From Nash to Nash's Game Theory

-- an explanation of the mathematics in the movie "A Beautiful Mind"and John Nash's Nobel-winning theory

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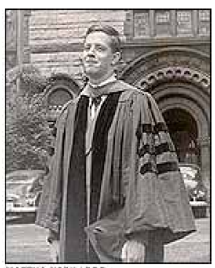
- Life of John Nash
- Introduction to Game Theory
- Nash's Nobel-Prize winning theory

Nash at HKU,2003

Who is John Nash ?

John Nash is an American mathematician who was born in 1928.

He earned a doctorate from Princeton University at the age of 22.



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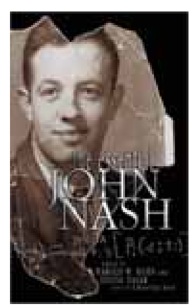
When Nash applied to graduate school at Princeton, his former teacher wrote only one line on his letter of recommendation: "This man is a genius".

- He began teaching at MIT in 1951.
- Soon after, Nash met Alicia Larde, a 21-year-old physics major at MIT.
- In 1957 they were married.

ALICIA NASH

- **In the late 1950s, Nash left MIT because of mental illness.**
- **It is a miracle that he can eventually recover twenty years later .**

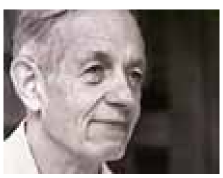


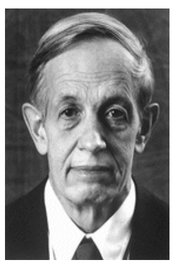
• In his 27 pages Ph.D thesis "**Non-cooperative Games**", Nash made very important contribution in establishing the mathematical principles of **Game Theory**.

• In this thesis, Nash greatly extended the work of **John von Neumann** whose is the founder of Game Theory.

- In 1994, Nash shared the Nobel Prize in Economics with John C. Harsanyi and Reinhard Selten.

1999 Steele prize for his works in pure mathematics.



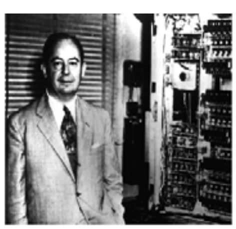


What is Game Theory ?

- Game theory is the study of **mathematical models** on conflicts and co-operations between rational individuals.
- It studies the behavior of decision makers whose decisions **affect each other.**
- Game theory provides the **language** and **framework** for the discussion of problems in economics, social sciences, evolutionary biology,etc.

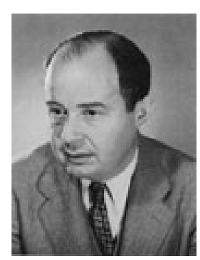
John von Neumann

- Game theory was first developed by the mathematician John von Neumann in 1928.
- Born in 1903, Hungary.
- Involved in the development of atomic bombs.
- Designed and built the first computer.



John von Neumann

- Possibly the last true polymath. Also did fundamental works in several branches of pure mathematics and theoretical physics.
- His memory and the speed with which his mind worked were astounding.



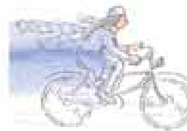
A story of von Neumann

- Someone asked him to solve the following problem.

10 m.p.h. 10 m.p.h.

Twenty miles apart

向右走 Speed of the bee is 15 m.p.h. 向左走



A story of von Neumann

- Question: What total distance did the bee cover when the people meet ?
- First method: Calculate the distance the bee covers on each of its trips between the two bicycles and finally sum the infinite series so obtained.
- Second method: Observe that the bicycles meet exactly an hour after they start so that fly had just an hour for his travel; the answer must therefore be 15 miles.

- When the question was put to von Neumann, he solved it in an instant, and therefore disappointed the questioner:
- “Oh, you must have heard the trick before !” “What trick”, asked von Neumann,”all I did was sum the infinite series.”

John von Neumann

- Game theory was first developed by John von Neumann in 1928. At that time, it was only considered as a branch of pure mathematics.
- In 1944, he and Oskar Morgenstern published the book“**Theory of Games and Economic Behavior**”.

