omega$ is a holomorphic function on Ω which is a $\mathbb{C}$-linear combination of $G_1, \ldots, G_{N'}$. Actually, we have defined the holomorphic functions h'_α on $\mathbb{C}^N$ because G_1, $\ldots$, $G_{N'}$ are defined on $\mathbb{C}^N$. Recall from Section 2.1 that the restriction of the minimal embedding $\iota : X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ to the dense open subset $\mathbb{C}^N \subset X_c$ may be written as $\iota(z) = [1, G_1(z), \ldots, G_{N'}(z)]$ in terms of the Harish-Chandra coordinates $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$. We write $h'_\alpha(z) = \sum_{j=1}^{N'} a_{\alpha,j} G_j(z)$ for $\alpha \in \mathbf{A}$ and some $a_{\alpha,j} \in \mathbb{C}$. Let

$$P = \left\{ [\xi_0, \xi_1, \ldots, \xi_{N'}] \in \mathbb{P}^{N'} : \sum_{j=1}^{N'} a_{\alpha,j} \xi_j = 0,\ \alpha \in \mathbf{A} \right\}$$

be a projective linear subspace in $\mathbb{P}^{N'}$. Then, it is clear that $H \cap \Omega = E$ and

$$\begin{aligned} P \cap \iota(\mathbb{C}^N) &= \left\{ [1, G_1(z), \ldots, G_{N'}(z)] \in \mathbb{P}^{N'} : \sum_{j=1}^{N'} a_{\alpha,j} G_j(z) = 0,\ z \in \mathbb{C}^N \right\} \\ &= \iota(H). \end{aligned}$$

Similarly, we have $P \cap \iota(\Omega) = \iota(E)$. $\square$

2.4 Holomorphic isometries in $\mathbf{HI}_1(\mathbb{B}^n, \Omega)$ arising from linear sections of the minimal embedding of the compact dual of Ω

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. For $n \geq 1$, the corresponding Kähler form of the Kähler metric $g_{\mathbb{B}^n}$ on $\mathbb{B}^n$ is given by

$$\omega_{g_{\mathbb{B}^n}} = \frac{1}{n+1}\omega_{ds^2_{\mathbb{B}^n}} = -\sqrt{-1}\partial\overline{\partial}\log\left(1-\sum_{j=1}^n |w_j|^2\right).$$

Recall that for a bounded domain U we denote by ds^2_U the Bergman metric on U. One may regard the following lemma as an analogue of Proposition 3.2 in [Ng11] or an assertion made in the proof of Theorem 1.3.1 in [Mok12, pp. 1634-1635].

Lemma 3 *Let $\Omega' = \Omega_1 \times \cdots \times \Omega_m \Subset \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_m} = \mathbb{C}^N$ be a bounded symmetric domain such that for $1 \leq l \leq m$, $m_{\Omega_l} = m'$ for some positive integer m' independent of l, where $m \geq 1$ is an integer and $\Omega_j \Subset \mathbb{C}^{N_j}$ is an irreducible bounded symmetric domain in its Harish-Chandra realization for $1 \leq j \leq m$. Let $f : (\mathbb{B}^n, \lambda ds^2_{\mathbb{B}^n}) \to (\Omega', ds^2_{\Omega'})$ be a holomorphic isometric embedding for some real constant $\lambda > 0$, where $n \geq 1$ is an integer. Then, $\lambda = \frac{km'}{n+1}$ for some positive integer k satisfying $1 \leq k \leq \frac{2}{C\cdot m'}$, where C is a positive real number such that $-C$ is the maximum of all holomorphic sectional curvatures of $(\Omega', ds^2_{\Omega'})$. In particular, if Ω' is irreducible, then we have $1 \leq k \leq \mathrm{rank}(\Omega')$.*

Proof Without loss of generality suppose that $f(\mathbf{0}) = \mathbf{0}$. We write $f = (f_1, \ldots, f_m)$ such that $f_j : \mathbb{B}^n \to \Omega_j$ is a holomorphic map, $1 \leq j \leq m$. Then, we have the polarized functional equation

$$\prod_{j=1}^m h_{\Omega_j}(f_j(w), f_j(\zeta)) = \left(1-\sum_{j=1}^n w_j\overline{\zeta_j}\right)^{\frac{\lambda(n+1)}{m'}}. \tag{14}$$

Note that from [Mok12] and [Mok16], at a general point $b = (b_1, \ldots, b_n) \in \partial\mathbb{B}^n$, there exists an open neighborhood U_b of b in $\mathbb{C}^n$ such that $f|_{U_b\cap\mathbb{B}^n}$ extends to a holomorphic embedding $f^\sharp = (f_1^\sharp, \ldots, f_m^\sharp) : U_b \to \mathbb{C}^N = \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_m}$ with $f^\sharp(U_b \cap \partial\mathbb{B}^n) \subset \partial\Omega'$. By composing with an automorphism of $\mathbb{B}^n$ and restricting to the unit disk $\{(\zeta, 0, \ldots, 0) \in \mathbb{B}^n : \zeta \in \Delta\}$ in $\mathbb{B}^n$, we assume without loss of generality that $b = (\zeta_0, 0, \ldots, 0)$ with $|\zeta_0|^2 = 1$. Now, we have

$$\prod_{j=1}^m h_{\Omega_j}(f_j^\sharp(\zeta, 0, \ldots, 0), f_j^\sharp(\zeta_0, 0, \ldots, 0)) = \left(1-\zeta\overline{\zeta_0}\right)^{\frac{\lambda(n+1)}{m'}} \tag{15}$$

for $\zeta \in U_b \cap \{(\zeta, 0, \ldots, 0) \in \mathbb{C}^n : \zeta \in \overline{\Delta}\}$ by Eq. (14) and continuity. Actually, both sides of Eq. (15) are holomorphic in ζ on some open neighborhood of ζ_0 in $\mathbb{C}$. As $\zeta \to \zeta_0$, the holomorphic function $\varphi(\zeta) := \prod_{j=1}^m h_{\Omega_j}(f_j^\sharp(\zeta, 0, \ldots, 0), f_j^\sharp(\zeta_0, 0, \ldots, 0))$ of ζ on the left-hand side of Eq. (15) vanishes to a certain order k, where k is a positive integer. This shows that $\frac{\lambda(n+1)}{m'}$ is a positive integer and we have $\lambda = \frac{km'}{n+1}$. By Ahlfors-Schwarz lemma (cf. [CCL79]), we have $f^* ds^2_{\Omega'} \le \frac{\frac{2}{n+1}}{C} \cdot ds^2_{\mathbb{B}^n}$. Therefore, we have $\frac{km'}{n+1} \cdot ds^2_{\mathbb{B}^n} = f^* ds^2_{\Omega'} \le \frac{2}{(n+1)C} \cdot ds^2_{\mathbb{B}^n}$, i.e., $k \le \frac{2}{C \cdot m'}$. If Ω' is irreducible, then we have $m' = m_{\Omega'} = p(\Omega') + 2$ and $C = \frac{2}{m' \mathrm{rank}(\Omega')}$ so that $k \le \mathrm{rank}(\Omega')$. The result follows. □

Lemma 4 *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. If $f : (\mathbb{B}^n, \lambda ds^2_{\mathbb{B}^n}; \mathbf{0}) \to (\Omega, ds^2_\Omega; \mathbf{0})$ is a germ of holomorphic isometry for some real constant $\lambda > 0$, then we have $\sum_{j=1}^N \overline{\frac{\partial f^j}{\partial w_\mu}(\mathbf{0})} f^j(\zeta) = \frac{\lambda(n+1)}{m_\Omega} \zeta_\mu$ for $1 \le \mu \le n$. Moreover,*

$$V_1' := \left\{ (\zeta, \xi) \in \mathbb{C}^n \times \mathbb{C}^N : \sum_{j=1}^N \overline{\frac{\partial f^j}{\partial w_\mu}(\mathbf{0})} \xi_j = \frac{\lambda(n+1)}{m_\Omega} \zeta_\mu, \;\; 1 \le \mu \le n \right\}$$

is a complex N-dimensional vector subspace of $\mathbb{C}^{n+N}$ such that $\mathrm{Graph}(f) \subset V \subseteq V_1'$. *In particular, we can write $V = V_1' \cap V_2'$, where*

$$V_2' := \left\{ (w, z) \in \mathbb{B}^n \times \Omega : \left. \frac{\partial^{|I|}}{\partial \overline{\zeta}^I} F_\zeta(w, z) \right|_{\zeta = \mathbf{0}} = 0 \quad \forall\, I,\; |I| \ge 2 \right\}.$$

