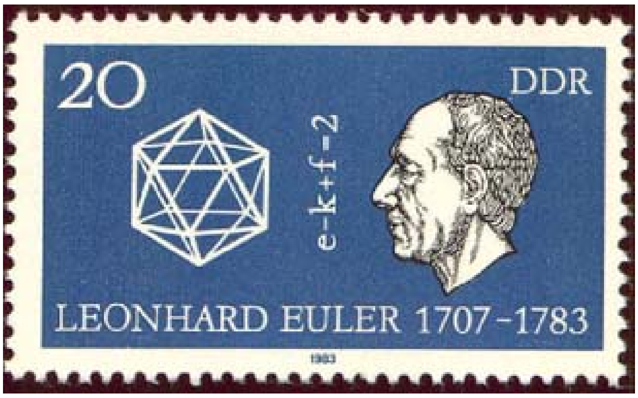
# Euler and his path (from the 18th century to the 21st century)

M.K. SIU
Department of Mathematics
The University of Hong Kong

# Leonhard Euler

(April 15, 1707 – September 18, 1783)







Leonhard Euler (1707-1783)

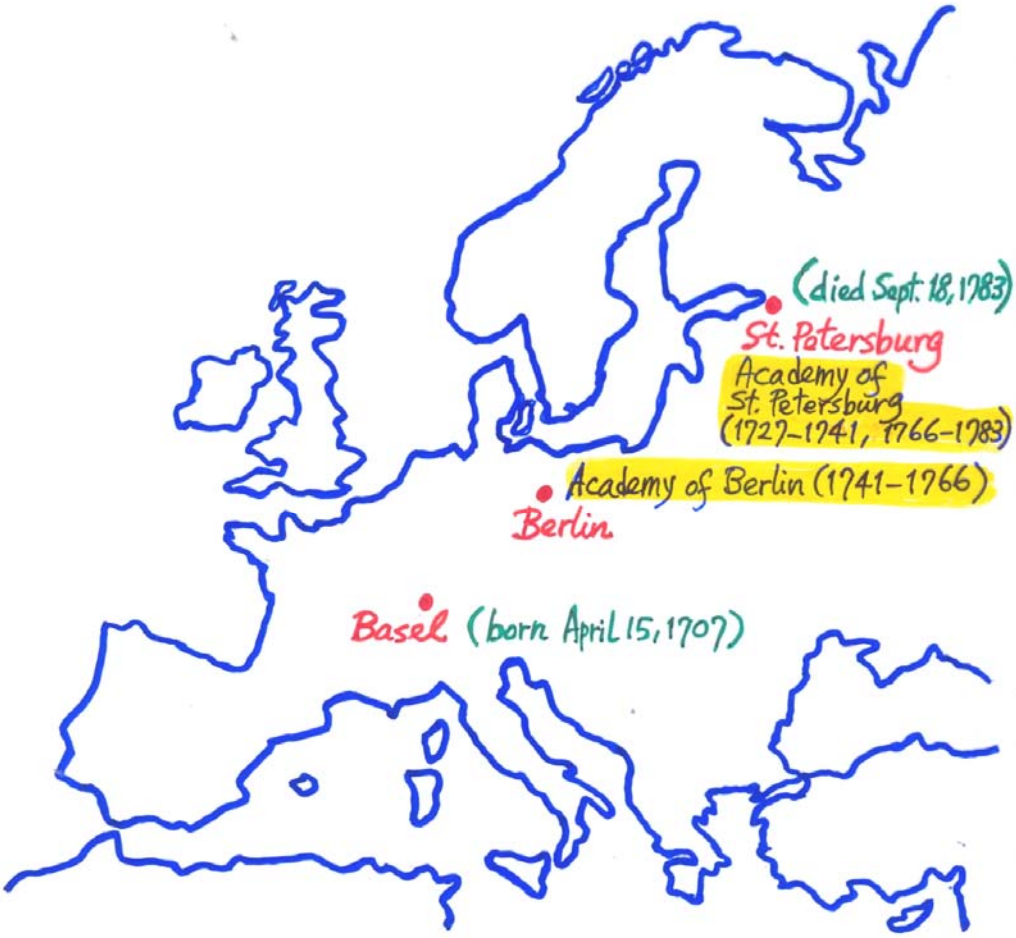
http://www.eulersociety.org


SCHWEIZERISCHE NATIONALBANK
BANCA NAZIUNALA SVIZRA
10
10
BANQUE NATIONALE SUISSE
BANCA NAZIONALE SVIZZERA
Dix Francs
87C1180753
10
Dieci Franchi
87C1180753


LEONHARD EULER
1707–1783
HELVETIA
e-k+f=2
130
ANGELO BOOG
2007
4000 BASEL
e-k+f=2
Leonhard Euler
1707-1783
AUSGABETAG 6. MÄRZ 2007


LEONHARD EULER
1707–1783
HELVETIA
e-k+f=2
130
ANGELO BOOG
2007

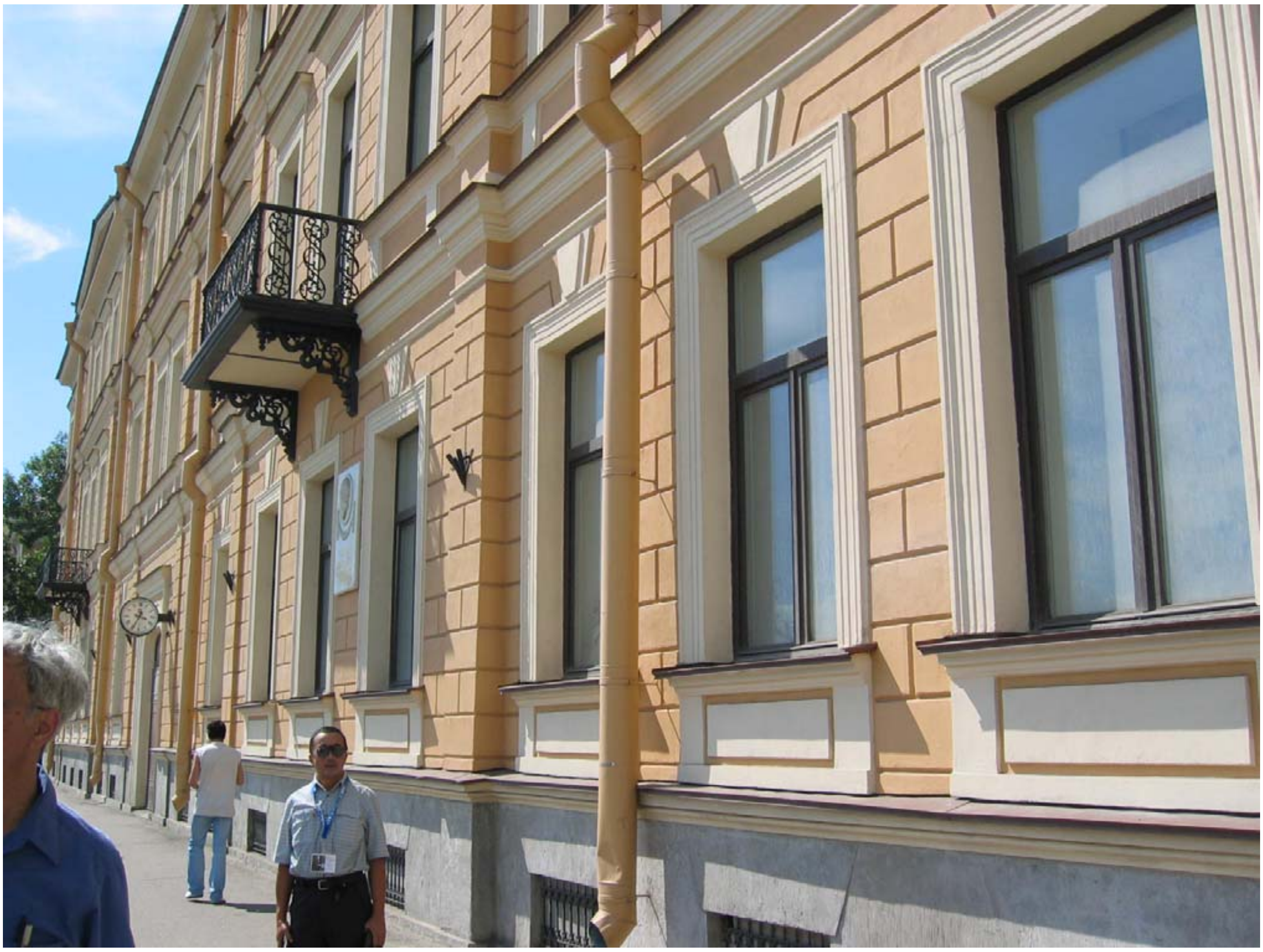
- “Euler“ in *Men of Mathematics,* E.T. Bell (1937)
- “Euler“(by A.P. Youschkevitch) in *Dictionary of Scientific Biography,* vol.4 (1970)
- *Euler : the Master of Us All,* W. Dunham (1999)
- M.K. Siu, Euler and heuristic reasoning, in *Learn From the Masters!* Edited by F. Swetz et al (1995), 145 -160.
- (Euler Society) www.eulersociety.org
- (Euler Archive) www.EulerArchive.org

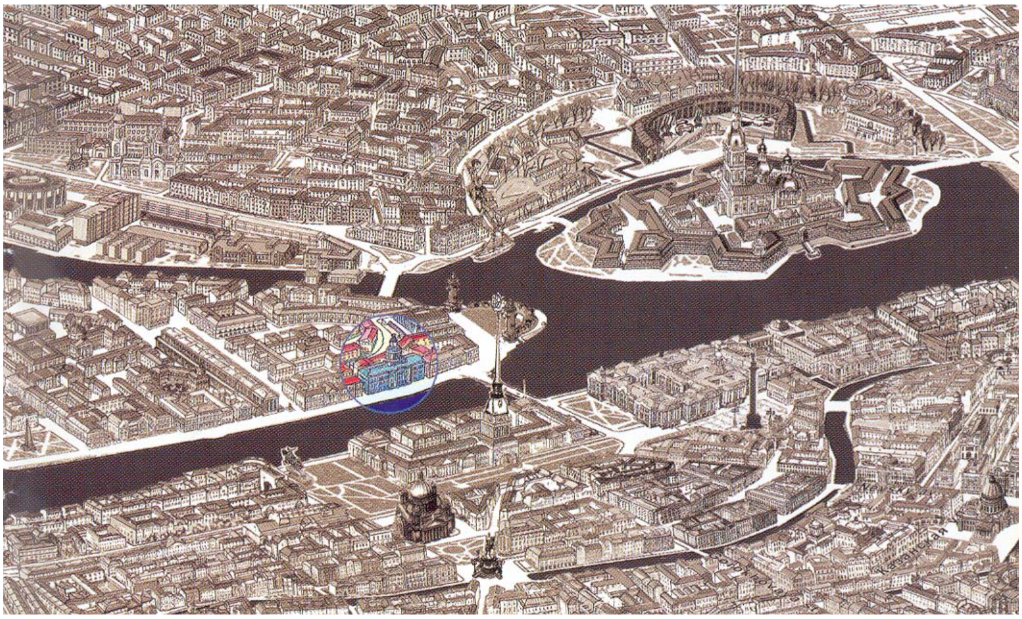


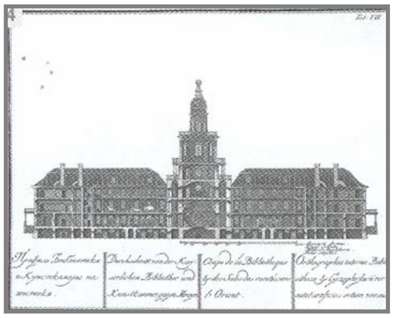

Leonhard Euler
(1707–1783)


LEONHARD EULER
1707-1783
MATHEMATIKER, PHYSIKER
INGENIEUR, ASTRONOM UND
PHILOSOPH, VERBRACHTE IN
RIEHEN SEINE JUGENDJAHRE.
ER WAR EIN GROSSER GELEHR-
TER UND EIN GÜTIGER
MENSCH.


ДОСТАВКА







# Kunstkamera, St. Petersburg

## (Peter the Great Museum of Anthropology and Ethnography)


HIER WOHNTE
VON 1743 BIS 1766
DER MATHEMATIKER
LEONHARD EULER
* 15. IV. 1707. † 18. IX. 1783.
SEINEM ANDENKEN
DIE STADT BERLIN
1907.


SANS, SOUCI.

"Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind."

François Arago (1786-1853)



"He ceased to calculate and to breathe."

*Eulogy of Euler*
Marquis de Condorcet
(1743-1794)

"He [Euler] preferred instructing his pupils to the little satisfaction of amazing them. He would have thought not to have done enough for science if he should have failed to add to the discoveries, with which he enriched science, the candid exposition of his ideas that led him to these discoveries."

**Eulogy of Euler**
Marquis de Condorcet
(1743 – 1794)

"Naturally enough, as any other author, he [Euler] tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself. …… We can learn from it a great deal about mathematics, or the psychology of invention, or inductive reasoning."

**George Pólya (1887-1985)**
*Mathematics and Plausible Reasoning*
*Volume I: Induction and Analogy*
*in Mathematics, 1954*

# “Read Euler, read Euler. He is the master of us all.”

Pierre-Simon Laplace
(1749-1827)

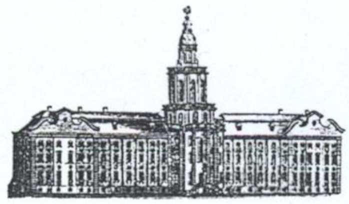
Vollständige
Anleitung
zur
Algebra
von
Hrn. Leonhard Euler.

Erster Theil.
Von den verschiedenen Rechnungs-Arten,
Verhältnißen und Proportionen.

St. Petersburg.
gedruckt bey der Kays. Acad. der Wissenschaften 1770.

Abb. 21
Titelblatt der «Algebra»,
St. Petersburg 1770.

TENTAMEN
NOVAE THEORIAE
MVSICAE
EX
CERTISSIMIS
HARMONIAE PRINCIPIIS
DILVCIDE EXPOSITAE.
AVCTORE
LEONHARDO EVLERO.



PETROPOLI, EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
cIↄIↄccxxxix.

Abb. 22
Titelblatt der «Musiktheorie»,
St. Petersburg 1739.

METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI
AUCTORE
LEONHARDO EULERO,
*Professore Regio, & Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

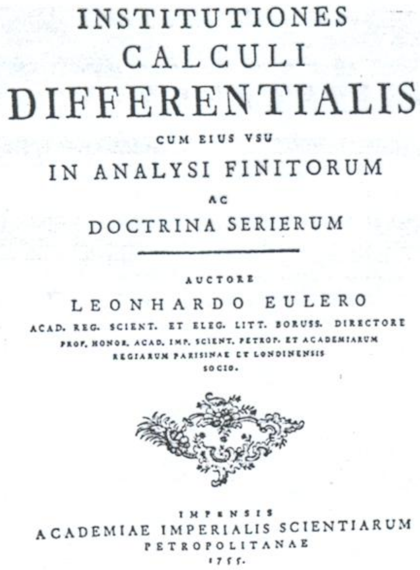
Abb. 23
Titelblatt der «Variationsrechnung»,
Lausanne und Genf 1744.

INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
*Professore Regio* BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLVIII.

Abb. 24
Titelblatt der «Introductio», Lausanne 1748.



INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS USU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

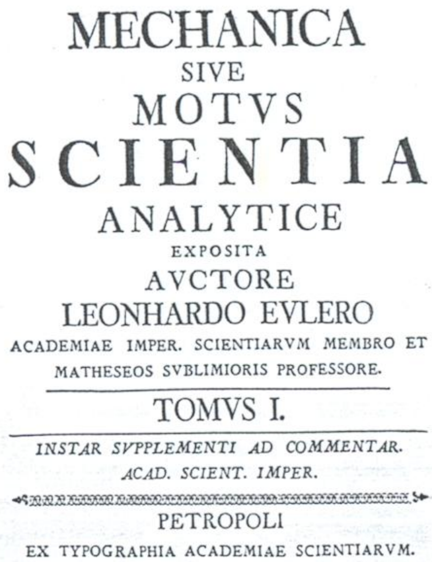
Abb. 25
Titelblatt der «Differentialrechnung», St. Petersburg 1755.

INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM
IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI
Impenſis Academiae Imperialis Scientiarum
1768.

Abb. 26
Titelblatt der «Integralrechnung», St. Petersburg 1768.



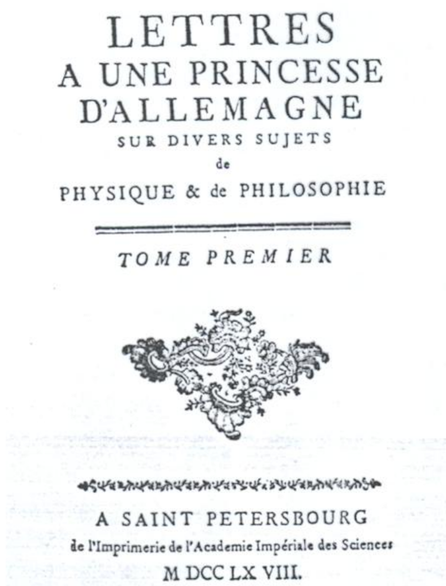
MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.

TOMVS I.

*INSTAR SVPPLEMENTI AD COMMENTAR.*
*ACAD. SCIENT. IMPER.*

PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

Abb. 27
Titelblatt der «Mechanik», St. Petersburg 1736.

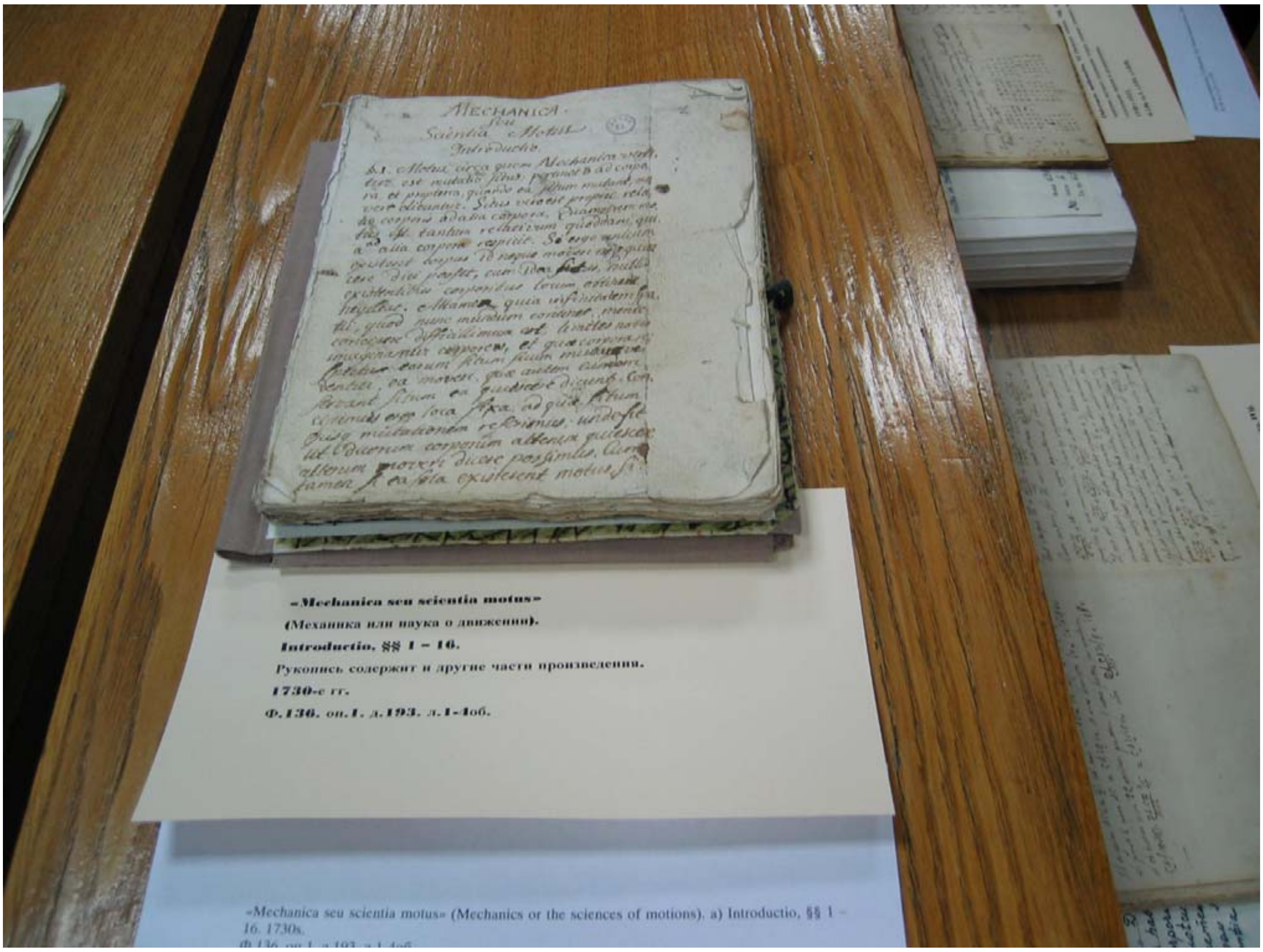


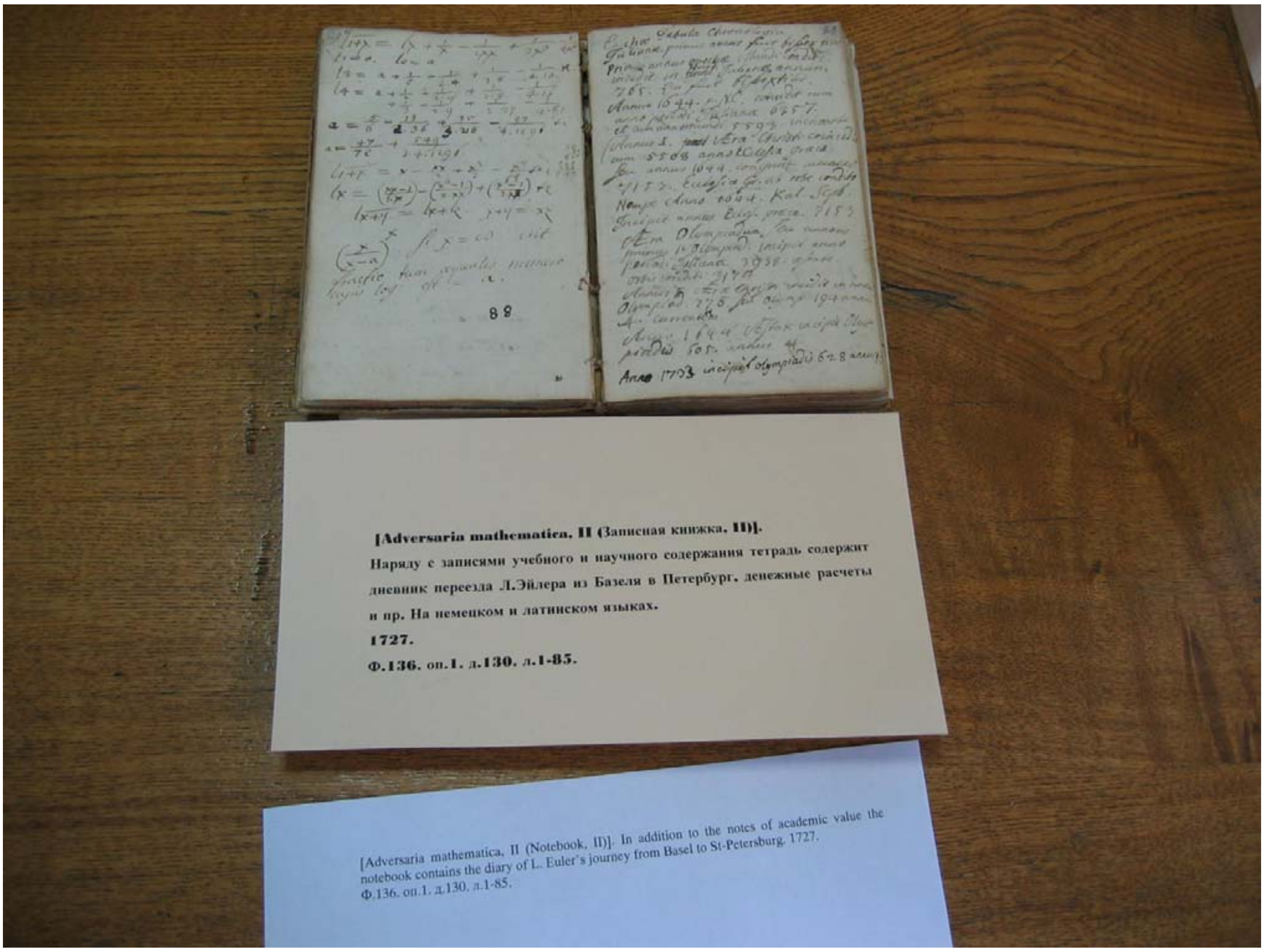
LETTRES
A UNE PRINCESSE
D'ALLEMAGNE
SUR DIVERS SUJETS
de
PHYSIQUE & de PHILOSOPHIE

*TOME PREMIER*

A SAINT PETERSBOURG
de l'Imprimerie de l'Academie Impériale des Sciences
M DCC LX VIII.

