Zero-sum games, non-archimedean convexity and sinuous central paths

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Based on work with Akian and Guterman (tropical geometry and games)
Allamigeon, Benchimol and Joswig (tropical linear programming).

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- application: geometry of the central path in LP

via tropical geometry

Part I.
Some open problems concerning zero-sum games and linear / SDP programming

The mean payoff problem

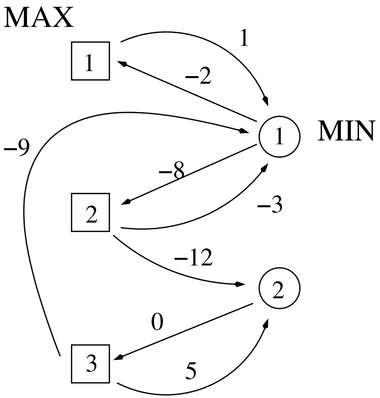
Mean payoff games

$G = (V, E)$ bipartite graph. $r_{ij} \in \mathbb{Z}$ price of the arc $(i, j) \in E$.

MAX and MIN move a token, alternatively (square states: MAX plays; circle states: MIN plays). n MIN nodes, m MAX nodes.

MIN always pays to MAX the price of the arc (having a negative fortune is allowed)

v_i^k value of MAX, initial state (i, MIN).



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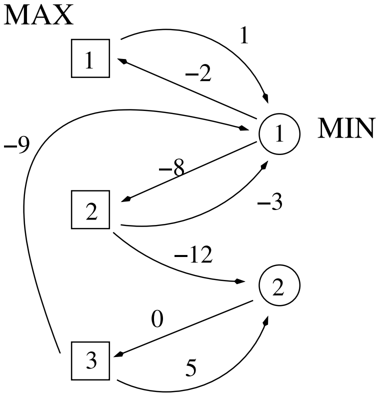

$$\lim_{k\to\infty} v^k/k = (-1, 5)$$

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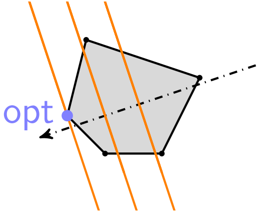
Zwick and Paterson [1996] showed that value iteration solves MPG in pseudo polynomial time $O((n+m)^5 W)$ where $W = \max_{ij} |r_{ij}| = O(2^L)$.

Complexity issues in linear programming

A linear program is an optimization problem:

$$\min c \cdot x; \; Ax \leqslant b, \; x \in \mathbb{R}^n \;\;,$$

where $c \in \mathbb{Q}^n$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.



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Can linear programming be solved in strongly polynomial time?

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$\neq$ strongly polynomial (arithmetic model): number of arithmetic operations bounded by poly(m, n), **and** the size of operands of arithmetic operations is bounded by poly(L).

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Consider the simplex and interior point methods in the light of Smale problem 9?

The simplex method (Dantzig, 1947)

Iterate over adjacent vertices (basic points) of the polyhedron while improving the objective function

$$c^\top v^1 \geqslant c^\top v^2 \geqslant \ldots \geqslant c^\top v^N$$

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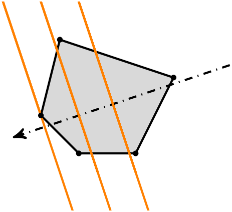

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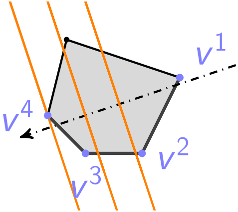

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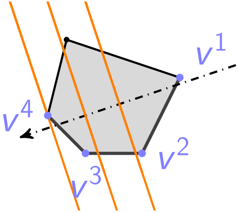

the algorithm is parametrized by a *pivoting rule*, which selects the next edge to be followed.

Complexity of pivoting algorithms?