Proof We have the polarized functional equation

$$h_\Omega(f(w), f(\zeta)) = \left(1 - \sum_{j=1}^n w_j \overline{\zeta_j}\right)^{\frac{\lambda(n+1)}{m_\Omega}} \tag{16}$$

Moreover, Lemma 1 asserts that $h_\Omega(z, \xi) = 1 - \sum_{j=1}^N z_j \overline{\xi_j} + \sum_{l=1}^{N''} (-1)^{\chi_l} \hat{G}_l(z)\, \overline{\hat{G}_l(\xi)}$, where $\hat{G}_l(z)$ is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree ≥ 2 for $1 \le l \le N''$. Therefore, differentiating both sides of Eq. (16) with respect to w_μ and evaluating at $w = \mathbf{0}$ gives $\sum_{j=1}^N \overline{\frac{\partial f^j}{\partial w_\mu}(\mathbf{0})} f^j(\zeta) = \lambda' \zeta_\mu$ for $1 \le \mu \le n$, where $\lambda' := \frac{\lambda(n+1)}{m_\Omega}$. Letting V_1' be as defined in the statement of the lemma, it is clear that $\mathrm{Graph}(f) \subset V_1'$. Moreover, the Jacobian matrix of V_1' at each point in V_1' is of full rank n, so $V_1' \subset \mathbb{C}^{n+N}$ is actually a complex N-dimensional vector subspace. The rest follows from the previous observations. □

The following characterizes the image of any holomorphic isometric embedding $f : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$, where $n \ge 1$. In short, the embedded image of $f(\mathbb{B}^n)$ in $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big)$ via the embedding ι is an irreducible component of the intersection of $\iota(\Omega)$ with the linear section $P \cap X_c$ for some projective linear subspace $P \subseteq \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big)$.

Theorem 1 *Let $\Omega \Subset \mathbb{C}^N \subset X_c$ be the standard embeddings of an irreducible bounded symmetric domain Ω of rank ≥ 2 in its Harish-Chandra realization $\Omega \Subset \mathbb{C}^N$ as a bounded domain and its Borel embedding $\Omega \subset X_c$ as an open subset of its dual Hermitian symmetric space X_c. Let n be a positive integer, and $f : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ be a holomorphic isometric embedding. Denote by $\iota : X_c \to \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ the minimal embedding of X_c defined by the positive generator $\mathcal{O}(1)$ of $\mathrm{Pic}(X_c) \cong \mathbb{Z}$. Then, $f(\mathbb{B}^n)$ is an irreducible component of some complex-analytic subvariety $\mathscr{V} \subseteq \Omega$ satisfying $\iota(\mathscr{V}) = P \cap \iota(\Omega)$, where P is some projective linear subspace of $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$.*

Proof The hypothesis on the mapping f is equivalently that $f : (\mathbb{B}^n, \lambda ds^2_{\mathbb{B}^n}) \to (\Omega, ds^2_\Omega)$ is a holomorphic isometry for $\lambda = \frac{m_\Omega}{n+1}$. We can suppose that $f(\mathbf{0}) = \mathbf{0}$ after composing with some $\Psi \in \mathrm{Aut}(\Omega)$. In the notation of Lemma 4 we define

$$V_1'' := V_1' \cap (\mathbb{B}^n \times \Omega) = \left\{ (\zeta, z) \in \mathbb{B}^n \times \Omega : \sum_{l=1}^{N} \overline{\frac{\partial f^l}{\partial w_j}(\mathbf{0})} z_l - \zeta_j = 0,\ 1 \leq j \leq n \right\},$$

which is a smooth affine linear section of $\mathbb{B}^n \times \Omega$ in $\mathbb{C}^{n+N} \cong \mathbb{C}^n \times \mathbb{C}^N$. From Lemma 4, one can compute $V_2' = \mathbb{B}^n \times V''$, where

$$V'' := \left\{ z \in \Omega : \frac{\partial^I}{\partial \overline{\zeta}^I} h_\Omega(z, f(\zeta)) \Big|_{\zeta = \mathbf{0}} = 0\ \forall\, I,\ |I| \geq 2 \right\}.$$

Therefore, $V = V_1'' \cap (\mathbb{B}^n \times V'')$.

If the system of functional equations is sufficiently non-degenerate, then any irreducible component S of $V = V_1'' \cap (\mathbb{B}^n \times V'')$ containing $\mathrm{Graph}(f)$ is of dimension $n = \dim \mathrm{Graph}(f)$ [Mok12, p. 1622]. Let S'' be an irreducible component of V'' containing $f(\mathbb{B}^n)$ and $S_0 \subset V_1'' \cap (\mathbb{B}^n \times S'')$ be an irreducible component of $V_1'' \cap (\mathbb{B}^n \times S'')$ containing $\mathrm{Graph}(f)$. Then, we have $n = \dim S \geq \dim S_0 \geq N + (n + \dim S'') - (n + N) = \dim S''$ so that $\dim S'' = n$ because $f(\mathbb{B}^n) \subset S''$. If the system of functional equations is not sufficiently non-degenerate, then we may take $E = \bigcap_{\alpha \in \mathbf{A}} \mathrm{Zero}(h_\alpha)$ for some family $\{h_\alpha\}_{\alpha \in \mathbf{A}}$ of extremal functions on Ω so that any irreducible component of $V \cap (\mathbb{B}^n \times E) = V_1'' \cap (\mathbb{B}^n \times (V'' \cap E))$ containing $\mathrm{Graph}(f)$ is of dimension $n = \dim \mathrm{Graph}(f)$ (by Proposition 1.1.2. and Theorem 1.1.1. in [Mok12]). By the same arguments as above, the irreducible component S' of $V'' \cap E$ containing $f(\mathbb{B}^n)$ is of dimension n.

Since by construction both V'' and E are complex-analytic subvarieties in $\mathbb{C}^N$, $V'' \cap E \subset \mathbb{C}^N$ is also a complex-analytic subvariety. Moreover, from the expression of $h_\Omega(z, f(\zeta)) = 1 + \sum_{l=1}^{N'} G_l(z) \overline{G_l(-f(\zeta))}$ and Proposition 2, it is clear that V'' (resp. E) can be viewed as the intersection of $\iota(\Omega)$ with the linear section $X_c \cap P_1$ (resp. $X_c \cap P_2$) for some projective linear subspace P_1 (resp. P_2) of $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$. Hence, $f(\mathbb{B}^n)$ lies inside an irreducible component S of $\mathscr{V} \cap \Omega$, where $\dim S = n$, $\mathscr{V} = V''$ (resp. $\mathscr{V} = V'' \cap E$) when the functional equation is sufficiently non-degenerate (resp. not sufficiently non-degenerate) such that $\iota(\mathscr{V}) = P \cap \iota(\Omega)$ for some projective linear subspace

$P \subseteq \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$. Since $f(\mathbb{B}^n)$ and S are both irreducible complex-analytic subvarieties of Ω having the same dimension n and $f(\mathbb{B}^n) \subset S$, we conclude that $f(\mathbb{B}^n) = S$ and the result follows. □

3 Holomorphic isometries of $\mathbb{B}^m$ into type-IV domains

In the previous section, we have shown that the image of a holomorphic isometric embedding in $\mathbf{HI}_1(\mathbb{B}^m, \Omega)$ is an irreducible component of the intersection of Ω with a linear section of the minimal embedding of the compact dual X_c of Ω (cf. Theorem 1). For the particular case where $\Omega = D_n^{IV} \Subset \mathbb{C}^n$ ($n \geq 3$) is an irreducible bounded symmetric domain of type IV, we completely characterize linear sections of $Q^n \subset \mathbb{P}^{n+1}$ intersecting with $D_n^{IV} \Subset \mathbb{C}^n \subset Q^n$ which correspond to holomorphic isometries $F \in \mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ for some $m \geq 1$. As a consequence, we have a complete classification of images of maps in $\mathbf{HI}_{\lambda'}(\mathbb{B}^m, D_n^{IV})$ for integers $n \geq 3$ and $m \geq 1$ and for any $\lambda' > 0$. Among these, we will provide explicit examples of nonstandard holomorphic isometric embeddings $(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ which are incongruent to those holomorphic isometric embeddings constructed in [Mok16].