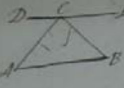
Abb. 28
Titelblatt der «Philosophischen Briefe», St. Petersburg 1768.


«Mechanica seu scientia motus»
(Механика или наука о движении).
Introductio, §§ 1 – 16.
Рукопись содержит и другие части произведения.
1730-е гг.
Ф.136. оп.1. д.193. л.1-4об.
«Mechanica seu scientia motus» (Mechanics or the sciences of motions). a) Introductio, §§ 1 – 16. 1730s.


88
[Adversaria mathematica, II (Записная книжка, II)].
Наряду с записями учебного и научного содержания тетрадь содержит дневник переезда Л.Эйлера из Базеля в Петербург, денежные расчеты и пр. На немецком и латинском языках.
1727.
Ф.136. оп.1. д.130. л.1-85.
[Adversaria mathematica, II (Notebook, II)]. In addition to the notes of academic value the notebook contains the diary of L. Euler's journey from Basel to St-Petersburg. 1727.
Ф.136. оп.1. д.130. л.1-85.

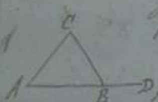
84

Prop. Die Winkel welche zwey [illegible] Winkel oder Winkel [illegible] die Figur.




Prop. In einem jeglichen Triangulo als ABC, halten die drey Winkel A, B, und C zusammen 180 grad, oder sind so gross als 2 rechte Winkel. nemlich: $A+B+ACB = 180$ gr.

Demonstratio. Durch das punct C werde eine Linie DE der Seite AB parallel gezogen, und da wird seyn $B = BCE$, und $A = ACD$. Daher $A+B+ACB = BCE+ACD+ACB$. Aber die drey Winkel [illegible] halten 180 grad, und also ist [illegible] $A+B+ACB = 180$ grad.




Prop. Wenn in einem Dreyeck ABC eine Seite AB verlängert wird, so wird der äussere Winkel CBD so gross seyn als die zwey innerlichen A und C zusammen genommen sind. Also $CBD = A+C$.

Demonst. $CBD + CBA = 180$ gr. Es ist auch $A+C+CBA = 180$ gr. Also ist $CBD+CBA = A+C+CBA$. Man nehme [illegible] den Winkel CBA weg, so wird $CBD = A+C$ [illegible].

## Von den Triangeln.

22

Die Triangel werden auch Dreyecke [illegible] eingetheilt in Absicht der Seiten, und [illegible] in Absicht der Winkel.

In Ansehung der Seiten ist ein Dreyeck entweder aequilaterum, oder Isosceles, oder Scalenum.


Ein Triangulum aequilaterum ist ein Dreyeck, dessen alle drey Seiten einander gleich sind. z.B. [illegible] ABC ist $AB = AC = BC$. Dieses [illegible] wird auch ein gleichseitiges Dreyeck und Triangulum Isopleuron genannt.


Ein Triangulum Isosceles oder Aequicrurum ist ein Dreyeck, dessen nur zwey Seiten einander gleich sind. Ein Beyspiel ist ABC, worin $AC=BC$. Die dritte Seite AB wird die Basis genannt.

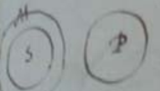

Ein Triangulum scalenum ist ein Dreyeck, dessen keine Seite der andern gleich, sondern alle unter sich [illegible] ungleich sind. In einem [illegible] als ABC ist AB die grösste, BC die mittlere, und AC die kleinste Seite.

[illegible]

Si t.S continetur in M et t.M continetur in P

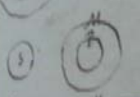
Conclusio universalis negans t.S non est P

I. Si t.S est intra M et t.M extra P.



II.

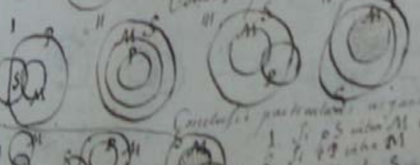
II. Si t.S est extra M. et totum P intra M



Conclusio particularis affirmans p.S est P

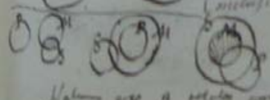
1. Si t.M intra P et p.S intra M
2. Si t.M intra S et p.P intra M

I II III IV



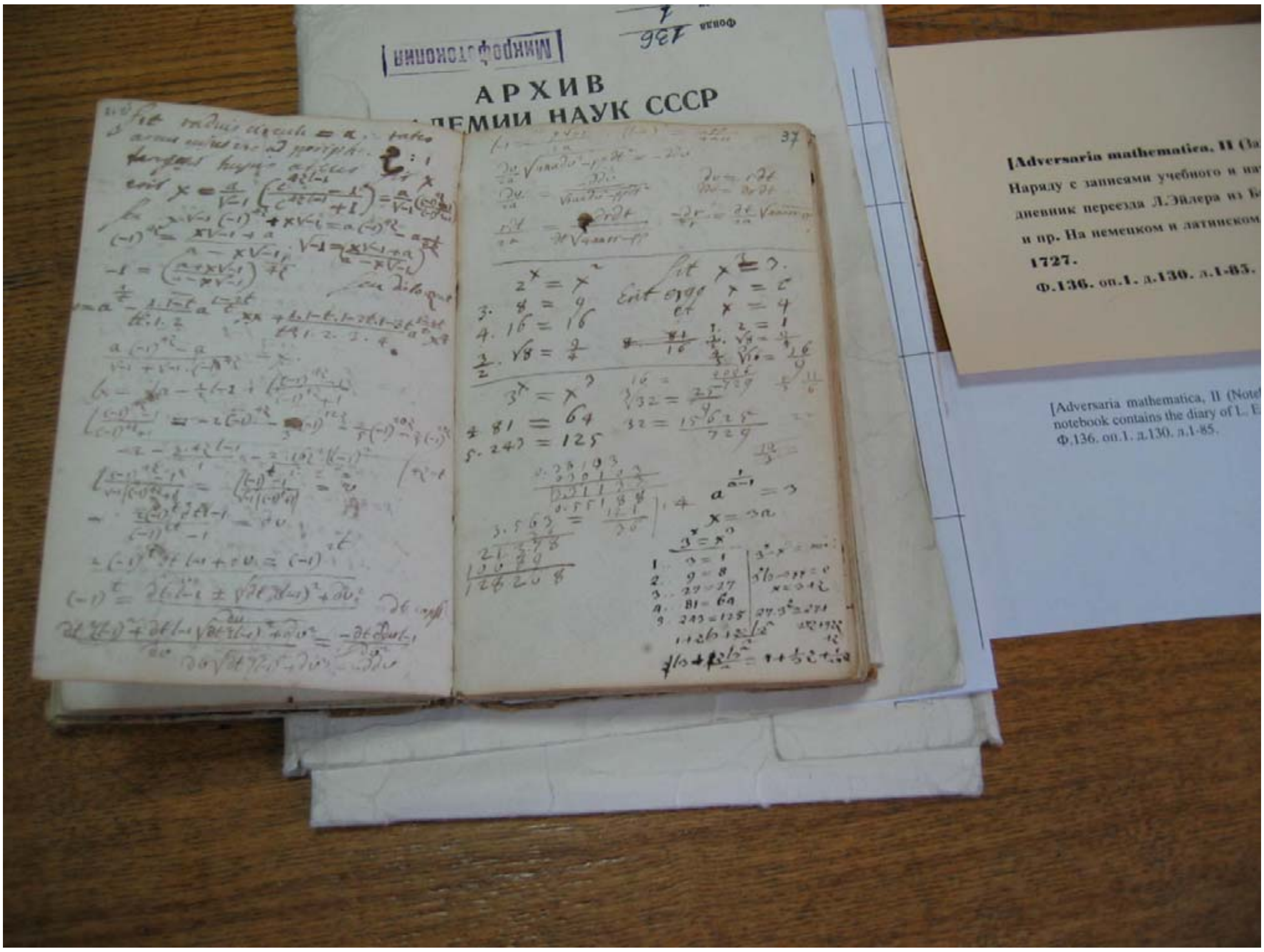
Conclusio particularis negans p.S non est P

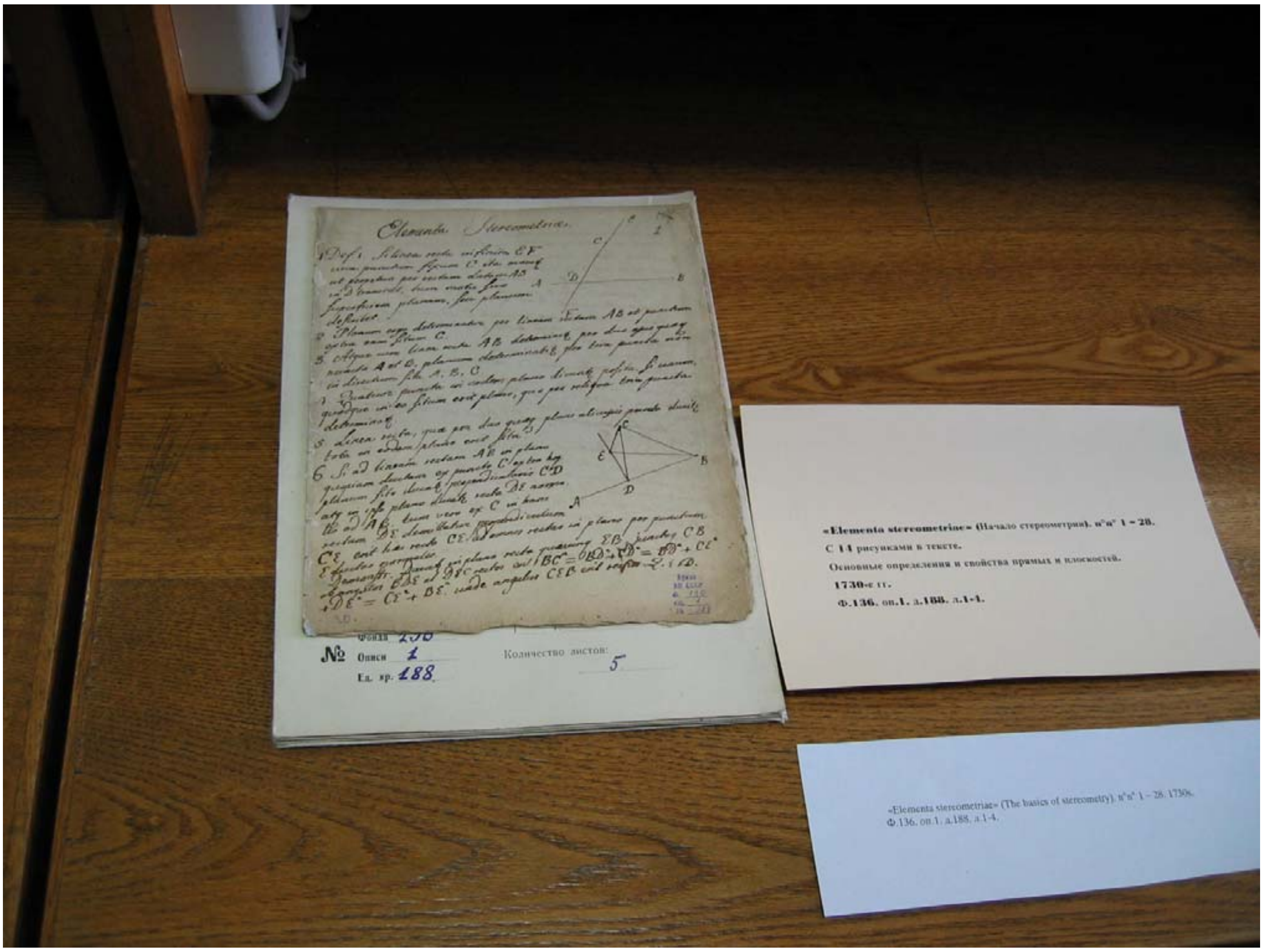
1. Si p.S intra M et t.M extra P
2. Si t.P intra M et p.S extra M
3. Si t.M intra S et t.M extra P

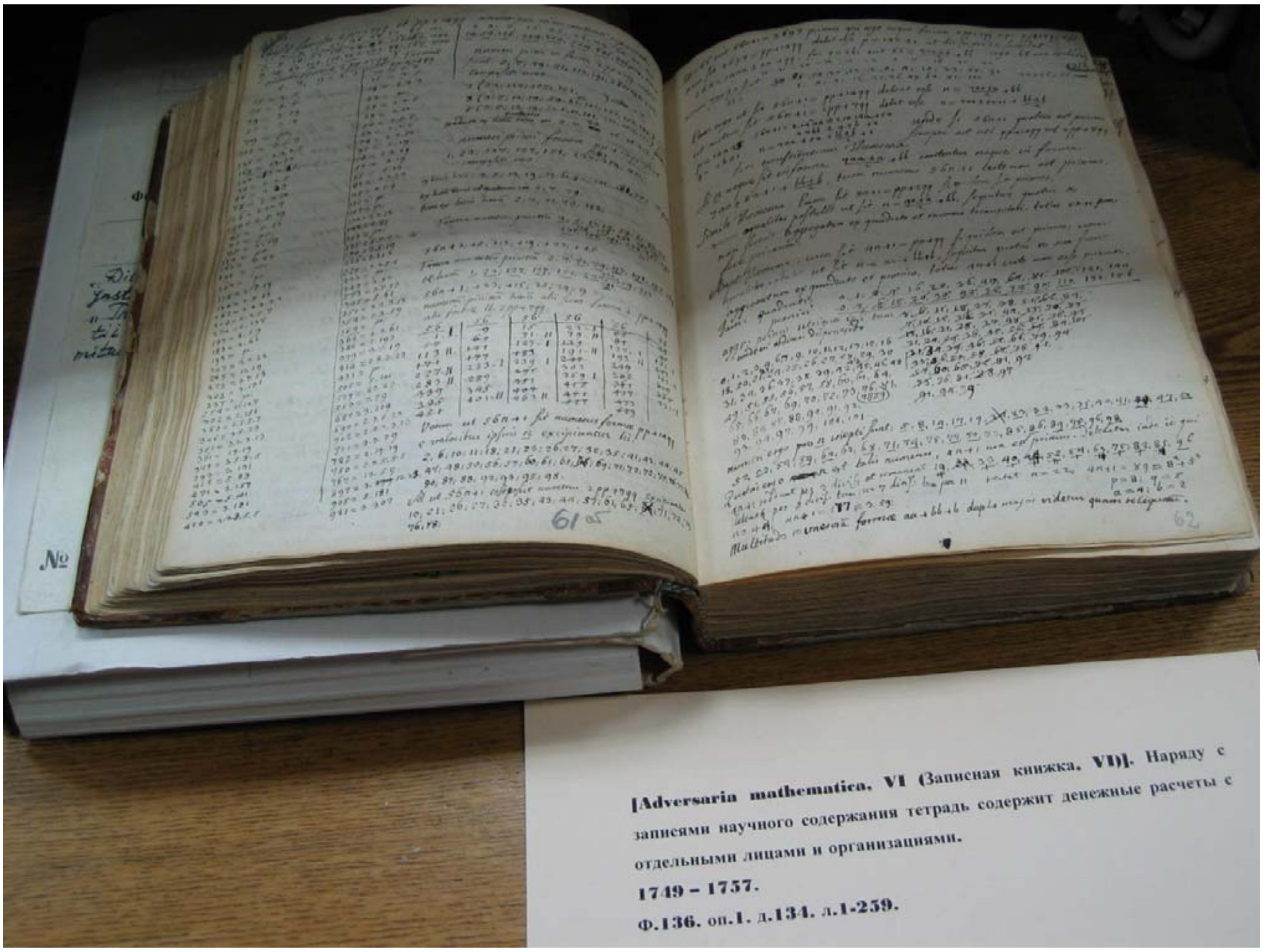


Habemus ergo 8 regulas concludendi

[illegible]


АРХИВ
ДЕМИИ НАУК СССР
[Adversaria mathematica, II
Наряду с записями учебного и
дневник переезда Л.Эйлера из
и пр. На немецком и латинском
1727.
Ф.136. оп.1. д.130. л.1-85.
[Adversaria mathematica, II (Note
notebook contains the diary of L. E
Ф.136. оп.1. д.130. л.1-85.


Elementa Stereometriae.
№ Описи 1
Ед. хр. 188
Количество листов: 5
«Elementa stereometriae» (Начало стереометрии). n°n° 1 – 28.
С 14 рисунками в тексте.
Основные определения и свойства прямых и плоскостей.
1730-е гг.
Ф.136. оп.1. д.188. л.1-4.
«Elementa stereometriae» (The basics of stereometry). n°n° 1 – 28. 1730s.
Ф.136. оп.1. д.188. л.1-4.


[Adversaria mathematica, VI (Записная книжка, VI)]. Наряду с записями научного содержания тетрадь содержит денежные расчеты с отдельными лицами и организациями.
1749 – 1757.
Ф.136. оп.1. д.134. л.1-259.

**Leonhard Euler**, *Opera Omnia*

Edited by the Euler Committee of the Swiss Academy of Science in collaboration with numerous specialist, Birkhaüser, Basel-Boston

**Series I** (*Opera Mathematica*)

Vol. **1** (1911)
to
Vol. **29** (1956)

**Series II** (*Opera Mechanica et Astronomica*)

Vol. **1** (1912)
to
Vol. **31** (1994)

**Series III** (*Opera Physica, Micellanea*)

Vol. **1** (1926)
to
Vol. **12** (1960)

**Series IV** (*Commercium Epistolicum*)

Vol. **1** (1975)
to
Vol. **6** (1986)

A glossary of 44 items on terms, formulae, equations and theorems which bear Euler's name, *Math. Magazine*, Vol. **56**, No. 5 (1983).

Euler Angles
Euler (-Poincaré) Characteristic
Euler Path/Circuit
Euler's Constant $\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log_e n\right]$
Euler (-Venn) Diagram
Euler Totient Function $\varphi(n)$
Euler's Identity $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\ \text{prime}}\left[1-\frac{1}{p^s}\right]^{-1}$
Euler's Formula for $e^{i\theta}$
Euler (-Lagrange) Equation
Euler's Equation in Hydrodynamics
Euler's Iteration Method
Euler's Criterion $a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \pmod p$, $p$ odd prime
Euler's Theorem on Congruence $a^{\varphi(n)} \equiv 1 \pmod n$, $(a,n)=1$
Euler (-Lagrange) Theorem
Euler's Theorem for Polyhedra $V-E+F=2$
Euler's Theorem on Rotation
Euler's Conjecture on Latin Squares
Euler's Theorem on Homogeneous Function
Euler's Theorem on Primes $\sum_{p\ \text{prime}}\frac{1}{p}$ and $\prod\left[1-\frac{1}{p}\right]^{-1}$ diverge

**etc.**

一位旅遊者，如果他必須等到親自檢查了橋的每一部份都穩固才肯過橋的話，他不會走得很遠。有時得冒一點險，即使從事數學研究亦復如是。

(A traveller who refuses to pass over a bridge until he has personally tested the soundness of every part of it is not likely to go far; something must be risked, even in mathematics.)

Horace Lamb
(1849-1934)


*Hakase no Aishita Sushiki* 博士熱愛的算式
*(The Professor's Beloved Formula)*
Yoko Ogawa (2003) 小川洋子
movie released in 2006
English translation: *The Gift of Numbers* (2006)

*The Professor's Beloved Formula*

Yoko Ogawa (2003)


$$e^{i\pi} + 1 = 0$$

Euler, "Introductio in analysin infinitorum" (1748)



Secantes autem et cosecantes ex tangentibus per solam subtractionem inveniuntur; est enim

$$\text{cosec.}\, z = \cot.\frac{1}{2}z - \cot.\, z$$

et hinc

$$\sec.\, z = \cot.\left(45^0 - \frac{1}{2}z\right) - \text{tang.}\, z.$$

Ex his ergo luculenter perspicitur, quomodo canones sinuum construi potuerint.

138. Ponatur denuo in formulis § 133 arcus $z$ infinite parvus et sit $n$ numerus infinite magnus $i$, ut $iz$ obtineat valorem finitum $v$. Erit ergo $nz = v$ et $z = \frac{v}{i}$, unde $\sin.\, z = \frac{v}{i}$ et $\cos.\, z = 1$; his substitutis fit

$$\cos.\, v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i + \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2}$$

atque

$$\sin.\, v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i - \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}.$$

In capite autem praecedente vidimus esse

$$\boxed{\left(1 + \frac{z}{i}\right)^i = e^z}$$

denotante $e$ basin logarithmorum hyperbolicorum; scripto ergo pro $z$ partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit

$$\cos.\, v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$

et

$$\sin.\, v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur, quomodo quantitates exponentiales imaginariae ad sinus et cosinus arcuum realium reducantur.[1]) Erit vero

1) Has celeberrimas formulas, quas ab inventore *Formulas Eulerianas* nominare solemus, Eulerus distincte primum exposuit in Commentatione 61 (indicis Enestroemiani): *De summis*

$$\boxed{\begin{aligned} e^{+v\sqrt{-1}} &= \cos. v + \sqrt{-1} \cdot \sin. v \\ \text{et} \qquad e^{-v\sqrt{-1}} &= \cos. v - \sqrt{-1} \cdot \sin. v. \end{aligned}}$$

(★)

139. Sit iam in iisdem formulis § 133 $n$ numerus infinite parvus seu $n = \frac{1}{i}$ existente $i$ numero infinite magno; erit

$$\cos. nz = \cos. \frac{z}{i} = 1 \quad \text{et} \quad \sin. nz = \sin. \frac{z}{i} = \frac{z}{i};$$

arcus enim evanescentis $\frac{z}{i}$ sinus est ipsi aequalis, cosinus vero $= 1$. His positis habebitur

$$1 = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}} + (\cos. z - \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}}}{2}$$

et

$$\frac{z}{i} = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}} - (\cos. z - \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$

Sumendis autem logarithmis hyperbolicis supra (§ 125) ostendimus esse

$$l(1+x) = i(1+x)^{\frac{1}{i}} - i \quad \text{seu} \quad y^{\frac{1}{i}} = 1 + \frac{1}{i} ly$$

posito $y$ loco $1 + x$. Nunc igitur posito loco $y$ partim $\cos. z + \sqrt{-1} \cdot \sin. z$ partim $\cos. z - \sqrt{-1} \cdot \sin. z$ prodibit

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*serierum reciprocarum ex potestatibus numerorum naturalium ortarum*, Miscellanea Berolin. 7, 1743, p. 172; *Leonhardi Euleri Opera omnia*, series I, vol. 14. Iam antea quidem cum amico Chr. Goldbach (1690—1764) formulas huc pertinentes, partim speciales partim generaliores, communicaverat. Sic in epistola d. 9. Dec. 1741 scripta invenitur haec formula

$$\frac{2^{+\sqrt{-1}} + 2^{-\sqrt{-1}}}{2} = \text{Cos. Arc. } l\,2$$

et in epistola d. 8. Maii 1742 scripta haec

$$a^{p\sqrt{-1}} + a^{-p\sqrt{-1}} = 2 \text{ Cos. Arc. } pla.$$

Vide *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. I, p. 110 et 123; *Leonhardi Euleri Opera omnia*, series III. Confer etiam Commentationem 170 nota 1 p. 35 laudatam, imprimis § 90 et 91. A. K.

