The capacity of the $(1, \infty)$-RLL Input-Constrained Erasure Channel with Feedback

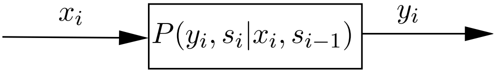
Oron Sabag, Haim Permuter and Navin Kashyap

Ben Gurion University and Indian Institute of Science

CAM16, Hong-Kong

Main Ingredients

1. Channel with memory



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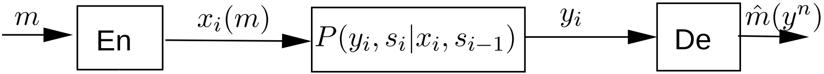

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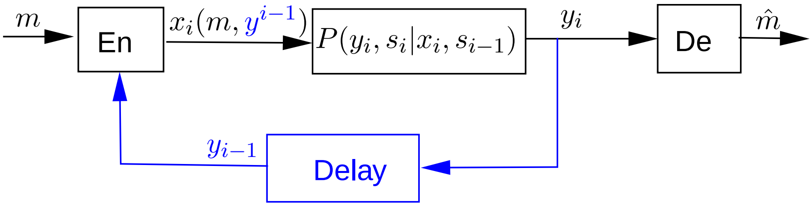

2. Feedback

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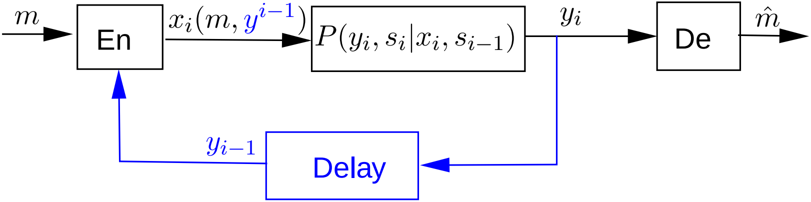

2. Feedback
3. Directed information

$$I(X^n \to Y^n)$$

Definition of directed information

$$I(X^n;Y^n) \triangleq H(Y^n) - H(Y^n|X^n)$$

$$H(Y^n|X^n) \triangleq E[-\log P(Y^n|X^n)]$$

$$P(y^n|x^n) = \prod_{i=1}^{n} P(y_i|x^n, y^{i-1})$$

$$X^n \triangleq (X_1, X_2, X_3, ..., X_n)$$

Definition of directed information

Directed Information [Massey90]

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Feedback capacity of FSC

Theorem

For any FSC <u>with</u> feedback *[P.&Weissman&Goldsmith09]*

$$C_{FB} \geq \frac{1}{n} \max_{P(x^n||y^{n-1})} \min_{s_0} I(X^n \to Y^n|s_0) - \frac{\log|\mathcal{S}|}{n}$$

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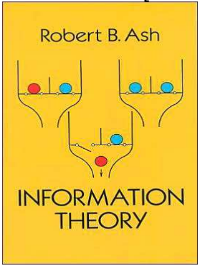
- $\max_{P(x^n||y^{n-1})} I(X^n \to Y^n)$ can be computed using an extension of Blahut-Arimoto algorithm. [Naiss.&P.13]
- Under mild conditions

$$C_{FB} = \lim_{n\to\infty} \frac{1}{n} \max_{P(x^n||y^{n-1})} I(X^n \to Y^n)$$

Trapdoor Channel [Blackwell61]



Ising Channel [Berger90]

$$y_i = \begin{cases} x_i, & \text{with prob. } \frac{1}{2} \\ x_{i-1}, & \text{with prob. } \frac{1}{2} \end{cases}$$

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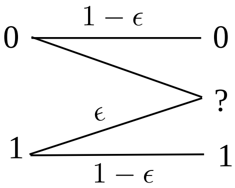
Erasure Channel with no repeated 1's



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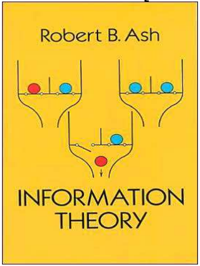

$$C_{fb} = \log \phi, \ \ \phi = \frac{\sqrt{5}+1}{2}$$

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$$C_{fb} = \max_p \frac{2H_2(p)}{3+p} \approx 0.575$$