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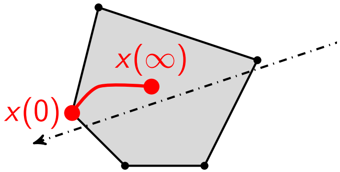
- is there a pivoting rule ensuring that the number of iterations in the worst case is polynomially bounded?
- It is not even known that the graph of the polyhedron has polynomial diameter (polynomial Hirsch conjecture), ie that the perfectly lucid pivoting rule makes a polynomial number of steps.

Interior points

For all $\mu > 0$, consider the *barrier problem*

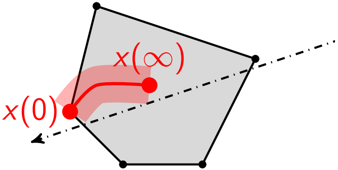
$$\min \mu^{-1} c \cdot x - \sum_{i=1}^{m} \log(b_i - A_i x), \quad b_i - A_i x > 0 \ i \in [m]$$

$\mu \mapsto x(\mu)$ optimal solution, is the central path. branch of an algebraic curve. $x(0)$ is the solution of the LP.



Inductive step of interior points:

- $x \leftarrow \mathrm{NewtonStep}(x, \mu)$;
- reduce μ so that x remains in an attraction bassin of Newton's method.



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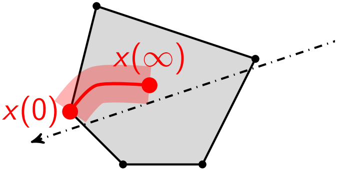


"the good convergence properties of Karmarkar's algorithm arise from good geometric properties of the set of trajectories", Bayer and Lagarias, 89.

Dedieu, Malajovich, and Shub considered the total curvature, an idealized complexity measure of the central path...

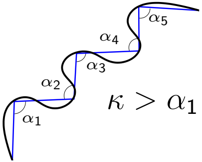
Total curvature

The total curvature of a path γ, parametrized by arc length, so that $\|\gamma'(s)\| = 1$, is given by

$$\kappa := \int_0^L \|\gamma''(s)\| ds$$

or

$$\kappa = \sup_{q \geqslant 2} \sup_{0 \leqslant \lambda_0 < \cdots < \lambda_q \leqslant L} \angle \gamma(\lambda_{k-1})\gamma(\lambda_k)\gamma(\lambda_{k+1})$$



$$\kappa > \alpha_1 + \cdots + \alpha_5$$

Continuous analogue of Hirsch's conjecture

Dedieu and Shub (2005) initially conjectured that the total curvature of the central path is $O(n)$ (n number of variables).

This was motivated by a theorem of Dedieu-Malajovich-Shub (2005): total curvature is $O(n)$, averaged over all 2^{n+m} LP's (cells of the arrangement of hyperplanes), $\epsilon_i A_i x \leqslant b_i$, $\eta_j x_j \geqslant 0$, $\epsilon_i, \eta_j = \pm 1$.

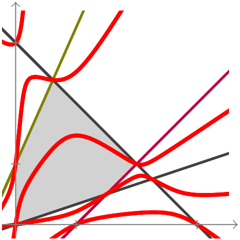


Illustration from Benchimol's Phd

Deza, Terlaky and Zinchenko (2008) constructed a redundant Klee-Minty cube, showing that a total curvature exponential in n is possible, and revised the conjecture of Dedieu and Shub:

Conjecture (Continuous analogue of Hirsch conjecture, [Deza, Terlaky, and Zinchenko, 2008])

The total curvature of the central path is $O(m)$, where m is the number of constraints.

Theorem (Allamigeon, Benchimol, SG, Joswig, arXiv:1405.4161)

There is a LP with $2r+2$ variables and $3r+4$ inequalities such that the central path has a total curvature in $\Omega(2^r)$.

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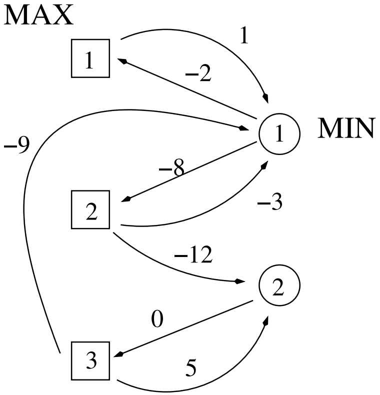
The pivoting rule must satisfy mild technical conditions, in particular, combinatorial rules, depending on signs of minors of $\begin{pmatrix} A & b \\ c & 0 \end{pmatrix}$ work.