Euler's formula was published in 1743, and even earlier, in letters to Goldbach of 1741 and 1742.

$$(\cos z \pm i \sin z)^n = \cos(nz) \pm i \sin(nz)$$

(Abraham de Moivre, 1707/1722)

$$\boxed{\begin{aligned} \cos(nz) &= \frac{1}{2}[(\cos z + i \sin z)^n + (\cos z - i \sin z)^n] \\ \sin(nz) &= \frac{1}{2i}[(\cos z + i \sin z)^n - (\cos z - i \sin z)^n] \end{aligned}}$$

$c = \cos z, s = \sin z.$

By the Binomial Theorem,

$$(c \pm is)^n = [c^n - \frac{n(n-1)}{1 \cdot 2} c^{n-2} s^2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} c^{n-4} s^4 - \cdots] \pm i\,[nc^{n-1}s - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} c^{n-3} s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} c^{n-5} s^5 - \cdots]$$

$$\boxed{\cos(nz) = c^n - \frac{n(n-1)}{1 \cdot 2} c^{n-2} s^2 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} c^{n-4} s^4 - \cdots}$$

$$\boxed{\sin(nz) = nc^{n-1}s - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} c^{n-3} s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} c^{n-5} s^5 - \cdots}$$

$c = \cos z, s = \sin z.$

By the Binomial Theorem,

$$(c \pm is)^n = [c^n - \frac{n(n-1)}{1\cdot 2}c^{n-2}s^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}c^{n-4}s^4 - \cdots]$$
$$\pm i\,[nc^{n-1}s - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}c^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}c^{n-5}s^5 - \cdots]$$

$$\cos(nz) = c^n - \frac{n(n-1)}{1\cdot 2}c^{n-2}s^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}c^{n-4}s^4 - \cdots$$

$$\sin(nz) = nc^{n-1}s - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}c^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}c^{n-5}s^5 - \cdots$$

$z$ gets infinitely small, $n$ gets infinitely large, but

keep $nz = v$ fixed, so "$\sin z = z$" $= \dfrac{v}{n}$, "$\cos z = 1$".

$$\boxed{\begin{aligned} \cos v &= 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \frac{v^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots \\ \sin v &= v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{v^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots \end{aligned}}$$

$$\cos v = \frac{(1+iv/n)^n + (1-iv/n)^n}{2}$$

$$\sin v = \frac{(1+iv/n)^n - (1-iv/n)^n}{2i}$$

$$\boxed{(1+z/n)^n = e^z, \text{ where } e = 2.7182818\ldots}$$

e.g.

| $n$ | $(1+\frac{1}{n})^n$ | $(1+\frac{3}{n})^n$ |
|---|---|---|
| 1 | 2 | 4 |
| 2 | 2.25 | 6.25 |
| 3 | 2.3703… | 8 |
| 4 | 2.4414… | 9.3789… |
| 5 | 2.4883… | 10.4857… |
| ⋮ | ⋮ | ⋮ |
| 10 | 2.5937… | 13.7858… |
| 100 | 2.7048… | 19.2186… |
| 1000 | 2.7169… | 19.9955… |
| 10000 | 2.7181… | 20.0765… |
| 100000 | 2.7182… | 20.0846… |
| ⋮ | ⋮ | ⋮ |
| ∞ | 2.7182818… | 20.085537… |

$$\cos v = \frac{1}{2}[e^{iv} + e^{-iv}], \ \sin v = \frac{1}{2i}[e^{iv} - e^{-iv}]$$

$$e^{\pm iv} = \cos v \pm i \sin v. \quad (\bigstar)$$

$e^{i\pi} = -1$ or $e^{i\pi} + 1 = 0$

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$= \boldsymbol{?}$$

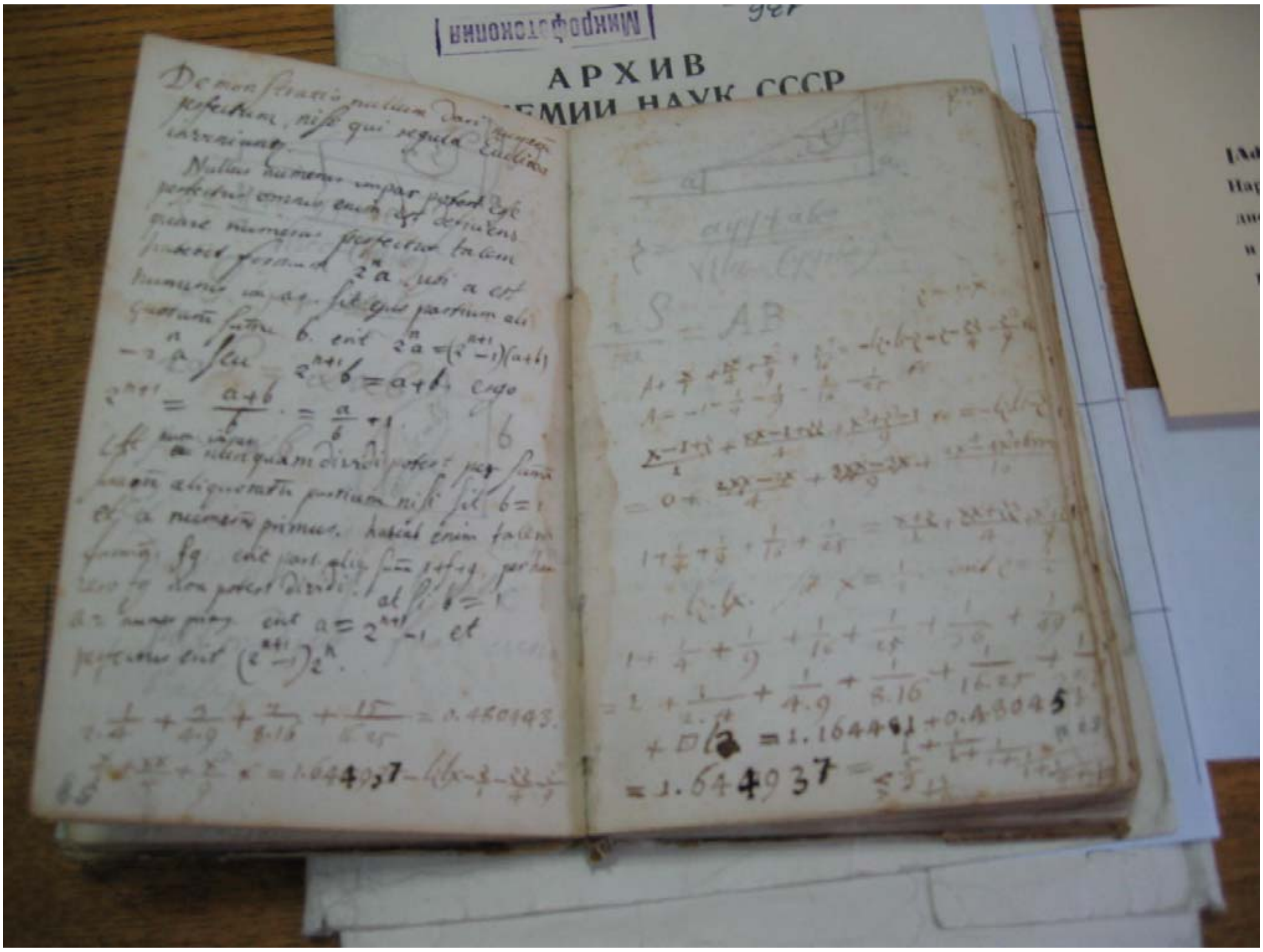
(Pietro Mengoli, 1650)

$S \approx 1.645$ (John Wallis, 1655)

$S \approx 1.644934$ (Leonhard Euler, 1731)

$S \approx 1.64493406684822643647$

(Leonhard Euler, 1735)


АРХИВ
Демонстрация nullum dari numerum perfectum, nisi qui regula Euclidea inveniatur.
=1.644937

$\alpha, \beta$ are the roots of $aX^2 + bX + c = 0$,

$(a \neq 0, c \neq 0)$.

$\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ are the roots of $cX^2 + bX + a = 0$.

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}.}$$

$\alpha, \beta, \gamma$ are the roots of $aX^3 + bX^2 + cX + d = 0$,

$(a \neq 0, d \neq 0)$.

$\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ are the roots of $dX^3 + cX^2 + bX + a = 0$.

$$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{c}{d}.}$$

$\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 = 0,$$

$(a_n \neq 0, a_0 \neq 0)$.

$\dfrac{1}{\alpha_1}, \dfrac{1}{\alpha_2}, \cdots, \dfrac{1}{\alpha_n}$ are the roots of

$$a_0 X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n = 0.$$

$$\boxed{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n} = -\frac{a_1}{a_0}.}$$

$$\sin v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots = 0$$

has roots $0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$.

$$\frac{\sin v}{v} = 1 - \frac{v^2}{1\cdot 2\cdot 3} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots = 0$$

has roots $\pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \cdots$.

$$1 - \frac{X}{1\cdot 2\cdot 3} + \frac{X^2}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots = 0$$

has roots $\pi^2, (2\pi)^2, (3\pi)^2, (4\pi)^2, \cdots$.

$$\frac{1}{\pi^2} + \frac{1}{(2\pi^2)} + \frac{1}{(3\pi^2)} + \frac{1}{(4\pi^2)} + \cdots = \frac{1}{1\cdot 2\cdot 3}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

# Riemann Zeta-Function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

$\zeta(2) = \dfrac{\pi^2}{6}$ (L. Euler, 1735)

Computation of $\zeta(2n)$ (L. Euler, 1739)

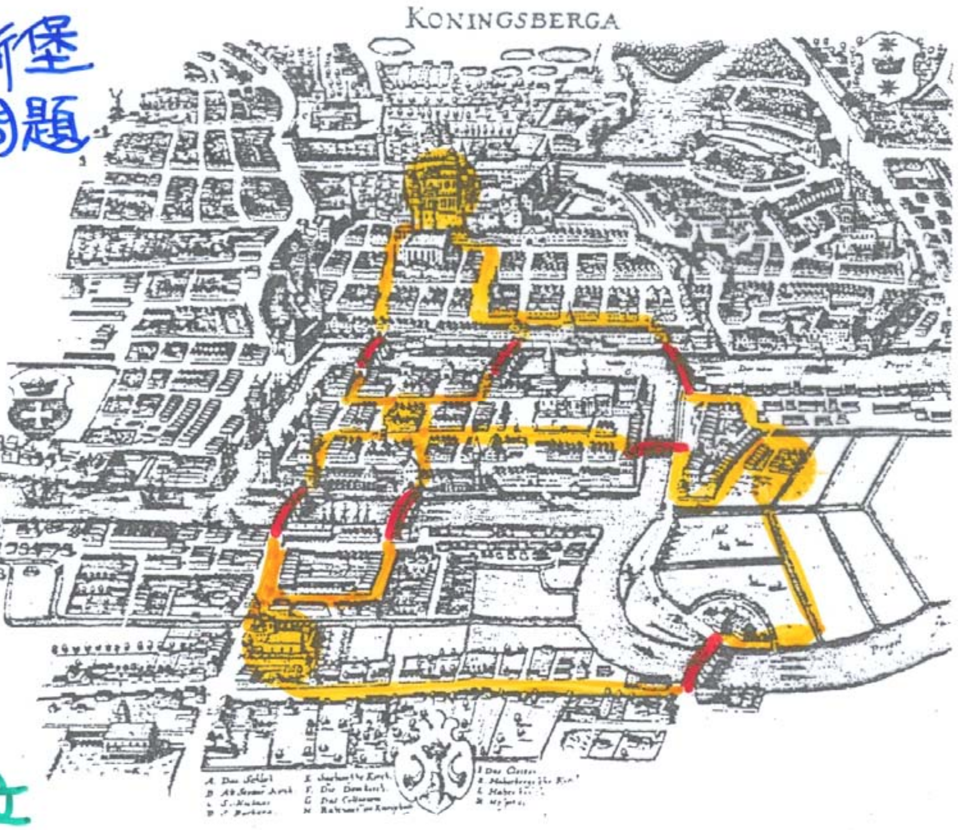
$\zeta(3)$ is irrational (R. Apéry, 1978)

Infinitely many $\zeta(2n+1)$ are irrational (T. Rivoal, 2000)

One of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational (V.V. Zudilin, 2001)

V.S. Varadarajan, *Euler Through Time: A New Look at Old Theorems* (2006)

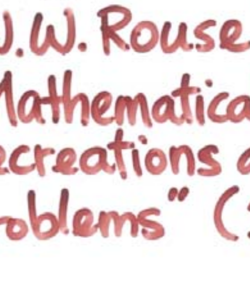
柯尼斯堡
七橋問題




歐拉

L. Euler, Solutio Problematis ad Geometriam Situs Pertinentis (Aug. 26, 1735)


First complete proof by C. Hierholzer (1873)



W. W. Rouse Ball, "Mathematical Recreations and Problems" (1892)

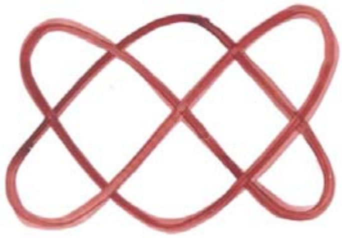
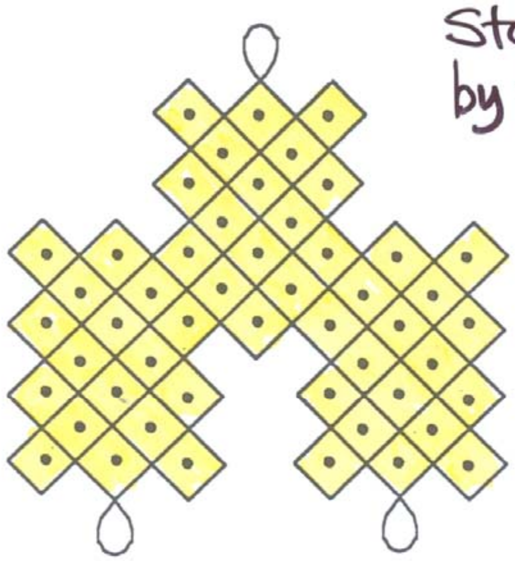
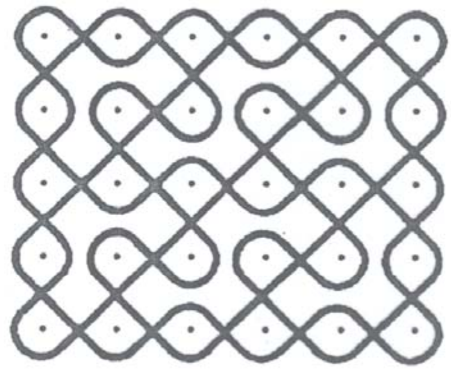
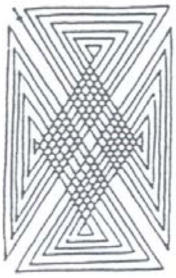


Figure drawn in the sand
by Bushoong children
(Central Africa)

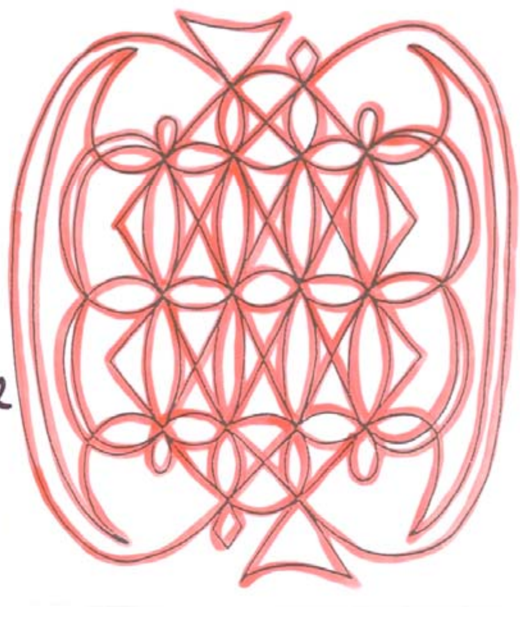
Storytelling figures in the sand
by the Tshokwe (Central Africa)

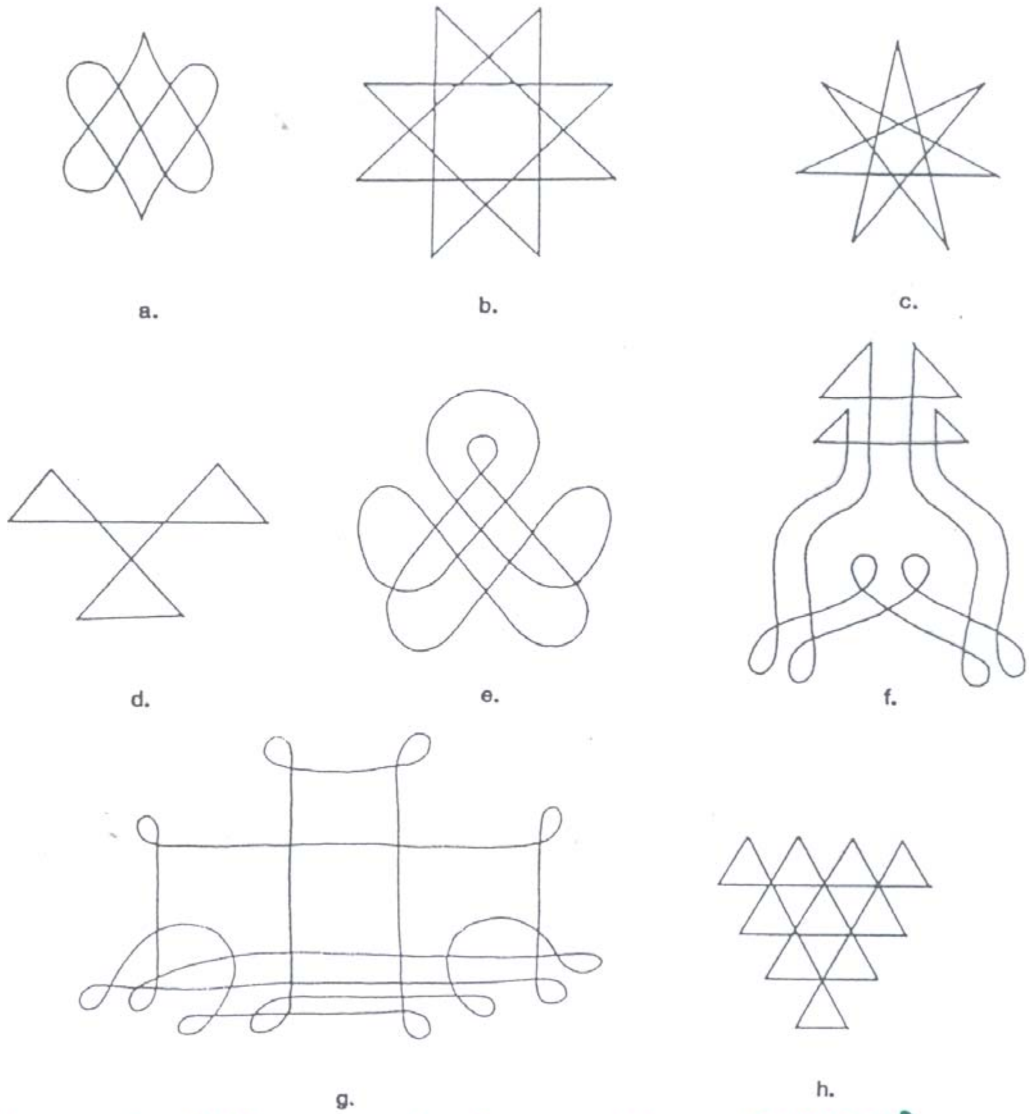






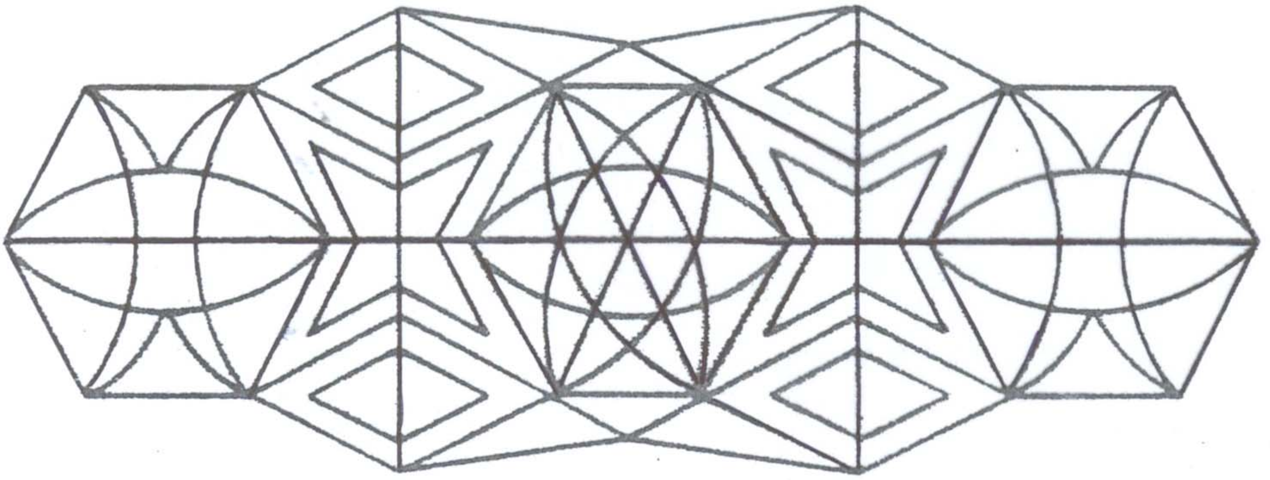
Figures in the
sand by people
of Malekula
(South Pacific)






Danish folk-puzzle by J. Kamp (1877)

J. B. Listing: Vorstudien zur Topologie (1847)



Can you trace this figure in one stroke?
(一筆畫)

This figure appeared in:

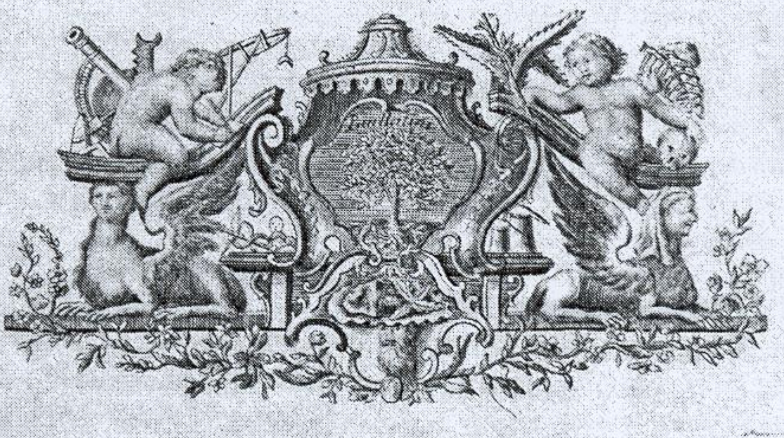
J. B. Listing, "Vorstudien Zur Topologie" (1847)

C. Hierholzer (1873 posthumously)

# COMMENTARII ACADEMIAE SCIENTIARVM IMPERIALIS PETROPOLITANAE.

TOMVS VIII.

AD ANNVM MDCCXXXVI.



PETROPOLI,
TYPIS ACADEMIAE cIↃIↃCCXLI.

Presented to the St. Petersburg Academy on August 26, 1735

L. EULER

SOLUTIO PROBLEMATIS AD GEOMETRIAM SITUS PERTINENTIS

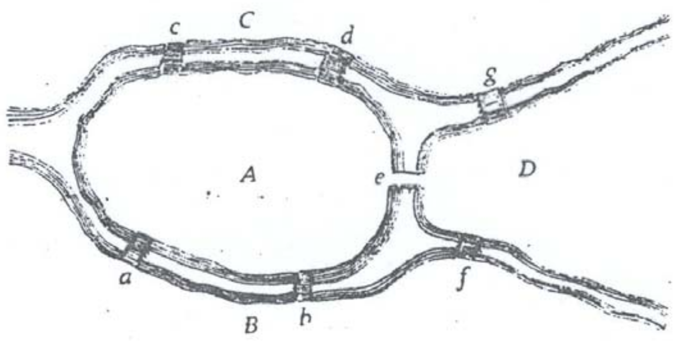
[The solution of a problem relating to the geometry of position]

*Commentarii Academiae Scientiarum Imperialis Petropolitanae* **8** (1736), 128–140.

1. In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the *geometry of position*. This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.

