Midterm Exam 1
Math 1013/1804 (2013-2014, Semester 2)

Lecture Section 2D

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Problem 1. (5 Points)

What is the limit

$$\lim_{x\to 0}\left(x\cos^2\left(\frac{1}{x}\right)\right)?$$

A:	∞	B:	0
C:	1	D:	Does not exist

Answer:____________

Solution.

The answer is B by the Squeeze Theorem. Here we use $|x|$ and $-|x|$ above and below $f(x) := x\cos^2\left(\frac{1}{x}\right)$. The relevant bound is

$$\left|\cos^2\left(\frac{1}{x}\right)\right| \leq 1.$$

Problem 2. (5 Points)

Suppose that

$$\begin{aligned} f(a) &= b,\\ f(b) &= c,\\ f(c) &= b-c,\\ f(d) &= b+c, \end{aligned}$$

and for every $r\in\mathbb{R}$ and every $s\in\mathbb{R}$ we have

$$f(rc+sd) = rf(c)+sf(d).$$

If f has an inverse, $c\neq d$ and $d\neq -2c$, then which of the following is $f(c-3d)$?
Hint: Compute $f\left(\frac{d-c}{2}\right)$.

A:	$-d-3c$	B:	$c+d$
C:	$-4c$	D:	None of the above

Answer:____________

Solution.

The answer is A. Since f has an inverse, it is also injective. However,

$$f\left(\frac{d-c}{2}\right) = \frac{1}{2}f(d) - \frac{1}{2}f(c) = \frac{b+c}{2} - \frac{b-c}{2} = c.$$

Since $f(b) = c$, we must therefore have

$$b = \frac{d-c}{2}.$$

Therefore

$$f(c) = b - c = \frac{d-c}{2} - c = \frac{d-3c}{2}$$

and

$$f(d) = b + c = \frac{d-c}{2} + c = \frac{d+c}{2}.$$

It follows that

$$f(c-3d) = f(c) - 3f(d) = \frac{d-3c}{2} - \frac{3d+3c}{2} = -d-3c.$$

This means that A is correct. Since $c \neq d$, $-d-3c \neq -4c$, so C is not correct. Since $d \neq -2c$, B is also not correct.

Problem 3. (5 Points)

Which of the following is/are true?

A: If a function is not continuous at $x = a$, then it is not differentiable at $x = a$.
B: All functions which are continuous at $x = a$ are also differentiable at $x = a$.
C: All functions which are differentiable at $x = a$ are also continuous at $x = a$.
D: If a function is not differentiable at $x = a$, then it is not continuous at $x = a$.

Answer:_______________

Solution.

The answer is A and C (this was given as a theorem stated in the form of A, but it was also explained in lecture that this equivalent to C). For B and D, think about $f(x) = |x|$. This function is continuous at $x = 0$, but it is not differentiable at $x = 0$. Therefore neither B nor D is satisfied.

Problem 4. (5 Points)

Which of the following is

$$\frac{d}{dx}\left(\sqrt{\frac{2x^3-5}{4x^2-1}}\right)?$$

A: $\frac{8x^4-6x^2-40x}{2\sqrt{2x^3-5}(4x^2-1)^{\frac{3}{2}}}$

B: $-\frac{\sqrt{4x^2-1}}{2\sqrt{2x^3-5}}\left(\frac{8x^4-6x^2+40x}{(4x^2-1)^2}\right)$

C: $\frac{\sqrt{4x^2-1}}{2\sqrt{2x^3-5}}\left(\frac{8x^4-6x^2+40x}{(4x^2-1)^2}\right)$

D: $\frac{\left(8x^4-6x^2+40x\right)\left(2x^3-5\right)^{-\frac{1}{2}}}{2(4x^2-1)^{\frac{3}{2}}}$

Answer:_______________

Solution.

The answer is C and D. One gets C directly from the chain rule and then the quotient rule (after simplifying). By rewriting

$$\frac{1}{\sqrt{x^3-4}} = \frac{1}{(x^3-4)^{\frac{1}{2}}} = \left(x^3-4\right)^{-\frac{1}{2}},$$

this is the same as D.

Note that B would be obtained if the quotient rule is used backwards (the minus sign is in the wrong place). For A, there is a minus sign incorrect next to 16x.

Problem 5. (5 Points)

What is

$$\lim_{x \to 0} \sin^2\left(\frac{1}{x}\right)?$$

A:	1	B:	∞
C:	0	D:	Does not exist.

Answer:_______________

Solution.

The answer is D. The function $f(x) := \sin^2\left(\frac{1}{x}\right)$ alternates between 0 and 1 infinitely often as x gets close to 0. It therefore does not get "closer and closer" to a given value, so the limit doesn't exist.