Although the word “tropical” appears in none of these statements, the proofs rely on tropical geometry in an essential way, through linear and semidefinite programming over non-archimedean fields.

Part II.
Operator approach to mean payoff games

v_i^k value of MAX, initial state (i, MIN).

$$v_1^k = \min(-2 + 1 + v_1^{k-1}, -8 + \max(-3 + v_1^{k-1}, -12 + v_2^{k-1}))$$
$$v_2^k = 0 + \max(-9 + v_1^{k-1}, 5 + v_2^{k-1})$$



$$v^1 = (0, 0)$$
$$v^2 = (-11, 5)$$
$$v^3 = (-15, 10)$$
$$v^4 = (-16, 15)$$
$$\lim_k v^k/k = (-1, 5)$$

Theorem (Shapley)

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$$[T(v)]_j = \min_{i \in [m],\, j \to i} \left(r_{ji} + \max_{k \in [n],\, i \to k} (r_{ik} + v_k) \right)$$

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$$[T^k(u)]_i$$

is the value of a modified game in horizon k with initial state i, in which MAX receives an additional payment of u_j in the terminal state j

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Known axioms in non-linear Markov semigroup / PDE viscosity solutions theory, eg Crandall and Tartar, PAMS 80

Theorem (Bewley, Kohlerg 76, Neyman 03)

The mean payoff vector

$$\lim_{k\to\infty} T^k(0)/k$$

does exist if $T : \mathbb{R}^n \to \mathbb{R}^n$ *is semi-algebraic and nonexpansive in any norm.*

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Finite action space and perfect information implies T piecewise linear.

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$T : \mathbb{R}^n \to \mathbb{R}^n$ extends continuously $(\mathbb{R} \cup \{-\infty\})^n \to (\mathbb{R} \cup \{-\infty\})^n$. (Burbanks, Nussbaum, Sparrow).

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- all states are winning.

The converse follows from a fixed point theorem of Kohlberg (a nonexpansive piecewise linear map has an invariant half-line).

Part III.
Tropical geometry

In the tropical world

$$\text{“}a+b\text{”} = \max(a,b) \qquad \text{“}a \times b\text{”} = a+b$$

The semifield of scalars is the *max-plus semifield*, $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$.

- “$2+3$” $=$

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- “2×3” $=5$
- “0” $= -\infty$, “1” $= 0$.

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For any totally ordered abelian group $(G, +, \leqslant)$, one can define $G_{\max}$.
$G = (\mathbb{R}^N, +, \leqslant_{\text{lex}})$ specially useful.

These structures are said to be idempotent ($a + a = a$) or of characteristic one (Connes, Consani), These algebras were invented by various schools in the world

- Cuninghame-Green $\sim$ 60 Vorobyev $\sim$65 ... Zimmerman, Butkovic; scheduling, combinatorital optimization
- Maslov $\sim$ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon $\sim$ 78- ... Hashiguchi, Pin, Krob (Schützenberger's school) ... Automata theory
- Gondran, Minoux $\sim$ 77 Operations research
- Cohen, Quadrat, Viot $\sim$ 83-, "Max Plus ", Olsder, Baccelli, S.G., Akian discrete event systems, optimal control, idempotent probabilities.
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney $\sim$00- max-plus approximation of HJB
- Del Moral $\sim$95 Puhalskii $\sim$99, idempotent probabilities.
- Since 2000' in pure maths, tropical geometry: Viro, Kapranov, Mikhalkin, Passare, Sturmfels ..., recent work by Connes, Consani

Tropical convex cones

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Since "$\lambda \geqslant 0$" is automatic tropically, modules = cones.

Tropical convex cones

Scalars act on vectors by "λx" $= \lambda e + x$.