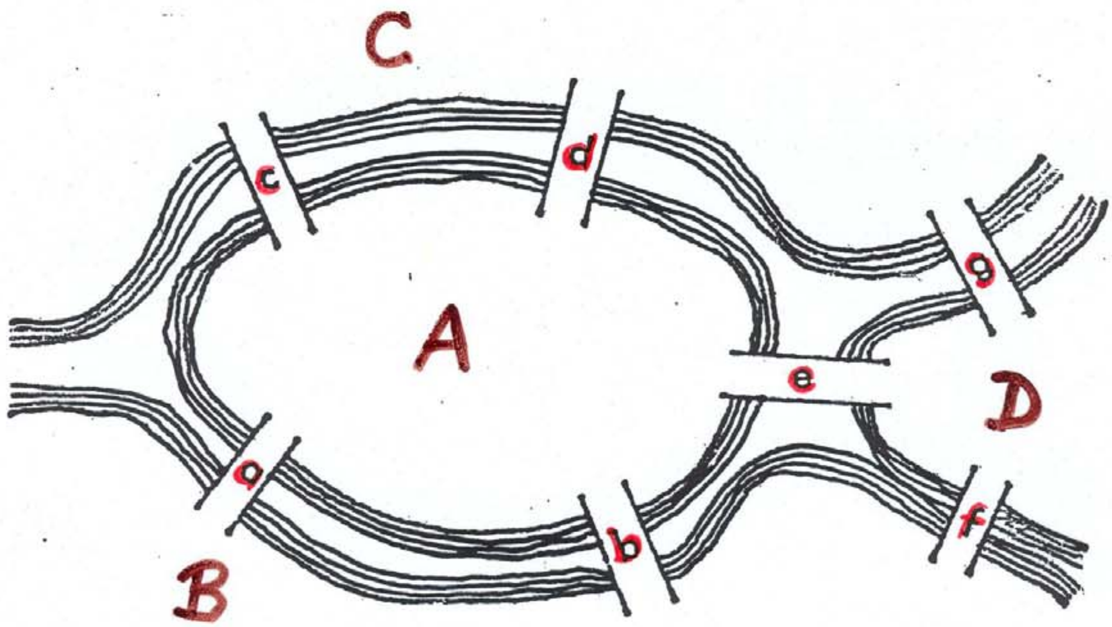
2. The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is an island *A*, called *the Kneiphof;* the river which surrounds it is divided into two branches, as can be seen in Fig. [1.2], and these branches are crossed by seven bridges, *a*, *b*, *c*, *d*, *e*, *f* and *g*. Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once. I was told that some people asserted that this was impossible, while others were in doubt; but nobody would actually assert that it could be done. From this, I have formulated the general problem: whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?

3. As far as the problem of the seven bridges of Königsberg is concerned, it can be solved by making an exhaustive list of all possible routes, and then finding

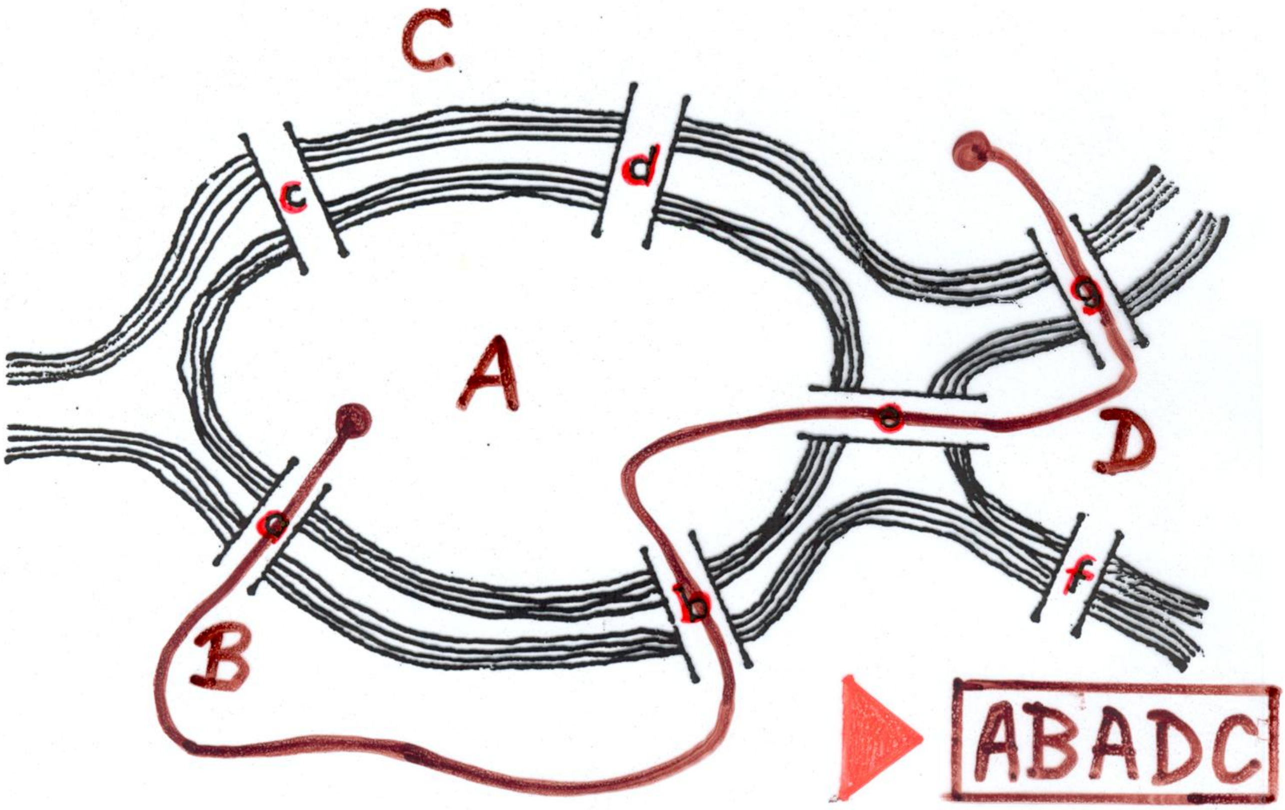



English translation in Chapter 1, *Graph Theory, 1736-1936* by N.L. Biggs, E.K. Lloyd, R.J. Wilson (1976)

2. ... From this, I have formulated the general problem: whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?




Generalization and Specialization complement each other.



C
c
d
A
g
D
B
f
b
ABADC

4. My whole method relies on the particularly convenient way in which the crossing of a bridge can be represented. ······

Good notation facilitates thinking.

7. The problem is therefore reduced to finding a sequence of eight letters, formed from the four letters $A, B, C$ and $D$, in which the various pairs of letters occur the required number of times. ······ to find out whether or not it is even possible to arrange the letters in this way, ······ to find a rule which will be useful in this case, and in others, for determining whether or not such an arrangement can exist.

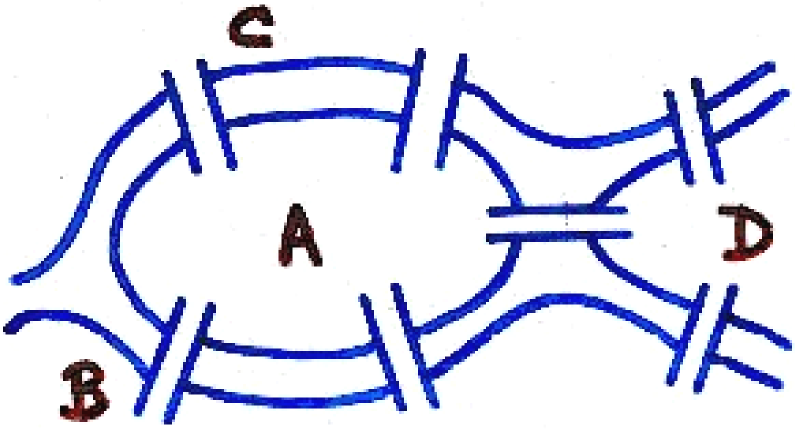
Transform a problem.

8. In order to try to find such a rule, I consider a single area $A$. ······

Sub-problem

Consider a simpler case.

7. The problem is therefore reduced to finding a sequence of eight letters, formed from the four letters $A, B, C$ and $D$, in which the various pairs of letters occur the required number of times.




$d_A = 5$ $\quad$ $\bar{n}_A = 3$

$d_B = 3$ $\quad$ $\bar{n}_B = 2$

$d_C = 3$ $\quad$ $\bar{n}_C = 2$

$d_D = 3$ $\quad$ $\bar{n}_D = 2$

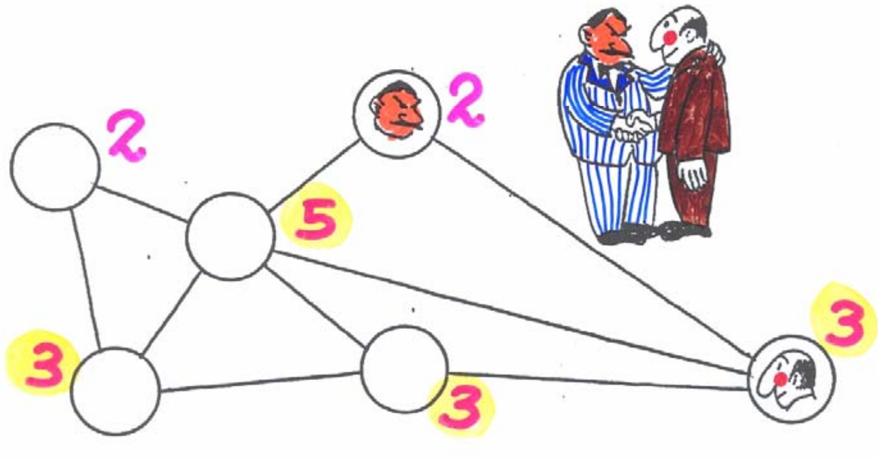
We cannot have a sequence of 8 letters from A,B,C,D with A appearing 3 times and B,C,D each appearing 2 times!

16. …… I shall, however, describe a much simpler method for determining this which is not difficult to derive from the present method, after I have first made a few preliminary observations. First, I observe that the number of bridges written next to the letters $A, B, C$, etc. together add up to twice the total number of bridges. ……

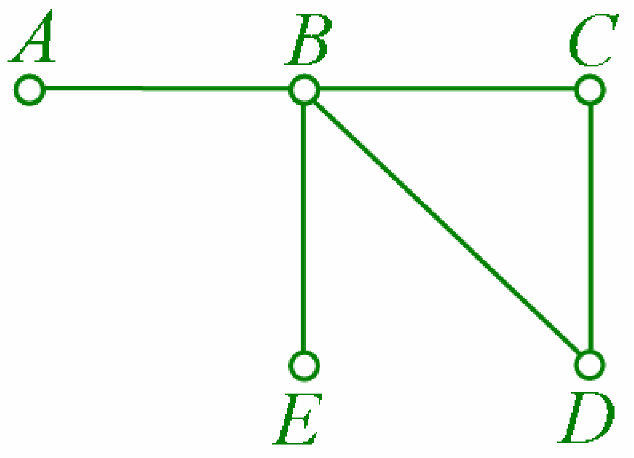
Looking back at your solution.

Handshaking Lemma

$\Sigma d = 2N$



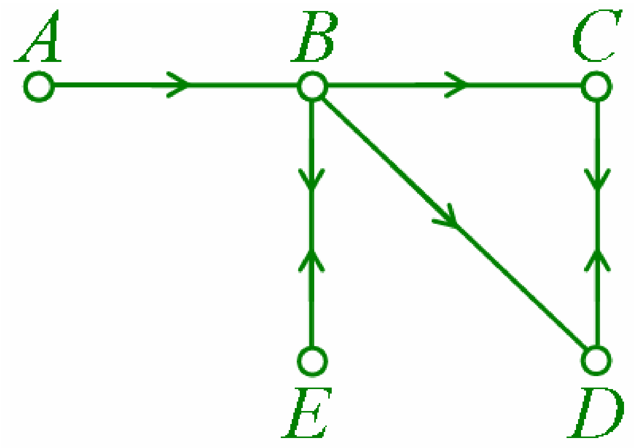

$$\text{Total number of hands stretched out} = 2\times \begin{pmatrix}\text{number of}\\ \text{handshakes}\end{pmatrix}$$




$$
\begin{array}{c c c}
 & \begin{matrix} A & B & C & D & E \end{matrix} & \begin{matrix} \text{Row sum} \\ \text{(degree)} \end{matrix} \\
\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 4 \\ 2 \\ 2 \\ 1 \end{matrix} \\
\begin{matrix} \text{Column sum} \\ \text{(degree)} \end{matrix} & \begin{matrix} 1 & 4 & 2 & 2 & 1 \end{matrix} &
\end{array}
$$

$$1 + 4 + 2 + 2 + 1 = 10 = \text{number of 1's}$$

sum of degrees = twice number of edges




$$
\begin{array}{c|ccccc|c}
 & A & B & C & D & E & \text{Row sum (out-degree)} \\
\hline
A & 0 & 1 & 0 & 0 & 0 & 1 \\
B & 0 & 0 & 1 & 1 & 1 & 3 \\
C & 0 & 0 & 0 & 1 & 0 & 1 \\
D & 0 & 0 & 1 & 0 & 0 & 1 \\
E & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline
\text{Column sum (in-degree)} & 0 & 2 & 2 & 2 & 1 &
\end{array}
$$

$$1 + 3 + 1 + 1 + 1 = 7$$

$$0 + 2 + 2 + 2 + 1 = 7$$

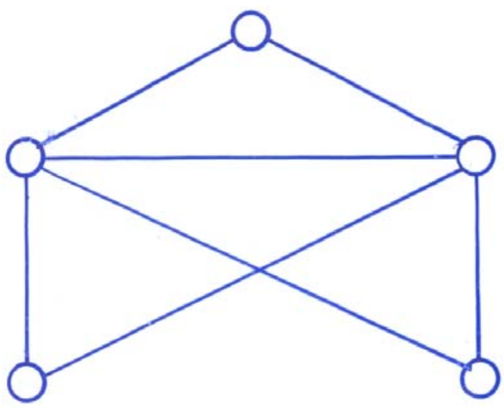
$G$ (**graph**) with its set $V(G)$ **of** vertices and set $E(G)$ of edges.

In plain English, a graph is a set of dots (vertices), some of which are joined by lines (edges).

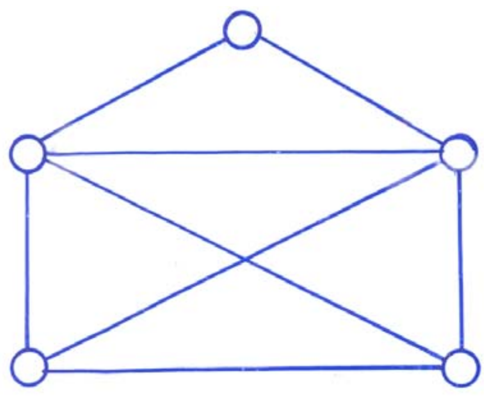
The degree of a vertex is the number of edges joined to it.


地鐵路綫圖
MTR system map
www.mtr.com.hk
新界
New Territories
九龍
Kowloon
港島
Hong Kong Island
大嶼山
Lantau Island
荃灣
Tsuen Wan
大窩口
Tai Wo Hau
葵興
Kwai Hing
葵芳
Kwai Fong
荔景
Lai King
美孚
Mei Foo
荔枝角
Lai Chi Kok
長沙灣
Cheung Sha Wan
深水埗
Sham Shui Po
太子
Prince Edward
旺角
Mong Kok
油麻地
Yau Ma Tei
佐敦
Jordan
尖沙咀
Tsim Sha Tsui
九龍塘
Kowloon Tong
石硤尾
Shek Kip Mei
樂富
Lok Fu
黃大仙
Wong Tai Sin
鑽石山
Diamond Hill
彩虹
Choi Hung
九龍灣
Kowloon Bay
牛頭角
Ngau Tau Kok
觀塘
Kwun Tong
藍田
Lam Tin
油塘
Yau Tong
調景嶺
Tiu Keng Leng
將軍澳
Tseung Kwan O
坑口
Hang Hau
寶琳
Po Lam
博覽館
AsiaWorld-Expo
機場
Airport
青衣
Tsing Yi
欣澳
Sunny Bay
迪士尼
Disneyland Resort
東涌
Tung Chung
纜車
Cable Car
南昌
Nam Cheong
奧運
Olympic
九龍
Kowloon
香港
Hong Kong
上環
Sheung Wan
中環
Central
金鐘
Admiralty
灣仔
Wan Chai
銅鑼灣
Causeway Bay
天后
Tin Hau
炮台山
Fortress Hill
北角
North Point
鰂魚涌
Quarry Bay
太古
Tai Koo
西灣河
Sai Wan Ho
筲箕灣
Shau Kei Wan
杏花邨
Heng Fa Chuen
柴灣
Chai Wan
九廣西鐵
KCR West Rail
九廣東鐵
KCR East Rail

A graph is Eulerian if there is a 'walk' (on the edges ) which covers all edges, each appearing exactly once, and ends at the starting vertex. A graph is semi-Eulerian if there is a 'walk' which covers all edges, each appearing exactly once.



Eulerian

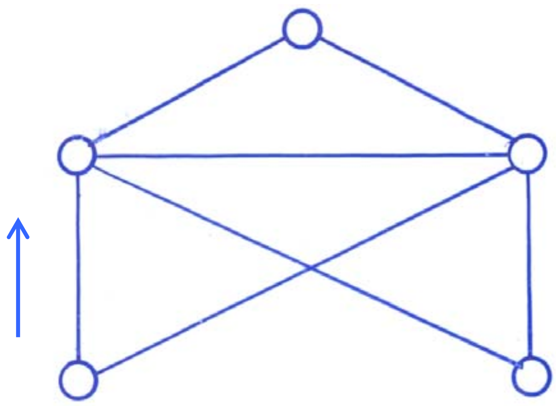


semi-Eulerian

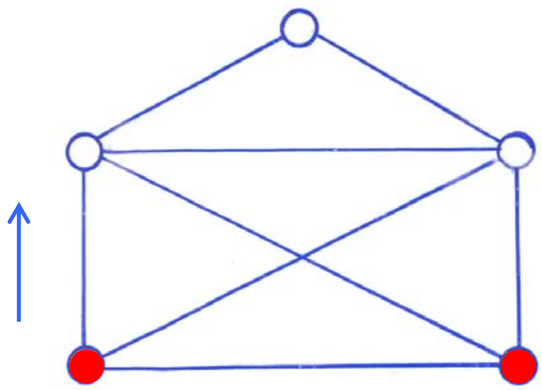
**Theorem** (Euler)

A finite graph is Eulerian if and only if it is connected and all vertices are of even degree (i.e. no vertex of odd degree). A finite graph is semi-Eulerian if and only if it is connected and there are exactly two vertices of odd degree.

A graph is Eulerian if there is a ‘walk’ (on the edges ) which covers all edges, each appearing exactly once, and ends at the starting vertex. A graph is semi-Eulerian if there is a ‘walk’ which covers all edges, each appearing exactly once.



Eulerian



semi-Eulerian

THEOREM 6.2 (Euler 1736). *A connected graph $G$ is* **Eulerian** *if and only if the* **degree of each vertex** *of $G$ is* **even**.

*Proof.* $\Rightarrow$ Suppose that $P$ is an Eulerian trail of $G$. Whenever $P$ passes through a vertex, there is a contribution of 2 towards the degree of that vertex. Since each edge occurs exactly once in $P$, each vertex must have even degree.

$\Leftarrow$ The proof is by induction on the number of edges of $G$. Suppose that the degree of each vertex is even. Since $G$ is connected, each vertex has degree at least 2 and so, by Lemma 6.1, $G$ contains a cycle $C$. If $C$ contains every edge of $G$, the proof is complete. If not, we remove from $G$ the edges of $C$ to form a new, possibly disconnected, graph $H$ with fewer edges than $G$ and in which each vertex still has even degree. By the induction hypothesis, each component of $H$ has an Eulerian trail. Since each component of $H$ has at least one vertex in common with $C$, by connectedness, we obtain the required Eulerian trail of $G$ by following the edges of $C$ until a non-isolated vertex of $H$ is reached, tracing the Eulerian trail of the component of $H$ that contains that vertex, and then continuing along the edges of $C$ until we reach a vertex belonging to another component of $H$, and so on. The whole process terminates when we return to the initial vertex (see Fig. 6.6). //

This proof can easily be modified to prove the following two results. We omit the details.

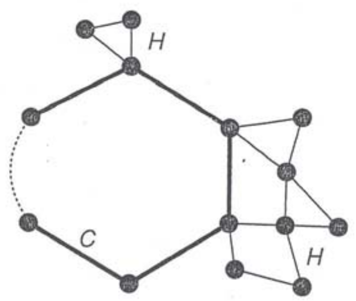



*Fig. 6.6*

[R.J. Wilson, "Introduction to Graph Theory", 4th ed., 1996.]

COROLLARY 6.3. *A connected graph is Eulerian if and only if its set of edges can be split up into disjoint cycles.*

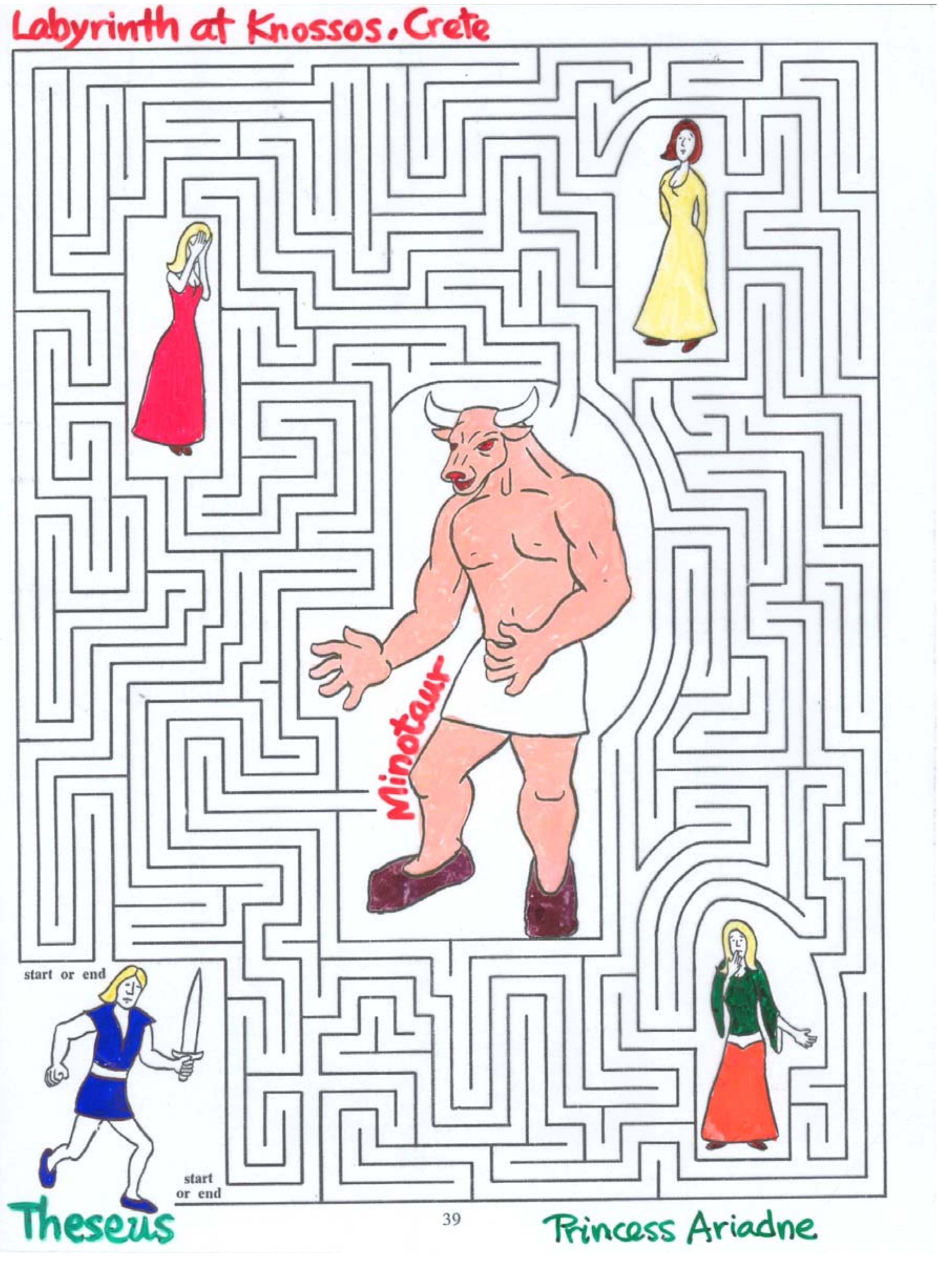
COROLLARY 6.4. *A connected graph is* **semi-Eulerian** *if and only if it has* **exactly two vertices** *of* **odd degree**.

