Polynomial time approximation of entropy of shifts of finite type

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Conference on Applied Mathematics
University of Hong Kong
August, 2016

$\mathbb{Z}^d$ Shifts of finite type

- Let $\mathcal{A}$ be a finite alphabet.
  $\mathcal{A}^{\mathbb{Z}^d} :=$ { all d-dimensional arrays of symbols from $\mathcal{A}$ }.
- **Shift of finite type (SFT):**
  Let $\mathcal{F}$ is a *finite* list of "forbidden" patterns on *finite* sets,
  $$X = X_{\mathcal{F}} =$$
  $$\{x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no translate of an element of } \mathcal{F}\}$$
- SFT's also known as "finite memory constraints."
- **Nearest neighbor (n.n.) SFT:** an SFT where all forbidden patterns are patterns on *edges* of $\mathbb{Z}^d$.
- Main Example ($d = 2$): **hard square SFT**
  $$\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{matrix}1\\1\end{matrix}\}$$

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Topological entropy

- d-dimensional cube: $B_n := [0, n-1]^d$
- for an SFT X,

$$L_n(X) = \{ \text{ legal configurations on } B_n \}$$

- **Topological entropy (noiseless capacity):**

$$h(X) := \lim_{n \to \infty} \frac{\log |L_n(X)|}{n^d}$$

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SFT's, $d = 1$

- A one-dimensional n.n. SFT $X = X_{\mathcal{F}}$ is a set of sequences specified by a directed graph G with vertices in $\mathcal{A}$ and an edge from a to b iff $ab \notin \mathcal{F}$.

  Golden Mean Shift ($(1, \infty)$ constraint): $\mathcal{F} = \{11\}$

- Adjacency matrix A of G is the square matrix indexed by $\mathcal{A}$:
  $A_{ab} = \left\{ \begin{array}{ll} 1 & ab \notin \mathcal{F} \\ 0 & ab \in \mathcal{F} \end{array} \right\}$
- $h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of A.
- Characterization of entropies for $d = 1$ (Lind):

$$\{\log \lambda^{1/q}\}$$

  where λ is a Perron number and $q \in \mathbb{N}$

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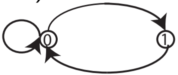

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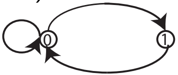

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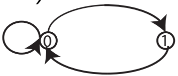

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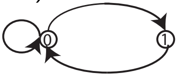

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Examples of $\mathbb{Z}^2$ SFTs: hard square

- **hard squares** $\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{matrix}1\\1\end{matrix}\}$
- h(hard square SFT) = ???
- (Baxter) h(hard hexagons) = log(λ) where λ is an algebraic integer of degree 24.

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Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- ***q*-checkerboard** $\mathcal{C}_q$: $\mathcal{A} = \{1, \ldots, q\}, \mathcal{F} = \{aa, \begin{matrix} a \\ a \end{matrix}\}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = (3/2)\log(4/3)$
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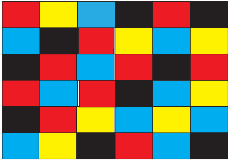

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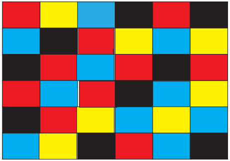

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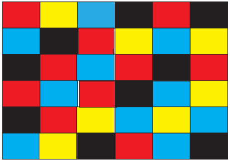

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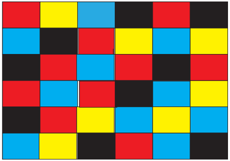

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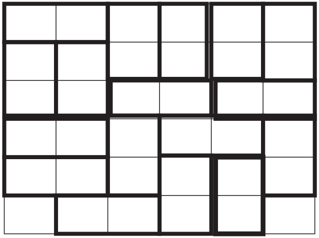

$$\mathcal{F} = \{LL, LT, LB, RR, TR, BR, \begin{matrix} T \\ L \end{matrix}, \begin{matrix} T \\ R \end{matrix}, \begin{matrix} T \\ T \end{matrix}, \begin{matrix} B \\ B \end{matrix}, \begin{matrix} L \\ B \end{matrix}, \begin{matrix} R \\ B \end{matrix}\}$$