$V \subset \mathbb{R}^n_{\max}$ is a submodule, aka tropical convex cone, if for all $x, y \in V, \lambda, \mu \in \mathbb{R}_{\max}$,

$$\text{“}\lambda x + \mu y\text{”} = \sup(\lambda e + x, \mu e + y) \in V \quad .$$

Since "$\lambda \geqslant 0$" is automatic tropically, modules = cones.

V is a tropical convex set if the same is true conditionnally to "$\lambda + \mu = 1$", i.e., $\max(\lambda, \mu) = 0$.

By the subharmonic certificates theorem, the game is winning for MAX, i.e., $\exists j, \lim_{k\to\infty}[T^k(0)]_j/k \geqslant 0$, iff

$$V = \{v \in \mathbb{R}^n_{\max} \mid T(v) \geqslant v\} \not\equiv \text{“}0\text{”} = (-\infty, \dots, -\infty) \enspace .$$

Proposition

If T is a Shapley operator, then V is a tropical convex cone of $\mathbb{R}^n_{\max}$, closed in the Euclidean topology.

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Proof.

T M, $u, v \in V$ implies $T(sup(u, v)) \geqslant \sup(T(u), T(v)) \geqslant \sup(u, v)$.
T AH, $v \in V$ implies $T(\lambda e + v) = \lambda e + T(v) \geqslant \lambda e + v$. □

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All closed tropical convex cones of $\mathbb{R}^n_{\max}$ arise from a Shapley operator T (infinite number of actions on one side allowed).

Tropical adjoints

Let $A \in \mathbb{R}_{\max}^{m\times n}, x \in \mathbb{R}_{\max}^{n},\ y \in \mathbb{R}_{\max}^{n}$

$$(Ax)_i = \max_{j\in[n]}(A_{ij} + x_j), \qquad i \in [m]$$

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The adjoint $A^\sharp$ is a priori defined as a self-map of the completion $(\mathbb{R} \cup \{\pm\infty\})^n$ of $\mathbb{R}_{\max}^n$, but it does preserve $\mathbb{R}^n$ as soon as the game has no states without actions.

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More on adjoints: Cohen, SG, Quadrat, LAA 04

The Shapley operator of a MPG can be written as

$$[T(v)]_j = \min_{i \in [m],\ j \to i} \left(-A_{ij} + \max_{k \in [n],\ i \to k} (B_{ik} + v_k) \right)$$

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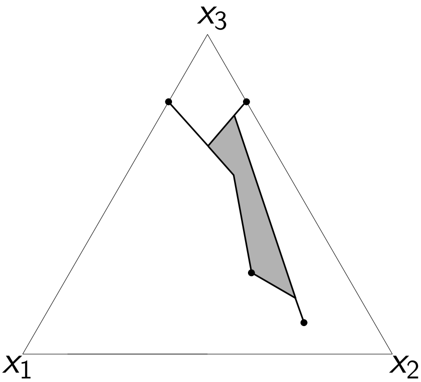
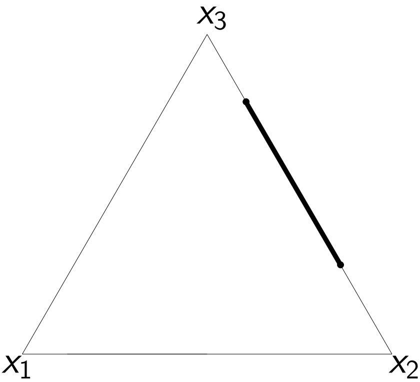
The set of subharmonic certificates $\{v \mid Av \leqslant Bv\}$ is a tropical convex polyhedral cone.