- A concept or definition in mathematics does not come out of the blue. It is man-made, but it is not arbitrary nor artificial.

- A proof in mathematics evolves with time. What the first proof may lack in completeness and polish, it makes up for in clarity, wealth of ideas, and revelation of the author's train of thought.

> "A good proof is one which makes us wiser."
>
> Yuri I. Manin

Yurii Ivanovich Manin
(1937 - )


Labyrinth at Knossos, Crete
Minotaur
start or end
start or end
Theseus
Princess Ariadne

# Cretan Labyrinth at Knossos

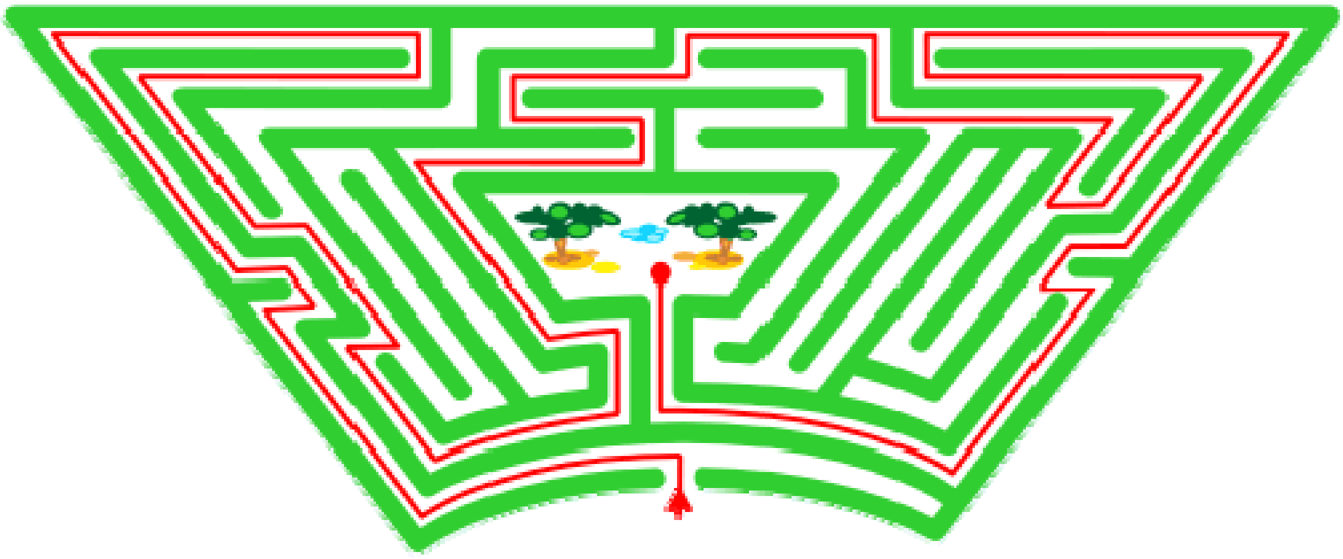
Cretan Labyrinth at Knossos

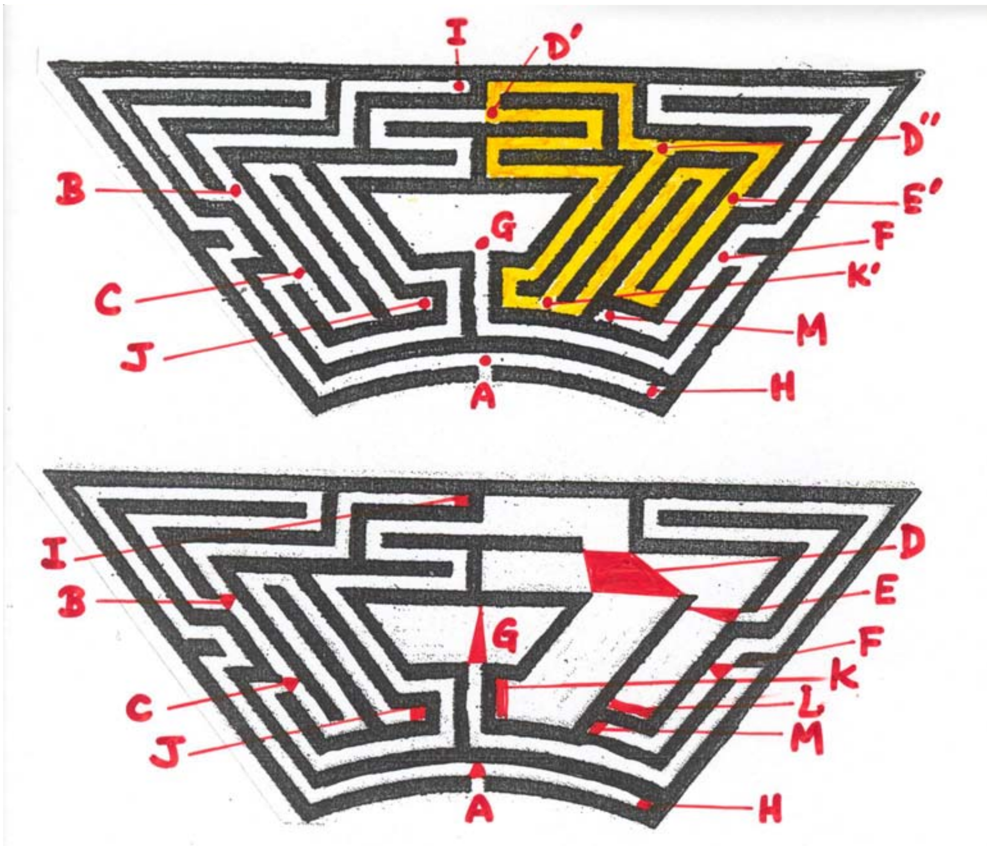
Mister, have you dropped something?

先生，你是否掉了這團線呢？

Hampton Court Palace

Hampton Court Maze

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D''
B
E'
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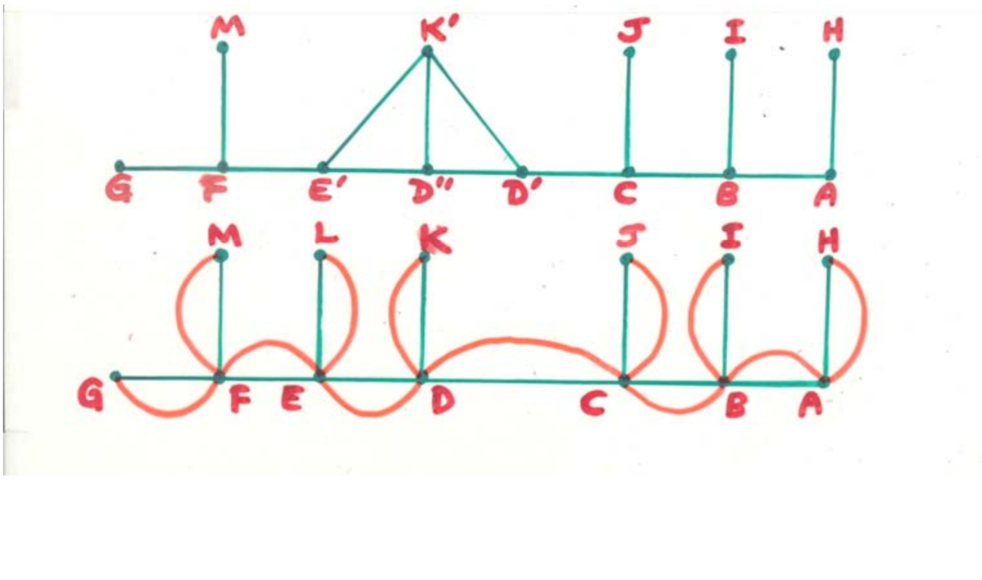
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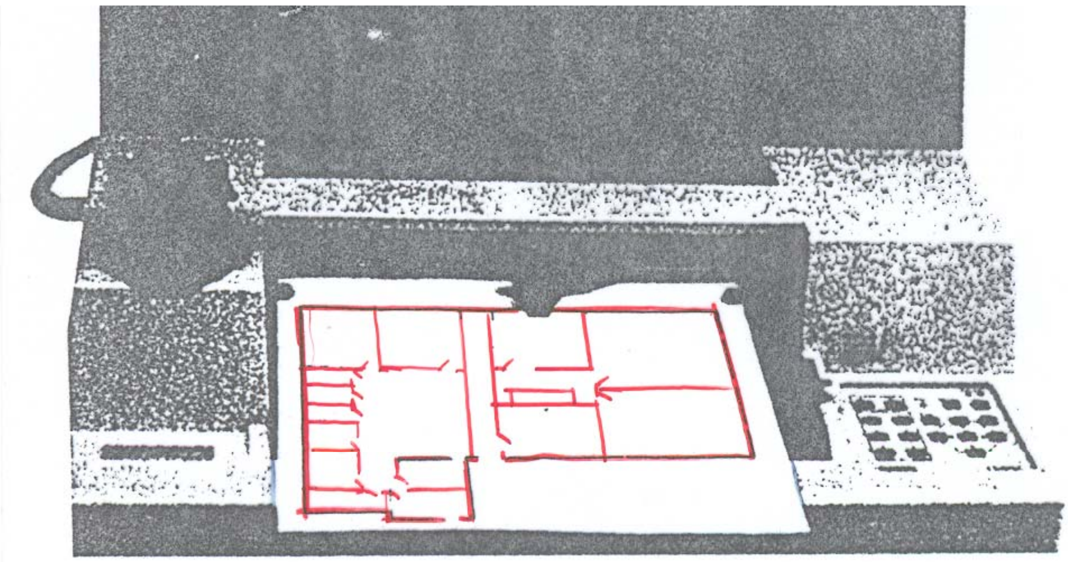
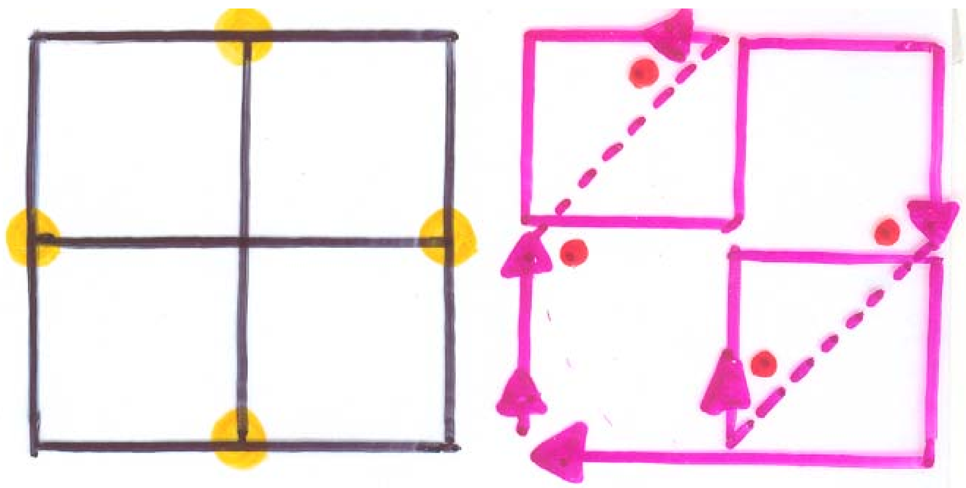
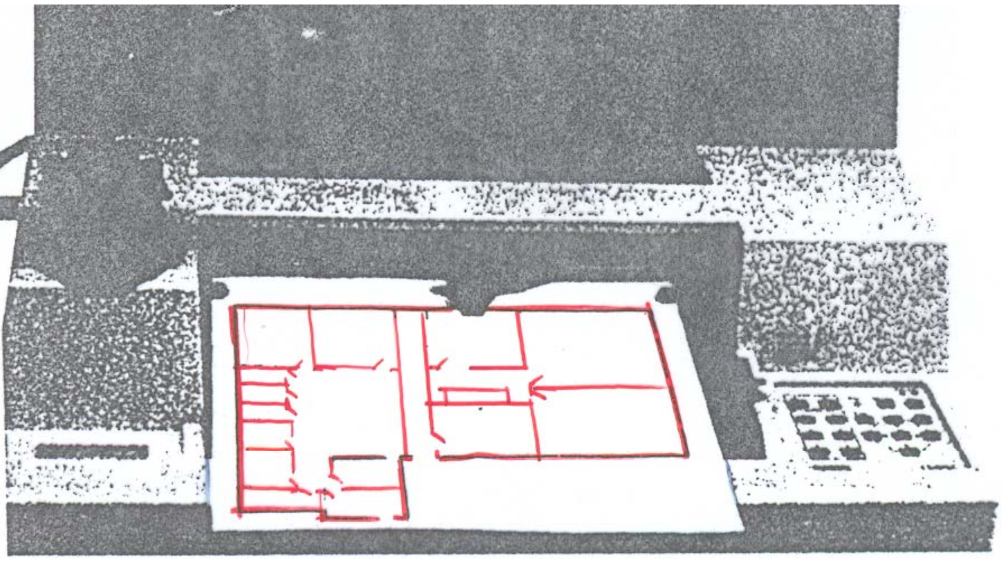
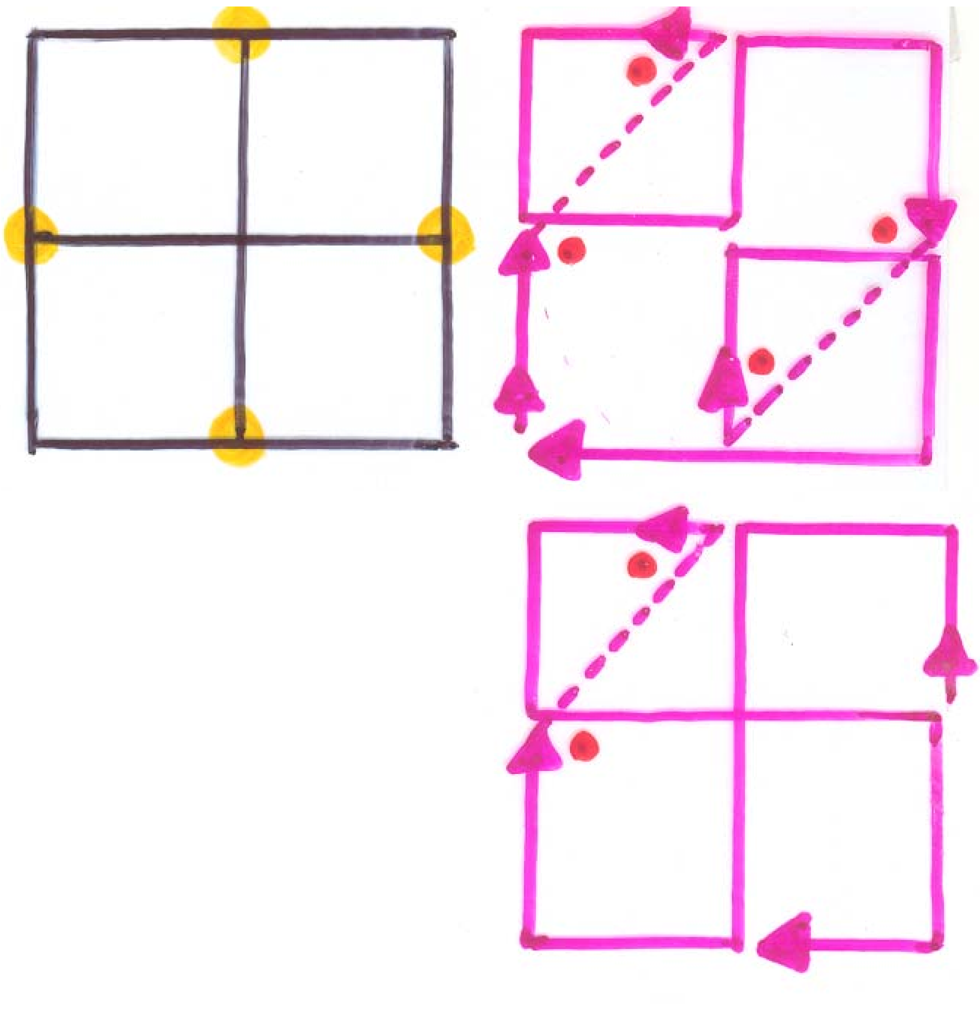


AN INK-PEN PLOTTER.





AN INK-PEN PLOTTER.



*Mathematical Programming 5 (1973) 88–124. North-Holland Publishing Company*

# 匹配、歐拉路及中國郵差

# MATCHING, EULER TOURS AND THE CHINESE POSTMAN

Jack EDMONDS
*University of Waterloo, Waterloo, Ontario, Canada*

and

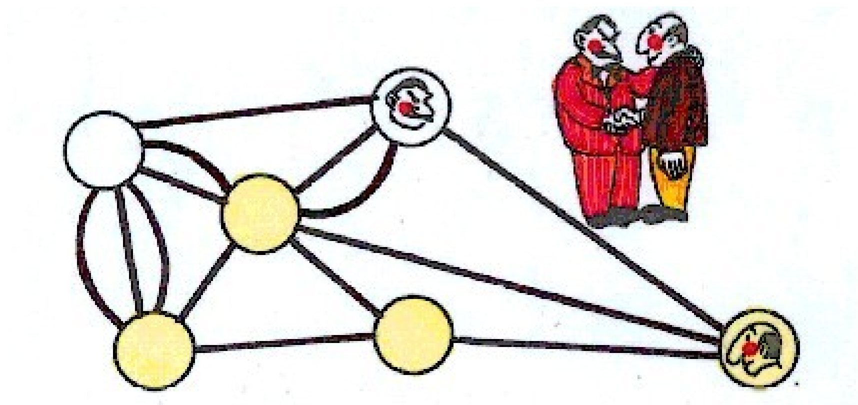
Ellis L. JOHNSON
*IBM Watson Research Center, Yorktown Heights, New York, U.S.A.*

Received 20 May 1972
Revised manuscript received 3 April 1973

The solution of the Chinese postman problem using matching theory is given. The convex hull of integer solutions is described as a linear programming polyhedron. This polyhedron is used to show that a good algorithm gives an optimum solution. The algorithm is a specialization of the more general $b$-matching blossom algorithm. Algorithms for finding Euler tours and related problems are also discussed.

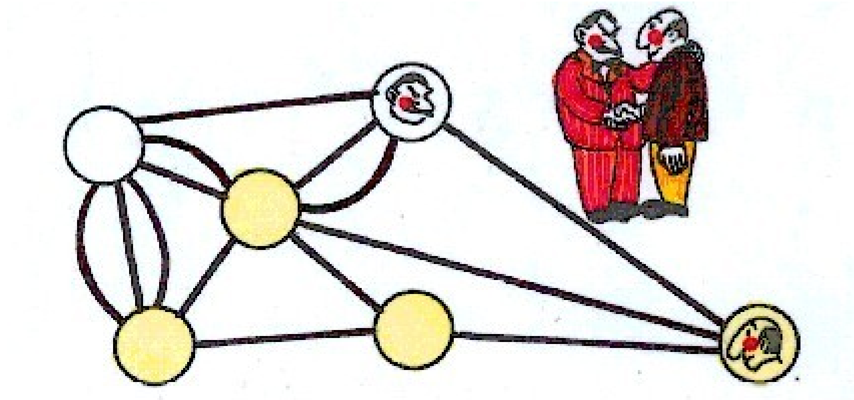
J. EDMONDS & E.L. JOHNSON, Math. Program., vol. 5 (1973), 88–124.

GUAN Meigu (管梅谷), Graphic programming using odd or even points (奇偶點圖上作業法), Chinese Math., vol. 1 (1962), 273–277.
J. Edmond, The Chinese Postman Problem, Oper. Res., vol. 13, Suppl. 1 (1965), 373.



Number of extended hands = 
2 × (number of handshakes)

? How many guests extend their hands an odd number of times?

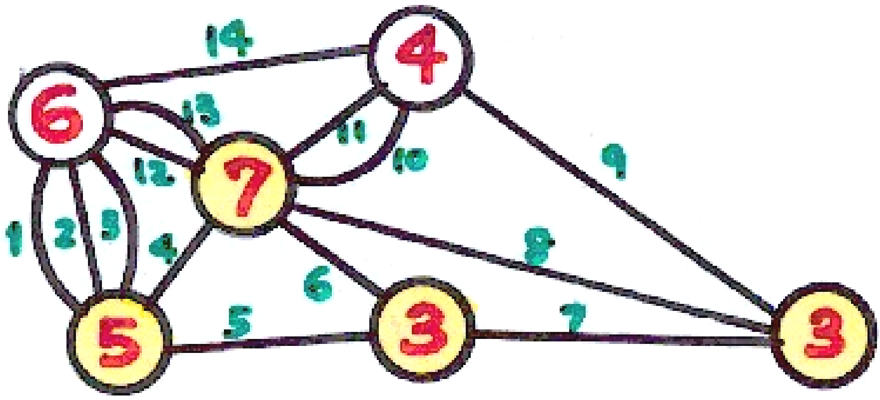


Number of extended hands = $2 \times$ (number of handshakes)

? How many guests extend their hands an odd number of times?

**Answer:** We don't know the exact number, but it must be an even number.

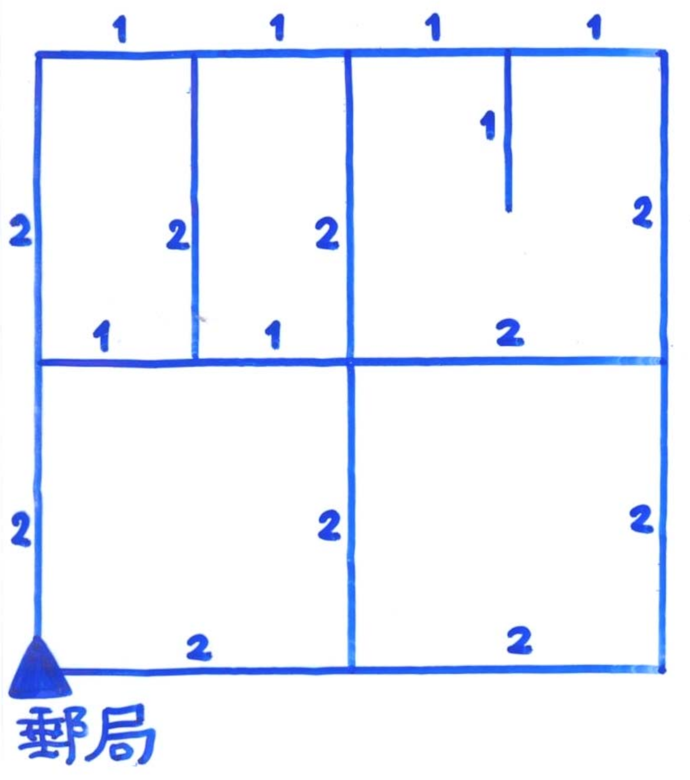
In a graph, there is an even number of vertices of odd degree.




$$6+4+7+5+3+3=28=2\times 14$$

# Chinese Postman Problem

A postman starts from the Post Office, covers every street and returns to the Post Office. Find a shortest route for him.



