Introduction

Structural Characterization

Forbidden Structures

Structure of Mobius-free tournament

Main Theorem

Future Work

A Min-Max Relation on Tournaments

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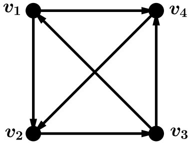
Background

Ranking Tournament Problem:

Consider a sports tournament in which n players meet pairwise in games, and assume that each game ends with a win or a loss (no ties). Use the results to find a ranking of all n players such that the number of upsets is minimized, where an *upset* occurs if a player ranked lower on the ranking beats a player ranked higher.

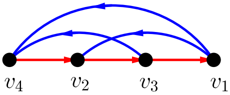
- n players, $\binom{n}{2}$ games, $n!$ possible rankings.

Example



Vertex : player
Arc : game
(v_i, v_j) : v_i **defeats** v_j

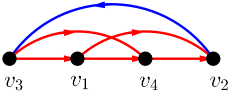
- A bad ranking



Ranking : (v_4, v_2, v_3, v_1)

Blue : 3 upsets

- An optimal ranking



Ranking : (v_3, v_1, v_4, v_2)

Blue : 1 upset

Background

- In combinatorial optimization, this problem is equivalent to finding *a feedback arc set* with minimum size on *tournaments* (the un-weighted feedback arc set problem on tournaments).

Theorem 1.1 (Alon, 2006)

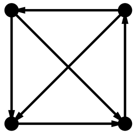
The un-weighted feedback arc set problem is NP-hard.

Question 1.1

Which tournaments can be ranked with no errors ?

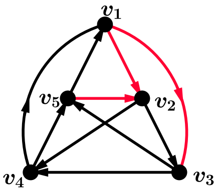
Preliminary

- A tournament is an orientation of a complete graph.



An orientation of K_4 (F_0)

- *A feedback arc set* in a digraph is a subset of arcs whose removal makes the digraph acyclic.



$S = \{(v_1, v_2), (v_1, v_3), (v_5, v_2)\}$ **is a feedback arc set.**

The (Fractional) Feedback Arc Set Problem

Let $G = (V, A)$ be a digraph with a nonnegative integral weight $w(e)$ on each arc e.

Definition 1.1

The problem of finding a feedback arc set with minimum total weight is called *the feedback arc set problem* (FASP).

- Let $\mathcal{C}_G$ be the set of all directed cycles of G.
- Let M_G be the $\mathcal{C}_G$-A incidence matrix.

Then FASP can be represented as an integer program:

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : M_G\boldsymbol{x} \geq \mathbf{1};\ \boldsymbol{x} \geq \mathbf{0} \text{ integral}\} \tag{1}$$

The (Fractional) Feedback Arc Set Problem

The fractional FASP $P(G, \boldsymbol{w})$ is given by the linear relaxation of (1):

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : M_G\boldsymbol{x} \geq \mathbf{1};\ \boldsymbol{x} \geq \mathbf{0}\}. \tag{2}$$

• Ranking tournament problem is the un-weighted (or $\boldsymbol{w} = \mathbf{1}$) FASP on tournaments.

The (Fractional) Cycle Packing Problem

• A collection $\mathcal{C}$ of directed cycles (repetition is allowed) of G is called a *cycle packing* if each arc e is used at most $w(e)$ times by members of $\mathcal{C}$.

Definition 1.2

The problem of finding a cycle packing with the maximum size is called *the cycle packing problem*(CPP).

The LP formulation of CPP is as follows:

$$\max\{\boldsymbol{y}^T\mathbf{1} : \boldsymbol{y}^T M_G \le \boldsymbol{w}^T;\ \boldsymbol{y} \ge \mathbf{0} \text{ integral}\} \tag{3}$$

The fractional CPP $D(G, \boldsymbol{w})$ is given by the linear relaxation of (3):

$$\max\{\boldsymbol{y}^T\mathbf{1} : \boldsymbol{y}^T M_G \le \boldsymbol{w}^T;\ \boldsymbol{y} \ge \mathbf{0}\}. \tag{4}$$

A Primal-Dual Pair of Linear Programs

Note that $P(G, \boldsymbol{w})$ and $D(G, \boldsymbol{w})$ forms a primal-dual pair of linear programs:

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : M_G\boldsymbol{x} \geq \mathbf{1};\ \boldsymbol{x} \geq \mathbf{0}\} = \max\{\boldsymbol{y}^T\mathbf{1} : \boldsymbol{y}^T M_G \leq \boldsymbol{w}^T;\ \boldsymbol{y} \geq \mathbf{0}\}$$

- Let $\tau^*_{\boldsymbol{w}}(G)$ and $\nu^*_{\boldsymbol{w}}(G)$ denote the optimal values of $P(G, \boldsymbol{w})$ and $D(G, \boldsymbol{w})$ respectively.
- Let $\tau_{\boldsymbol{w}}(G)$ denote the minimum total weight of a feedback arc set.
- Let $\nu_{\boldsymbol{w}}(G)$ denote the maximum size of a cycle packing.

