the degree of any vertex in ~~the~~ the subgraph induced on ~~(V,~~ $(M-N)\cup(N-M))$ is at most 2, we conclude that ~~the~~ any connected component must be one of the types.

<u>Theorem</u>: M is a maximum matching in G if and only if it admits no augmenting path with respect to M.

<u>proof</u>: If P is an augmenting path with respect to ~~the~~ M, then $M \oplus P$ is a matching whose cardinality exceeds that of M. Thus M is not maximum, which is a contradiction.

Conversely, suppose there is a matching N with $|N| > |M|$. Now consider the induced graph on $M \oplus N$. Each connected component is either an alternating path or an alternating cycle. Note that each cycle must contain the same number of edges from M and from N. Since $|M| < |N|$, there exists at least one alternating path that contains more edges from N than from M. ~~And that~~ Such a path necessarily extends two vertices that are exposed by M, there is an augmenting path with respect to M.

Let $G(S, T, E)$ be a bipartite graph, M be a non-maximum matching in G. The following algorithms finds an augmenting path with respect to M.

Step 0. Label "I" to every exposed vertex in S.

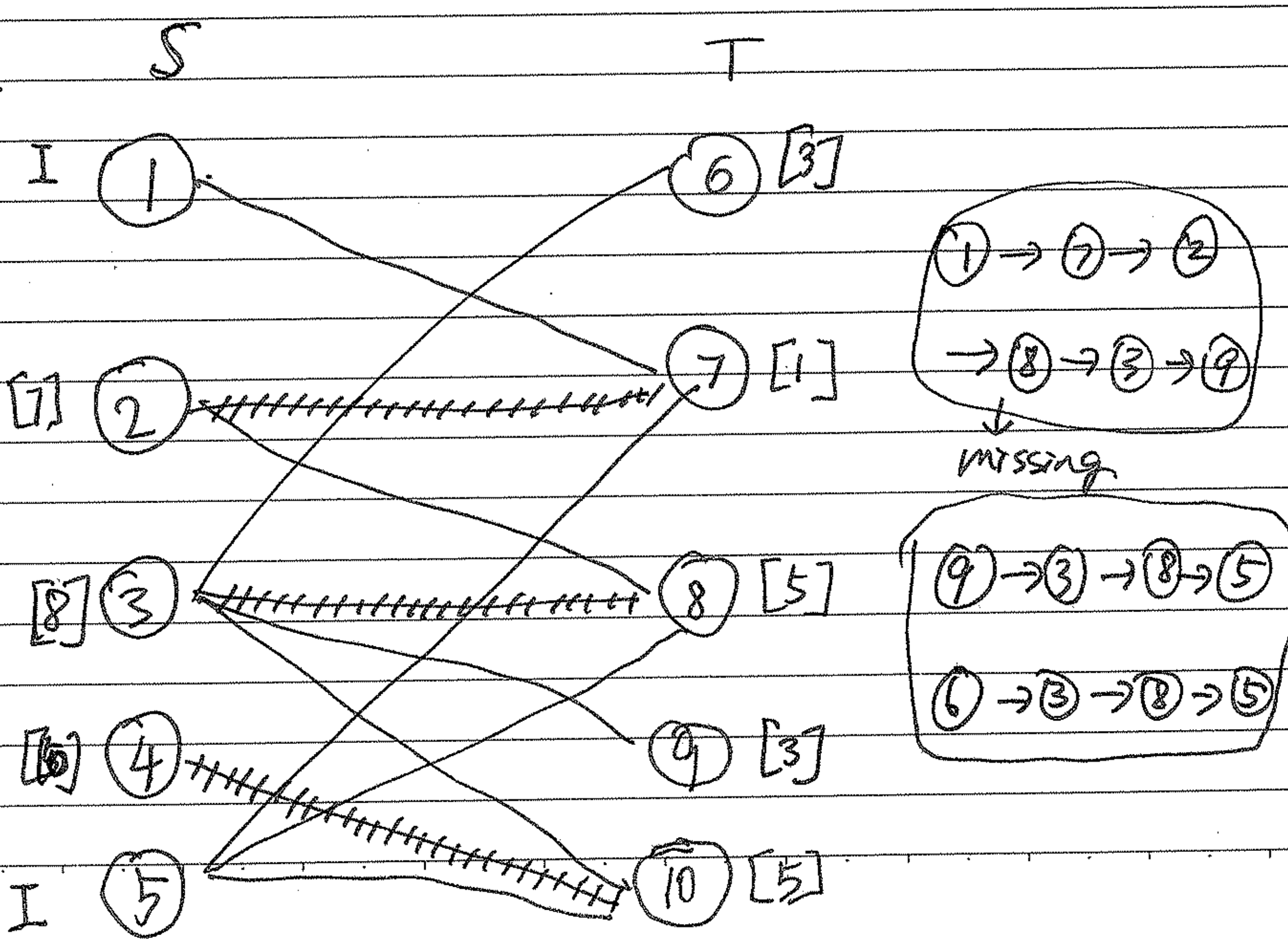
Step 2. For any unscanned labeled vertex i ($i \in S$), and for each edge (i, j) outside M incident to vertex j, give the vertex j the label "[i]" unless vertex j has already been labeled. Then go to Step 1.