- (Fisher-Kastelyn-Temperley): $h(\text{ Dimers }) = \frac{1}{16\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \log(4 + 2\cos\theta + 2\cos\phi)\, d\theta d\phi$
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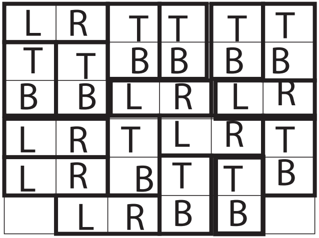

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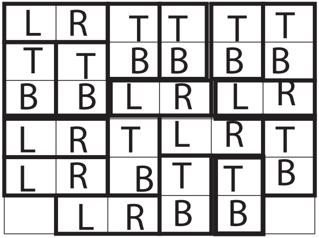

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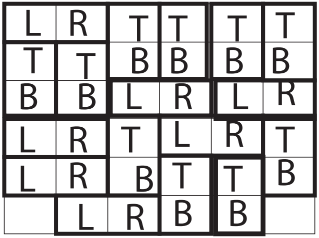

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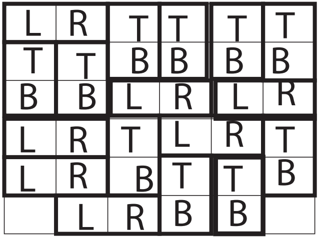

$$\mathcal{F} = \{LL, LT, LB, RR, TR, BR, \begin{array}{c}T\\L\end{array}, \begin{array}{c}T\\R\end{array}, \begin{array}{c}T\\T\end{array}, \begin{array}{c}B\\B\end{array}, \begin{array}{c}L\\B\end{array}, \begin{array}{c}R\\B\end{array}\}$$

- (Fisher-Kastelyn-Temperley):
  $h(\text{ Dimers }) = \frac{1}{16\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\log(4 + 2\cos\theta + 2\cos\phi)\ d\theta d\phi$
- h(Monomers-Dimers) = ???

Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$ (Hochman-Meyerovitch):

  $$\{\textbf{right recursively enumerable} \text{ (RRE) numbers } h \geq 0\}$$

  i.e, there is an algorithm that produces a sequence $r_n \geq h$ s.t. $r_n \to h$.
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  - Necessity: Let $r_n := \frac{\log |L_n|}{n^d}$.
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  - Sufficiency (hard): Emulate Turing machine with an SFT.
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polynomial time approximation

- A **polynomial time approximation algorithm**: on input n, produces r_n s.t. $|r_n - h| < 1/n$ and r_n can be computed in time poly(n).
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Given a shift-invariant Borel probability measure μ on $\mathcal{A}^{\mathbb{Z}^d}$,

- For finite $S \Subset \mathbb{Z}^d$,

$$H_\mu(S) := \sum_{x \in \mathcal{A}^S} -\mu(x) \log \mu(x) = \int -\log \mu(x) d\mu(x)$$

- For finite disjoint S, T,

$$H_\mu(S \mid T) := \sum_{x \in \mathcal{A}^S, y \in \mathcal{A}^T:\ \mu(y)>0} -\mu(x,y) \log \mu(x \mid y)$$

- Extend to finite S and infinite T:

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Entropy (entropy rate) of μ

- $h(\mu) := \lim_{n\to\infty} \frac{H_\mu(B_n)}{n^d}$
- $d = 1$: Theorem: $h(\mu) = H_\mu(0 \mid \{-1, -2, -3, \dots\})$
- $d = 2$: Let $\prec$ denotes lexicographic order: $(i, j) \prec (i', j')$ iff either $j < j'$ or ($j = j'$ and $i < i'$).