To see this slightly more formally, suppose that the limit existed. Let's call the limit

$$L := \lim_{x \to 0} \sin^2\left(\frac{1}{x}\right).$$

Since $\sin^2\left(\frac{1}{x}\right) = 0$ whenever $\frac{1}{x} = \pi n$ with $n \in \mathbb{Z}$, for

$$x_n = \frac{1}{\frac{\pi}{2}}$$

we have

$$\sin^2\left(\frac{1}{x_n}\right) = 0.$$

Since x_n gets smaller and smaller (closer and closer to 0) as n gets bigger, $\sin^2\left(\frac{1}{x_n}\right)$ must get closer and closer to L (by the definition of the limit).

However, for $y_n = \frac{1}{\frac{\pi}{2}+2\pi n}$ (for every $n \in \mathbb{Z}$), we have

$$\sin^2\left(\frac{1}{y_n}\right) = 1.$$

Since y_n gets smaller and smaller as n gets bigger and bigger, $\sin^2\left(\frac{1}{y_n}\right)$ must also get closer and closer to L. But now both 0 and 1 must be close to L, which is impossible. It follows that our original assumption that the limit exists must be incorrect/wrong/faulty. Therefore, the limit does not exist.

Problem 6. (5 Points)

Which of the following statements is/are true?

A The vertical line test determines whether a graph is the graph of a function or not.
B The horizontal line test determines whether a function has an inverse.
C The vertical line test determines whether a function is injective.
D The horizontal line test determines whether a graph is the graph of a function or not.

Answer:________________

Solution.

The answer is A and B. Part A is correct because the vertical line test makes sure that there is a unique (i.e., exactly one) output for each input. The answer B is correct because the horizontal line test checks whether a function is injective, and we saw in lecture that this is equivalent to a function having an inverse.

Part C is incorrect. For example $f(x) := x^2$ passes the vertical line test, but is not injective (for example, $f(1) = f(-1)$).

Part D is incorrect. It is the vertical line test that checks whether a function is the graph of a function or not.

Problem 7. (5 Points)

Suppose that

$$f(x) := \left(\sqrt{x^4 + 24x + 4}\right)\left(2x^7 + (r-3)x + 3\right).$$

For which of the following values of $r \in \mathbb{R}$ is

$$f'(0) = 0?$$

A: 2　　B: 3
C: -6　　D: 12

Answer:________________

Solution.

The correct answer is C. By the product rule, we have

$$f'(x) = \left(\sqrt{x^4 + 24x + 4}\right)\left(14x^6 + r - 3\right) + \frac{4x^3 + 24}{2\sqrt{x^4 + 4}}\left(2x^7 + (r-3)x + 3\right).$$

Therefore

$$f'(0) = \sqrt{4}(r-3) + \frac{24}{2\sqrt{4}}(3) = 2r + 12.$$

Therefore $f'(0) = 0$ if and only if $r = -6$.

The answer B is what you would get if you incorrectly thought that the product rule was

$$\frac{d}{dx}\left(g(x)h(x)\right) = g'(x)h'(x).$$

The answer A is what you would get if you incorrectly thought that the product rule was

