ON THE BOUNDARY BEHAVIOR OF AUTOMORPHIC FORMS

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ABSTRACT. We investigate the boundary behavior of modular forms f on the full modular group. We first show that $\left\{x \in [0,1] | \lim_{y\to 0^+} y^{k/2}|f(x+iy)| \text{ exists}\right\}$ is contained in a set of Lebesgue measure 0. In particular, we recover the well-known fact that the real axis is a natural boundary of definition for f. On the other hand, using the Rankin-Selberg Dirichlet series attached to f, we show that taking the limit over the "average" over all $x \in [0,1]$ behaves "well". Our results also apply to Maass wave forms.

1. Introduction and statement of results

For k a positive integer, let M_k be the space of modular forms of weight k on the full modular group $\Gamma(1) = SL_2(\mathbb{Z})$ and denote by S_k the subspace of cusp forms. If $f, g \in M_k$ and at least one of them is cuspidal, we put as usual

$$\langle f, g\rangle := \int_{\mathcal{F}} f(z)\overline{g(z)}y^{k-2}dxdy,$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma(1)$ and we write $z = x + iy$ for $z \in \mathbb{H}$, the complex upper half-plane.

Let $f \in S_k$. Using the transformation law of f, one easily sees that for $x \in \mathbb{Q}$

$$\lim_{y\to 0^+} |f(x+iy)| = 0.$$

Moreover, for $x \in \mathbb{R}$,

$$\lim_{y\to 0^+} y^{k/2+\varepsilon}|f(x+iy)| = 0 \qquad (\forall \varepsilon > 0). \tag{1.1}$$

This follows from the fact that the function $y^{\frac{k}{2}}|f(z)|$ is bounded in $\mathbb{H}$. That gives rise to the question of which exponent of y is minimal such that (1.1) is still true. In this paper, we will show that $k/2 + \varepsilon$ is indeed optimal.

For $f \in M_k$, put

$$h(z) := y^k|f(z)|^2 \qquad (z \in \mathbb{H}). \tag{1.2}$$

Then more generally we shall prove

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Theorem 1.1. *Let $f \in M_k$, $f \neq 0$. Then the set*

$$S := \{x \in [0,1] | \lim_{y \to 0^+} h(x+iy) \text{ exists}\}$$

is contained in a set of Lebesgue measure 0.

Remark. With Theorem 1.1, we slightly improve upon a result of [11] in which the author states the nonexistence of the limit

$$\lim_{y \to 0^+} f(x+iy)$$

under certain conditions on x and f (see also [7] and [8]).

Clearly, from Theorem 1.1 we obtain the following well-known result.

Corollary 1.2. *If $f \neq 0$ is in M_k, then f cannot be analytically continued to the real line.*

Next we show that the limit behaves at least "well" in average which we describe in the following. For $f, g \in M_k$ and $y \in \mathbb{R}^+$, let us define

$$H_{f,g}(y) := y^k \int_{[0,1]} f(x+iy) \cdot \overline{g(x+iy)}\, dx. \tag{1.3}$$

Remark. This function also plays an important role in the "Semi-Hull" problem in the more general context of Siegel modular forms (cf. [4]).

As an example, let us consider $f = \Delta$, where

$$\Delta(z) := q \prod_{n=1}^{\infty} (1-q^n)^{24} \qquad (q = e^{2\pi i z})$$

is the usual Δ-function in S_{12}. Computational evidence, obtained using MAGMA, suggests that

$$\lim_{y \to 0^+} \left(y^{12} \int_0^1 |\Delta(x+iy)|^2 dx \right) \approx 9.886 \times 10^{-7}.$$

Observe that

$$9.886 \times 10^{-7} \approx \frac{\|\Delta\|^2}{V(\mathcal{F})},$$

where $V(\mathcal{F}) = \frac{\pi}{3}$ is the volume of $\mathcal{F}$.

This example illustrates a general phenomenon which describes the asymptotic behavior of the m-th derivative of $H_{f,g}(y)$ as $y \to 0^+$.