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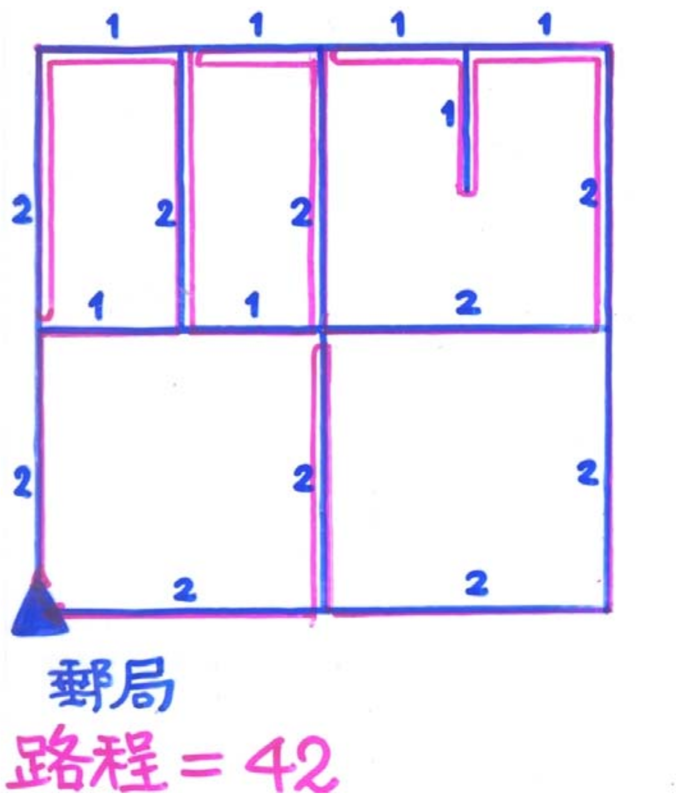

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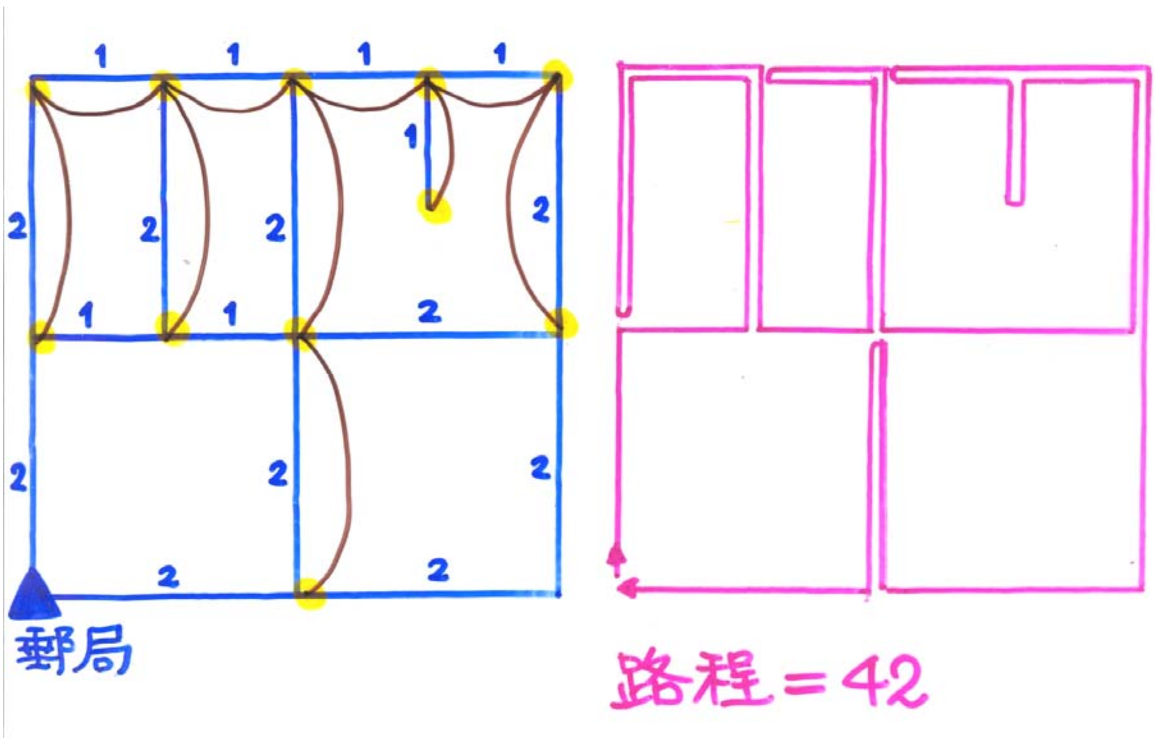

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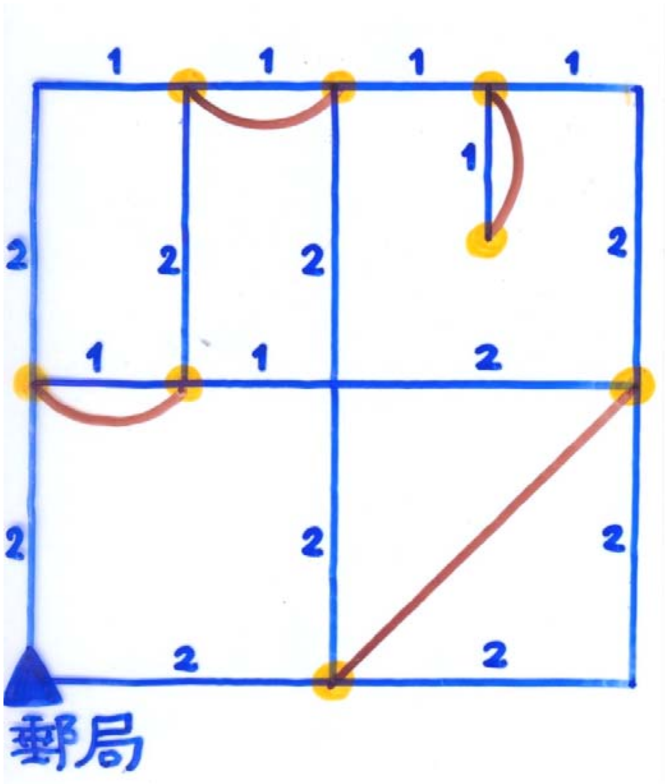

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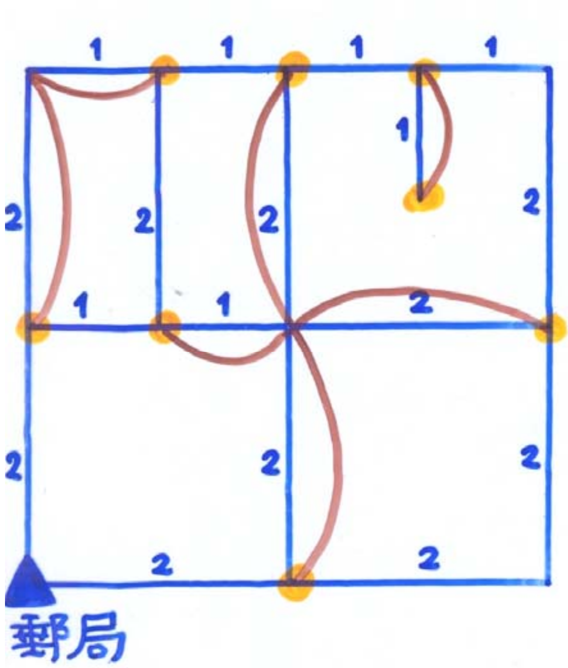


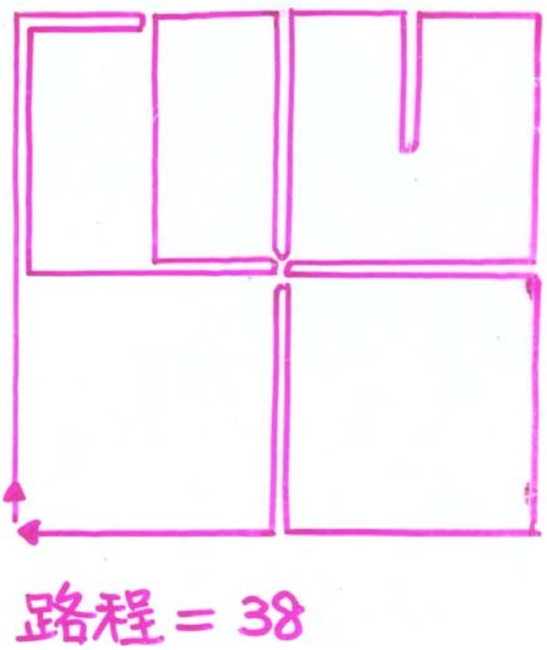

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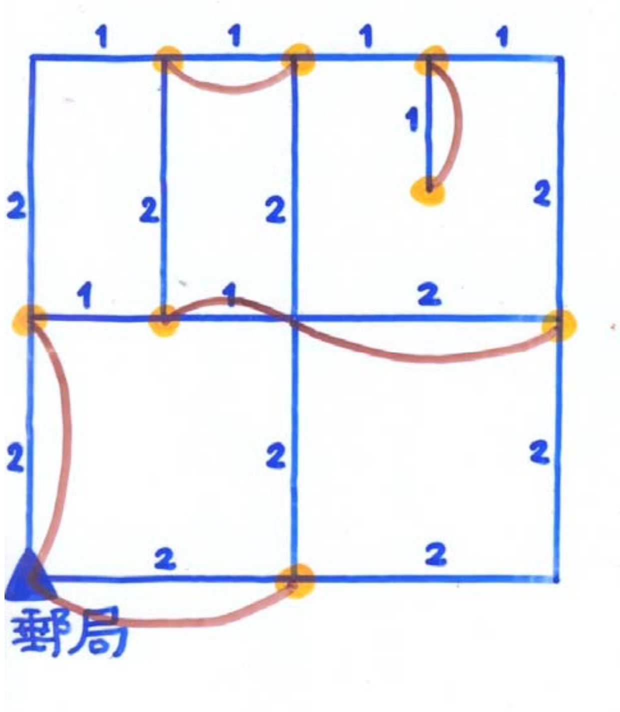


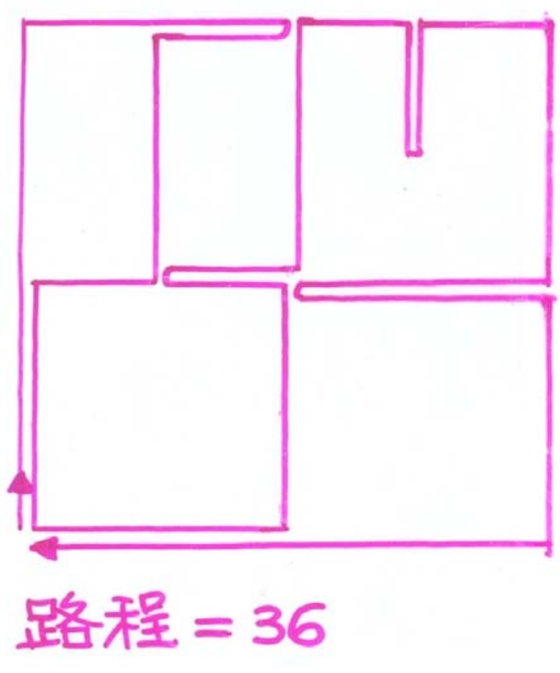

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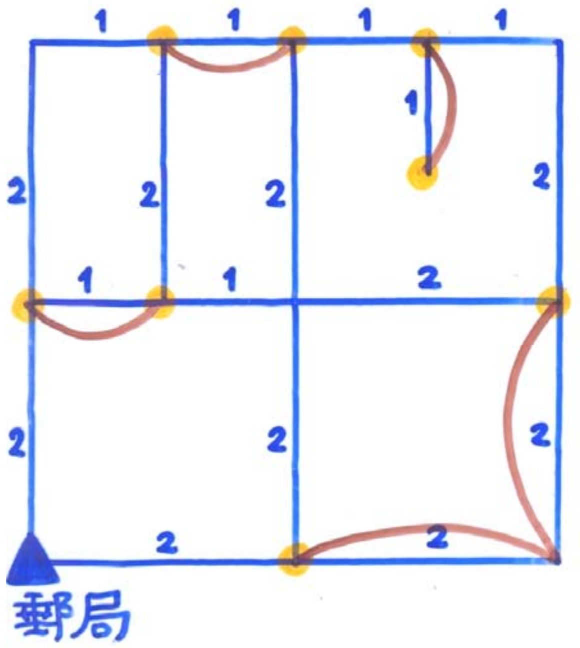
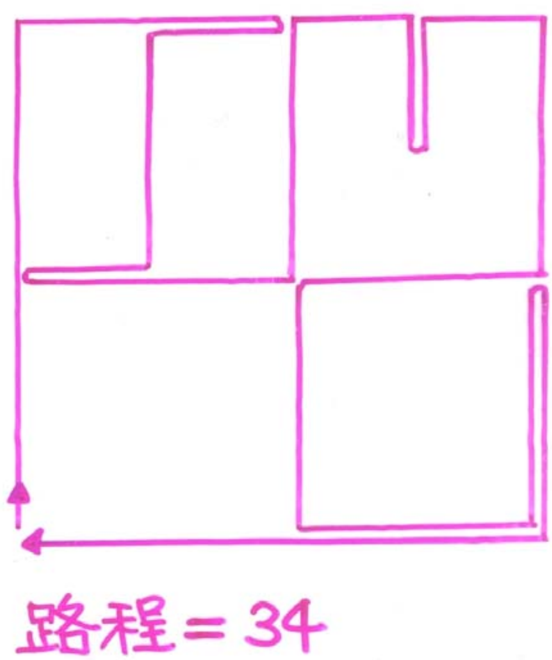


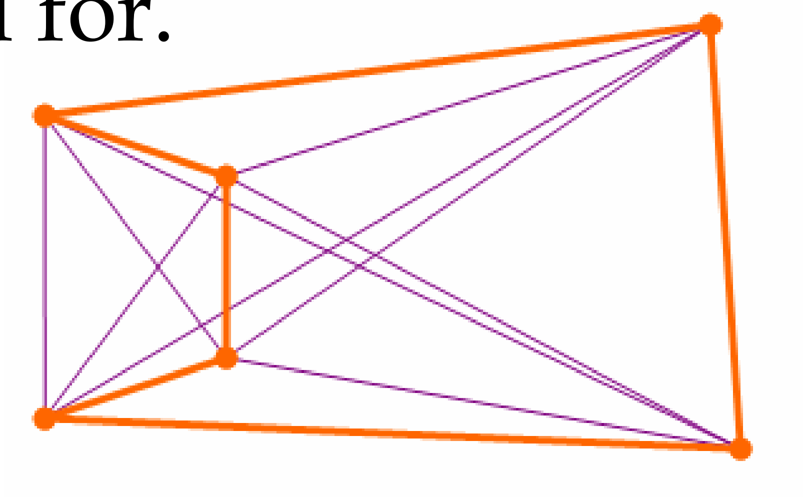
# Travelling Salesman Problem (TSP)

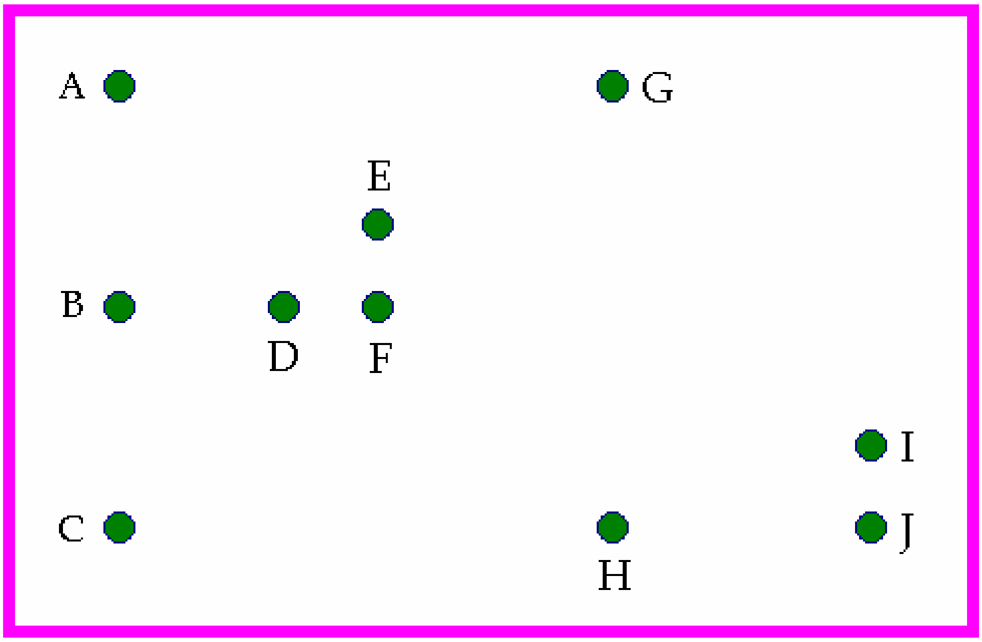
A travelling salesman must visit every city (exactly once) and return to the city from where he begins his trip. What is the shortest route?

## Travelling Salesman Problem (TSP)

A travelling salesman must visit every city (exactly once) and return to the city from where he begins his trip. What is the shortest route?

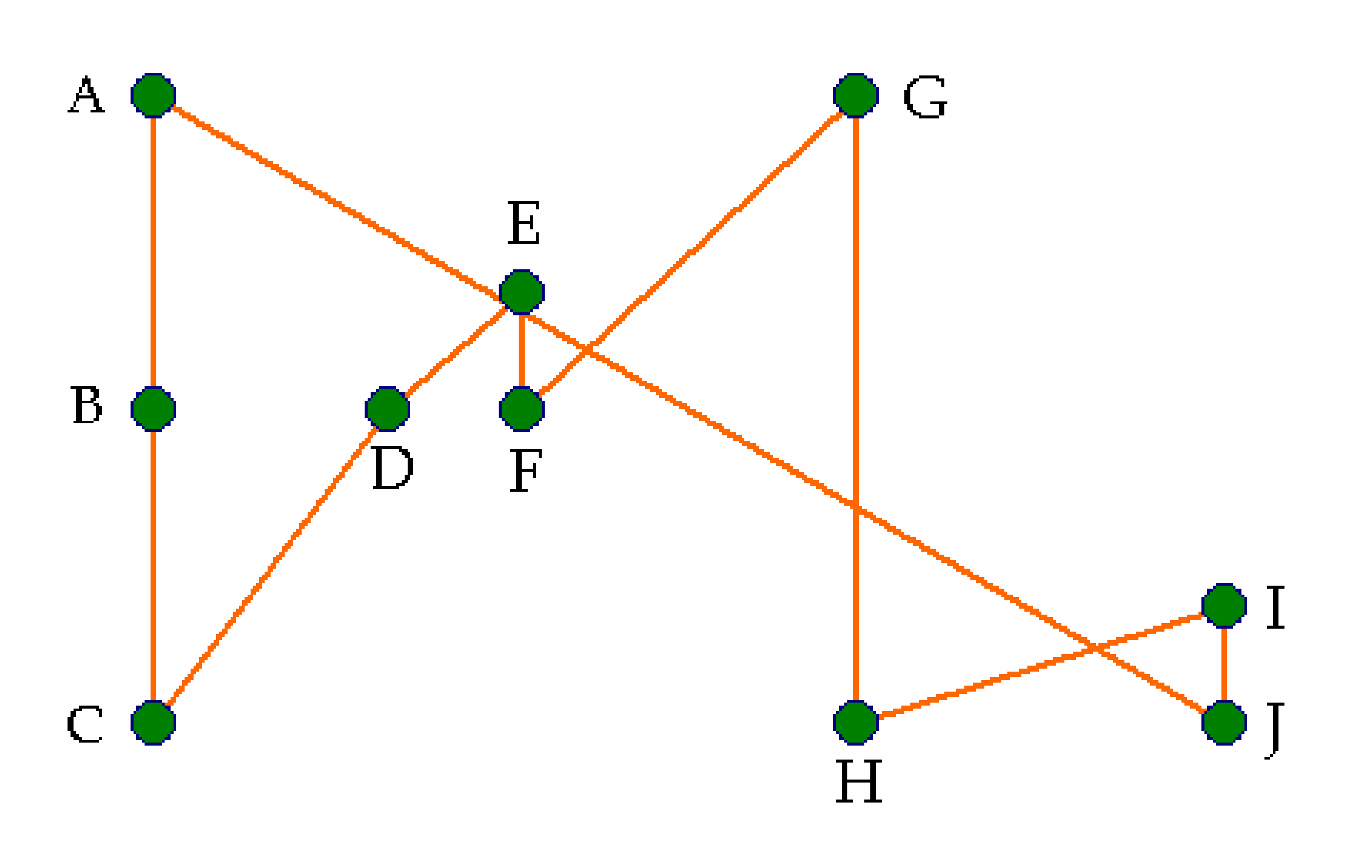
More precisely, this is called the Euclidean TSP, because (Euclidean) distances satisfy the triangle inequality. In general, distances can be replaced by costs that may not satisfy the triangle inequality, and the cheapest route is asked for.



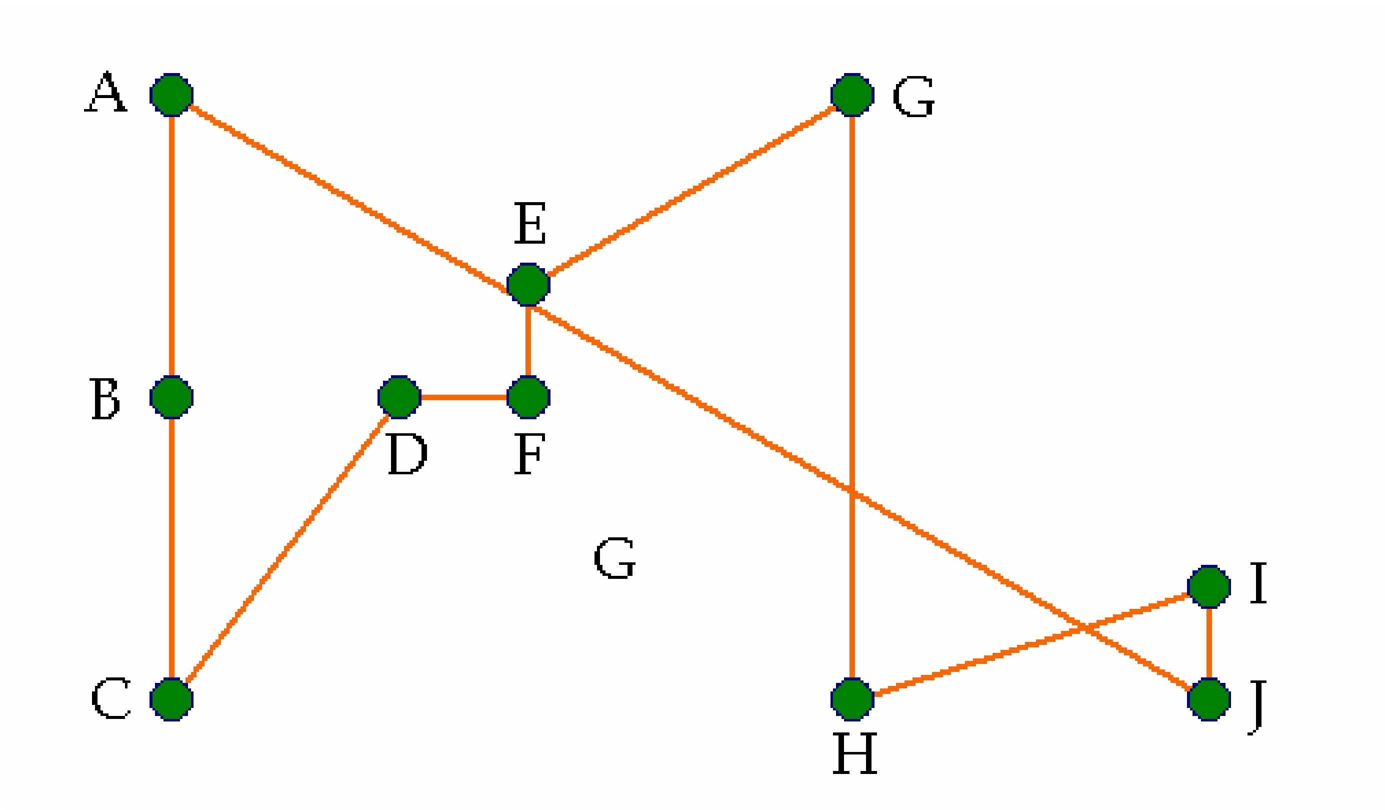



| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| **A** | - | 8 | 16 | 10.63 | 12.08 | 13.6 | 21 | 26.4 | 34.54 | 35.78 |
| **B** | 8 | - | 8 | 7 | 11.4 | 11 | 22.47 | 22.47 | 32.39 | 32.98 |
| **C** | 16 | 8 | - | 10.63 | 15.56 | 13.6 | 26.4 | 21 | 32.14 | 32 |
| **D** | 10.63 | 7 | 10.63 | - | 5 | 4 | 16.12 | 16.12 | 25.49 | 26.25 |
| **E** | 12.08 | 11.4 | 15.56 | 5 | - | 3 | 11.18 | 14.87 | 22.47 | 23.71 |
| **F** | 13.6 | 11 | 13.6 | 4 | 3 | - | 12.81 | 12.81 | 21.59 | 22.47 |
| **G** | 21 | 22.47 | 26.4 | 16.12 | 11.18 | 12.81 | - | 16 | 17.03 | 19.42 |
| **H** | 26.4 | 22.47 | 21 | 16.12 | 14.87 | 12.81 | 16 | - | 11.4 | 11 |
| **I** | 34.54 | 32.39 | 32.14 | 25.49 | 22.47 | 21.59 | 17.03 | 11.4 | - | 3 |
| **J** | 35.78 | 32.98 | 32 | 26.25 | 23.71 | 22.49 | 19.42 | 11 | 3 | - |