For $\overline{z} \in \mathbb{Z}^2$, let $\mathcal{P}(\overline{z}) := \{\overline{z}' \in \mathbb{Z}^2 : \ \overline{z}' \prec \overline{z}\}$ the lexicographic past of $\overline{z}$, and $\mathcal{P} := \mathcal{P}(0)$

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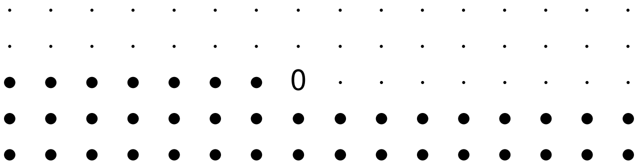

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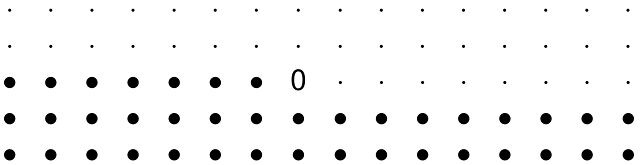

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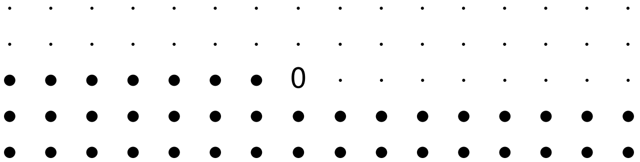

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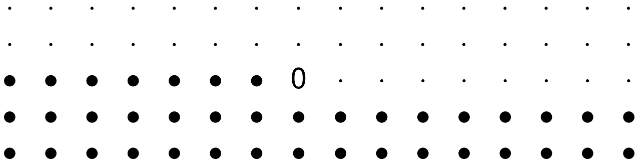

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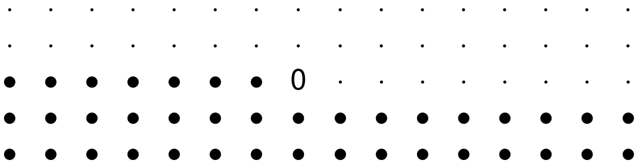

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Variational Principle for Topological Entropy

- For an SFT X,
$$h(X) = \sup_{\mu} h(\mu)$$
where the sup is taken over all shift-invariant Borel probability measures μ s.t. support$(\mu) \subseteq X$.
- Fact: The sup is always achieved. A measure which achieves the sup is called a **measure of maximal entropy (MME).**
- So for an MME μ, $h(X) = h(\mu) = \int I_\mu(x) d\mu(x)$
- Under certain conditions, $h(X) = h(\mu) = \int I_\mu(x) d\nu(x)$ for some other invariant measures ν
- If this holds for $\nu =$ the δ-measure on a fixed point $s^{\mathbb{Z}^d}$, then

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Rough Idea for showing $h(X) = I_\mu(s^{\mathbb{Z}^d})$

An MME μ should be "nearly uniform". So, μ captures entropy:

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This is an average of n^d terms of two types:

- *Bulk terms*: Terms that are far from the boundary of B_n
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Bulk terms are close to $I_\mu(s^{\mathbb{Z}^d})$. All terms are uniformly bounded. Most terms are bulk terms. So, $h(X) = I_\mu(s^{\mathbb{Z}^d})$.

