Additive Equations in Primes and Almost Primes

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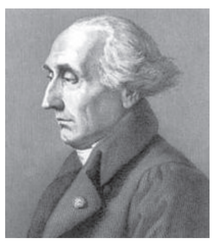
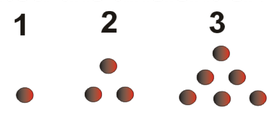
Joseph Louis Lagrange proved in 1770 the celebrated theorem that

every natural number can be represented as a sum of four square numbers.



Examples: $31 = 5^2 + 2^2 + 1^2 + 1^2$, $46 = 5^2 + 4^2 + 2^2 + 1^2$,
$299 = 12^2 + 9^2 + 7^2 + 5^2$, $6676 = 77^2 + 25^2 + 11^2 + 1^2$.

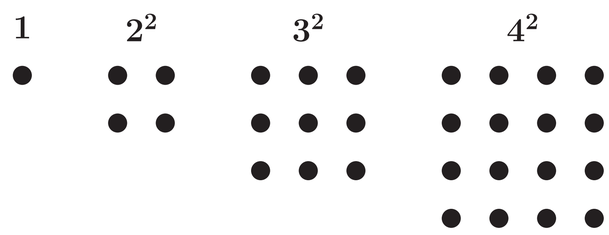
The *geometric numbers* have long fascinated the ancient civilizations. For instances, *each triangular number*



, etc. is equal to the sum of the first k natural numbers

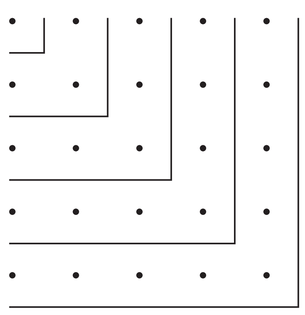
$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$$

The square numbers are, arguably, the most aesthetic geometric numbers:

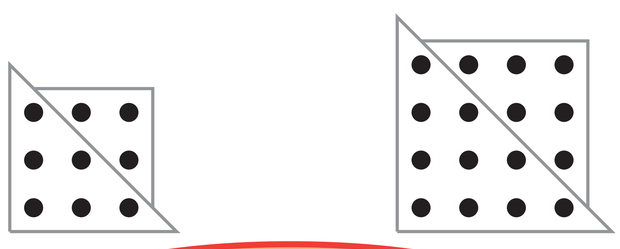


The sum of the first k consecutive odd numbers is a square number:

$$1 + 3 + 5 + \cdots + (2k-3) + (2k-1) = k^2$$



Sum of every consecutive pair of triangular numbers is a square number:



$$\frac{1}{2}(r-1)r + \frac{1}{2}r(r+1) = r^2$$

The representation $6676 = 77^2 + 25^2 + 11^2 + 1^2 = 7^2 11^2 + 5^4 + 11^2 + 1^2$ actually involves some complicated higher powers of natural numbers. It is natural to ask whether one can use in such representations only squares of the most simple numbers, namely, the prime numbers.

As members of the family of natural numbers, the *prime numbers* (or simply *primes*) are the simpliest fundamental units that build all the natural numbers by multiplication (this is the Fundamental Theorem of Arithmetic). This is analogous to that of the *fundamental particles* in our physical world.

One of the main goals of this GRF project is to prove the following

Conjecture.

Every large positive integer of the form $24k + 4$ can be represented as sum of four squares of prime numbers.

Example: $6676 = 5^2 + 7^2 + 19^2 + 79^2$, $220 = 5^2 + 5^2 + 7^2 + 11^2$.

This is a notoriously difficult problem which has resisted the attempts of numerous mathematicians in the past two hundred years. One fruitful approach to this problem is to use instead of the primes slightly wider classes of numbers, namely, the *almost primes* P_r, numbers having no more than r prime factors. P_1 is simply the family of the primes, and the smaller the r, the better. There have been substantial progress along this path of attack, as shown in the following.

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad n \equiv 4 \pmod{24},$$

G. Greaves (1976)	x_1, x_2 are primes, x_3, x_4 are natural numbers
J. Brüdern, E. Fouvry (1994)	x_1, x_2, x_3, x_4 are P_{34} – numbers
D.R. Heath-Brown, I. Tolev(2003)	x_1 is prime, x_2, x_3, x_4 are P_{101} – numbers and x_1, x_2, x_3, x_4 are P_{25} – numbers
K.M. Tsang, L. Zhao*(Rpg student) (2016)	x_1 is prime, x_2, x_3, x_4 are P_5 – numbers and x_1, x_2, x_3, x_4 are P_4 – numbers
T.W. Ching# (Rpg student) (2016)	x_1 is prime, x_2 is P_2 – number and $x_3 x_4$ is P_{183} – number

The latest advancements result from recent development on deep and powerful techniques in analytic number theory, including sieve methods, circle method, estimates of exponential sums over quadratics, and combined with the famous Chen's switching principle. We are now within a stone's throw from achieving this 200+ years feat. It is hoped that such advancements in our techniques would shed light on a wider circle of related problems in analytic number theory.

*K.M. Tsang, L. Zhao, *On Lagrange's Equation with Almost Prime Variables*, to appear in J. Reine Angew. Math.

Preprint.