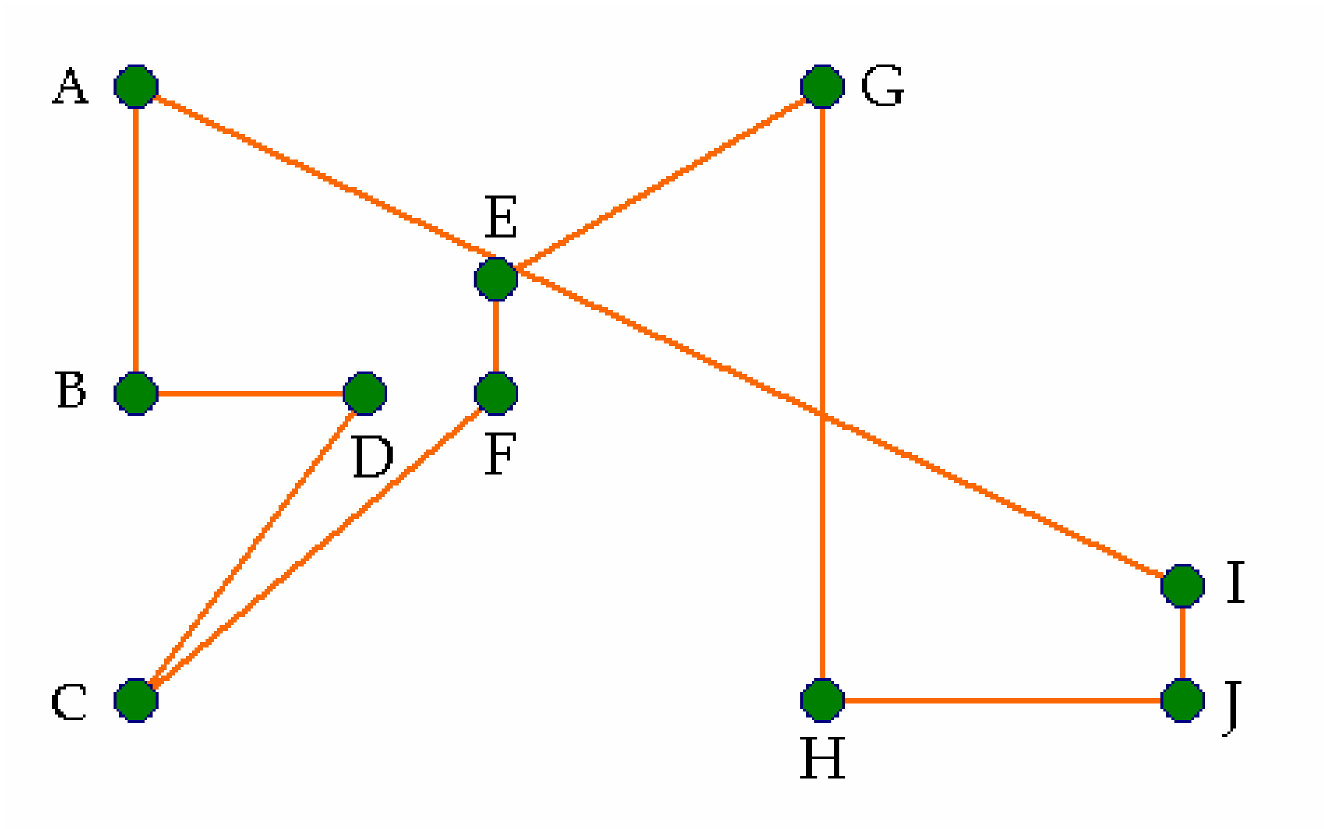
This example is taken from Chapter 17 in
*Combinatorial Optimization: Algorithms and Complexity*,
Christos H. Papadimitriou and Kenneth Steiglitz (1982)



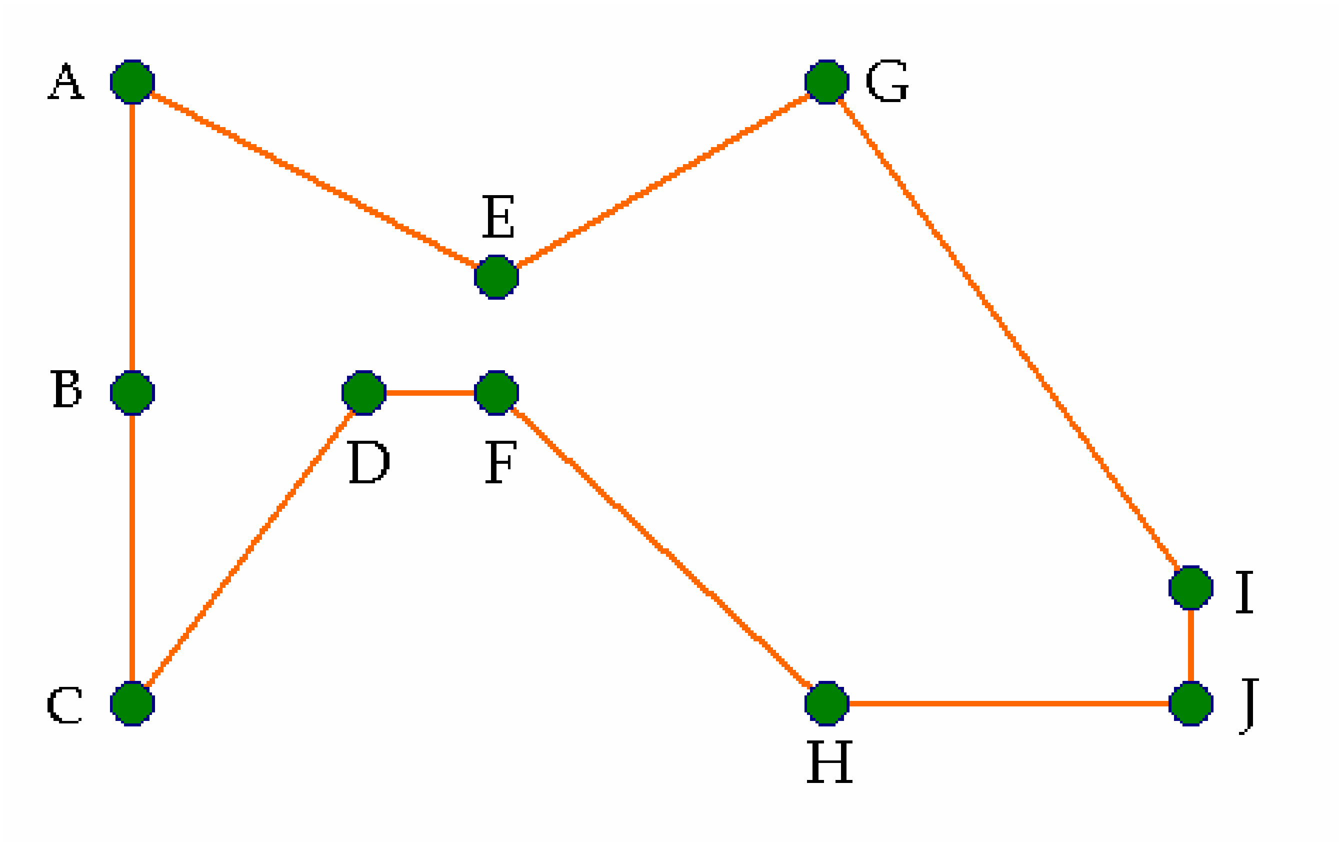

Total distance = 113.62



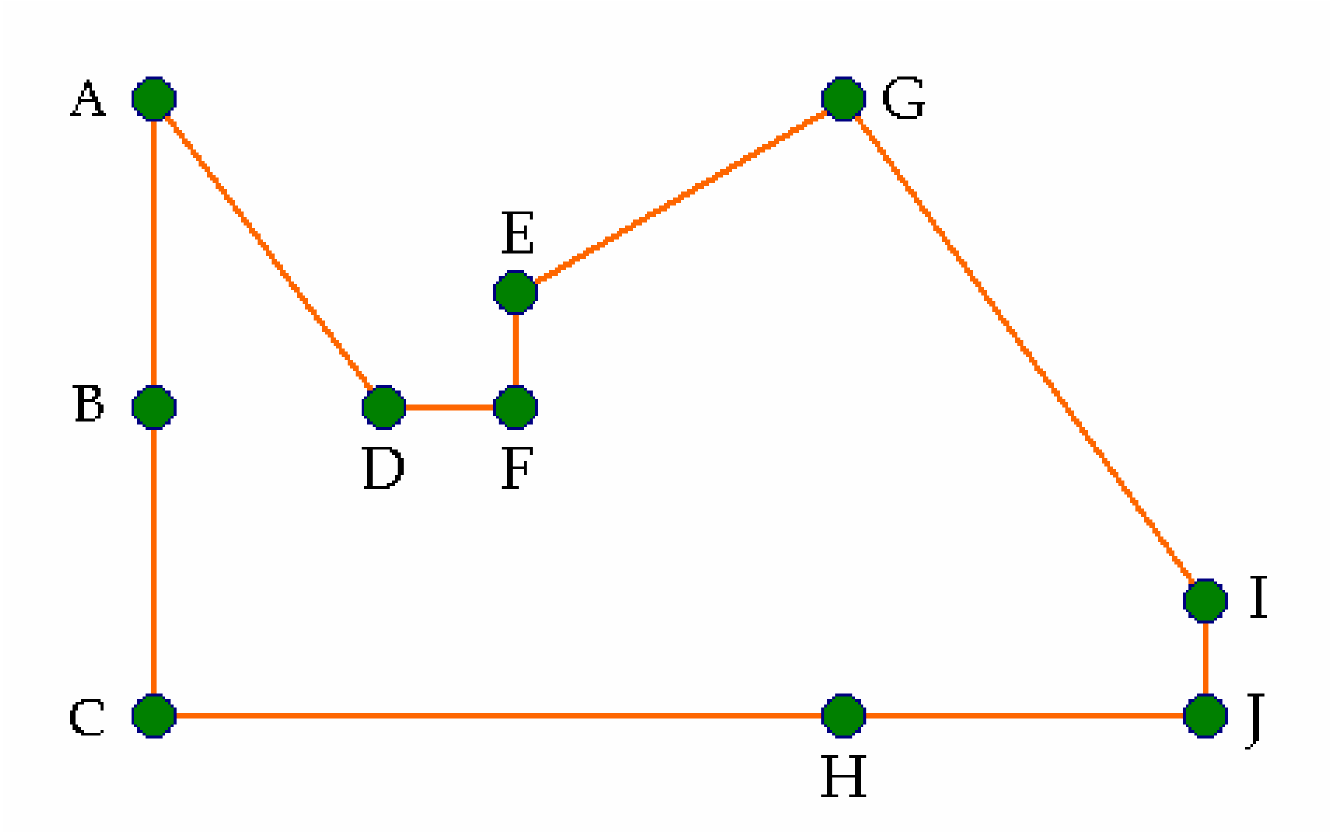

Total distance = 110.99



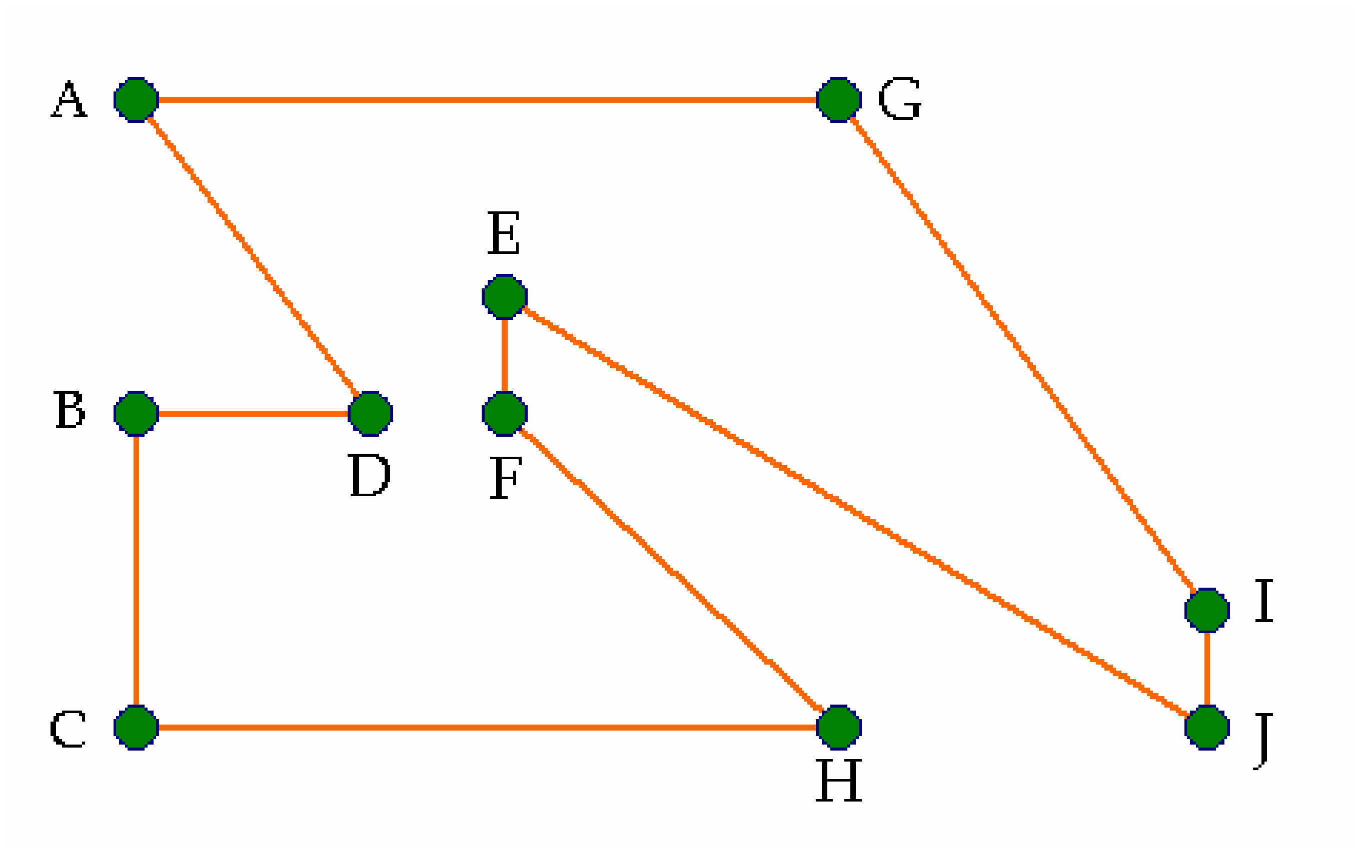

Total distance = 114.95




Total distance = 97.73




Total distance = 96.84




Total distance = 127.18

An exhaustive search would mean the checking of all possible cycles (cycles in reversed direction to each other are regarded as the same).

If there are $N$ cities, then there are

$$C = \frac{1}{2} N(N\text{-}1)\,(N\text{-}2)\;...3.2.1 \text{ cycles to check.}$$

Suppose $10^8$ cycles can be checked per second, then the time $T$ to check all is given by:

| N | C | T |
|---|---|---|
| 10 | 181440 | 0.002 sec. |
| 12 | 19958400 | 0.2 sec. |
| 14 | 3113510400 | 31.1 sec. |
| 16 | $6.54 \times 10^{11}$ | 1 hr. 49 min. |
| 18 | $1.78 \times 10^{14}$ | 20 days 14 hr. |
| 20 | $6.1 \times 10^{16}$ | 19 years 2 months |
| 25 | ? | ? |

An exhaustive search would mean the checking of all possible cycles (cycles in reversed direction to each other are regarded as the same).

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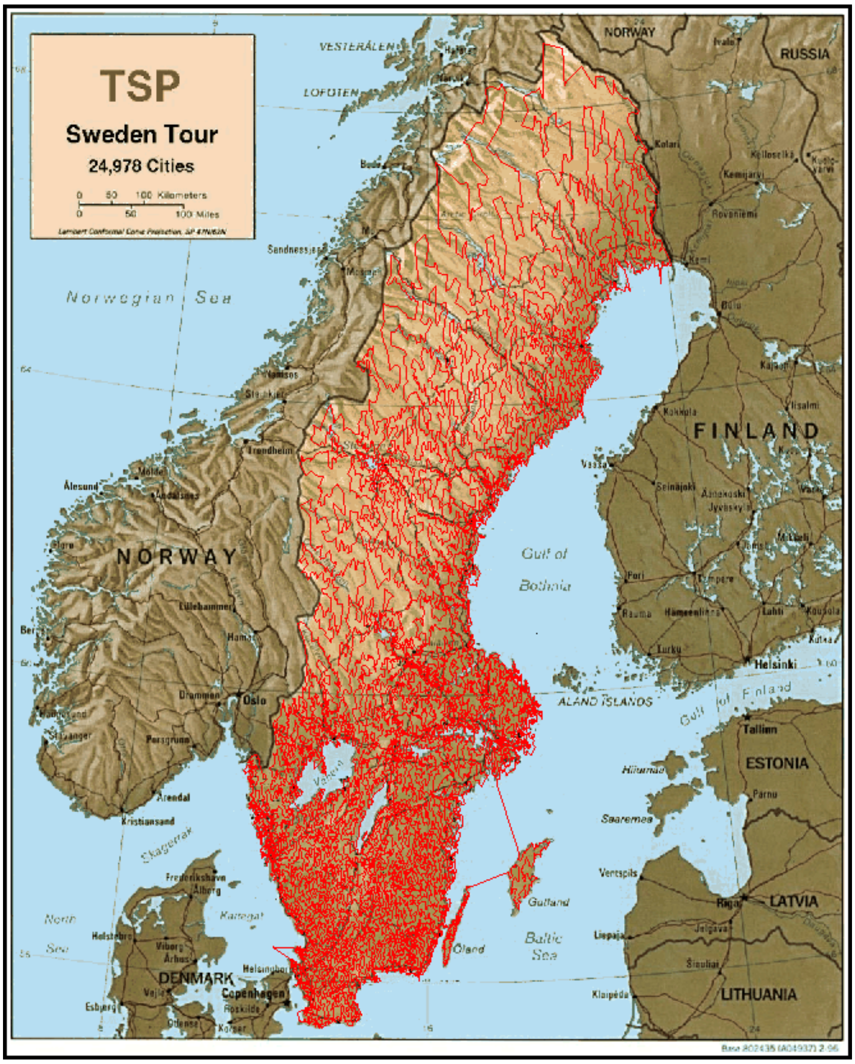
| 18 | $1.78 \times 10^{14}$ | 20 days 14 hr. |
| 20 | $6.1 \times 10^{16}$ | 19 years 2 months |
| 25 | $3.1 \times 10^{23}$ | About $10^9$ years |



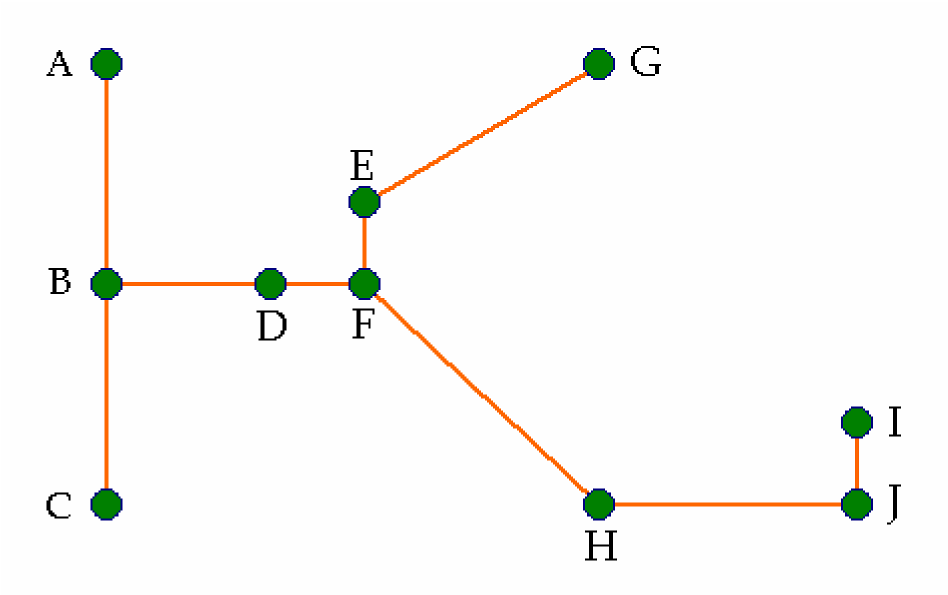

[www.tsp.gatech.edu/sweden/index.html]

In May 2004 the TSP of visiting 24,978 cities in Sweden was solved. The optimal tour comes to approximately 72,500 Km. In March 2005 the TSP of visiting 33,810 points in a circuit board was solved.

# An (efficient) approximate algorithm for the TSP

1. Find a minimal spanning tree $T_0$.
2. Duplicate all edges of $T_0$ to obtain an Eulerian graph $G$.
3. Find an Eulerian cycle in $G$ and obtain an embedded tour $\tau$ (by skip-when-necessary).

Total dist. of $\tau \leq 2 \times$ Total dist. of optimal tour.