From the LP duality theorem, the following inequalities hold.

$$\nu_{\boldsymbol{w}}(G) \leq \nu^*_{\boldsymbol{w}}(G) = \tau^*_{\boldsymbol{w}}(G) \leq \tau_{\boldsymbol{w}}(G). \tag{5}$$

• If $\nu_{\boldsymbol{w}}(G) = \tau_{\boldsymbol{w}}(G)$, then all inequalities in (5) hold with equations.

Cycle Mengerian Digraphs

Definition 1.3

$G = (V, A)$ is called *cycle Mengerian* if $\nu_{\boldsymbol{w}}(G) = \tau_{\boldsymbol{w}}(G)$ for all weight function $\boldsymbol{w} \in \mathbb{Z}_+^A$.

- For a cycle Mengerian digraph G, FASP $\rightarrow$ the fractional FASP and CPP $\rightarrow$ the fractional CPP, and hence both are solvable in polynomial time.

Known Cycle Mengerian Digraphs

Theorem 1.2 (Lucchesi and Younger Theorem, 1978)

Planar digraphs are cycle Mengerian.

- An extension of Lucchesi-Younger theorem is given by Applegate et al. in 1991 and by Barahona et al. in 1994.

Theorem 1.3 (Applegate et al., 1991 and Barahona et al., 1994)

Any digraph without $K_{3,3}$ minor is cycle Mengerian.

- If G is a digraph without $K_{3,3}$ minor, then G is obtained from a planar digraph and a tournament on 5 vertices by identifying at most two vertices.

TDI System

Definition 1.4

A rational system $A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$ is called *totally dual integral* (*TDI*) if the maximum in the LP-duality equation
$\min\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \max\{\boldsymbol{y}^T\boldsymbol{b} : \boldsymbol{y}^T A \leq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \boldsymbol{0}\}$ has an integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the minimum is finite.

Theorem 1.4 (Edmonds and Giles, 1977)

If $A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ is TDI and $\boldsymbol{b}$ is integral, then both programs in the LP-duality equation
$\min\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \max\{\boldsymbol{y}^T\boldsymbol{b} : \boldsymbol{y}^T A \leq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \boldsymbol{0}\}$
have integral optimal solutions.

Equivalence of Cycle Mengerian and TDI System

From the definition of cycle Mengerian and Edmonds and Giles' theorem, the following three statements are equivalent.

- G is cycle Mengerian.
- Both $P(G, \boldsymbol{w})$ and $D(G, \boldsymbol{w})$ have integral optimal solutions for any $\boldsymbol{w} \in \mathbb{Z}_+^A$.
- System $M_G \boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}$ is TDI.

Our Problem

- In this work, we study the feedback arc set problem (FASPT) and the cycle packing problem on tournaments (CPPT) and we will present a complete characterization of all cycle Mengerian tournaments.

Main Result

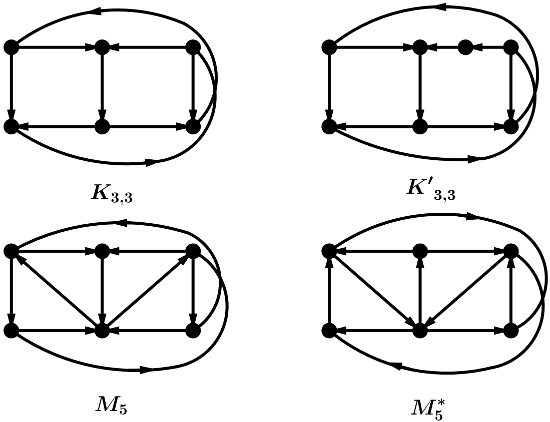
We prove that a tournament $T = (V, A)$ is cycle Mengerian iff it is Möbius-free iff system $M_T \boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}$ is TDI.

• Our result implies that both FASP and CPP on Möbius-free tournaments are solvable in polynomial time.

Forbidden Structures

Definition 2.1

A tournament is called *Möbius-free* if it contains none of $K_{3,3}$, $K'_{3,3}$, M_5 and M_5^* as a subdigraph.



Why Möbius-free tournaments ?

Theorem 2.1

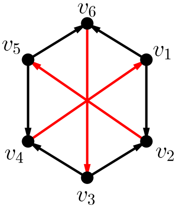
Let T be a cycle Mengerian tournament. Then T must be Möbius-free.