3.1 Preliminaries

Recall that for a bounded domain U we denote by ds_U^2 the Bergman metric on U. Let m, n be integers satisfying $m \geq 1$ and $n \geq 3$. The irreducible bounded symmetric domain in $\mathbb{C}^n$ of type IV can be written as

$$D_n^{IV} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 2,\ \sum_{j=1}^n |z_j|^2 < 1 + \left| \frac{1}{2} \sum_{j=1}^n z_j^2 \right|^2 \right\}$$

and the Kähler form corresponding to the Bergman metric $ds^2_{D_n^{IV}}$ on D_n^{IV} is given by $\omega_{ds^2_{D_n^{IV}}} = -n\sqrt{-1}\partial\overline{\partial} \log \left(1 - \sum_{j=1}^n |z_j|^2 + \left| \frac{1}{2} \sum_{j=1}^n z_j^2 \right|^2 \right)$. Moreover, the corresponding Kähler form of the Kähler metric $g_{D_n^{IV}}$ on D_n^{IV} is given by $\omega_{g_{D_n^{IV}}} = \frac{1}{n} \omega_{ds^2_{D_n^{IV}}}$. Let $F : (\mathbb{B}^m, \lambda' g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ be a holomorphic isometric embedding. In this section, we suppose that all component functions of F are non-constant; otherwise, we could reduce the study to holomorphic isometries $F' \in \mathbf{HI}_{\lambda'}(\mathbb{B}^m, D_{n'}^{IV})$ for some $n' \leq n-1$. From Lemma 3, if $F : (\mathbb{B}^m, \lambda ds^2_{\mathbb{B}^m}) \to (D_n^{IV}, ds^2_{D_n^{IV}})$ is a holomorphic isometric embedding, where $\lambda > 0$ is a positive real number, then $\lambda' := \lambda(m+1)/n$ is a positive integer and $1 \leq \lambda' \leq 2$. In particular, $F \in \mathbf{HI}_{\lambda'}(\mathbb{B}^m, D_n^{IV})$. On the other hand, the following shows that $\mathbf{HI}_2(\mathbb{B}^m, D_n^{IV})$ is empty for $m \geq 2$ and $n \geq 3$.

Proposition 3 *Let $F : (\mathbb{B}^m, \lambda' g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ be a holomorphic isometric embedding, where $m \geq 2$ and $n \geq 3$. Then, $\lambda' = 1$.*

Proof Write $S := F(\mathbb{B}^m)$. Note that $\mathscr{C}_y(Q^n) \cong Q^{n-2}$ is a hyperquadric in $\mathbb{P}(T_y(D_n^{IV})) \cong \mathbb{P}^{n-1}$ for any $y \in D_n^{IV}$, where $\mathscr{C}_x(Q^n)$ is the VMRT of Q^n at $x \in Q^n$ (see Section 2.1). For $m \geq 2$, $\mathbb{P}(T_y(S))$ is a projective linear subspace in $\mathbb{P}(T_y(D_n^{IV})) \cong \mathbb{P}^{n-1}$ of complex dimension $m-1 \geq 1$. Therefore, $\mathbb{P}(T_y(S)) \cap \mathscr{C}_y(Q^n) \neq \varnothing$ for any $y \in S$. Let $[\alpha] \in \mathbb{P}(T_y(S)) \cap \mathscr{C}_y(Q^n)$ with $\|\alpha\|^2_{g_{D_n^{IV}}} = 1$. Then, it follows from the Gauss equation that $-\frac{2}{\lambda'} = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(S, g_{D_n^{IV}}|_S) \leq R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(D_n^{IV}, g_{D_n^{IV}}) = -2$ so that $\lambda' \leq 1$ and thus $\lambda' = 1$. □

The following corollary is a direct consequence of Proposition 3 and Proposition 1.

Corollary 1 *Let $F : (\mathbb{B}^m, 2g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ be a holomorphic isometric embedding. Then, $m = 1$ and F is totally geodesic.*

Thus, among $\mathbf{HI}_\lambda(\mathbb{B}^m, D_n^{IV})$, where $m \geq 1$ and $n \geq 3$, it remains to consider $\mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ with $1 \leq m \leq n-1$ and $n \geq 3$. Note that if $(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ is a totally geodesic holomorphic isometric embedding, then its image is clearly the intersection of D_n^{IV} with some projective linear subspace $\mathbb{P}^m \cong P \subset \mathbb{P}^{n+1}$ which lies inside Q^n entirely. The latter is the case since totally geodesic embeddings $(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ extend to totally geodesic embeddings $(\mathbb{P}^m, g_{\mathbb{P}^m}) \to (Q^n, g_{Q^n})$.

3.2 Characterization of images of holomorphic isometries of complex unit balls into type-IV domains

3.2.1 Basic settings

For $1 \leq m \leq n-1$ and $n \geq 3$, let $\mathbf{A}'' \in M(n-m, n; \mathbb{C})$ be a matrix of rank $n-m$. If $1 \leq m \leq n-2$, we define

$$V_{\mathbf{A}''} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \mathbf{A}'' \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sum_{j=1}^n z_j^2 \\ \mathbf{0} \end{pmatrix} \right\}.$$

If $m = n-1$, then $\mathbf{A}'' = \mathbf{v} = \big(v_1, \cdots, v_n\big) \in M(1, n; \mathbb{C})$ and we let $V_{\mathbf{v}} \subseteq \mathbb{C}^n$ be the affine-algebraic subvariety defined by $\sum_{j=1}^n v_j z_j - \frac{1}{2}\sum_{j=1}^n z_j^2 = 0$. Moreover, we define $\Sigma_{\mathbf{A}''} := V_{\mathbf{A}''} \cap D_n^{IV}$.

We write $Q^n = \big\{[z_1, \ldots, z_{n+2}] \in \mathbb{P}^{n+1} : \sum_{j=1}^n z_j^2 - 2z_{n+1}z_{n+2} = 0\big\}$. Let $\iota : D_n^{IV} \subset \mathbb{C}^n \to Q^n \subset \mathbb{P}^{n+1}$ be the Borel embedding defined by $\iota(z) = \Big[z_1, \ldots, z_n, 1, \frac{1}{2}\sum_{j=1}^n z_j^2\Big]$, where $n \geq 3$. Then, we have $\iota(\Sigma_{\mathbf{A}''}) = (P_{\mathbf{A}''} \cap Q^n) \cap \iota(D_n^{IV})$, where $P_{\mathbf{A}''} \subseteq \mathbb{P}^{n+1}$ is the projective linear subspace defined by $\widetilde{\mathbf{A}}'' \big(z_1 \cdots z_{n+2}\big)^T = \mathbf{0}$ and $\widetilde{\mathbf{A}}''$ is given by $\begin{bmatrix} \mathbf{A}'' \begin{bmatrix} 0 & -1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{bmatrix}$ (resp. $\big[\mathbf{A}''\ 0\ {-1}\big]$) if $1 \leq m \leq n-2$ (resp. $m = n-1$). By computing the (complex) Jacobian matrix of the projective subvariety $P_{\mathbf{A}''} \cap Q^n \subset \mathbb{P}^{n+1}$ at $o = [0, \ldots, 0, 1, 0]$, $P_{\mathbf{A}''} \cap Q^n$ is smooth at o. From linear algebra, we have the following lemma.

