Geometry on subvarieties of bounded symmetric domains and their quotient manifolds and on fibered spaces over them

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Objective

Bounded symmetric domains and their quotient manifolds are of great importance to Complex Differential Geometry and in Algebraic Geometry. They are dual to Hermitian symmetric spaces S of the compact type, which are Fano manifolds. In this proposal among other things we propose to use methods developed in the geometric theory of uniruled projective manifolds basing on varieties of minimal rational tangents (VMRTs) to study the geometry of bounded symmetric domains and their finite-volume quotients. One of the problems is to characterize images of holomorphic isometries of $\mathbf{B}^m$ of maximal dimension into irreducible bounded symmetric domains Ω. Earlier the PI has given examples of such maps constructed from VMRTs.

Research Findings

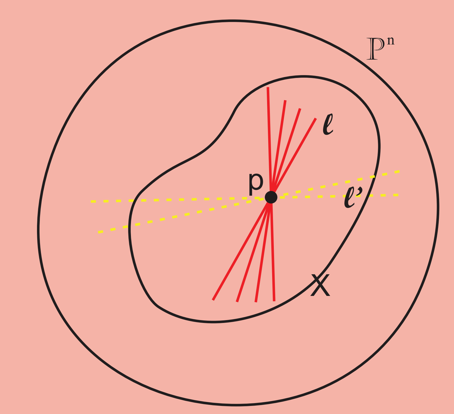
When D is a Lie sphere of dimension $n \geq 3$ (dual to the hyperquadric), Chan-Mok showed the existence of non-standard holomorphic isometries of $\mathbf{B}^{n-1}$ into D incongruent to VMRT-type examples, classifying all holomorphic isometries of any $\mathbf{B}^m$ into D. For Ω of any type and for isometries wrt normalized canonical Kähler-Einstein metrics, they proved that images of holomorphic isometries are intersections with Ω of linear sections of the dual manifold $S = G^{\mathbb{C}}/P$ wrt minimal embeddings. Mok-Yang proved the uniqueness of holomorphic isometries of $\mathbf{B}^m$ of maximal dimension into Ω belonging to some series of classical domains and for exceptional domains Ω. There isomorphism classes modulo $G^{\mathbb{C}}$ of tangents to images Z of holomorphic isometries are identified using the Gauss equation and duality and Z is reconstructed by adjoining chains of minimal rational curves.

Varieties of Minimal Rational Tangents of VMRTs Hermitian Symmetric Spaces $S = G/K$

Type	G	K	$G/K = S$	$\mathcal{C}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2,n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

Uniruling of Fano hypersurfaces by lines

Examples of spaces with a lot of lines



ℓ is a line that lies on X

ℓ' is a line tangent to X but not lying on X

Example:

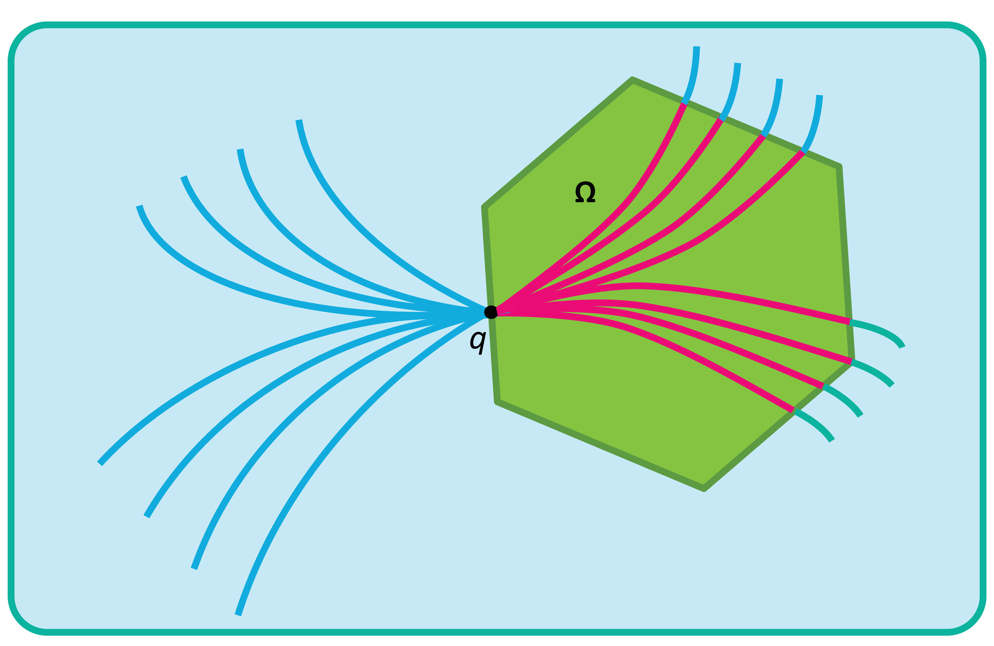
$$X \subset \mathbb{P}^{n+1} \quad \text{defined by} \quad z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$$

Highlights

Our project highlights geometric structures when one identifies an irreducible bounded symmetric domain Ω of rank ≥ 2 as an open subset of its compact dual manifold $S = G/K = G^{\mathbb{C}}/P$. These are L-structures defined by reductive complex Lie groups L, e.g., holomorphic conformal structures and Grassmann structures, and they arise from the fact that tangent vectors fall into different isomorphism types under the isotropy action of the parabolic subgroup P of $G^{\mathbb{C}}$. We highlight the interplay between canonical Kähler metrics and these geometric structures, and in the study of subvarieties on Ω we consider moreover new geometric structures arising from examples of holomorphic isometric embeddings of the complex unit ball into Ω. As such we establish a link from Complex Differential Geometry to the theory of VMRTs on uniruled projective manifolds developed by Hwang-Mok.

