Distribution and twist moments of the values of automorphic L-functions at the boundary point 1

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Introduction. Prime numbers $2, 3, 5, \cdots$ play an atomic role in natural numbers. The widely known Prime Number Theorem was firstly proved by using the nonvanishing property of the innovative device – Riemann's zeta function – on the line $\Re e\, s = 1$. Since then, new models of such an innovative device were constructed for various purpose, which are collectively called L-functions.

P.G.L. Dirichlet
(German, 1805-1859)

Associated to a character χ_d is Dirichlet's L-function $L(s, \chi_d)$. Among the many problems, the distribution of the family of the values $L(1, \chi_d)$ over different characters χ_d,

$$\left\{ L(1, \chi_d) : d \text{ } {\scriptstyle \text{fundamental} \atop \text{discriminant}} \right\},$$

drew good attentions.

Particularly, H.L. Montgomery and R.C. Vaughan raised three conjectures by the end of last century. A. Granville and K. Soundararajan (2003) made a great progress and solved a substantial part of them. The Riemann zeta function and the Dirichlet L-functions are known as $GL(1)$ members in the family of automorphic L-functions. A natural question is to study the analogous problem for $GL(n)$ automorphic L-functions.

Findings for the $GL(2)$ case. Last decade there were several investigations toward this problem by various research groups. In [1], we extended the Montgomery-Vaughan conjectures to the $GL(2)$ case of varying weight. More specifically we considered the distribution of the values $L(1, \mathrm{sym}^m f)$ (where $L(s, \mathrm{sym}^m f)$ is an $GL(m+1)$ automorphic L-function). We are able to solve part of them, for which a new tool – Elliott-Montgomery-Vaughan (EMV) large sieve inequality – is introduced.

$$\text{(EMV)}: \quad \sum_f \left| \sum_{P<p\le 2P} b_p \frac{\lambda_{\mathrm{sym}^m f}(p)}{p} \right|^{2j} \ll k \left(\frac{A(m+1)j}{P \log P} \right)^j$$

In addition it was found that the values of $L(1, \mathrm{sym}^2 f)$ twisted by the central value $L(1/2, f)$ exhibit interesting combinatorial properties. Statistically the $(-\ell)$-th twisted moment is related to the number of (partial) Riordan paths.