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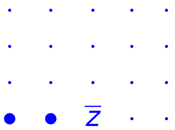

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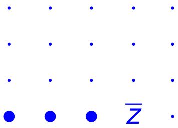

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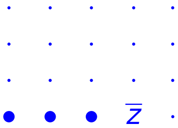

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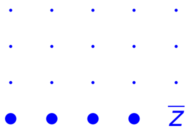

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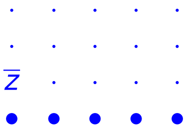

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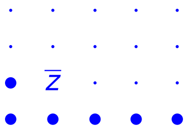

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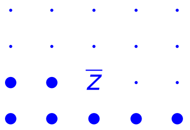

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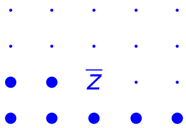

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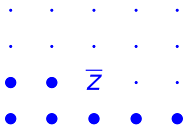

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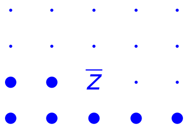

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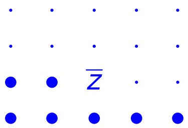

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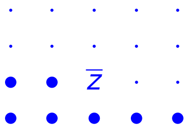

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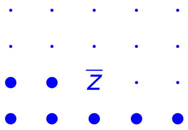

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A n.n. SFT X has a **safe symbol** s if it is legal with every configuration of nearest neighbours:

$$\begin{array}{c|c|c} & \star & \\ \hline \star & s & \star \\ \hline & \star & \end{array}$$

Examples: Yes: Hard squares ($s = 0$)
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Let $R_{a,b,c} := [-a,-1] \times [1,c] \cup [0,b] \times [0,c]$

Example: $R_{3,4,3}$:

Theorem: Let X be a n.n. $\mathbb{Z}^d$ SFT and μ an MME on X. If

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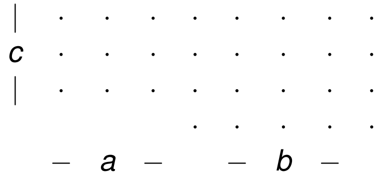

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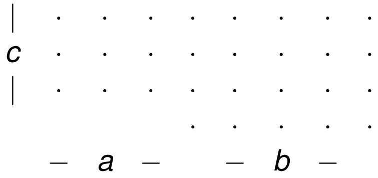

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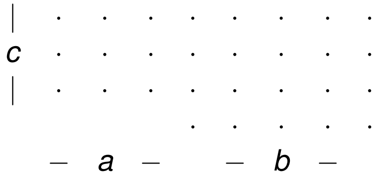
Let $R_{a,b,c} := [-a,-1] \times [1,c] \cup [0,b] \times [0,c]$
Example: $R_{3,4,3}$:



Theorem: Let X be a n.n. $\mathbb{Z}^d$ SFT and μ an MME on X. If

1. X has a safe symbol s – and –
2. (for $d=2$)

$$L := \lim_{a,b,c \to \infty} \mu(s^0 \mid s^{\partial R_{a,b,c}}) \text{ exists}$$

Then

$$h(X) = -\log L$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $h(X)$.

Proof of Moreover: Approximate L by $\mu(s^0 \mid s^{\partial R_{n,n,n}})$.

- Accuracy is $e^{-\Omega(n)}$
- Claim: Computation time is $e^{O(n)}$
- Trade exponential accuracy in exponential time for linear accuracy $(1/n)$ in polynomial time. $\square$

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Proof of Claim, via transfer matrices

$$\mu(s^0 \mid s^{\partial R_{n,n,n}}) = \frac{\#\begin{array}{ccccccc} & s & s & s & s & s & \\ s & \cdot & \cdot & \cdot & \cdot & \cdot & s \\ s & \cdot & \cdot & \cdot & \cdot & \cdot & s \\ s & s & s & \color{blue}{s} & \cdot & \cdot & s \\ & & & s & s & s & \end{array}}{\#\begin{array}{ccccccc} & s & s & s & s & s & \\ s & \cdot & \cdot & \cdot & \cdot & \cdot & s \\ s & \cdot & \cdot & \cdot & \cdot & \cdot & s \\ s & s & s & \cdot & \cdot & \cdot & s \\ & & & s & s & s & \end{array}}$$

$$= \frac{(\prod_{i=-n}^{-1} M_i)\hat{M}_0(\prod_{i=1}^{n-1} M_i)}{(\prod_{i=-n}^{-1} M_i)M_0(\prod_{i=1}^{n-1} M_i)}$$