Theorem 1.3. *If $f, g \in M_k$, with f or g in S_k, then we have*

$$\lim_{y \to 0^+} H_{f,g}^{(m)}(y) = \begin{cases} V(\mathcal{F})^{-1} \cdot \langle f, g \rangle & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. The case $m = 0$ of Theorem 1.3 can be interpreted as a “limit formula.” It gives the value of the Petersson inner product as the limiting value of a single integral. This is of computational value since as is well known and will be recalled below one can write $H_{f,g}(y)$ as a sum just depending on the Fourier coefficients of f and g.

Theorem 1.4. *If $f, g \in M_k \setminus S_k$, then we have*

$$\lim_{y\to 0^+} H_{f,g}(y) = \infty.$$

Remark. The results of this section are also valid for Maass wave forms (mutatis mutandis for the proofs). They can also be generalized to modular forms on congruence subgroups, modular forms of half integral weight, and Hilbert or Siegel modular forms.

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2. Basic facts about modular forms and Dirichlet series.

For general information about Dirichlet series and modular forms, we refer the reader to [1, 2, 3].

Since $M_k = \{0\}$ for k odd or $k < 4$, we may assume in the following that $k \geq 4$ is an even integer. In this case, we define the Eisenstein series

$$E_k(z) := \sum_{M\in\Gamma/\Gamma_\infty} (cz+d)^{-k},$$

where $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | \, n \in \mathbb{Z}\}$. Recall that $\langle f, E_k\rangle = 0$ for $f \in S_k$.

For $f, g \in M_k$ with Fourier coefficients $a_f(n)$ and $a_g(n)$, respectively, we define the Rankin-Selberg convolution

$$D_{f,g}(s) := \sum_{n=1}^{\infty} a_f(n) \cdot \overline{a_g(n)} \cdot n^{-s} \qquad (\mathrm{Re}(s) > 2k-1). \tag{2.1}$$

We set

$$\mathcal{D}_{f,g}(s) := (2\pi)^{-2s} \cdot \Gamma(s) \cdot \Gamma(s+1-k) \cdot \zeta(2s+2-2k) \cdot D_{f,g}(s).$$

Then we have the following (cf. [2])

Theorem 2.1. *If $f, g \in M_k$, then the function $\mathcal{D}_{f,g}(s)$ has meromorphic continuation to the entire complex plane. Moreover,*

$$\mathcal{D}_{f,g}(s) - \sum_{s' \textit{ all poles}} \frac{\mathrm{Res}_{s=s'}\left(\mathcal{D}_{f,g}(s)\right)}{s-s'}$$

is bounded in vertical strips and the functional equation

$$\mathcal{D}_{f,g}(2k-1-s) = \mathcal{D}_{f,g}(s) \tag{2.2}$$

is valid. If either f or g is in S_k, then $\mathcal{D}_{f,g}(s)$ has at most possible simple poles at $s=k$ and $s=k-1$ with residues $\frac{\pi^{1-k}}{2}\langle f,g\rangle$ and $-\frac{\pi^{1-k}}{2}\langle f,g\rangle$, respectively.

The function $\mathcal{D}_{E_k,E_k}(s)$ has simple poles exactly at $s=0$, $s=k-1$, $s=k$, and $s=2k-1$.

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we show the following

Lemma 3.1. *Let $f\in M_k$, $f\neq 0$, and define h by (1.2). Let $\zeta=[a_0;a_1,a_2,\dots]\in\mathbb{R}$ be an infinite continued fraction (in the usual notation) such that the a_n are unbounded. Then there exist sequences $z_n=\zeta+iy_n$ and $w_n=\zeta+iu_n$ $(n\in\mathbb{N})$ with $z_n\to\zeta$, $w_n\to\zeta(n\to\infty)$ such that:*
(i) $h(z_n)$ converges to a nonzero constant,
(ii) $h(w_n)$ converges either to 0 or ∞, depending on whether $f\in S_k$ or not.

Remark. Similar techniques as in the proof of Lemma 3.1 where used in [7], [8], and [11].