Modules of subharmonic vectors



states 1,2,3 winning



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Part VI.
Tropical convexity

Tropical half-spaces

Given $a, b \in \mathbb{R}^n_{\max}$, $a, b \not\equiv -\infty$,

$$H := \{x \in \mathbb{R}^n_{\max} \mid \text{“}ax \leqslant bx\text{”}\}$$

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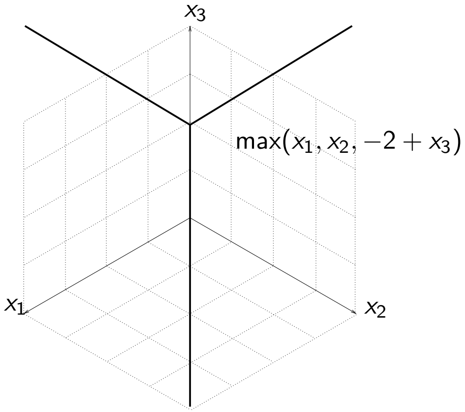

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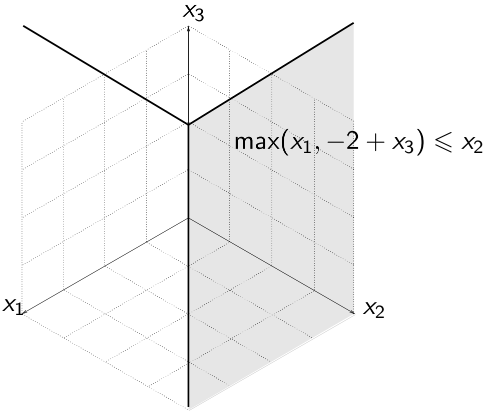

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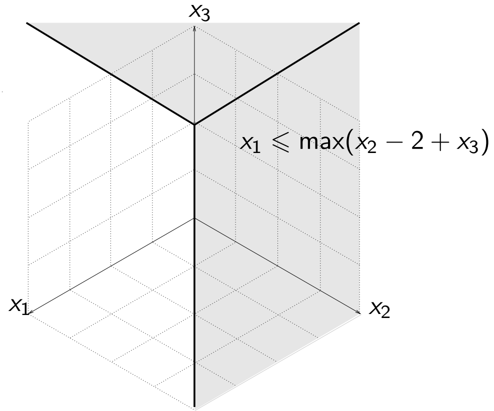

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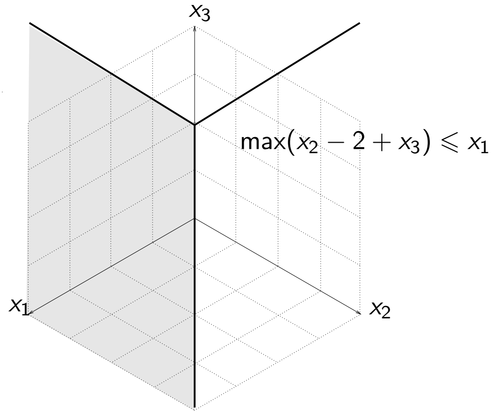
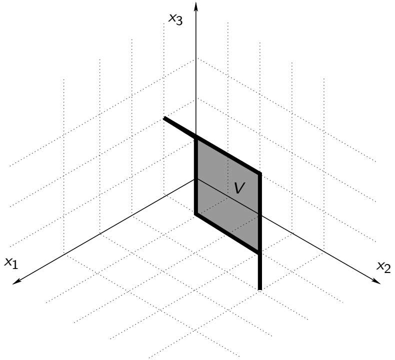


Tropical polyhedral cones

can be defined as intersections of finitely many half-spaces



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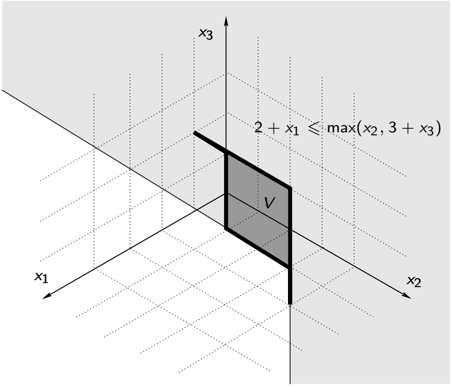