M_i is transition matrix from column i to column $i+1$ compatible with $s^{\partial R_{n,n,n}}$ and

$\hat{M}_0$ is matrix obtained from M_0 by forcing s at origin. □

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Extensions

- Weaken fixed point $s^{\mathbb{Z}^d}$ to periodic orbit
- Weaken safe symbol to topological strong spatial mixing
- Applies to
  - hard squares
  - monomer-dimers
  - q-checkerboard SFT with $q \geq 6$
- Generalize results from entropy to *pressure* of n n. interactions on n.n. SFT's
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End of talk

The following slides form a hodge-podge of topics that were not included in the talk.

Defn of TSSM with gap g:
for any disjoint $U, S, V \in Z^d$ s.t. $d(U, V) \geq g$,
if $u \in A^U$, $s \in A^S$, $v \in A^V$, s.t. us and sv are allowed, then so is usv.

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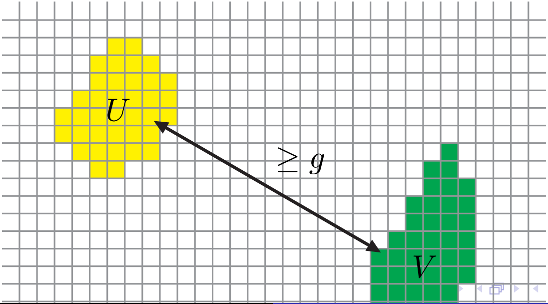

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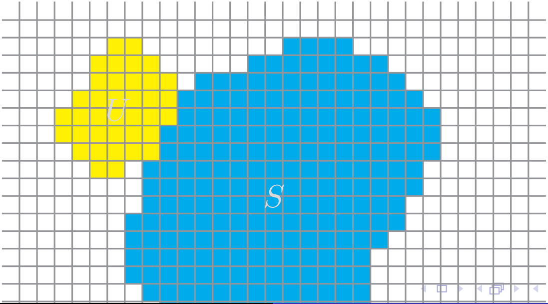

Topological Strong Spatial Mixing (TSSM)

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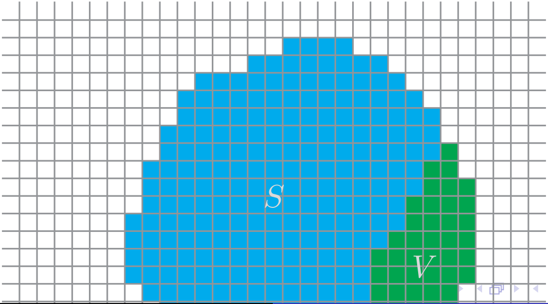

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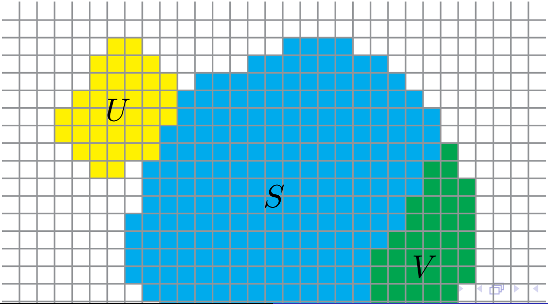

Let $R_{a,b,c} := [-a,-1] \times [1,c] \cup [0,b] \times [0,c]$

Example: $R_{3,4,3}$:

Theorem: Let X be a $\mathbb{Z}^d$ n.n. SFT and μ an MME on X. If

1. X satisfies TSSM
2. (for $d=2$) For some periodic orbit O in X and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c})) \text{ exists}$$

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$$h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)$$

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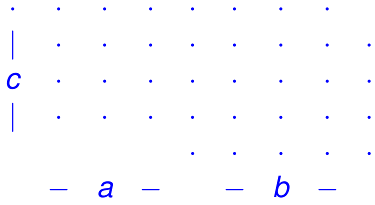