Lemma 5 *Let n', m' be integers such that $1 \le m' < n'$ and $\mathbf{A}'' \in M(m', n'; \mathbb{C})$ be a matrix such that $\mathbf{A}''\overline{\mathbf{A}''}^T = \mathbf{I_{m'}}$. Then, there exists a matrix $\mathbf{U}' \in M(n'-m', n'; \mathbb{C})$ such that $\begin{bmatrix}\mathbf{U}' \\ \mathbf{A}''\end{bmatrix} \in U(n')$, where $U(n')$ is the group of $n' \times n'$ unitary matrices.*

Proposition 4 *Under the above assumptions, for any matrix $\mathbf{A}' \in M(n-m, n; \mathbb{C})$ satisfying $\mathbf{A}'\overline{\mathbf{A}'}^T = \mathbf{I_{n-m}}$, the irreducible component of $\Sigma_{\mathbf{A}'}$ containing $\mathbf{0}$ is smooth and actually $\Sigma_{\mathbf{A}'} \subset D_n^{IV}$ is a smooth complex-analytic subvariety of dimension m.*

Proof Let $\mathbf{A}' \in M(n-m, n; \mathbb{C})$ be a matrix such that $\mathbf{A}'\overline{\mathbf{A}'}^T = \mathbf{I_{n-m}}$. We write $\mathbf{A}' = \begin{bmatrix}\mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{n-m}\end{bmatrix}$, where $\mathbf{v}_j \in M(1, n; \mathbb{C})$ is a row vector for $1 \le j \le n-m$. Then, $\mathbf{v}_j\overline{\mathbf{v}_i}^T = \delta_{ij}$ for $1 \le i, j \le n-m$. It suffices to show that the (complex) Jacobian matrix of $\Sigma_{\mathbf{A}'}$ is of full rank $n-m$ at any point $z_0 \in \Sigma_{\mathbf{A}'}$. Assume the contrary that the Jacobian matrix of $\Sigma_{\mathbf{A}'}$ at some point $(z_1^0, \ldots, z_n^0) \in \Sigma_{\mathbf{A}'} \subset D_n^{IV}$ is not of full rank $n-m$. Note that the Jacobian matrix of $\Sigma_{\mathbf{A}'}$ at $(z_1^0, \ldots, z_n^0)$ equals $\mathbf{A}' - \begin{bmatrix} z_1^0 & \cdots & z_n^0 \\ \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}$, which is of rank at least $n-m-1$. Then, $\mathbf{A}' - \begin{bmatrix} z_1^0 & \cdots & z_n^0 \\ \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}$ is of rank $n-m-1 < n-m$ by the assumption. If $n-m = 1$, then we have $\mathbf{A}' = \begin{pmatrix} z_1^0 & \cdots & z_n^0\end{pmatrix}$ so that $\sum_{j=1}^n (z_j^0)^2 = 0$. In particular, $1 - \sum_{j=1}^n |z_j^0|^2 + \left|\frac{1}{2}\sum_{j=1}^n (z_j^0)^2\right|^2 = 0$ and $\sum_{j=1}^n |z_j^0|^2 = 1$, which contradicts with the fact that $(z_1^0, \ldots, z_n^0) \in \Sigma_{\mathbf{A}'} \subset D_n^{IV}$. Supposing $n-m \ge 2$, we have $\mathbf{v}_1 - \sum_{j=2}^{n-m} c_j\mathbf{v}_j = \left(z_1^0, \cdots, z_n^0\right)$ for some $c_j \in \mathbb{C}$, $2 \le j \le n-m$. Since $(z_1^0, \ldots, z_n^0) \in \Sigma_{\mathbf{A}'}$, we have

$$\sum_{j=1}^n (z_j^0)^2 = \mathbf{v}_1 \begin{pmatrix} z_1^0 \\ \vdots \\ z_n^0 \end{pmatrix} - \sum_{j=2}^{n-m} c_j\mathbf{v}_j \begin{pmatrix} z_1^0 \\ \vdots \\ z_n^0 \end{pmatrix} = \frac{1}{2}\sum_{j=1}^n (z_j^0)^2$$

so that $\sum_{j=1}^n (z_j^0)^2 = 0$. Moreover, we have

$$\sum_{j=1}^n |z_j^0|^2 = \left(\mathbf{v}_1 - \sum_{j=2}^{n-m} c_j\mathbf{v}_j\right)\left(\overline{\mathbf{v}_1}^T - \sum_{j=2}^{n-m} \overline{c_j\mathbf{v}_j}^T\right) = 1 + \sum_{j=2}^{n-1} |c_j|^2.$$

But then $1 - \sum_{j=1}^n |z_j^0|^2 + \left|\sum_{j=1}^n (z_j^0)^2\right|^2 = -\sum_{j=2}^{n-1} |c_j|^2 \le 0$, a plain contradiction. Thus, $\Sigma_{\mathbf{A}'}$ is smooth and of dimension m. Let $V \subseteq \Sigma_{\mathbf{A}'}$ be an irreducible component containing $\mathbf{0}$. Then, $V \subset \Sigma_{\mathbf{A}'}$ is the connected component of $\Sigma_{\mathbf{A}'}$ containing $\mathbf{0}$ and V is smooth. □

3.2.2 Basic results

In what follows we are going to show that the functional equation for a holomorphic isometry $F : (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ provides sufficient defining equations for an affine-algebraic subvariety V in $\mathbb{C}^n$ which extends $F(\mathbb{B}^m)$.

Proposition 5 *Let $F : (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ be a holomorphic isometric embedding, where $n \geq 3$, $m \geq 1$. Then, $\Psi(F(\mathbb{B}^m))$ is the irreducible component of the complex-analytic subvariety $\Sigma_{\mathbf{A'}}$ containing $\mathbf{0}$ and $\iota(\Sigma_{\mathbf{A'}}) = P_{\mathbf{A'}} \cap \iota(D_n^{IV})$ for some matrix $\mathbf{A'} \in M(n-m, n; \mathbb{C})$ satisfying $\mathbf{A'}\overline{\mathbf{A'}}^T = \mathbf{I_{n-m}}$ and some $\Psi \in \mathrm{Aut}(D_n^{IV})$ satisfying $\Psi(F(\mathbf{0})) = \mathbf{0}$.*

Proof Since $\mathbb{B}^n$ and D_n^{IV} are not biholomorphic to each other, we have $m \leq n-1$. Let $\Psi \in \mathrm{Aut}(D_n^{IV})$ be such that $\Psi(F(\mathbf{0})) = \mathbf{0}$. Write $\widetilde{F} := \Psi \circ F$. Then, we have the functional equation $1 - \sum_{j=1}^n |\widetilde{F}^j(w)|^2 + \left|\frac{1}{2}\sum_{j=1}^n (\widetilde{F}^j(w))^2\right|^2 = 1 - \sum_{l=1}^m |w_l|^2$. By a well-known result of Calabi [Ca53, Theorem 2 (Local Rigidity)], there exists $\mathbf{U} \in U(n)$ such that

$$\mathbf{U}\begin{pmatrix}\widetilde{F}^1(w)\\ \vdots \\ \widetilde{F}^n(w)\end{pmatrix} = \begin{cases}\left(w_1, \cdots, w_m, \frac{1}{2}\sum_{j=1}^n (\widetilde{F}^j(w))^2, \mathbf{0}\right)^T & \text{if } 1 \leq m \leq n-2 \\ \\ \left(w_1, \cdots, w_{n-1}, \frac{1}{2}\sum_{j=1}^n (\widetilde{F}^j(w))^2\right)^T & \text{if } m = n-1\end{cases} \tag{17}$$