Proof Sketch of Theorem 2.1

- None of $K_{3,3}$, $K'_{3,3}$, M_5 and M_5^* is cycle Mengerian.
- If T is cycle Mengerian, then any subdigraph of T is also cycle Mengerian.

Example

We will demonstrate why $K_{3,3}$ is not cycle Mengerian.



$K_{3,3}$

The weight of each arc is 1.

Set $x(e) = \dfrac{1}{2}$ for $e = (v_4, v_1), (v_2, v_5), \text{or} (v_6, v_3)$, and otherwise $x(e) = 0$.

Set $y(C) = \dfrac{1}{2}$ for $C = v_1v_2v_5v_4, v_3v_2v_5v_6$, or $v_1v_6v_3v_4$, and otherwise $y(C) = 0$.

- $\boldsymbol{x}$, $\boldsymbol{y}$ are optimal solutions to $P(K_{3,3}, \mathbf{1})$ and $D(K_{3,3}, \mathbf{1})$ respectively, with common optimal value 3/2.
- So $K_{3,3}$ is not cycle Mengerian.

Internally 2-strong Tournaments

A *dicut* of T is a partition (X, Y) of V such that all arcs between X and Y are directed from X to Y.

A dicut (X, Y) is *trivial* if $|X| \leq 1$ or $|Y| \leq 1$.

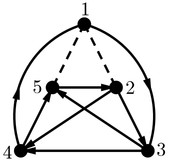
Definition 2.2

T is called *internally strong* if every dicut of T is trivial, and is called *internally 2-strong* $(i2s)$ if T is strong and $T \backslash v$ is internally strong for every $v \in V$.

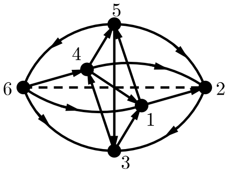
Internally 2-strong Tournaments

Lemma 2.1

Let $T = (V, A)$ be $i2s$. Then one of the following statements holds:
- $|V| \leq 4$;
- $|V| = 5$ *and* $T \in \{F_1, F_2, F_3\}$;
- $|V| = 6$ *and either* T *has a vertex* v *with* $T \backslash v \in \{F_1, F_2, F_3\}$ *or* $T \in \{F_4, F_5\}$;
- $|V| \geq 7$ *and* T *has a vertex* v *such that* $T \backslash v$ *is* $i2s$.



F_1, F_2, F_3



F_4, F_5

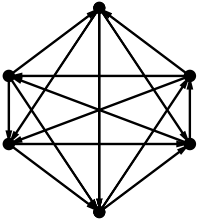
Remark: $(1, 2), (5, 1) \in F_1$; $(2, 1), (1, 5) \in F_2$; $(2, 1), (5, 1) \in F_3$. $(6, 2) \in F_4$, $(2, 6) \in F_5$.

$i2s$ and Möbius-free Tournaments

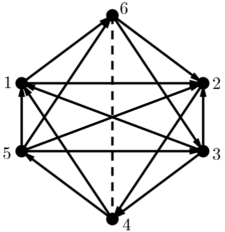
Lemma 2.2

An $i2s$ tournament $T = (V, A)$ is Möbius-free if and only if one of the following holds:

- $|V| \leq 5$;
- $T = G_1$, G_2, G_3 *or* F_4.



G_1



$G_2, G_3 : (6,4) \in G_2, (4,6) \in G_3$

Structure of Möbius-free Tournaments

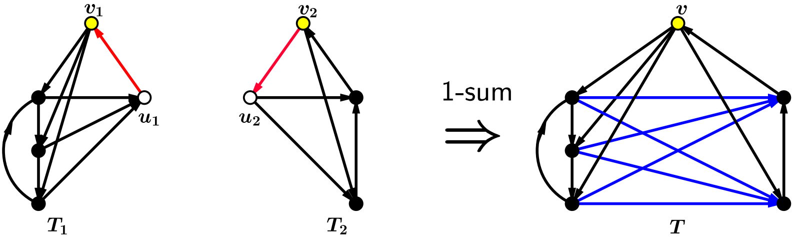
Definition 2.3

An arc $e = (u, v)$ of T is called *special* if e is the only arc in T leaving u or the only arc entering v.

Definition 2.4 (1-sum operation)

Let T_1 and T_2 be two tournaments. Suppose that T_1 and T_2 have special arcs (u_1, v_1) and (v_2, u_2), respectively, such that u_1 has out-degree one in T_1 and u_2 has in-degree one in T_2. Then the *1-sum* of T_1 and T_2 over (u_1, v_1) and (v_2, u_2) is obtained from the disjoint union of $T_1 \backslash u_1$ and $T_2 \backslash u_2$ by identifying v_1 with v_2 and then adding all arcs from $T_1 \backslash \{v_1, u_1\}$ to $T_2 \backslash \{v_2, u_2\}$.