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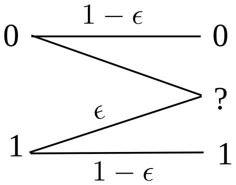

$$C_{fb} = \max_p \frac{H_2(p)}{p + \frac{1}{1-\epsilon}}$$

Feedback capacity computation

For a unifilar channel we have

$$C_{FB} = \lim_{n\to\infty} \frac{1}{n} \max_{Q(x^n||y^{n-1})} I(X^n \to Y^n)$$

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Dynamic programming (DP) formulation

- state: β_{t-1}
- action: a_t
- disturbance: w_t

$$P(w_t|\beta^{t-1}, w^{t-1}, a^t) = P(w_t|\beta_{t-1}, a_t); \qquad \beta_t = F(\beta_{t-1}, a_t, w_t)$$

- reward per unit time:

$$g(\beta_{t-1}, a_t)$$

- objective:

$$\sup_{\{a_t\}_{t\geq 1}} \liminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \mathbb{E}\left[g(\beta_{t-1}, a_t)\right]$$

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Dynamic programming operator, T

- The dynamic programming operator T is given by

$$(T \circ J)(\beta) = \sup_{a \in \mathcal{A}} \left(g(\beta, a) + \sum_{w} P(w|\beta, a) J(F(\beta, a, w)) \right)$$

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- In our case: $J : [0,1] \to \mathbb{R}$ and

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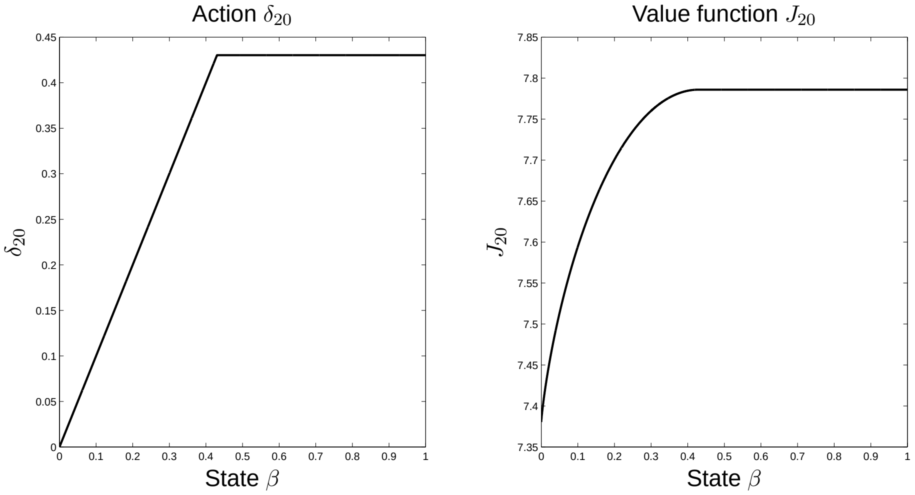
- Value iteration algorithms: executed 20 iterations

$$J_{k+1} = T \circ J_k$$

$$C_{FB} \approx 0.405 \text{ bits}$$

Result of value iteration



Dynamic programming- Bellman equation

Theorem (Bellman Equation)

If there exist a function $J(\beta)$ and a constant ρ that satisfy

$$T \circ J(\beta) = J(\beta) + \rho$$

then ρ is the optimal infinite horizon average reward.

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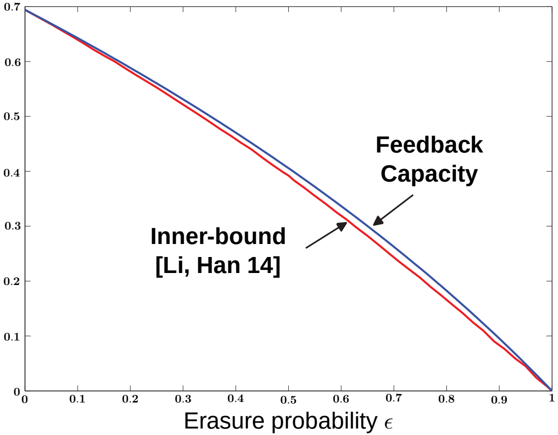
$$J^*(z) = \begin{cases} \bar{\epsilon} H_b(z) - z\bar{\epsilon}\frac{H_b(p_\epsilon)}{p_\epsilon + \frac{1}{1-\epsilon}} & \text{if } 0 \leq z \leq p_\epsilon \\ \frac{H_b(p_\epsilon)}{p_\epsilon + \frac{1}{1-\epsilon}} & \text{if } p_\epsilon \leq z \leq 1. \end{cases}$$

$$\rho^*_\epsilon = \max_{0 \leq p \leq 1} \frac{H_b(p)}{p + \frac{1}{1-\epsilon}},$$

We showed that $J^*(z)$ and ρ^*_ϵ solve the Bellman equation.