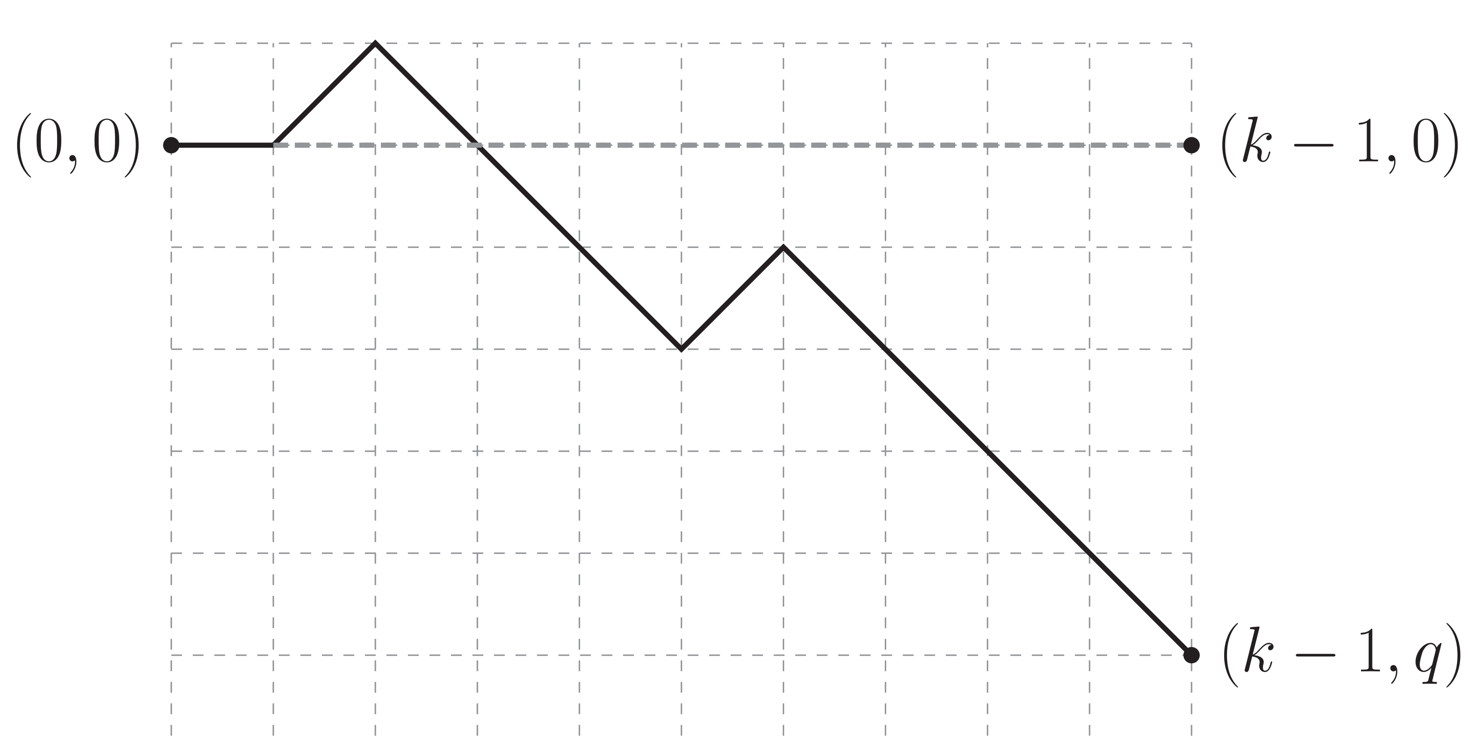

Figure 1. A partial Riordan path

Alternatively the result is expressed in terms of local integrals over the group $SU(2)$. In [2], the $(-\ell)$-th twisted moment by the square of the central value is shown to be asymptotically

$$4 \log k \prod_p (1-p^{-1}) \int_{SU(2)} \det(I - p^{-1/2}\mathrm{St}\, g)^{-2} \det(I - p^{-1}\mathrm{sym}^2\, g)^{\ell}\, dg$$

Research plan & method. As part of GRF project from the RGC, Dr YK Lau from HKU Department of Mathematics is investigating the more challenging $GL(n)$ case. In general for $n \ge 3$, the $GL(n)$ automorphic L-functions $L(s, F)$ are less understood and are lack of tools. Our objectives are to derive some large sieve inequalities to facilitate the investigation, and to study the distribution of the values $L(1, F)$. An interesting question is to understand the statistical behaviour of the twisted moments from the combinatorial viewpoint and the representation theoretic viewpoint .

To fix ideas, we confine to the case $n = 3$. Our investigation relies heavily on the Hecke relations of the Fourier coefficients $A_F(m_1, m_2)$:

$$A_F(n,1)A_F(m_1,m_2) = \sum_{\substack{abc=n \\ a|m_1,\, b|m_2}} A_F\left(\frac{m_1 c}{a}, \frac{m_2 a}{b}\right),$$

$$A_F(1,n)A_F(m_1,m_2) = \sum_{\substack{abc=n \\ a|m_1,\, b|m_2}} A_F\left(\frac{m_1 b}{a}, \frac{m_2 c}{b}\right),$$

and the Kuznetsov trace formula amongst other tools. The Hecke relations yield the multiplicativity of $A_F(m, n)$ and boils $A_F(p^r, p^s)$ down to $A_F(1, p^j)$ and $A_F(p^j, 1)$ which eventually lead to the study of combinatorial questions. The Kuznetsov trace formula explores the orthogonality between $A_F(m, n)$ and $A_F(m', n')$. Then we shall establish an EMV large sieve inequality and extend the methods for the $GL(2)$ case to this context.

Research results & Discussion. A large sieve inequality and zero density theorem have been established, which are two preliminary tools to proceed the research plan. An asymptotic result for the moment problem is obtained and its main term is under exploration. In summary, this investigation advances our knowledge on the distribution of the values of the L-function at the boundary point 1 and brings into more tools in the study of $GL(n)$ automorphic L-functions.

References.

[1] Y.-K. Lau & J. Wu, A large sieve inequality of Elliott-Montgomery-Vaughan type for automorphic forms and two applications. Int. Math. Res. Not. IMRN 2008, Art. ID rnm 162, 35 pp.

[2] Y.-K. Lau, E. Royer & J. Wu, Twisted moments of automorphic L-functions. J. Number Theory 130 (2010), 2773–2802.