for any $w = (w_1, \ldots, w_m) \in \mathbb{B}^m$. Writing $\mathbf{U} = \left(u_{ij}\right)_{1 \leq i,j \leq n}$, we have $w_l = \sum_{j=1}^n u_{lj}\widetilde{F}^j(w)$ for $1 \leq l \leq m$. We write $\mathbf{U} = \begin{bmatrix}\mathbf{U'}\\ \mathbf{A'}\end{bmatrix}$, where $\mathbf{U'} \in M(m, n; \mathbb{C})$ and $\mathbf{A'} \in M(n-m, n; \mathbb{C})$ are matrices. Then, it follows from Eq. (17) that $S := \widetilde{F}(\mathbb{B}^m) \subseteq \Sigma_{\mathbf{A'}}$. It is clear that $\iota(\Sigma_{\mathbf{A'}}) = P_{\mathbf{A'}} \cap \iota(D_n^{IV}) \subset P_{\mathbf{A'}} \cap Q^n$. Since $\Sigma_{\mathbf{A'}}$ is smooth by Proposition 4, letting S' be the irreducible component of $\Sigma_{\mathbf{A'}}$ containing S, we have $\dim S' = m$ and S' is a connected complex submanifold of D_n^{IV}. Since $S \subseteq S'$ are irreducible complex-analytic subvarieties of D_n^{IV} and $\dim S' = m = \dim S$, we have $S' = S = \widetilde{F}(\mathbb{B}^m)$. □

Proposition 6 *Let n, m be integers satisfying $1 \leq m \leq n-1$ and $n \geq 3$. Let $\mathbf{A'} \in M(n-m, n; \mathbb{C})$ be a matrix satisfying $\mathbf{A'}\overline{\mathbf{A'}}^T = \mathbf{I_{n-m}}$. Then, the irreducible component $\widetilde{W}$ of $\Sigma_{\mathbf{A'}}$ containing $\mathbf{0}$ is the image of some holomorphic isometric embedding $F : (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$.*

Proof Let $\widetilde{W}$ be the irreducible component of $\Sigma_{\mathbf{A'}}$ containing $\mathbf{0}$ for some matrix $\mathbf{A'} \in M(n-m, n; \mathbb{C})$ satisfying $\mathbf{A'}\overline{\mathbf{A'}}^T = \mathbf{I_{n-m}}$. Then, there exists a matrix $\mathbf{U'} \in M(m, n; \mathbb{C})$ such that $\begin{bmatrix}\mathbf{U'}\\ \mathbf{A'}\end{bmatrix} \in U(n)$ by Lemma 5. From Lemma 4, $\widetilde{W}$ is a connected complex m-dimensional submanifold of D_n^{IV} and $\mathbf{0} \in \widetilde{W}$. Writing $\left(\widetilde{G}_1(z), \cdots, \widetilde{G}_m(z)\right)^T = \mathbf{U'}\left(z_1, \cdots, z_n\right)^T$, we have $\sum_{l=1}^m |\widetilde{G}_l(z)|^2 +$

$\left|\frac{1}{2}\sum_{j=1}^n z_j^2\right|^2 = \sum_{j=1}^n |z_j|^2$ for $(z_1,\ldots,z_n)\in\widetilde{W}$. This implies that for $(z_1,\ldots,z_n)$ $\in\widetilde{W}$, we have

$$1-\sum_{j=1}^n |z_j|^2 + \left|\frac{1}{2}\sum_{j=1}^n z_j^2\right|^2 = 1-\sum_{l=1}^m |\widetilde{G}_l(z)|^2. \tag{18}$$

Then, it follows from Eq. (18) that $-\log(1-\sum_{l=1}^m |\widetilde{G}_l(z)|^2)$ is a local Kähler potential of $(\widetilde{W}, g_{D_n^{IV}}|_{\widetilde{W}})$ for $z\in\widetilde{W}\subset D_n^{IV}$, which is the restriction of the Kähler potential of $(D_n^{IV}, g_{D_n^{IV}})$ to a neighborhood of $\mathbf{0}$ in $\widetilde{W}$. Hence, $(\widetilde{W}, g_{D_n^{IV}}|_{\widetilde{W}})$ is locally holomorphically isometric to $(\mathbb{B}^m, g_{\mathbb{B}^m})$. By Theorem 2.1.2. in [Mok12], $\widetilde{W}\subset D_n^{IV}$ is the image of some holomorphic isometric embedding $F:(\mathbb{B}^m, g_{\mathbb{B}^m})\to(D_n^{IV}, g_{D_n^{IV}})$. □

Remark 2 The map $(\widetilde{G}_1,\ldots,\widetilde{G}_m)$ gives a holomorphic isometry from $\widetilde{W}$ onto $\mathbb{B}^m$. Inverting this map, we can get an explicit formula for the holomorphic isometry $F:(\mathbb{B}^m, g_{\mathbb{B}^m})\to(D_n^{IV}, g_{D_n^{IV}})$, but we are not concerned with such an explicit formula in this article.

Note that the following proposition characterizes those holomorphic isometries $F\in\mathbf{HI}_1(\mathbb{B}^{n-1}, D_n^{IV})$ $(n\ge 3)$ which are congruent to nonstandard holomorphic isometries constructed in [Mok16].

Proposition 7 *Let $n\ge 3$ be an integer and $\widetilde{W}$ be the irreducible component of $\Sigma_{\mathbf{A}'}$ containing $\mathbf{0}$, where $\mathbf{A}'=\big(c_1,\cdots,c_n\big)\in M(1,n;\mathbb{C})$ is a row vector such that $\mathbf{A}'\overline{\mathbf{A}'}^T=1$. Then, $\widetilde{W}=V_q$ for some $q\in E_1=\mathrm{Reg}(\partial D_n^{IV})$ if and only if $\mathbf{A}'\cdot(\mathbf{A}')^T=\mathbf{0}$, where V_q is defined in* [Mok16].

Proof For $q=(q_1,\ldots,q_n)\in E_1=\mathrm{Reg}(\partial D_n^{IV})$, Mok [Mok16] defined the subvariety $V_q=\mathcal{V}_q\cap D_n^{IV}\subset D_n^{IV}$, where

$$\mathcal{V}_q := \bigcup\{\ell : \ell \text{ is a minimal rational curve on } Q^n \text{ through } q\}$$

(cf. [Mok16, p. 4518]). Suppose that $q\in E_1$ is chosen so that $\mathbf{0}\in V_q$. Then, we have $V_q=V_q'\cap D_n^{IV}$, where $V_q':=\mathcal{V}_q\cap\mathbb{C}^n=\big\{(z_1,\ldots,z_n)\in\mathbb{C}^n:\sum_{j=1}^n q_jz_j-\frac{1}{2}\sum_{j=1}^n z_j^2=0\big\}$. If $\mathbf{A}'\cdot(\mathbf{A}')^T=\mathbf{0}$, then $1-\sum_{j=1}^n|c_j|^2+\left|\frac{1}{2}\sum_{j=1}^n c_j^2\right|^2=0$ and $\sum_{j=1}^n|c_j|^2=1$ so that $c:=(c_1,\ldots,c_n)\in E_1$. Thus, we have $\widetilde{W}=\Sigma_{\mathbf{A}'}=V_c$. Conversely, if $\widetilde{W}=V_q$ for some $q\in E_1=\mathrm{Reg}(\partial D_n^{IV})$, then we have $\widetilde{W}=V_q=\Sigma_{\mathbf{A}'}$ so that $\mathbf{A}'=\big(q_1\ \cdots\ q_n\big)$ satisfies $\mathbf{A}'\cdot(\mathbf{A}')^T=\mathbf{0}$. □

Remark 3 Let $\mathbf{A}'\in M(1,n;\mathbb{C})$ and $f\in\mathbf{HI}_1(\mathbb{B}^{n-1}, D_n^{IV})$ $(n\ge 3)$ be such that $\mathbf{A}'\overline{\mathbf{A}'}^T=1$, $\mathbf{A}'\cdot(\mathbf{A}')^T\ne 0$ and $f(\mathbb{B}^{n-1})$ is the irreducible component of $\Sigma_{\mathbf{A}'}=V_{\mathbf{A}'}\cap D_n^{IV}$ containing $\mathbf{0}$. Then, $V_{\mathbf{A}'}$ is a smooth affine-algebraic subvariety of $\mathbb{C}^n$ so that $f(\mathbb{B}^{n-1})$ can in particular be extended as a complex submanifold of some neighborhood of D_n^{IV} in $\mathbb{C}^n$.