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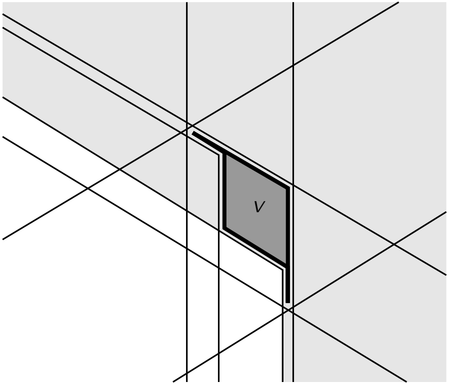
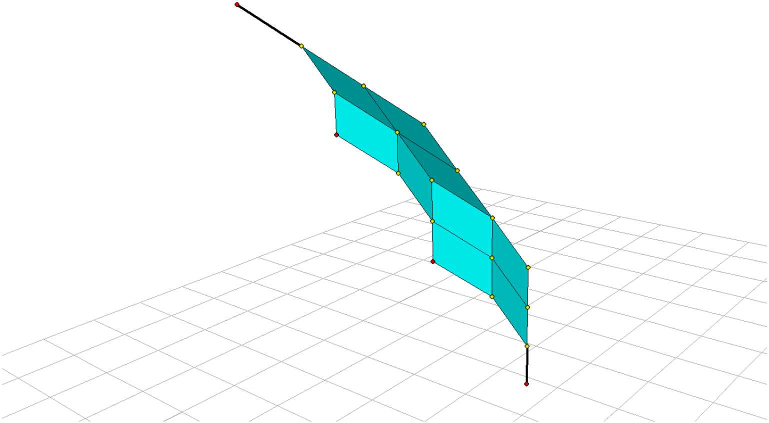


A tropical polytope with four vertices



Structure of the polyhedral complex: Develin, Sturmfels

Tropical objects arise by considering non-archimedean valuations.

There is a convenient choice of non-archimedean field in tropical geometry . . .

Puiseux series with real exponents $\mathbb{C}\{\{t^{-\mathbb{R}}\}\}$,

$$f(t) = \sum_{k \in \mathbb{N}} a_k t^{b_k}$$

were $a_k \in \mathbb{C}$, and the sequence $b_k \in \mathbb{R}$ tends to $-\infty$.

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$\mathbb{K} := \mathbb{R}\{\{t^{-\mathbb{R}}\}\}_{\mathrm{cvg}}$, absolutely convergent real generalized Puiseux series constitute a real closed field, van den Dries and Speissegger (TAMS) (definable functions in a variant of the o-minimal structure $\mathbb{R}_{an,*}$).

Non-archimedean valuation of

$$f(t) = \sum_{k\in\mathbb{N}} a_k t^{b_k}$$

$$\operatorname{val} f = \max_{k, a_k\neq 0} b_k = \lim_{t\to\infty} \frac{|\log f(t)|}{\log t}$$

$$\operatorname{val}(f+g) \leqslant \max(\operatorname{val} f, \operatorname{val} g), \qquad \text{and } = \text{ holds if } f,g \geqslant 0$$
$$\operatorname{val}(fg) = \operatorname{val} f + \operatorname{val} g \ .$$

Theorem (Combines Develin and Yu and Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

1. *Every tropical polyhedron P can be written as $P = \operatorname{val} \mathcal{P}$ where $\mathcal{P}$ is a polyhedron in $\mathbb{K}_+^n$, here $\mathbb{K} = \mathbb{R}\{\{t^{-\mathbb{R}}\}\}_{cvg}$.*
2. *Moreover, P is the uniform (Hausdorff) limit of*

$$\log_t \mathcal{P} := \{\frac{\log z}{\log t} \mid z \in \mathcal{P}\}$$

as $t \to \infty$.

Related deformation in Briec and Horvath.