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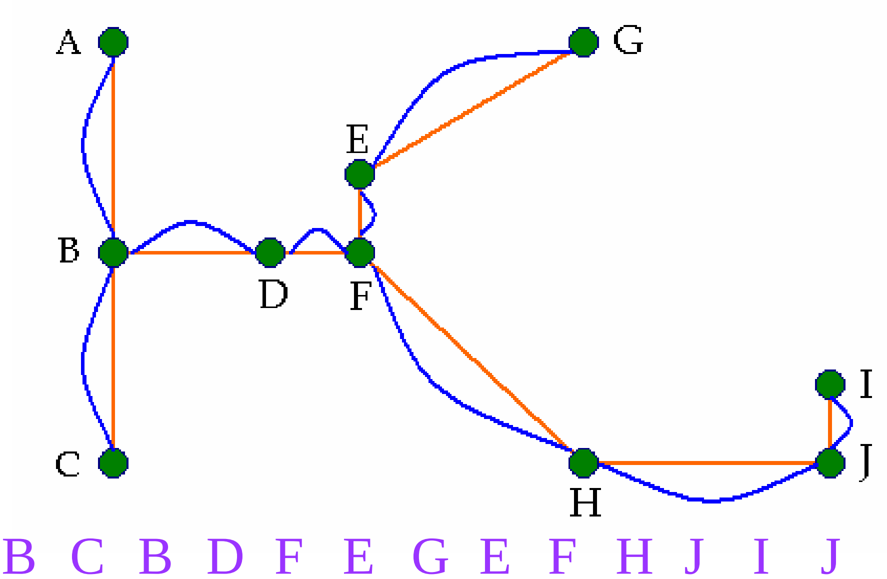


B A B C B D F E G E F H J I J H F D B

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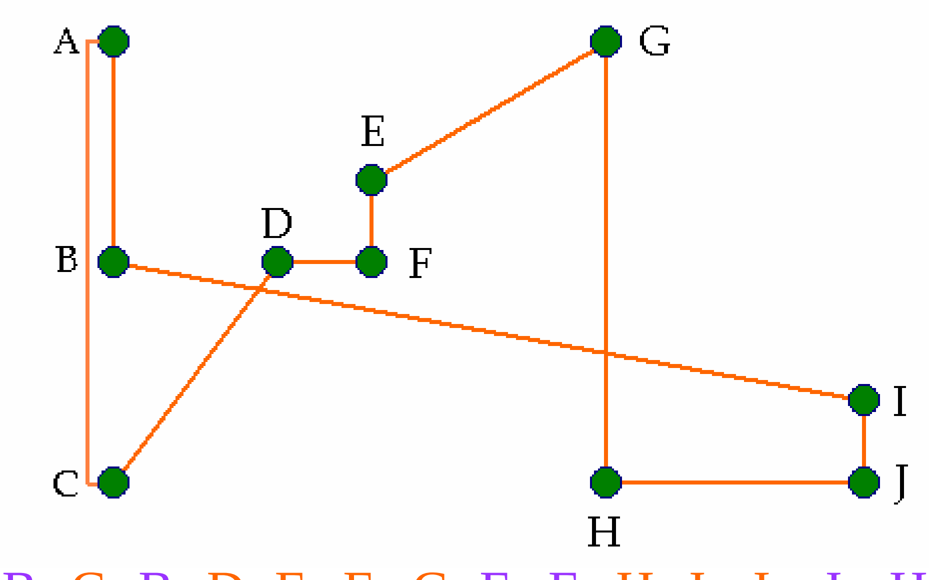
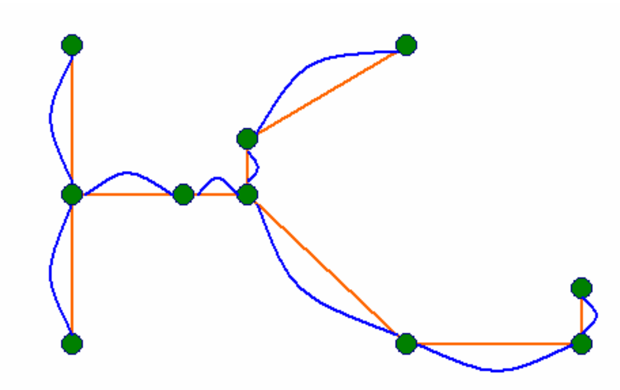



B A B C B D F E G E F H J I J H F D B

Total distance = 115.2



$T_0 =$ a minimal spanning tree

$\tau =$ tour obtained by algorithm

$\tau_0 =$ optimal tour

$d(*) =$ total distance of tour $(*)$

$$d(\tau) \le d(G) = 2d(T_0).$$

$$d(T_0) \le d(\text{any spanning tree}),$$
$$d(\text{some spanning tree}) \le d(\tau_0),$$
$$\therefore \quad d(T_0) \le d(\tau_0)\,.$$

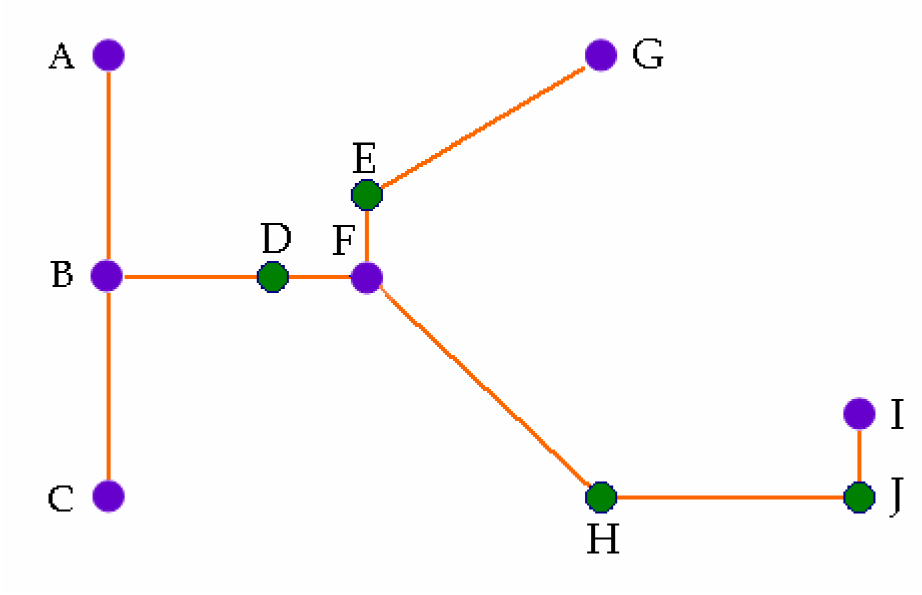
$$\boxed{\therefore \quad d(\tau) \le 2 \times d(\tau_0)}$$

# A better approximate algorithm for the TSP

[Nicos Christofides, Worst-case analysis of a new heuristic for the Travelling Salesman Problem, *Technical Report*, *GSIA*, Carnegie-Mellon University, 1976]

1. Find a minimal spanning tree $T_0$.
2. Locate all the vertices of **odd** degree in $T_0$ and find an **optimal matching** *M* of them.
3. Duplicate the edges in *M* to obtain an Eulerian graph *G*.
4. Find an Eulerian cycle in *G* and obtain an embedded tour $\tau$ (by skip-when-necessary).

Total dist.of $\tau \leq 1.5 \times$ (Total dist. of optimal tour).



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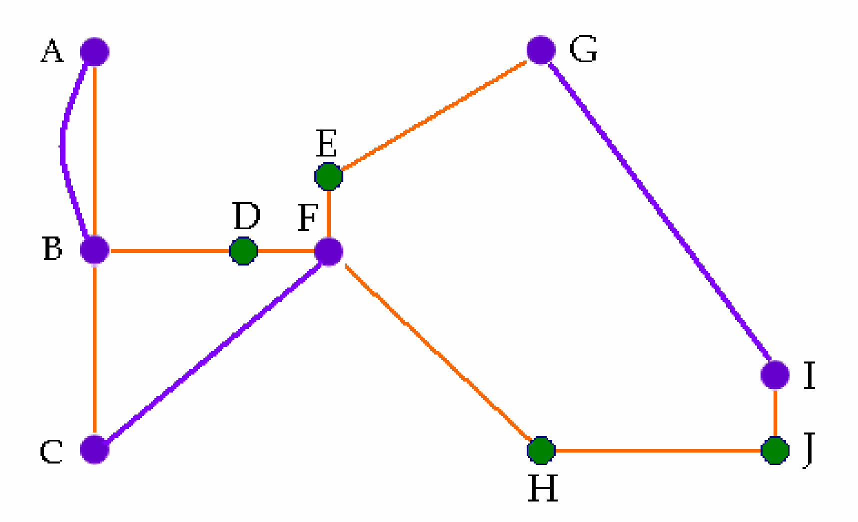


B A B C F H J I G E F D B

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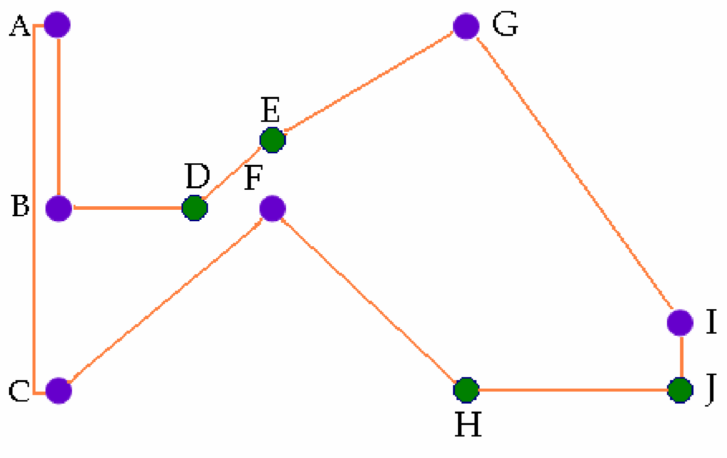
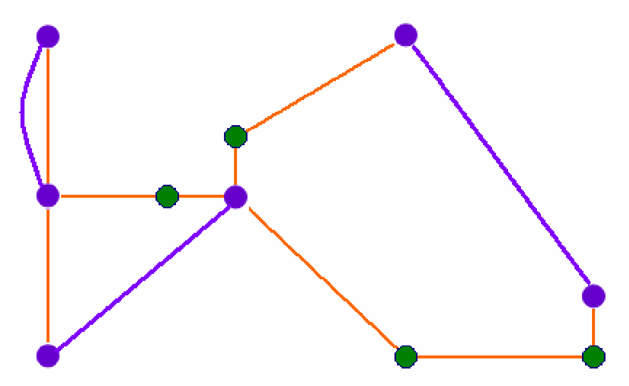



B A B C F H J I G E F D B

Total distance = 104.62

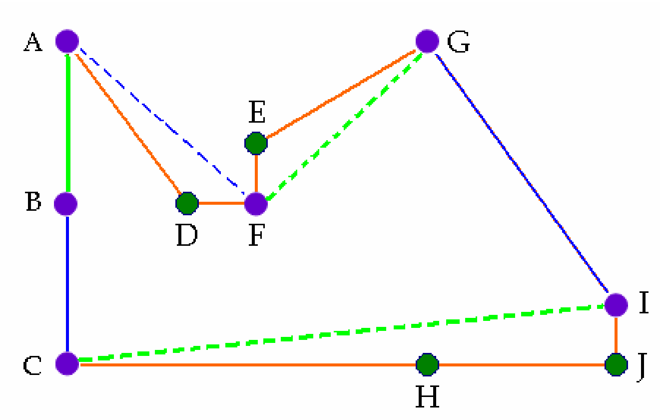


$T_0$ = a minimal spanning tree

$\tau$ = tour obtained by algorithm

$\tau_0$ = optimal tour

$d(*)$ = total distance of tour $(*)$




$M$ **optimal matching**

$M_1$

$M_2$

$$d(\tau) \le d(G) = d(T_0) + d(M).$$

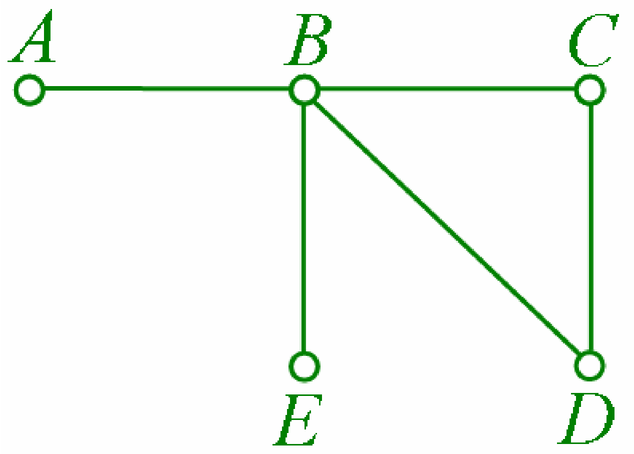
Similarly, $d(T_0) \le d(\tau_0)$.

$$d(\tau_0) \ge d(M_1) + d(M_2) \ge 2d(M),$$

because $d(M_1) \ge d(M)$, $d(M_2) \ge d(M)$.

$$\therefore \quad d(\tau) \le d(\tau_0) + \frac{1}{2} d(\tau_0)$$

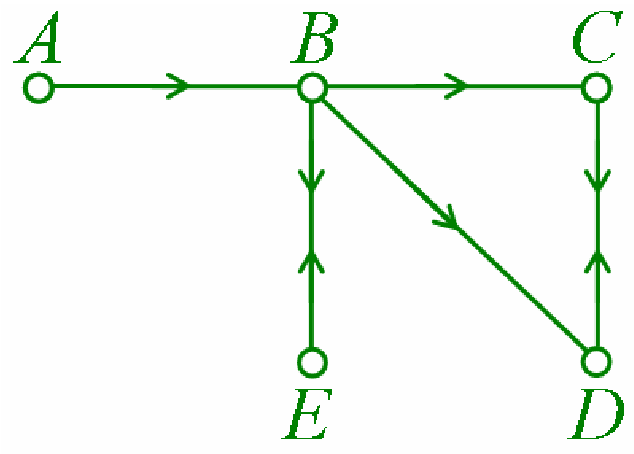
$$\boxed{d(\tau) \le \frac{3}{2} \times d(\tau_0)}$$




$$
\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ \text{Column sum (degree)} \end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
\begin{array}{ccccc} 1 & 4 & 2 & 2 & 1 \end{array}
\end{array}
\begin{array}{c} \text{Row sum (degree)} \\ 1 \\ 4 \\ 2 \\ 2 \\ 1 \\ \\ \end{array}
$$

$1 + 4 + 2 + 2 + 1 = 10$ = number of 1's

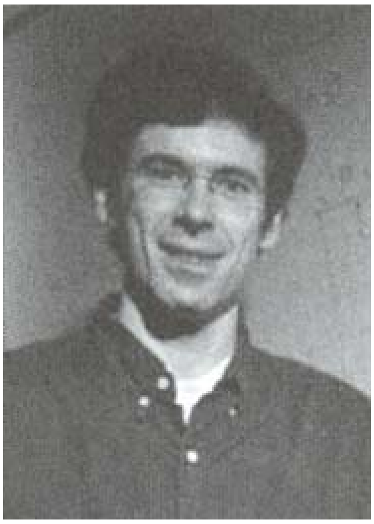
sum of degrees = twice number of edges




$$
\begin{array}{ccl}
 & \begin{matrix} A & B & C & D & E \end{matrix} & \begin{array}{l}\text{Row sum}\\\text{(out-degree)}\end{array} \\
\begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 3 \\ 1 \\ 1 \\ 1 \end{matrix} \\
\begin{array}{l}\text{Column sum}\\\text{(in-degree)}\end{array} & \begin{matrix} 0 & 2 & 2 & 2 & 1 \end{matrix} &
\end{array}
$$

$$1 + 3 + 1 + 1 + 1 = 7$$

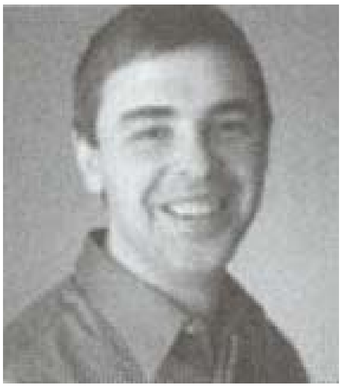
$$0 + 2 + 2 + 2 + 1 = 7$$

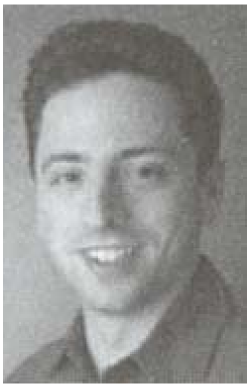


Jon Kleinberg

$9^{th}$ Annual ACM-SIAM Symposium on Discrete Algorithms (San Francisco, 1998)

HITS (Hypertext Induced Topic Search)

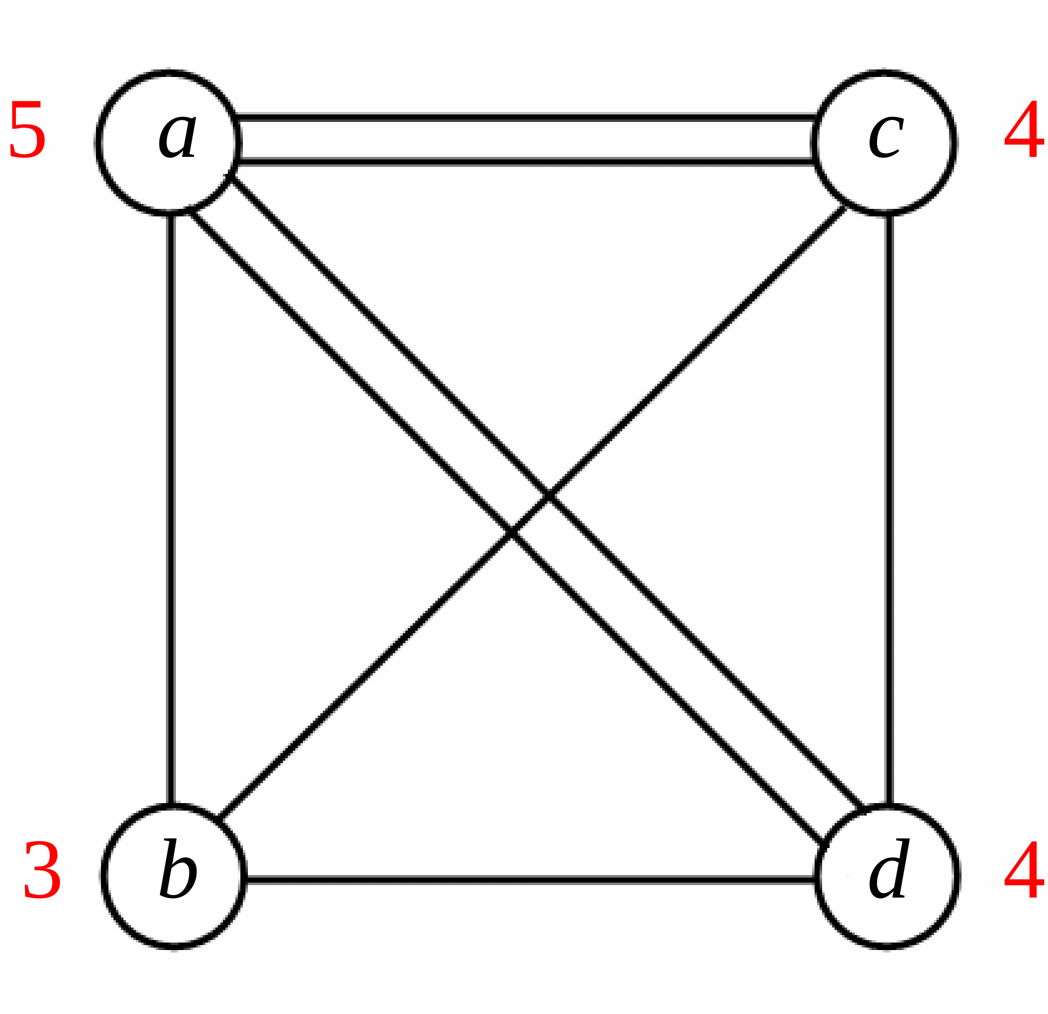




World Wide Web Conference 1998 (Brisbane, 1998)

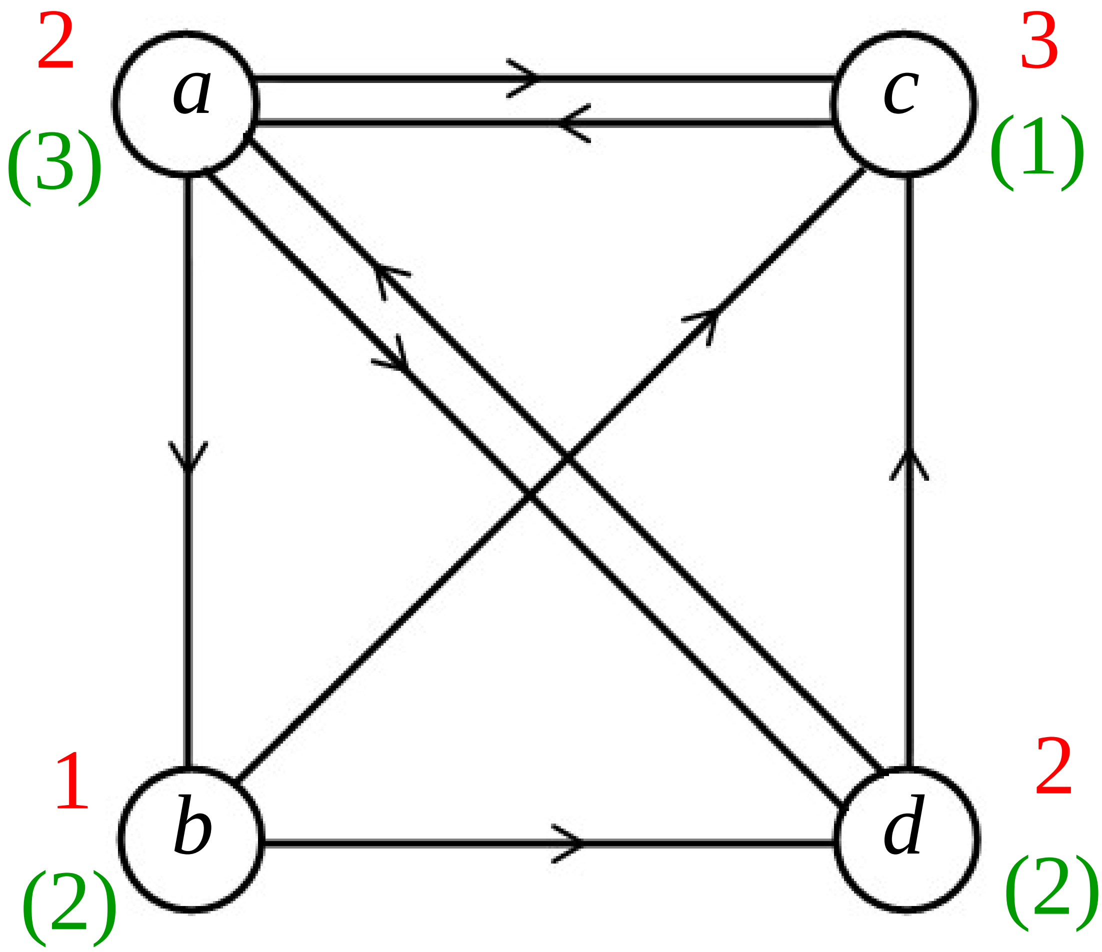
Larry Page Sergey Brin

PageRank




**Graph**

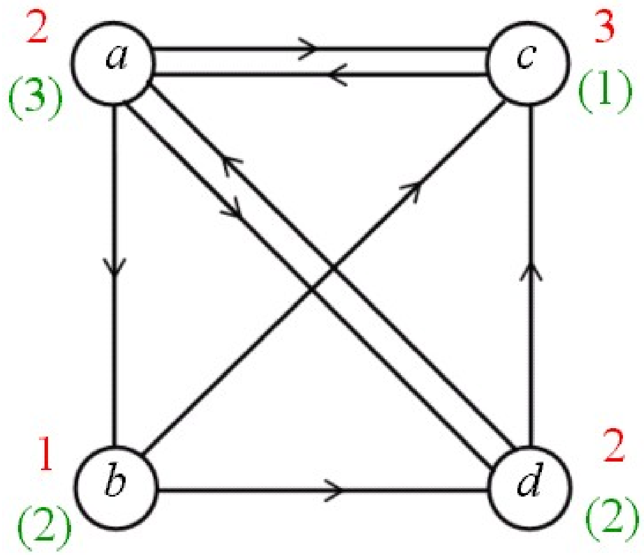
degree of a vertex




**Directed Graph**

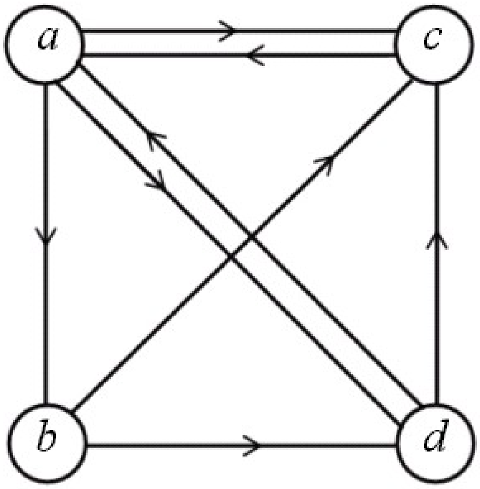
in-degree (out-degree)

of a vertex




$$\boxed{\begin{aligned} a &= \frac{c}{1} + \frac{d}{2} \\ b &= \frac{a}{3} \\ c &= \frac{a}{3} + \frac{b}{2} + \frac{d}{2} \\ d &= \frac{a}{3} + \frac{b}{2} \end{aligned}}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad \boldsymbol{AX} = \boldsymbol{X}$$




$$\boldsymbol{X} = \begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

$$\boxed{\begin{aligned} a &= \frac{12}{31} = 0.387.... \\ b &= \frac{4}{31} = 0.129.... \\ c &= \frac{9}{31} = 0.290.... \\ d &= \frac{6}{31} = 0.193.... \end{aligned}}$$

**PageRank algorithm** (Google)

Kurt Bryan, Tanya Leise, The $25,000,000 Eigenvector: The linear algebra behind Google, SIAM Review, 48(3) (2006), 569-581.


LEONHARDO EULERO
ACADEMIA PETROPOLITANA
MDCCCXXXVII

**Euler was a genius, whose height very few can hope to reach. But we can all learn from his great zest for life, work and study, his insatiable curiosity to know and to probe, his determination to procure deeper and deeper understanding, his industry, his modesty, his generosity, and his toughness in facing adversity with tranquility.**

「高山仰止， 景行行止。」
(The high mountain I look up at it. The great road I travel on it)

*Book VII, Ode IV, Book of Odes* 《詩經》


# To Euler, Happy 300th Anniversary!

The speaker likes to thank the Faculty of Science and the Department of Mathematics of HKU for their support, special thanks to Ms. Mimi Lui for working on the powerpoint file.