The Numerical Range

Dr. J.T. Chan, Department of Mathematics

Introduction

Let H be a Hilbert space over the complex numbers and let $B(H)$ be the Banach space of all bounded linear operators on H. For each T in $B(H)$, associate it with the set

$$W(T) = \{\langle Tx, x\rangle : x \in H \text{ and } \|x\| = 1\}.$$

It is called the *numerical range* of T.

The numerical range is a closed and bounded subset of the complex plane. But it turns out that this "small" set gives us many useful information about the operator T. Many studies are centred around this object.

Objective

In one area of research, Pellegrini obtained a characterisation of linear surjections on $B(H)$ that preserve the numerical range of each T in $B(H)$, i.e., those $\varphi : B(H) \to B(H)$ such that

$$W(\varphi(T)) = W(T) \quad \text{for all } T \in B(H).$$

It enabled us to gain better understanding of the role played by the numerical range in $B(H)$. Since then, mathematicians have tried to improve the result in different directions. For every T in $B(H)$, let

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

This number is called the numerical radius of T. We were able to describe those linear surjections $\varphi : B(H) \to B(H)$ such that

$$w(\varphi(T)) = w(T) \quad \text{for all } T \in B(H)$$

and later extended the result to the more general setting of C^*-algebras.

Together with my former PhD student, Dr Kong Chan, we have been studying problems relating to the numerical range and numerical radius.

Details of Research

Inspired by a series of papers of Molnár and his collaborators, we considered the diameter of the numerical range of T in $B(H)$. We were able to describe those linear surjections $\varphi : B(H) \to B(H)$ such that

$$D_W(\varphi(T)) = D_W(T) \quad \text{for all } T \in B(H).$$

We are now trying to extend the result to the more general setting of C^*-algebras. Actually, the results of Molnár mentioned above can be viewed as the result in the commutative C^*-algebra case.

One difficulty in dealing with $D_W(\cdot)$ is that it is not a norm on $B(H)$. Indeed $D_W(T) = 0$ if and only if T is a scalar operator. We therefore had to consider the quotient space of $B(H)$ mod the scalar operators and then the bounded linear functionals on this quotient space.

To extend the result to C^*-algebras, we proceeded with the same strategy as in the original proof. We were able carry some of the results over to the new situation. For example, the description of bounded linear functionals on the analogous quotient space. However, some other arguments were more $B(H)$ specific. Further investigation is needed.

In another direction, we studied mappings $\varphi : B(H) \to B(H)$ such that

$$w(\varphi(S) - \varphi(T)) = w(S - T),$$

where $w(\cdot)$ denotes the radius of various generalized numerical ranges. We have settled the problem for a couple of generalized numerical ranges. We are currently studying the so-called q-numerical range, which is defined for $q \in [0, 1]$. The problem has also been studied by other mathematicians, but we believe that their results can be improved.

Plan of Research

For the problem relating to the diameter of the numerical range, the techniques used for $B(H)$ is not directly applicable to the C^*-algebra case. We are trying to use the GNS-representation theorem to write a C^*-algebra into a direct sum of $B(H)$'s so that we may first work on each of the summands and then try to combine the results together to get the desired result. There are still hurdles to overcome.

Closing Remark

There are mathematicians all over the world working on the numerical range and related topics. The research has been fruitful. Many applications, both in the pure and applied field, were found. This is a field worthy of studying.

References

1. JT Chan, Numerical radius preserving operators on C^*-algebras, Arch. Math. (Basel) 70 (1998), 486-488.
2. JT Chan and K Chan, Two distance preservers of generalized numerical radii, Linear Multilinear Algebra 62 (2014), 674-682.
3. L Molnár, Selected preserver problems on algebraic structures of linear operators and on function spaces, Springer-Verlag, Berlin, 2007.