Step 1. Find a vertex i ~~with~~ unscanned label. If $i \in S$, go to step 2; If $i \in T$, go to step 3. If ~~no unscanned~~ all labeled nodes have been scanned, there is no augmenting path.

Step 3. ~~Scan~~ If vertex i is exposed, an augmenting path has been found, terminating at vertex i. Can be found through back-tracing labels.

The vertex preceding vertex i on the path are identified by "back-tracing": using ~~to~~ the labels on vertices: ~~vertex i~~, if vertex i is labeled $[j]$, then the second last vertex on the path is vertex j, if vertex j is labeled $[k]$, the third last vertex on the path is vertex k, and so on, continue the back-tracing procedure until we reach a vertex labeled "I". If vertex i is not exposed, identify the unique edge $(i,j) \in M$ and label vertex j "$[j]$", ~~then~~ Go to Step 2.

<u>Example:</u>



Remark: For an non-maximum matching M in a bipartite graph G, one can find an augmenting path P using the algorithm above. Then reset $M = M \oplus P$, (here note that new M covers all vertices covered by old M) and continue ~~to~~ the procedure until M is ~~the~~ a maximum matching. We thus proved the following theorem:

Theorem: Let M be a matching in a bipartite graph. Then there exists a maximum matching that covers all vertices covered by M.

The above algorithm and theorem can be generalized to general graphs.

Mendelsohn-Dulmage Theorem:

Let M ~~and N~~ be ~~two~~ a matching~~s~~ in a general graph G. Then there exists a maximum matching X ~~[illegible]~~, such that X covers all the vertices covered by M.

A graph G without perfect matching is called <u>Saturated non-factorizable</u> if a new graph G' obtained by adding any single edge has a perfect matching.

<u>Lemma</u>: Let G be a saturated non-factorizable graph, and let S be the vertex subset of G adjacent to every other vertex of G, then the components of $G-S$ are complete graphs.

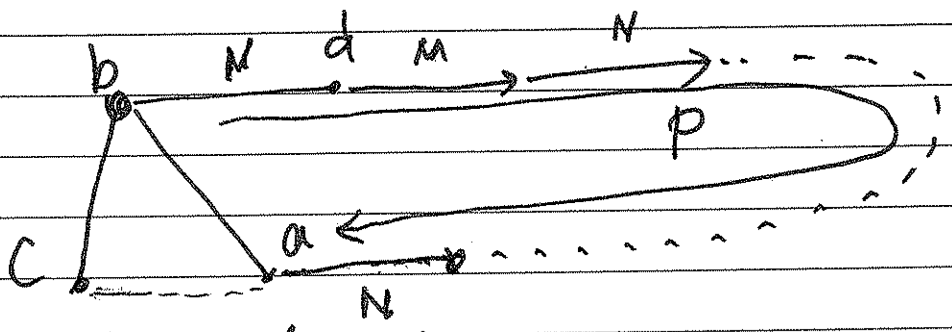
<u>proof</u>: Let (a,b) and (b,c) be any two adjacent edges of $G-S$. It suffices to prove that a and c are adjacent too. Suppose that, by contradictions, they are not.