3.2.3 Slicing of complex unit balls

By Theorem 2 in [Mok16], if $F:(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ is a holomorphic isometric embedding, then $m \le n-1$. We show that any holomorphic isometric embedding $F:(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ with $1 \le m \le n-2$ and $n \ge 3$ is obtained from some holomorphic isometric embedding $f:(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ by slicing $\mathbb{B}^{n-1}$ by an affine linear subspace.

Theorem 2 *Let $F:(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ be a holomorphic isometric embedding, where $1 \le m \le n-2$ and $n \ge 3$. Then, $F = \widetilde{f} \circ \rho$ for some holomorphic isometries $\widetilde{f}:(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ and $\rho:(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$.*

Proof Let $\Psi \in \mathrm{Aut}(D_n^{IV})$ be such that $(\Psi \circ F)(\mathbf{0}) = \mathbf{0}$. Write $\widetilde{F} := \Psi \circ F$. Then, $\widetilde{F}(\mathbb{B}^m)$ is an irreducible component of $\Sigma_{\mathbf{A'}}$ for some matrix $\mathbf{A'} \in M(n-m, n; \mathbb{C})$ satisfying $\mathbf{A'}\overline{\mathbf{A'}}^T = \mathbf{I_{n-m}}$ by Proposition 5.

We write $\mathbf{A'} = \begin{bmatrix} \mathbf{v} \\ \mathbf{A'_1} \end{bmatrix}$, where $\mathbf{v} \in M(1, n; \mathbb{C})$ is a row vector and $\mathbf{A'_1} \in M(n-m-1, n; \mathbb{C})$ is a matrix. Then, $\mathbf{v}\overline{\mathbf{v}}^T = 1$ so that the irreducible component of $\Sigma_{\mathbf{v}}$ containing $\mathbf{0}$ is the image of some holomorphic isometric embedding $f:(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ by Proposition 6. We can suppose that $f(\mathbf{0}) = \mathbf{0}$ after composing with some element in $\mathrm{Aut}(\mathbb{B}^{n-1})$. Since $\Sigma_{\mathbf{A'}} \subseteq \Sigma_{\mathbf{v}}$ and $\widetilde{F}(\mathbb{B}^m) \subset \Sigma_{\mathbf{A'}}$ is the irreducible component containing $\mathbf{0}$, $\widetilde{F}(\mathbb{B}^m)$ lies in the irreducible component of $\Sigma_{\mathbf{v}}$ containing $\mathbf{0}$ so that $S := \widetilde{F}(\mathbb{B}^m) \subseteq f(\mathbb{B}^{n-1}) =: S'$. Note that both $\left(S, g_{D_n^{IV}}|_S\right) \cong (\mathbb{B}^m, g_{\mathbb{B}^m})$ and $\left(S', g_{D_n^{IV}}|_{S'}\right) \cong (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$ are of constant holomorphic sectional curvature -2. Therefore, $\left(S, g_{D_n^{IV}}|_S\right) \subset \left(S', g_{D_n^{IV}}|_{S'}\right)$ is totally geodesic so that $\widetilde{F} = f \circ \rho$ for some (totally geodesic) holomorphic isometric embedding $\rho:(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$. The result follows. □

Recall that for a bounded domain U we denote by ds_U^2 the Bergman metric on U. Combining the previous results, we state the theorem as follows:

Theorem 3 *Let $F:(\mathbb{B}^m, \lambda ds_{\mathbb{B}^m}^2) \to (D_n^{IV}, ds_{D_n^{IV}}^2)$ be a holomorphic isometric embedding, where $n \ge 3$ and $m \ge 1$ are integers, and $\lambda > 0$ be s real constant. Then, either $\lambda = \frac{n}{m+1}$ or $\lambda = \frac{2n}{m+1}$ and we have the following:*

1. *If $\lambda = \frac{n}{m+1}$, then $1 \le m \le n-1$ and $F = \widetilde{f} \circ \rho$ for some holomorphic isometry $\widetilde{f}:(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ and some (totally geodesic) holomorphic isometry $\rho:(\mathbb{B}^m, g_{\mathbb{B}^m}) \to (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$.*
2. *If $\lambda = \frac{n}{m+1}$ and $m = n-1$, then F is congruent to a nonstandard holomorphic isometry $\hat{F}:(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ such that $\hat{F}(\mathbb{B}^{n-1})$ is the irreducible component of $\Sigma_{\mathbf{c}}$ containing $\mathbf{0}$ for some row vector $\mathbf{c} \in M(1, n; \mathbb{C})$ satisfying $\mathbf{c}\overline{\mathbf{c}}^T = 1$. In addition, F is congruent to the nonstandard holomorphic isometric embedding constructed in* Mok [Mok16] *if and*

only if there is such a holomorphic isometry $\hat{F}$ *congruent to* F *such that* $\mathbf{c}\mathbf{c}^T = 0$.

3. *If* $\lambda = \frac{2n}{m+1}$, *then* $m = 1$ *and* F *is totally geodesic.*

Proof According to Lemma 3, $\lambda' := \frac{\lambda(m+1)}{n}$ is a positive integer such that $\lambda' = 1$ or $\lambda' = 2$. In particular, we have either $\lambda = \frac{n}{m+1}$ or $\lambda = \frac{2n}{m+1}$. If $\lambda = \frac{n}{m+1}$, i.e., $\lambda' = 1$, then $1 \le m \le n-1$ by [Mok16, Theorem 2]. Thus, if $\lambda' = 1$, then part 1 follows from Theorem 2. If $m = n-1$, then part 2 follows from Proposition 5 and Proposition 7. Moreover, F and $\hat{F}$ are nonstandard because of the identity $m = n-1 = p(D_n^{IV})+1$ and the arguments in [Mok16, p. 4519]. If $\lambda = \frac{2n}{m+1}$, i.e., $\lambda' = 2$, then $m = 1$ and F is totally geodesic by Corollary 1. □

Remark 4

1. Recall from Proposition 6 that the irreducible component of $\Sigma_{\mathbf{c}}$ containing $\mathbf{0}$ in part 2 of Theorem 3, for *any* $\mathbf{c} \in M(1,n;\mathbb{C})$ satisfying $\mathbf{c}\overline{\mathbf{c}}^T = 1$ is indeed the image of a holomorphic isometry from $(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$ to $(D_n^{IV}, g_{D_n^{IV}})$, where $n \ge 3$.
2. For any $f \in \mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ $(n \ge 3)$, f is totally geodesic if and only if f is congruent to some $\hat{f} \in \mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ such that $\hat{f}(\mathbb{B}^m)$ is the irreducible component of $\Sigma_{\mathbf{A}''}$ containing $\mathbf{0}$ for some matrix $\mathbf{A}'' \in M(n-m,n;\mathbb{C})$ satisfying $\mathbf{A}''\overline{\mathbf{A}''}^T = \mathbf{I_{n-m}}$ and the condition that $\mathbf{A}'' \cdot \mathbf{v} = \mathbf{0}$ implies $\mathbf{v}^T\mathbf{v} = 0$ for any column vector $\mathbf{v} \in M(n,1;\mathbb{C}) \cong \mathbb{C}^n$.
3. For each row vector $\mathbf{c} \in M(1,n;\mathbb{C})$ satisfying $\mathbf{c}\overline{\mathbf{c}}^T = 1$, let $\hat{F}_{\mathbf{c}} \in \mathbf{HI}_1(\mathbb{B}^{n-1}, D_n^{IV})$ be such that $\hat{F}_{\mathbf{c}}(\mathbb{B}^{n-1})$ is the irreducible component of $\Sigma_{\mathbf{c}}$ containing $\mathbf{0}$. For any row vector $\mathbf{c} \in M(1,n;\mathbb{C})$ satisfying $\mathbf{c}\overline{\mathbf{c}}^T = 1$ and $\mathbf{c}\mathbf{c}^T = 0$, there is a sequence $\{\mathbf{c}_j\}_{j=1}^{+\infty}$ of row vectors in $M(1,n;\mathbb{C})$ satisfying $\mathbf{c}_j\overline{\mathbf{c}_j}^T = 1$ and $\mathbf{c}_j{\mathbf{c}_j}^T \ne 0$ such that $\lim_{j\to+\infty} \mathbf{c}_j = \mathbf{c}$ and the family $\{\hat{F}_{\mathbf{c}_j}\}_{j=1}^{+\infty}$ of holomorphic isometric embeddings in $\mathbf{HI}_1(\mathbb{B}^{n-1}, D_n^{IV})$ converges to $\hat{F}_{\mathbf{c}} \in \mathbf{HI}_1(\mathbb{B}^{n-1}, D_n^{IV})$ which is congruent to the nonstandard holomorphic isometric embedding constructed in Mok [Mok16].