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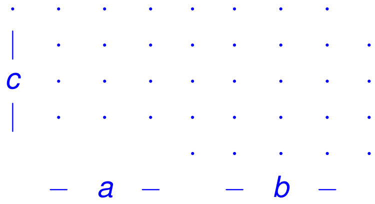

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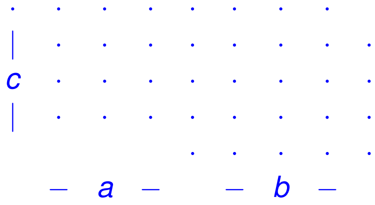

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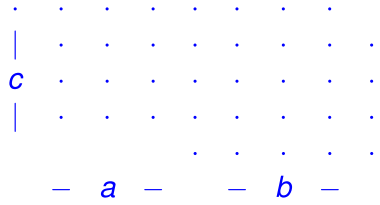

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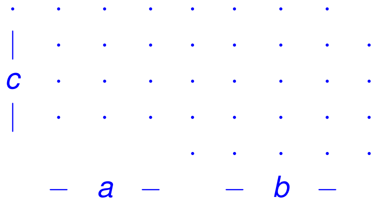

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A **Markov random field (MRF)** is a shift-invariant Borel probability measure μ on $\mathcal{A}^{\mathbb{Z}^d}$ such that for any choice of:

- $S \Subset \mathbb{Z}^d$,
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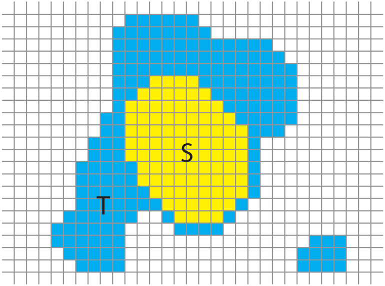

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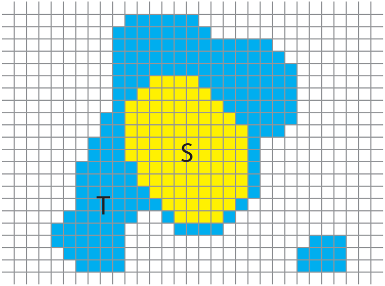

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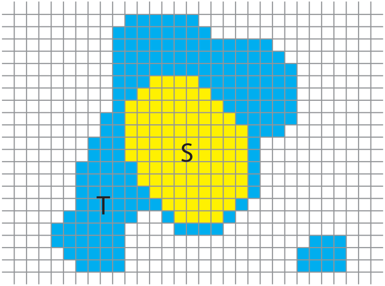

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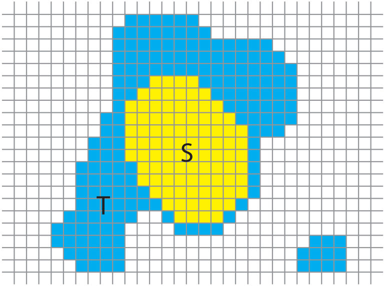

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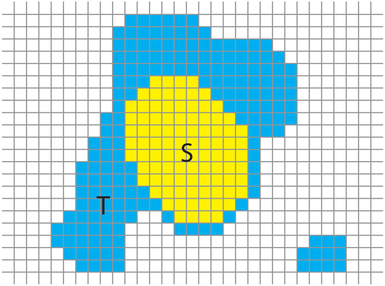

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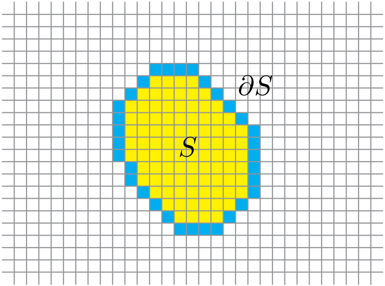