Image of holomorphic isometry of the unit ball $f : B^n \hookrightarrow \Omega$.

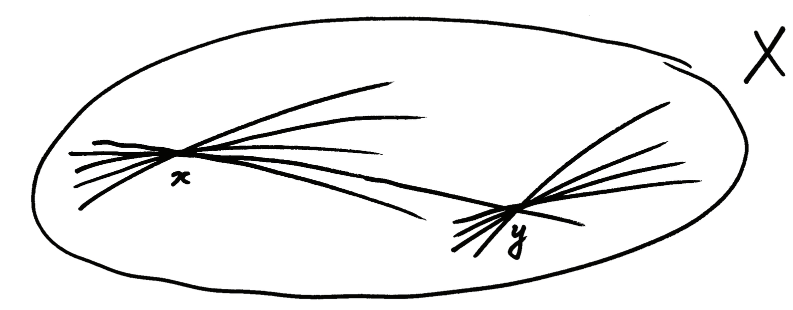
$$\mathcal{V}_q = \bigcup \{\text{lines } \ell \text{ on } S = G^{\mathbb{C}}/P,\ q \in \ell\}\,;$$
$$V_q = \mathcal{V}_q \cap \Omega = f(B^n).$$



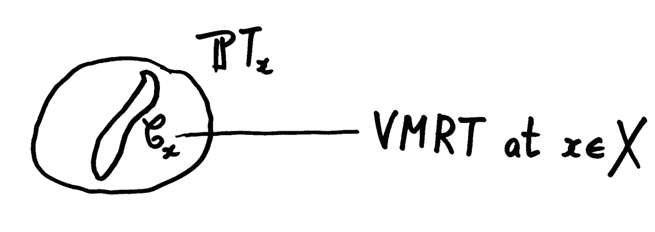
Research Plan

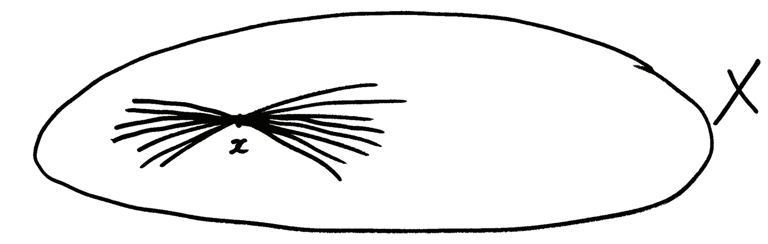
We study isomorphism classes modulo P of tangent subspaces of images S of holomorphic isometric embeddings of the complex unit ball using methods of Complex Differential Geometry. On the other hand, we consider the process of re-construction of S by adjoining chains of minimal rational curves using methods of the geometric theory of VMRTs of Hwang-Mok imitating the reconstruction of Fano manifolds of Picard number 1 by the adjunction of chains of minimal rational curves.

Minimal Rational Curves



Variety of Minimal Rational Tangents (VMRT)





Outlook

Our project reveals the interplay between Complex Differential Geometry and the geometric theory of uniruled projective manifolds basing on VMRTs introduced by the PI and J.-M. Hwang. Especially, we were led to study uniruled projective subvarieties of Fano manifolds. In the study of holomorphic isometries of $\mathbf{B}^m$ into Ω we have established a link to VMRT theory by examining geometric substructures defined by images of such holomorphic isometries. The interface between Complex Differential Geometry and VMRT theory promises to be fertile soil for ongoing and further investigation involving problems in Functional Transcendence Theory in the interface of Complex Geometry and Arithmetic Geometry.

Further Reading

(J.-M. Hwang & N. Mok) Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation, *Invent. Math.* **131** (1998), 393-418.

(N. Mok) Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric, *J. Eur. Math. Soc.* **14** (2012), 1617-1656.

(N. Mok) Holomorphic isometries of the complex unit ball into irreducible bounded symmetric domains, to appear in *Proc. A.M.S.*

(N. Mok & Y. Zhang) Rigidity of pairs of rational homogeneous spaces of Picard number 1 and analytic continuation of geometric substructures on uniruled projective manifolds. Preprint, IMR 2015.

(N. Mok & S.T. Chan) Holomorphic isometric embeddings of $\mathbf{B}^n$ into irreducible bounded symmetric domains arise from linear sections of minimal embeddings of their compact duals. Preprint, IMR 2016.