By the definition of S, we can find another vertex d of G which is not adjacent to b. ($b \notin S$)

Now let M be a perfect matching for $G+(a,c)$ containing (a,c), and let N be a perfect matching of $G+(b,d)$ containing (b,d). Then $M \oplus N$ consisting of alternating cycles, and (a,c), (b,d) each lie on such a cycle, say C, D, respectively. We then have ~~the~~ two cases:

① $C \neq D$. Let $K = M \oplus C$, where C is treated as a set of edges. Then K is a perfect matching of G, a contradiction.

② $C = D$. Traverse C from b through d and continue until one of a and c is reached. Without loss of generality, assume a is reached first; let P be the b-a path

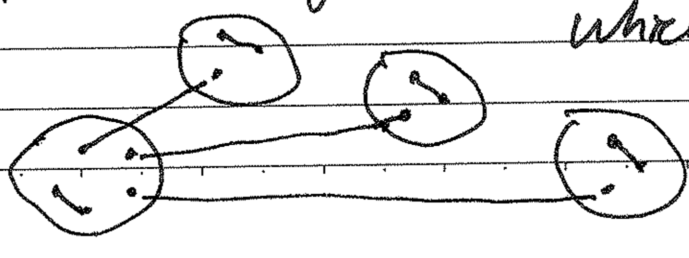


just traversed. Then $P+(a,b)$ is an alternating cycle with respect to N, thus $N \oplus (P+(a,b))$ is a perfect matching of G, a contradition.

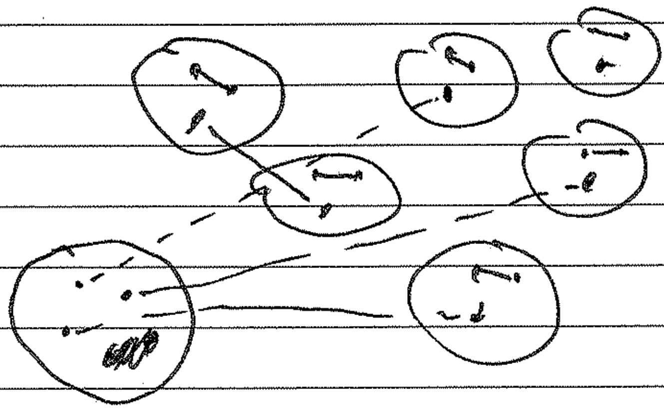
<u>Theorem</u>: A graph $G(V, E)$ is saturated non-factorizable if and only if G has even number of vertices and G consists of vertex-disjoint complete subgraphs $S_0, G_1, G_2, \ldots, G_K$ such that $K = |S_0| + 2$, $G_1, G_2, \ldots, G_K$ are of odd order and every vertex of every G_i is connected to every vertex of S_0.

<u>proof</u>: Let S be the set defined in the previous Lemma, and let $G_1, G_2, \ldots, G_K$ be the connected components of $G-S$. Then by the previous Lemma, $G_1, G_2, \ldots, G_K$ are all complete graphs; by the definition of S, S spans a complete graph, and every vertex of S is adjacent to every vertex in G.