Uniform MRF

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$$L_S^y(X) := \{x \in \mathcal{A}^S : \ xy \text{ is legal } \}$$

An MRF on X is **uniform** if whenever $\mu(y) > 0$, then for $x \in L_S^y(X)$

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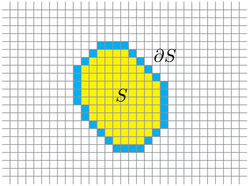
Let X be a n.n. SFT. For $S \Subset \mathbb{Z}^d$ and $y \in \mathcal{A}^{\partial S}$, let

$$L_S^y(X) := \{x \in \mathcal{A}^S : \ xy \text{ is legal } \}$$



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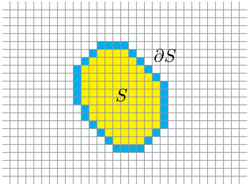

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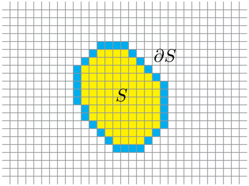

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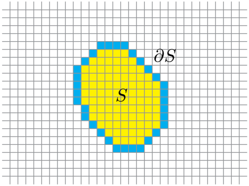

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- Since μ is an MME, μ must be a uniform MRF.
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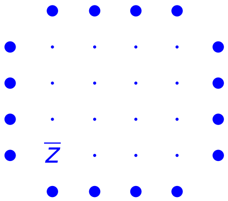

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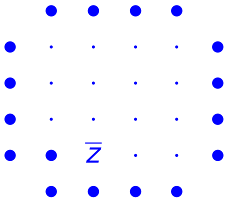

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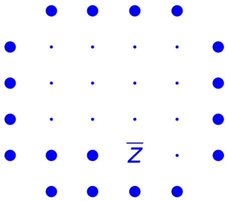

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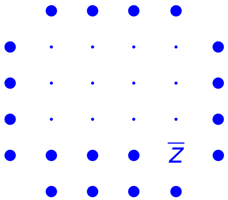

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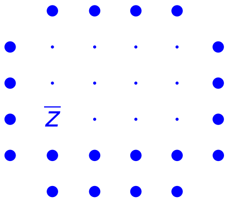

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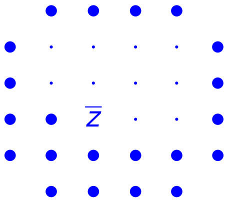

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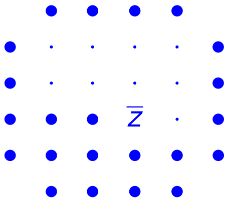

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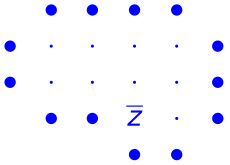

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Theorem: Let X be a n.n. $\mathbb{Z}^2$ SFT and μ an MME on X. If

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- An MRF μ satisfies **strong spatial mixing (SSM)** at rate $f(n)$

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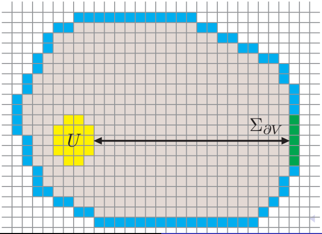

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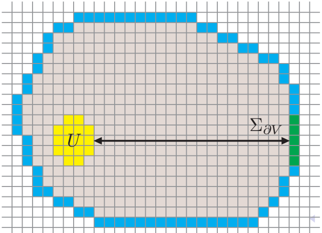

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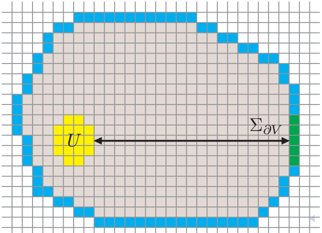

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Applies to:

- hard squares
- q-checkerboard with $q \geq 6$

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Nearest-Neighbour interactions and Gibbs measures

