On the Poisson geometry of flag varieties and Bott-Samelson resolutions

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Background.

Flag varieties of reductive algebraic groups and their Schubert subvarieties play important roles in representation theory. The theory of quantum groups gives rise to natural Poisson structures on many spaces related to flag varieties. We study these Poisson structures systematically. Our work lies at the crossroad of representation theory, geometric combinatorics, algebraic geometry, and symplectic geometry. Our findings add to the richness of both Poisson/symplectic geometry and Lie theory.

Let G be a connected complex semisimple Lie group, equipped with a so-called standard multiplicative Poisson structure π_{st} defined by a choice of a Borel subgroup B and an opposite Borel subgroup B_-. If $\mathbf{u} = (s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n})$ is a sequence of simple roots, with $P_{\alpha_j} = Bs_{\alpha_j}B$ the corresponding Borel subgroup of G, the Bott-Samelson variety

$$Z_{\mathbf{u}} = P_{\alpha_1} \times_B P_{\alpha_2} \times_B \cdots \times_B P_{\alpha_1}/B$$

is the quotient of $P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_n}$ by the group B^n via the action

$$(g_1, g_2, \ldots, g_n)\cdot(b_1, b_2, \ldots, b_n) = (g_1b_1, b_1^{-1}g_2b_2, \ldots, b_{n-1}^{-1}g_nb_n), \quad (1)$$

where $g_j \in P_{\alpha_j}$ and $b_j \in B$. The n-fold direct product Poisson structure π_{st}^n on G^n projects to a well-defined Poisson structure, denoted by π_n, on $Z_{\mathbf{u}}$. The project aims at a systematic study of the Poisson structure π_n on the Bott-Samelson varieties $Z_{\mathbf{u}}$.

Research Findings.

It turned out that a better way for studying the Poisson geometry of the Bott-Samelson varieties is to regard them as Poisson subvarieties of the Poisson variety (F_n, π_n), where

$$F_n = G \times_B G \times_B \cdots \times_B G/B,$$

a quotient of G^n by B^n via an action defined similarly as in (1). In fact, we introduced *three other series* of Poisson varieties. To carry out a systematical study of the four series, we also developed several general theories and applied the general theories to these four series.

- In a joint paper with a former student V. Mouquin, the four series of Poisson varieties were put in the general framework of mixed product Poisson structures and Poisson structures defined by quasi-triangular r-matrices;.
- In [2], the T-leaves, where T is a maximal torus of G, of all the four series were determined after developing a general theory on T-leaves of Poisson structures defined by quasi-triangular r-matrices, which is also applicable to other classes of varieties such as those studied in [1].
- In [3], explicit formulas for the standard Poisson structures on Bott-Samelson varieties are given in terms of the root strings and structure constants of the Lie algebra of G. In particular, it is shown in [3] that for each of the affine coordinate charts, the Poisson structure on $Z_{\mathbf{u}}$ gives rise to an iterated Poisson-Ore polynomial algebra which can in fact be defined over any field.

Further Reading.

1. J.-H. Lu, On the T-leaves and the ranks of a Poisson structure on twisted conjugacy classes, Indag. Math. **25** (5) (2014) 1102 – 1121.
2. J.-H. Lu and V. Mouquin, On the T-leaves of some Poisson structures related to products of flag varieties, arXiv:1511.02559; submitted.
3. B. Elek and J.-H. Lu, On a Poisson structure on Bott-Samelson varieties: computations in coordinates, arXiv:1601.00047; submitted.