3.3 Explicit examples and their applications

Recall that any holomorphic isometry $f \in \mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ arises from a linear section of the hyperquadric $Q^n \subset \mathbb{P}^{n+1}$. Moreover, a linear section of Q^n in $\mathbb{P}^{n+1}$ can be a complex projective subspace lying entirely in Q^n, a smooth quadric or a singular quadric in $Q^n \subset \mathbb{P}^{n+1}$. On top of the images of nonstandard holomorphic isometries $F \in \mathbf{HI}_1(\mathbb{B}^{n-1}, D_n^{IV})$ constructed in Mok [Mok16] which extend as singular quadrics in $Q^n \subset \mathbb{P}^{n+1}$, by Theorem 3 and by slicing we also have holomorphic isometries in $\mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ which arise from smooth linear sections Λ of $Q^n \subset \mathbb{P}^{n+1}$. In what follows, we write down explicit examples of images of such isometries in $\mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ which are not congruent to singular slices of the nonstandard holomorphic isometries constructed by Mok [Mok16]. Let $Q^m \subset Q^n$ be a smooth hyperquadric passing through $o = [0, \ldots, 0, 1, 0] \in \mathbb{P}^{n+1}$ such that $Q^m \cap \mathbb{C}^n$ is a linear subspace, where $\mathbb{C}^n$ is identified with its image in Q^n via the map $\xi : D_n^{IV} \Subset \mathbb{C}^n \hookrightarrow Q^n \subset \mathbb{P}^{n+1}$ mentioned in Section 2.1. Our examples are of the form $\gamma(Q^m) \cap D_n^{IV}$ for certain automorphisms γ of Q^n fixing o. The explicit parametrizations of $\gamma(Q^m)$ are taken from [Zh15].

Example 1 Let $n \geq 3$ and $1 \leq m \leq n-1$ be integers. For $a_{m+1}, \ldots, a_n \in \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, let $M(a_{m+1}, \ldots, a_n) \subset Q^n \cong G^{\mathbb{C}}/P$ be a non-flat standard model of Q^m passing through $o = [0, \ldots, 0, 1, 0]$, i.e., the image of the quadric $Q^m := \{[z_1, \ldots, z_n, z_{n+1}, z_{n+2}] \in Q^n : z_{m+1} = \cdots = z_n = 0\} \subseteq Q^n$ under a non-trivial element in $M^- \subset \mathrm{Aut}(Q^n)$ on Q^m [Zh15]. Here M^- is the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{m}^-$ being the $(-\sqrt{-1})$-eigenspace of the complex structure J at o on $\mathfrak{m}^{\mathbb{C}}$, where $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ is the Lie algebra of $G^{\mathbb{C}}$ (cf. [Mok89]). Then, it follows from direct computation in [Zh15] that $D(a_{m+1}, \ldots, a_n) := M(a_{m+1}, \ldots, a_n) \cap D_n^{IV} = \big\{(z_1, \ldots, z_n) \in D_n^{IV} : z_l = \frac{a_l}{\sqrt{2}} \sum_{j=1}^n z_j^2,\ m+1 \leq l \leq n\big\}$ for some $a_j \in \mathbb{C}^*$, $m+1 \leq l \leq n$. Moreover, $D(a_{m+1}, \ldots, a_n)$ is smooth at $\mathbf{0}$, thus we can consider the germ of open subset $\mathbf{S} = \mathbf{S}(a_{m+1}, \ldots, a_n)$ of $D(a_{m+1}, \ldots, a_n)$ at $\mathbf{0}$, which is the germ of complex submanifold in D_n^{IV} at $\mathbf{0}$. By computing the holomorphic sectional curvature of $(\mathbf{S}, g_{D_n^{IV}}|_{\mathbf{S}})$ at $\mathbf{0}$ and requiring that $(\mathbf{S}, g_{D_n^{IV}}|_{\mathbf{S}})$ is of constant holomorphic sectional curvature -2, it is necessary that $\sum_{l=m+1}^n |a_l|^2 = \frac{1}{2}$. Now, we consider all complex submanifolds $\mathbf{S}(a_{m+1}, \ldots, a_n) \subset D_n^{IV}$ with $a_{m+1}, \ldots, a_n \in \mathbb{C}^*$ satisfying $\sum_{l=m+1}^n |a_l|^2 = \frac{1}{2}$. Therefore, we assume that $a_{m+1}, \ldots, a_n \in \mathbb{C}^*$ satisfy $\sum_{l=m+1}^n |a_l|^2 = \frac{1}{2}$. Then, we can provide explicit examples in $\mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ as follows such that the image of any such isometry is precisely the irreducible component of $D(a_{m+1}, \ldots, a_n)$ containing $\mathbf{0}$.

(a) We always consider $(z_1, \ldots, z_n) \in D(a_{m+1}, \ldots, a_n)$. If $\sum_{l=m+1}^n a_l^2 \neq 0$, then

$$\sqrt{2}a_{l'} - \sqrt{2a_{l'}^2 - 4a_{l'}^2 \left(\sum_{l=m+1}^n a_l^2\right) \sum_{j=1}^m z_j^2} = 2\left(\sum_{l=m+1}^n a_l^2\right) z_{l'}$$

for $m+1 \leq l' \leq n$. Thus, we have defined

$$\begin{aligned} z_{l'} &= \upsilon_{l'}(z_1, \ldots, z_m) \\ &= \frac{1}{2\sum_{l=m+1}^n a_l^2} \left(\sqrt{2}a_{l'} - \sqrt{2a_{l'}^2 - 4a_{l'}^2 \left(\sum_{l=m+1}^n a_l^2\right) \sum_{j=1}^m z_j^2}\right) \end{aligned}$$

for $m+1 \leq l' \leq n$ (cf. [Zh15]). Then, we have

$$\begin{aligned} 1 - \sum_{j=1}^n |z_j|^2 + \left|\frac{1}{2}\sum_{j=1}^n z_j^2\right|^2 &= 1 - \sum_{j=1}^m |z_j|^2 + \left(\frac{1}{4} - \frac{1}{2}\sum_{l=m+1}^n |a_l|^2\right) \left|\sum_{j=1}^n z_j^2\right|^2 \\ &= 1 - \sum_{j=1}^m |z_j|^2. \end{aligned}$$