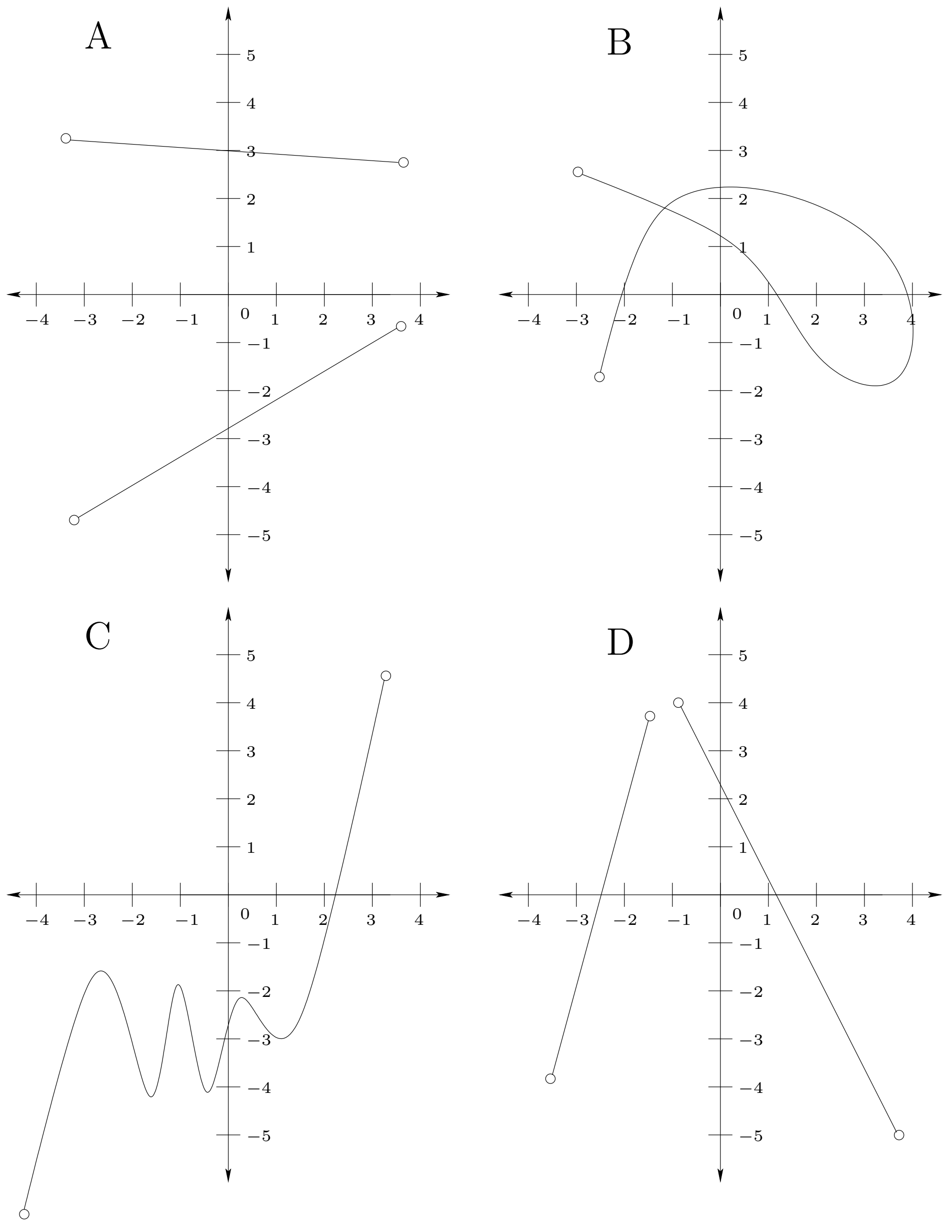
$$\frac{d}{dx}\left(g(x)h(x)\right) = g(x)h(x) + g'(x)h'(x).$$

The answer D is what you would get if you incorrectly thought that the product rule was

$$\frac{d}{dx}\left(g(x)h(x)\right) = g(x)h'(x) - h(x)g'(x).$$

Problem 8. (5 Points)

Which of the following graphs is the graph of a function? The entire graph is what you can see (lines do NOT continue forever, for example)!



Answer:________________

Solution.

By the vertical line test, we see that both C and D are graphs of functions, while A and B are not.

Problem 9. (10 Points)

Show that there exists at least one $x \in \mathbb{R}$ for which

$$x = 1 - \sin(x).$$

Solution.

Consider the function $f(x) = x - 1 + \sin(x)$. Since $x - 1$ and $\sin(x)$ are both continuous, $f(x)$ is also continuous (for all $x \in \mathbb{R}$). Furthermore,

$$f(0) = 0 - 1 + \sin(0) = -1.$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 + \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 1 + 1 = \frac{\pi}{2}.$$

Therefore $f(0) < 0$ and $f\left(\frac{\pi}{2}\right) > 0$. By the Intermediate Value Theorem, we conclude that there exists at least one $c \in \left(0, \frac{\pi}{2}\right)$ for which $f(c) = 0$. However, if $f(c) = 0$, then

$$0 = c - 1 + \sin(c) \implies c = 1 - \sin(c).$$

Therefore, this c satifies the conditions required.

Problem 10. (10 Points)
Suppose that y is a function of x which satisfies

$$xy - cos(x)sin(y) = 0.$$

(a) Solve for $\frac{dy}{dx}$ as a function of x and y (that is to say, you should give a function so that if someone gives you x and y, you can tell them $\frac{dy}{dx}$).

(b) Find $\frac{dy}{dx}$ at the point where $x = 0$ and $y = 0$.

Solution.
(a) We use implicit differentiation. By the product rule, we have

$$x\frac{dy}{dx} + y + \sin(y)\sin(x) - \left(\cos(y)\frac{dy}{dx}\right)\cos(x) = 0.$$

We now solve for $\frac{dy}{dx}$. Combining the terms with $\frac{dy}{dx}$ yields

$$\frac{dy}{dx}\left(x - \cos(y)\cos(x)\right) = -y - \sin(y)\sin(x).$$

Therefore

$$\frac{dy}{dx} = \frac{-y - \sin(y)\sin(x)}{x - \cos(y)\cos(x)}.$$

(b) For $x = 0$ and $y = 0$, we have

$$\frac{dy}{dx} = \frac{-0 - 0}{0 - 1 \cdot 1} = 0.$$