- A *nearest-neighbor interaction* is a shift-invariant function Φ from a set of configurations on vertices and edges in $\mathbb{Z}^d$ to $\mathbb{R} \cup \infty$
- For a nearest-neighbor interaction Φ, the *underlying SFT*:

$$X = X_\Phi := \{x \in \mathcal{A}^{\mathbb{Z}^d} \;:\; \Phi(x(\{v, v'\})) \neq \infty, \text{ for all } v \sim v'\}.$$

- A *nearest neighbour (n.n.) Gibbs measure* μ corresponding to Φ is an MRF on X such that for $S \Subset \mathbb{Z}^d$, $\delta \in \mathcal{A}^{\partial S}$, $\mu(\delta) > 0$, $w \in \mathcal{A}^S$:

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Examples of n.n. Gibbs measures

- uniform MME on n.n. SFT
- hard square model with activities
- ferromagnetic Ising model with no external field.

Equilibrium states versus n.n. Gibbs measures

- Pressure of n.n. interaction Φ:

$$P(\Phi) := \lim_{n\to\infty} \frac{\log Z^{\Phi}(B_n)}{n^d}$$

where $Z^{\Phi}(B_n)$ is the "free boundary" normalization.
- Let $A_\Phi(x) := -\Phi(x(0)) - \sum_{i=1}^{d} \Phi(x(0), x(e_i))$.
- Fact: $P_{X_\Phi}(A_\Phi) = P(\Phi)$.
- Lanford-Ruelle Theorem: Every equilibrium state for A_Φ is a Gibbs measure for Φ.
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Theorem (Adams, Briceno, Marcus, Pavlov): Let μ a Gibbs measure for a n.n. interaction Φ with underlying $\mathbb{Z}^d$ n.n. SFT X. If

1. X satisfies TSSM
2. For some periodic orbit O in X and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c\to\infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c})) \text{ exists}$$

Then

$$P(\Phi) = \frac{1}{|O|} \sum_{\omega\in O} -\log L(\omega) + A_\Phi(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$.

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- there exist sequences of finite subsets (Λ_n), (M_n) of $\mathbb{Z}^d$ such that $\Lambda_n \nearrow \infty$, $\Lambda_n \subseteq M_n$, $\frac{|M_n|}{|\Lambda_n|} \to 1$, such that
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Connection with Thermodynamic Formalism

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MME, $d = 1$

- Assuming adjacency matrix A is irreducible and aperiodic, there is a unique MME $\mu_{\max}$, which is a Markov chain given by transition matrix

$$P_{ij} = \left\{ \begin{array}{cc} \frac{r_j}{\lambda r_i} & ij \notin \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{array} \right\}$$

  where $\lambda = \lambda(A)$ and r is a right eigenvector for λ, and stationary vector $r_i \ell_i$ where ℓ is a left eigenvector for λ (suitably normalized)

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Entropy representation for MME, $d = 1$

- $$\begin{aligned} I_\mu(x) &= -\log \mu(x(0)|\ x(\mathcal{P})) \\ &= -\log P_{x_0 x_{-1}} \\ &= \log \lambda + \log r_{x_{-1}} - \log r_{x_0} \end{aligned}$$
- So, for *all* invariant measures ν,

$$\begin{aligned} \int I_\mu(x) d\nu(x) &= \int (\log \lambda + \log r_{x_{-1}} - \log r_{x_0}) d\nu(x) \\ &= \log \lambda \\ &= h(X) \end{aligned}$$

  In particular, if the SFT has a fixed point $x^* := a^{\mathbb{Z}}$ and ν is the delta measure on x^*, then on

$$h(X) = \int I_\mu(x) d\nu(x) = I_\mu(x^*) = -\log \mu(x^*)$$

  and so $h(X)$ can be computed from the value of the information function at only one point.
- In this case, $I_\mu(x)$ is defined everywhere.

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