Define $F = (F^1, \ldots, F^n) : \mathbb{B}^m \to D_n^{IV}$ by

$$F^j(w) = \sum_{l=1}^m u_{jl} w_l, \quad 1 \le j \le m,$$
$$F^{l'}(w) = \frac{\sqrt{2}a_{l'} + (-1)^{\chi_{l'}}\sqrt{2a_{l'}^2 - 4a_{l'}^2\left(\sum_{l=m+1}^n a_l^2\right)\sum_{j=1}^m (F^j(w))^2}}{2\sum_{l=m+1}^n a_l^2}$$

for $m+1 \le l' \le n$, where $\mathbf{U} = \left(u_{ij}\right)_{1\le i,j\le m} \in U(m)$, $\chi_{l'} = 0$ (resp. 1) if $\sqrt{2a_{l'}^2} = -\sqrt{2}a_{l'}$ (resp. $\sqrt{2a_{l'}^2} = \sqrt{2}a_{l'}$) for $m+1 \le l' \le n$. For each l', $m+1 \le l' \le n$, we specify the branch of $F^{l'}$ as follows: For each l', $m+1 \le l' \le n$, consider the term $2a_{l'}^2 - 4a_{l'}^2 \sum_{l=m+1}^n a_l^2 \sum_{j=1}^m (F^j(w))^2$ for $w \in \mathbb{B}^m$. We have

$$\begin{aligned}\left|4a_{l'}^2 \sum_{l=m+1}^n a_l^2 \sum_{j=1}^m (F^j(w))^2\right| &\le 4|a_{l'}|^2 \sum_{l=m+1}^n |a_l|^2 \sum_{j=1}^m |F^j(w)|^2 \\ &= 2|a_{l'}|^2 \sum_{j=1}^m |w_j|^2 < 2|a_{l'}|^2.\end{aligned}$$

Then, for $w \in \mathbb{B}^m$, we have

$$2a_{l'}^2 - 4a_{l'}^2\left(\sum_{l=m+1}^n a_l^2\right)\sum_{j=1}^m (F^j(w))^2 \in \{\zeta \in \mathbb{C} : |\zeta - 2a_{l'}^2| < 2|a_{l'}|^2\} \not\ni 0.$$

Therefore, we have really used a single branch for each component function $F^{l'}$ of F $(m+1 \le l' \le n)$ by choosing a branch cut of the square root function, which is either $\{z = x + \sqrt{-1}y \in \mathbb{C} : y = 0,\ x \ge 0\}$ or $\{z = x + \sqrt{-1}y \in \mathbb{C} : y = 0,\ x \le 0\}$. Then, we have

$$1 - \sum_{j=1}^n |F^j(w)|^2 + \left|\frac{1}{2}\sum_{j=1}^n (F^j(w))^2\right|^2 = 1 - \sum_{j=1}^m |F^j(w)|^2 = 1 - \sum_{j=1}^m |w_j|^2$$

so that $F = (F^1, \ldots, F^n) : (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ is a holomorphic isometric embedding.

(b) If $\sum_{l=m+1}^n a_l^2 = 0$ and $m < n-1$, then for $(z_1, \ldots, z_n) \in D(a_{m+1}, \ldots, a_n)$, we have $\frac{a_{l'}}{\sqrt{2}}\sum_{j=1}^n z_j^2 = z_{l'} = \frac{a_{l'}}{\sqrt{2}}\sum_{j=1}^m z_j^2$ and we define $z_{l'} = \upsilon_{l'}(z_1, \ldots, z_m) := \frac{a_{l'}}{\sqrt{2}}\sum_{j=1}^m z_j^2$ for $m+1 \le l' \le n$. Then, we have

$$\sum_{l'=m+1}^n |\upsilon_{l'}(z_1, \ldots, z_m)|^2 = \sum_{l'=m+1}^n \frac{|a_{l'}|^2}{2}\left|\sum_{j=1}^m z_j^2\right|^2 = \left|\frac{1}{2}\sum_{j=1}^m z_j^2\right|^2 = \left|\frac{1}{2}\sum_{j=1}^n z_j^2\right|^2.$$

Thus, for $(z_1, \ldots, z_n) \in D(a_{m+1}, \ldots, a_n)$, we have

$$1 - \sum_{j=1}^{n} |z_j|^2 + \left| \frac{1}{2} \sum_{j=1}^{n} z_j^2 \right|^2 = 1 - \sum_{j=1}^{m} |z_j|^2.$$

Define $F = (F^1, \ldots, F^n) : \mathbb{B}^m \to D_n^{IV}$ by

$$F^j(w) = \sum_{l=1}^{m} u_{jl} w_l, \quad 1 \leq j \leq m,$$

$$F^l(w) = \frac{a_l}{\sqrt{2}} \sum_{j=1}^{m} (F^j(w))^2, \quad m+1 \leq l \leq n,$$

for $(w_1, \ldots, w_m) \in \mathbb{B}^m$, where $\mathbf{U} = \big(u_{ij}\big)_{1 \leq i,j \leq m} \in U(m)$. Thus, we have $1 - \sum_{j=1}^{n} |F^j(w)|^2 + \left| \frac{1}{2} \sum_{j=1}^{n} (F^j(w))^2 \right|^2 = 1 - \sum_{j=1}^{m} |F^j(w)|^2 = 1 - \sum_{j=1}^{m} |w_j|^2$ so that $F = (F^1, \ldots, F^n) : (\mathbb{B}^m, g_{\mathbb{B}^m}) \to (D_n^{IV}, g_{D_n^{IV}})$ is a holomorphic isometric embedding.

Remark 5

1. Example 1 gives the first examples of holomorphic isometric embeddings of $\mathbb{B}^{n-1}$ into D_n^{IV} which are incongruent to those constructed in [Mok16] that we discovered without using the general theory based on functional equations as given in Theorem 1. In fact, if there is any example $f : (\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}}) \to (D_n^{IV}, g_{D_n^{IV}})$ incongruent to those in [Mok16], the image $S = f(\mathbb{B}^n) \subset D_n^{IV}$ must inherit a sub-VMRT structure modeled on (Q^{n-1}, Q^n) in the sense of Mok-Zhang [MZ15]. In other words, the restriction of the holomorphic conformal structure on Q^n to S must be non-degenerate. For any $x \in S$ and any $[\alpha] \in \mathbb{P}(T_x(S)) \cap \mathscr{C}_x(Q^n)$, from $R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(S, g_{D_n^{IV}}|_S) = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(D_n^{IV}, g_{D_n^{IV}}) = -2$ and the Gauss equation it follows that $\sigma(\alpha, \alpha) = 0$ for the second fundamental form of $(S, g_{D_n^{IV}}|_S) \hookrightarrow (D_n^{IV}, g_{D_n^{IV}})$. Thus, for the unique minimal disk $\Delta_\alpha \subset D_n^{IV}$ such that $x \in \Delta_\alpha$ and $T_x(\Delta_\alpha) = \mathbb{C}\alpha$, Δ_α must be tangent to S to the order ≥ 2 at x, from which it follows that all such minimal disks lie on S, i.e., S is rationally saturated. By Zhang [Zh15], S must be an open subset of a smooth hyperplane section Q^{n-1} of Q^n.
2. Note that examples in $\mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ obtained in part (a) of Example 1 cannot extend to any open neighborhood of $\mathbb{B}^m$ while examples in $\mathbf{HI}_1(\mathbb{B}^m, D_n^{IV})$ $(1 \leq m \leq n-2$ and $n \geq 3)$ obtained in part (b) of Example 1 are holomorphically extendible to $\mathbb{C}^m$.
3. Note that part (b) of Example 1 actually yields nonstandard holomorphic isometries $F : (\Delta, \frac{n}{2} ds_\Delta^2) \to (D_n^{IV}, ds_{D_n^{IV}}^2)$ $(n \geq 3)$ other than the square-root embedding composed with a totally geodesic holomorphic isometric embedding $\Delta^2 \to D_n^{IV}$ because F can be extended holomorphically to an open neighborhood of $\overline{\Delta}$ while the square-root embedding cannot be extended to any open neighborhood of $\overline{\Delta}$. In particular, this also provides an answer to Problem 5.2.5. in [Mok11].

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