Proof. The proof of Lemma 3.1 depends on the following well-known formula (see e.g. [5])

$$\frac{1}{q_l^2(a_{l+1}+2)}<\left|\zeta-\frac{p_l}{q_l}\right|\leq\frac{1}{q_l^2a_{l+1}}, \tag{3.1}$$

where as usual, $\frac{p_l}{q_l}$ is the l-th convergent of ζ. From the assumptions of Lemma 3.1, we find that

$$q_{n_j}^2\left|\zeta-\frac{p_{n_j}}{q_{n_j}}\right|\to 0$$

for some subsequence n_j.

Set $y_{n_j}:=\frac{1}{c\cdot q_{n_j}^2}$, where $c>0$ is chosen such that $f(x+ic)\neq 0$. Our assumption on the sequence a_n implies that ζ is irrational, hence q_n goes to infinity and so z_{n_j} $(j\to\infty)$ tends to ζ. Applying the modular transformation $M_{n_j}:=\left(\begin{smallmatrix} * & * \\ q_{n_j} & -p_{n_j}\end{smallmatrix}\right)$ and using the invariance of h under $\Gamma(1)$ we conclude that $h(z_{n_j})$ $(j\to\infty)$ converges to a non-zero constant.

Next we let $u_{n_j}:=\left|\zeta-\frac{p_{n_j}}{q_{n_j}}\right|$. Clearly $w_{n_j}\to\zeta$ $(j\to\infty)$. Applying again M_{n_j} gives

$$h(w_{n_j})=\left|\zeta-\frac{p_{n_j}}{q_{n_j}}\right|^{-k}q_{n_j}^{-2k}\left|f\left(\zeta+\frac{i}{2q_{n_j}^2|\zeta-p_{n_j}/q_{n_j}|}\right)\right|^2$$

and so one can easily see using standard estimates for modular forms and (3.1) that $h(w_{n_j})$ goes either to 0 or to ∞ depending on whether $f\in S_k$ or not. □

From Lemma 3.1, we directly obtain Theorem 1.1 since the set of all ζ that satisfy the assumptions of Lemma 3.1 contains the set of continued fractions in which every finite sequence of integers occurs. Myrberg shows in [9] and [10] that the latter is measurable and has full measure. Hence its intersection with the interval [0.1] has measure 1.

4. Proof of Theorem 1.3

Clearly, if $f, g \in M_k$ with Fourier coefficients $a_f(n)$ and $a_g(n)$, respectively, then we have

$$H_{f,g}(y) = y^k \sum_{n \in \mathbb{N}_0} a_f(n) \cdot \overline{a_g(n)} \cdot e^{-4\pi n y}. \tag{4.1}$$

Indeed, we insert the Fourier expansions of f and g into (1.3) and interchange summation and integration (which is allowed due to the local uniform convergence of the Fourier expansions of f and g) to get

$$H_{f,g}(y) = y^k \cdot \sum_{n,m \in \mathbb{N}_0} a_f(n) \cdot \overline{a_g(m)} \cdot e^{-2\pi y(n+m)} \int_{[0,1]} e^{2\pi i (n-m)x} dx.$$

Using the fact that the integral vanishes unless $n = m$, in which case it equals 1, gives us identity (4.1).

Proof of Theorem 1.3. To prove Theorem 1.3, we take the m-th derivative of (4.1),

$$H_{f,g}^{(m)}(y) = \sum_{j=0}^{m} \binom{m}{j} \cdot [k]_{m-j} \cdot (-4\pi)^j \cdot y^{k-m+j} \sum_{n>0} a_f(n) \cdot \overline{a_g(n)} \cdot n^j \cdot e^{-4\pi n y},$$

where, for $x \in \mathbb{R}$,

$$[x]_j := \prod_{i=1}^{j} (x - j + i).$$

For $y \in \mathbb{R}^+$ we set

$$G_{f,g,j}(y) := \sum_{n>0} a_f(n) \cdot \overline{a_g(n)} \cdot n^j \cdot e^{-4\pi n y}.$$