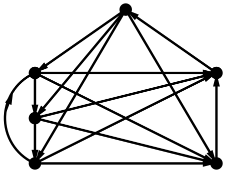
Example of 1-sum



- (u_1, v_1) and (v_2, u_2) are special arcs.
- T is the 1-sum of T_1 and T_2 over (u_1, v_1) and (v_2, u_2).
- v is obtained by identifying v_1 and v_2.

Structure of Möbius-free Tournaments

Let F_6 be the tournament as shown below, which is not $i2s$. Let $\mathcal{T} = \{F_0, F_2, F_3, F_4, G_2, G_3, F_6\}$.



$\boldsymbol{F_6}$

Lemma 2.3

Any Möbius-free tournament T with $|V| \geq 4$ can be decomposed into the 1-sum of two tournaments T_1, T_2 such that T_2 is a member in $\mathcal{T}$, unless $T = F_1$ or G_1.

Main Theorem

Theorem 3.1

A tournament T is cycle Mengerian iff it is Möbius-free.

Proof Outline of Main Theorem

- Necessity follows from Theorem 2.1.
- For the Sufficiency, it remains to show that every Möbius-free tournament $T = (V, A)$ is cycle Mengerian.

• Recall that $\mathcal{C}_T$ is the set of all cycles and M_T is the $\mathcal{C}_T$-A incidence matrix.

• To prove T is cycle Mengerian, it suffices to show that

- $M_T \boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}$ is TDI; or equivalently
- The fractional CPP $D(T, \boldsymbol{w})$: $\max\{\boldsymbol{y}^T\mathbf{1} : \boldsymbol{y}^T M_T \leq \boldsymbol{w}^T;\ \boldsymbol{y} \geq \mathbf{0}\}$ has an integral optimal solution for any weight function $\boldsymbol{w} \in \mathbb{Z}_+^A$.

Proof Outline of Main Theorem

When $|V| = 3$, T is a triangle which is cycle Mengerian. So we may assume that $|V| \geq 4$. By Lemma 2.3,

(i) $T = F_1, G_1$. Since F_1 is a sub-digraph of G_1, it suffices to show G_1 is cycle Mengerian.

- We first prove that $P(G_1, \boldsymbol{w})$ has an integral optimal solution $\boldsymbol{x}$ and then use the integrality of $\boldsymbol{x}$ to show that $D(G_1, \boldsymbol{w})$ has an integral optimal solution.

(ii) $T \neq F_1, G_1$ and T can be decomposed into the 1-sum of T_1 and T_2 such that $T_2 \in \mathcal{T}$.

Proof Outline of Main Theorem

Recall that $\mathcal{C}_{T_2\backslash u_2}$ is the set of all cycles in $T_2\backslash u_2$. Let l be the length of a longest cycle in $T_2\backslash u_2$. To prove (ii),

- Let $\boldsymbol{y}$ be an optimal solution of $D(T, \boldsymbol{w})$ satisfying
  - (a) $y(\mathcal{C}_{T_2\backslash u_2}) = \sum_{C \in \mathcal{C}_{T_2\backslash u_2}} y(C)$ is maximized;
  - (b) subject to (a), $\Big(\sum_{\substack{|C|=l \\ C \in \mathcal{C}_{T_2\backslash u_2}}} y(C), \sum_{\substack{|C|=l-1 \\ C \in \mathcal{C}_{T_2\backslash u_2}}} y(C), \ldots, \sum_{\substack{|C|=3 \\ C \in \mathcal{C}_{T_2\backslash u_2}}} y(C)\Big)$ is lexicographically minimum.
- For each member $T_2 \in \mathcal{T}$, we first show that either the optimal value $\nu_{\boldsymbol{w}}^*(T) = \boldsymbol{y}^T\mathbf{1}$ is an integer or the restriction of $\boldsymbol{y}$ on $\mathcal{C}_{T_2\backslash u_2}$ is optimal and integral. Based on this, we can further show that in either case there exists an integral optimal solution of $D(T, \boldsymbol{w})$.

Future Work

We show that FASP and CPP on Möbius-free tournaments are solvable in polynomial time. However, this result relies on the fact that there exists a polynomial-time algorithm for linear programs.

Problem 1

Give a strongly polynomial-time combinatorial algorithm for finding a feedback arc set with minimum total weight and a maximum cycle packing on Möbius-free tournaments.

Thank you!