If at most $|S|$ components of $G-S$ are odd, then a perfect matching can be easily found, which is an contradiction



So at least $|S|+1$, in fact, by parity, at least $|S|+2$ odd components of $G-S$ are odd. If $G-S$ has more than $|S|+2$ odd components, then by connecting two ~~of these~~ with a new edge, we have a graph G_1 with at least $|S|+1$ odd components, which does not admit a perfect matching, a contradiction. (to the fact that G is saturated non-factorizable).

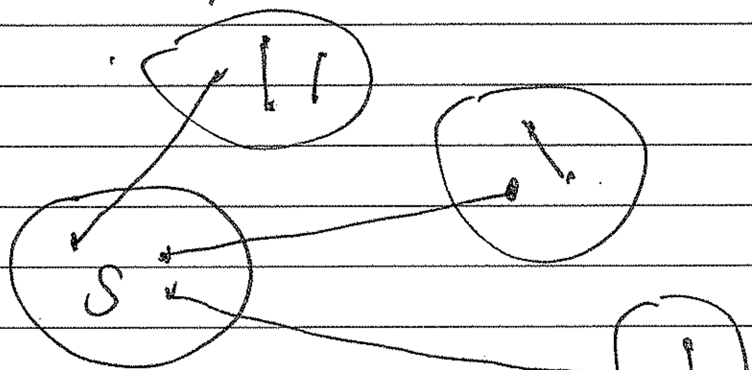


So $G-S$ has exactly $|S|+2$ odd components.

If $G-S$ has at least one even components, then by adding a new edge connecting ~~the~~ a even component to an odd component, we have a new graph with at least $|S|+1$ odd components, which does not admit perfect matching, a contradiction.
So $G-S$ does not have even components.

<u>Tutte's Theorem</u>: A graph $G(V, E)$ has perfect matching if and only if G has even vertices and for every vertex subset $S \subseteq V$, the subgraph $G-S$ consists of no more than $|S|$ odd components.

<u>proof</u>: $\Rightarrow$ Suppose M is a perfect matching of G. For any $S \subseteq V$, every odd component of $G-S$ "send" at least one edge of M to S, and every vertex in S "receives" at most one of these edges, which implies $G-S$ has at most $|S|$ odd components.



$\Leftarrow$ Suppose, by contradictions, that G has ~~even vertices~~ no perfect matchings. Add edges to G as long as the resulting graphs have no perfect matchings. Let G' be an extremal graph (which is saturated non-factorizable) resulted from this procedure.

Let S' be the set of all vertices, each of which is adjacent to every vertex of G'. Then $G'-S' \neq \phi$, since G' does not have a perfect matching. By the previous lemma, $G'-S' = G_1' \cup G_2' \cup \cdots \cup G_k'$, where the G_i's are vertex-disjoint complete graphs with odd ~~other~~ number of vertices, and $k = |S'|+2$.

So $G'-S'$ has ~~at~~ more than $|S'|$ odd components. If we remove all the edges inserted during the procedure, each of those odd components of $G'-S'$ gives rise to at least one odd components of $G-S'$, which implies S' violates the hypothesis.

Remark: Tutte's theorem's conditions for bipartite graphs are equivalent to Hall's theorem's conditions, thus Tutte's theorem is a generalization of Hall's theorem.

The *deficiency* of G is defined by ~~def(G)~~

$$def(G) = |V| - 2\nu(G),$$ i.e., $def(G)$ is the number of vertices of G exposed by any maximum matching of G.

Berge's Theorem: For any graph G,

$$def(G) = \max\{Co(G-X) - |X| : X \subseteq V\},$$

where $Co(G-X)$ denotes the number of odd components in the subgraph $G-X$.

proof: Denote $\delta(G) = \max\{Co(G-X) - |X| : X \subseteq V\}$. We shall prove $def(G) = \delta(G)$.

$\geq$: Let M be a maximum matching of G, X be a ~~any~~ vertex subset, such that ~~and~~ $G_1, G_2, \dots, G_K$ ~~be the~~ odd components of $G-X$ there are $K = Co(G-X)$.