Lemma 4.1. *If $f, g \in M_k$, with f or g in S_k, then we have*

$$G_{f,g,j}(y) = \frac{[k+j-1]_j}{(4\pi)^j} \cdot \frac{3}{\pi} \langle f, g \rangle \, y^{-j-k} + F_{f,g,j}(y),$$

where for $y \in \mathbb{R}^+$ we put

$$F_{f,g,j}(y) := y^{-2k+1-j} \cdot (4\pi)^{-j} \cdot \pi^{2k-1} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(2k+j-1-s)\mathcal{D}_{f,g}(s) y^s}{\pi^s \Gamma(2k-1-s)\Gamma(k-s)\zeta(2k-2s)} ds,$$

with c a sufficiently large positive constant.

Proof. For $\mathrm{Re}(s)$ sufficiently large, by taking the Mellin transform, we obtain

$$\int_0^\infty G_{f,g,j}(t)\cdot t^s\,\frac{dt}{t} \quad = \quad (4\pi)^{-s}\cdot\Gamma(s)\cdot D_{f,g}(s-j)$$

and hence by Mellin inversion and a change of variables

$$\begin{aligned} G_{f,g,j}(y) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(4\pi)^{-s}\cdot\Gamma(s)\cdot D_{f,g}(s-j)\cdot y^{-s}\,ds \\ &= (4\pi)^{-j}y^{-j}\cdot\frac{1}{2\pi i}\int_{c-j-i\infty}^{c-j+i\infty}(4\pi)^{-s}\cdot\Gamma(s+j)\cdot D_{f,g}(s)\cdot y^{-s}\,ds, \end{aligned}$$

where $c>0$ is sufficiently large.

We substitute $s\mapsto 2k-1-s$ and then shift the line of integration to the right. Using Theorem 2.1 and the fact that the integrand is of rapid decay if the imaginary part is large, we then deduce in the usual way the identity claimed.

□

From Lemma 4.1, we obtain

$$\begin{aligned} H_{f,g}^{(m)}(y) &= y^{-m}\cdot\frac{3}{\pi}\cdot\langle f,g\rangle\cdot\sum_{j=0}^{m}\binom{m}{j}[k]_{m-j}\cdot[k+j-1]_j\cdot(-1)^j \\ &+ \sum_{j=0}^{m}\binom{m}{j}[k]_{m-j}\cdot(-4\pi)^j\cdot y^{k-m+j}\cdot F_{f,g,j}(y). \end{aligned}$$

Since we can choose c sufficiently large in the definition of $F_{f,g,j}(y)$, the second summand vanishes for $y\to 0^+$. Here we are allowed to interchange limit and integration since the integrand (without the factor y^s) in the definition of $F_{f,g,j}(y)$ is absolutely convergent. Now Theorem 1.3 follows using that

$$\sum_{j=0}^{m}\binom{m}{j}[k]_{m-j}\cdot[k+j-1]_j\cdot(-1)^j$$

equals zero unless $m=0$, in which case it equals 1.

□

Proof of Theorem 1.4. To prove Theorem 1.4, we write $f,g\in M_k$ as $f=f_1+c_1E_k$, $g=g_1+c_2E_k$ with $f_1,g_1\in S_k$ and $c_1,c_2\in\mathbb{C}\backslash\{0\}$. Clearly, we have

$$H_{f,g}(y)=H_{f_1,g_1}(y)+\overline{c_2}H_{f_1,E_k}(y)+c_1H_{E_k,g_1}(y)+c_1\overline{c_2}H_{E_k,E_k}(y).$$

Thus, due to Theorem 1.3, we may assume $f=g=E_k$. The proof of Theorem 1.4 follows easily from the following Lemma.

Lemma 4.2. *We have*

$$G_{E_k,E_k,0}(y) = c_1' y^{1-2k} + c_2' y^{-k} + c_3' + F_{E_k,E_k,0}(y),$$

where c_1', c_2', and c_3' are certain nonzero constants.

Proof. Lemma 4.2 follows exactly as Lemma 4.1 does, using the fact that $\mathcal{D}_{E_k,E_k}$ has simple poles at $s = 0$, $s = k-1$, $s = k$, and $s = 2k-1$. □

□

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