Game Theory and Economics

- Game theory studies how people behave in strategic situations, where the outcome for each player depends on the actions of all the players.
- Since strategic interaction characterizes many economic situations, game theory has proved very useful in economic analysis.

An example of a strategic situation in economics

- Apple Daily and Oriental Daily are considering whether to have a price war.

Vs

- Both newspapers can choose to cut price or don't cut.
- If both newspapers choose don't cut then each will earn 20 million dollars.
- If one chooses don't cut and the other chooses cut price, then the newspaper choosing don't cut will only earn 5 million, while the one choosing cut price can earn 30 million dollars.

- If both newspapers choose cut price, then each can earn 10 million dollars.

		Oriental Daily	
		Cut price	Don't cut
Apple Daily	Cut price	10,10	30,5
	Don't cut	5,30	20,20

Prisoner's Dilemma

- **John and Peter have been arrested for possession of guns. The police suspects that they are going to commit a major crime.**
- **If no one confesses, they will both be jailed for 2 years.**
- **If only one confesses, he'll go free and his partner will be jailed for 10 years.**
- **If they both confess, they both get 5 years .**

Matrix Representation of Prisoner's Dilemma

		Peter	
		Confess	Don't confess
John	Confess	5,5	0,10
	Don't confess	10,0	2,2

Common features of the two examples

- Both have two players, say A and B.
- Each player has two strategies, say C and D.
- There are four possible outcomes of the moves : (C,C),(C,D)(D,C),(D,D).
- Payoffs of each possible outcome are known to each player.

		Oriental Daily (B)	
		Cut price(C)	Don't cut (D)
Apple Daily (A)	Cut price (C)	10,10	30,5
	Don't cut (D)	5,30	20,20

		Peter (B)	
		Confess (C)	Don't Confess (D)
John (A)	Confess (C)	5,5	0,10
	Don't Confess (D)	10,0	2,2

- In both examples, based on the payoffs, A and B have the same preferences of the outcomes.
- Player A : (*C*, *D*) > (*D*, *D*) > (*C*, *C*) > (*D*, *C*)
- Player B : (*D*, *C*) > (*D*, *D*) > (*C*, *C*) > (*C*, *D*)

- Player A : (*C*, *D*) > (*D*, *D*) > (*C*, *C*) > (*D*, *C*)
- Player B : (*D*, *C*) > (*D*, *D*) > (*C*, *C*) > (*C*, *D*)
- If we can analyze what A and B are going to choose based on the above information, then we can apply the result to both of the examples.
- We shall see later that both A and B will choose C even though they can have a better outcome (D,D) > (C,C).

What is a Game?

- The previous two situations are the examples of the games considered in Game Theory.
- We shall only focus on those games with the following information.

Four Elements of a Game

1. The number of players.
2. A complete description of the possible strategies of each player.

   - when each player moves, what are the possible moves? what is known to each player before moving?

Four Elements of a Game

3. A description of the outcome of the moves.

4. Payoff of each possible outcome

   - how much money each player receive for any specific outcome.

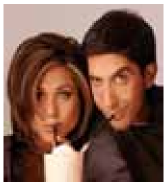
Dating Game

Ross and Rachel would like to go out on Friday night.

Ross prefers to see football, while Rachel prefers to have a drink.

However,they would rather go out together than alone.

Should Ross make a sacrifice?



		Rachel	
		Football	Drink
Ross	Football	2,1	0,0
	Drink	0,0	1,2

French or German?

Sa and Gil would like to take a language course.

Sa prefers to study German, while Gil prefers to study French.

They would rather take a course together than alone.

		Gil (B)	
		German (C)	French (D)
Sa (A)	German (C)	3,2	1,1
	French (D)	0,0	2,3

		Rachel (B)	
		Football (C)	Drink (D)
Ross (A)	Football (C)	2,1	0,0
	Drink (D)	0,0	1,2

A two person game is called a *Dating Game* if each of two players has two actions, say *C* and *D*.

Player A: (*C, C*) > (*D,D*) > (*C, D*) = (*D, C*)

Player B: (*D, D*) > (*C, C*) > (*C, D*) = (*D, C*)

Solution of a Game

- Given a game, we would like to predict the actual decision of each player.
- These decisions of the players are called a solution of the game.
- Let's consider the Prisoner's Dilemma Game again to find out the solution of this game.