Non-Feedback capacity

Comparing to achievable rate of non-feedback [Li, Han14].



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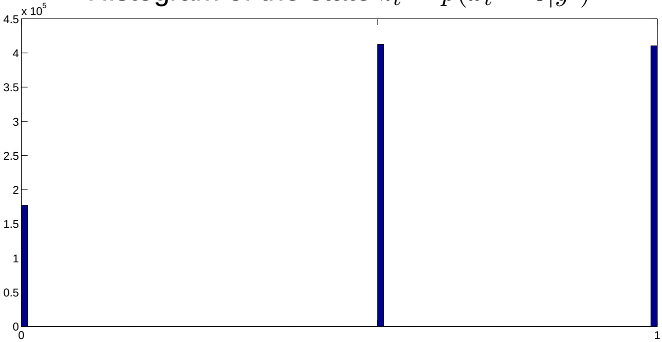
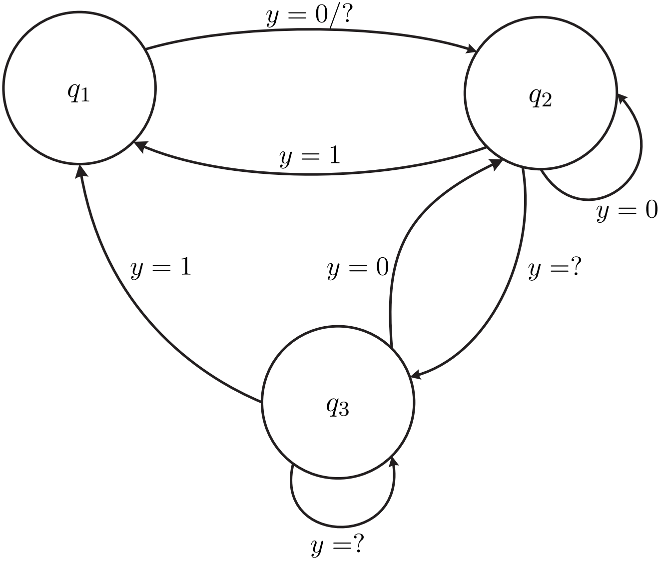
Result of Value iteration

Histogram of the state $z_t \triangleq p(x_t = 0|y^t)$



The DP optimal policy [Sabag/P./Kashyap15]



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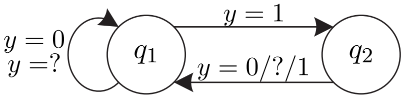

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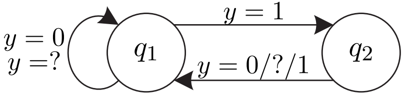

Theorem [Sabag/P./Pfiser16]

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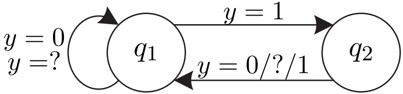

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For BEC with no repeated 1's

$$C_{fb} \leq \max_p \frac{H_2(p)}{\frac{1}{1-\epsilon} + p}$$

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Thank you !

Sufficient condition

- The channel state estimation (DP state):

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Theorem (Lower bound)

The feedback capacity satisfies

$$C_{\text{fb}} \geq I(X, S; Y|Q),$$

for all BCJR-invariant inputs.

Upper bound with sufficient condition

Theorem [Sabag/P./Pfiser16]

The feedback capacity of a unifilar FSC is bounded by

$$C_{fb} \leq \max_{p(x|s,q)} I(X,S;Y|Q), \quad \forall Q\text{-graph}$$

and if $p^*(x|s,q)$ is BCJR-invariant input, equality holds.