We want to predict the outcome of the game

Suppose that John decides to confess. What is the best decision for Peter?

		Peter	
		Confess	Do not Confess
John	Confess	5,5	0,10
	~~Do not Confess~~	~~10,0~~	~~2,2~~

We want to predict the outcome of the game

Suppose that John decides not to confess. What is the best decision for Peter?

		Peter	
		Confess	Do not Confess
John	~~Confess~~	~~5,5~~	~~0,10~~
	Do not Confess	10,0	2,2

Dominant Strategy

- Dominant Strategy: a strategy which is always better than any other strategy, regardless of what opponents may do.
- Confess is the dominant strategy of Peter.
- Since John has the same preference, confess is also the dominant strategy of John.
- Assume, John and Peter are rational, then both of them will choose to confess.

Outcome of the Game

		Peter	
		Confess	Do not Confess
John	Confess	5,5	0,10
	Do not Confess	10,0	2,2

Dominant Strategy Equilibrium

- If every player has a dominant strategy, every player will choose that strategy and we can therefore predict the outcome of the game.
- The decisions of the players form a dominant strategy equilibrium if every player chooses his dominant strategy.

Dominant Strategy Equilibrium

- At dominant strategy equilibrium, every player will not change his decision any more.
- If the dominant strategy equilibrium exists, it will be the solution (outcome) of the game.
- However, in most of games (e.g. the Dating Game), it does not exist. Therefore, we cannot use this method to predict the outcome of the game.

Existence of Dominant Strategy

- John von Neumann proved that for any zero-sum game, dominant (mixed) strategy always exists.
- However, in most of the games (e.g. the Dating Game), it does not exist. Therefore, we cannot use this method to predict the outcome of the game.
- **In zero-sum games, one player's loss is the other player's gain, e.g. chess.**

Suppose Sa has chosen German, what should Gil choose ?

		Gil (B)	
		German (C)	French (D)
Sa (A)	German (C)	3, 2	1, 1
	~~French (D)~~	~~0, 0~~	~~2, 3~~

Suppose Sa has chosen French, what should Gil choose ?

		Gil (B)	
		German (C)	French (D)
Sa (A)	German (C)	3, 2	1, 1
	French (D)	0, 0	2, 3

- **Therefore, Gil does not have any dominant strategy and hence this game does not have any dominant strategy equilibrium.**

No Dominant Strategy Equilibrium

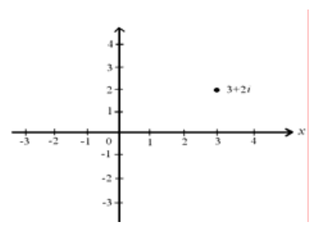
- We are facing a situation that most of the games do not have a solution, what should we do ?

How to resolve this problem?

- Let's look at a similar problem in mathematics.
- A number a is a zero of a polynomial p(x), (e.g. $p(x)=x^2 - 2x + 1$) if p(a)=0, e.g. 1 is a zero of $x^2 - 2x + 1$.
- Not every polynomial has a real zero. For example, x^2+4 has no zero as $x^2+4 >0$ for all real number x.

- To resolve this problem, try to let a more general type of numbers to be zeros. These numbers are called complex numbers,

  $a + bi$ ($i = \sqrt{\ }$, $i^2=-1$) .



How to resolve this problem?

- a = 2i ($=2\sqrt{\ }$) is a zero of x^2+4 as $(2i)^2 = 4i^2 = -4$.
- (Gauss) Every non-constant polynomial must have at least one zero.

Try to introduce a more general solution concept of a game and show that every game has at least one such solution.

How did John Nash win a Nobel -Prize in Economics?

- In his thesis, John Nash introduced the so-called **mixed Nash equilibrium** which is a more general solution concept of a game.
- He proved that for most of the games, at least one mixed Nash equilibrium exists.

A Beautiful Mind

ALICIA NASH

What should be the new solution concept ?

- Suppose we have a theory which can predict the outcome (solution) of a game. What property should this solution has?
- Suppose we have a theory which predicts that player 1,2,3 must choose A, B, C respectively, i.e. (A,B,C) is the solution of the game.

What should be the new solution concept ?

- Now if player 1 knows that player 2 and player 3 are going to choose B and C respectively, will he change his choice A?
- No, if the theory works. Similarly player 2 and player 3 will not change their choices if the others don't.

Nash Equilibrium

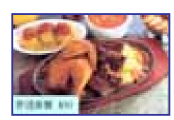
- The decisions of the players is a **Nash Equilibrium** if no individual will change his position, given that the others **do not change** their positions.

An Example of a Nash Equilibrium

- In a very boring lecture, all of the students would like to leave the classroom early, but no one is willing to leave before the end of the lecture unless someone else leaves first.
- Therefore, every student will choose to stay until the end of the lecture and this is a **Nash Equilibrium**.

Splitting the bill

- **10 classmates are having dinner together. Each one has the choice of a cheaper meal for 50 dollars or an excellent meal for 80 dollars. As usual, they decide to split the bill.**
- **Everyone chooses the cheaper meal is not a Nash equilibrium.**
- **How about everyone chooses the excellent meal?**



Dominant Vs Nash

- At a dominant strategy equilibrium, every player will not change his position any more.
- At a Nash Equilibrium, every player will not change his position provided that the others do not change their positions.
- Therefore, a dominant strategy equilibrium must be a Nash equilibrium.

Nash Equilibrium for Prisoner's Dilemma Game

		Peter	
		Confess	Do not Confess
John	Confess	5,5	0,10
	Do not Confess	10,0	2,2

What is not a Nash equilibrium?

- In the bar scene, Nash suggested everyone should go for his second choice.
- This is **not** a Nash equilibrium because if you know all the others are going to choose their second choices, then you will certainly change your choice to the most beautiful girl.

What is a mixed Nash equilibrium?

- To understand what a mixed Nash equilibrium is, let's play the following three person game.

A three-person game

- Each one can show either **one finger** or **two fingers**.
- If you are the only one showing one finger, then you get **one dollar** from me.
- If you are the only one showing two fingers, then you get **two dollars**.
- Otherwise, everyone gets **nothing** !
- Can you find the **Nash equilibria** for this game ?

Exercise

- Does this game have any dominant strategy equilibrium ?
- How many possible outcomes for this game? How many of them are Nash equilibria? List all of them.

Mixed Strategy

- Suppose we repeat this game many times, then each player need to decide for what percentage of the games should he choose one finger in order to win more, e.g. 10% one finger and 90 % two fingers, i.e. (10%,90%).
  These percentages a called a mixed strategy.On the other hand, the mixed strategy (0%, 100%) is also called a pure strategy.

Mixed Strategy

- Definition: A mixed strategy of a player in a repeated game is a probability distribution over the player's strategies.
- Definition: A mixed strategy is called a pure strategy is the probability of choosing one strategy is 1 (and hence 0 for all the other strategies) .
- For repeated games, we can also define a Nash equilibrium for mixed strategies and get what is called a mixed Nash equilibrium.

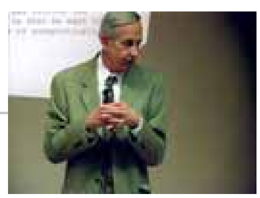
Mixed Nash Equilibrium

- The decisions of players forms a **mixed Nash equilibrium** if no individual will change his mixed strategy, given that the others **do not change** their mixed strategies.
- For the three person game, the mixed Nash equilibrium is ((41%,59%),(41%,59%),(41%,59%)) or simply (41%,41%,41%).

Nash's Nobel-Prize-winning Theorem

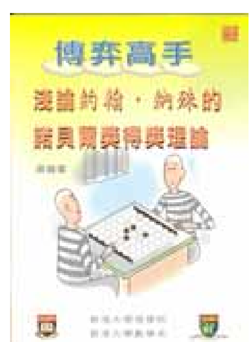
Every finite n-player non-cooperative game has a mixed Nash equilibrium.

Conclusion



- Who is John Nash ?
- Basic Game Theory: Prisoner's Dilemma, Dominant Strategy, mixed Nash Equilibrium.
- How did Nash win a Nobel-Prize?
- http://hkumath.hku.hk/~ntw/pub_lec.html

http://hkumath.hku.hk/~ntw/NgTWbook.pdf



Thank you