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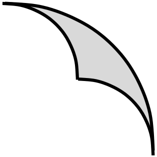

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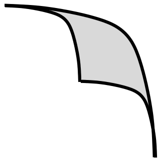

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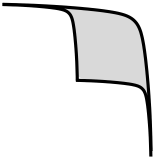

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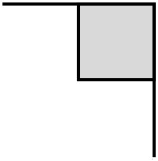


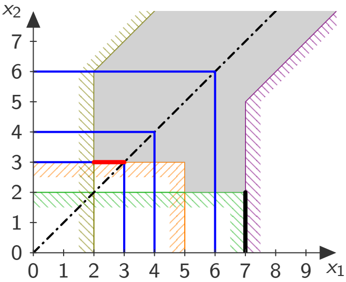
Related deformation in Briec and Horvath.

Tropical linear program

$$\min \text{“}c^\top x\text{”}; \qquad \text{“}A^+ x + b^+ \geqslant A^- x + b^-\text{”}$$

$$\min \max_j c_j + x_j$$

$$\max(\max_j(A^+_{ij} + x_j), b^+_i) \geqslant \max(\max_j(A^-_{ij} + x_j), b^-_i) \enspace .$$



Correspondence classical $\leftrightarrow$ tropical LP

Theorem (Allamigeon, Benchimol, SG, Joswig, arXiv:1308.0454, SIAM J. Disc. Math)

Suppose that $\mathcal{P} = \{x \in \mathbb{K}^n \mid \mathbf{A}x + \mathbf{b} \geqslant 0\}$ *is included in the positive orthant of* $\mathbb{K}^n$ *and that the tropicalization of* $(\mathbf{A}, \mathbf{b})$ *is sign generic (to be defined soon). Then,*

$$\mathrm{val}(\mathcal{P}) = \{x \in \mathbb{R}^n_{\max} \mid \text{“}A^+x + b^+ \geqslant A^-x + b^-\text{”}\} ,$$

where $(A^+ \; b^+) = \mathrm{val}(\mathbf{A}^+\mathbf{b}^+)$ *and* $(A^- \; b^-) = \mathrm{val}(\mathbf{A}^- \; \mathbf{b}^-)$. *Moreover the classical and tropical polyhedron have the same combinatorics: valuation sends basic points to basic points, edges to edges, etc.*

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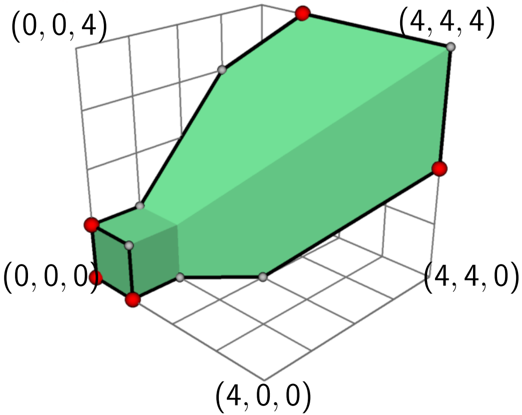
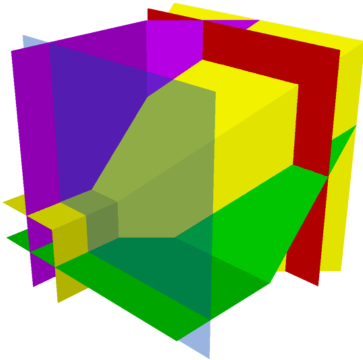
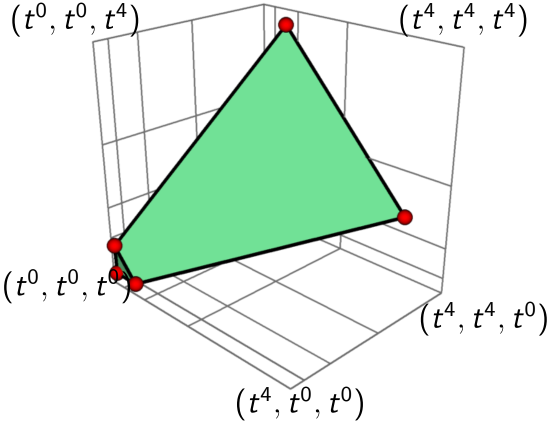
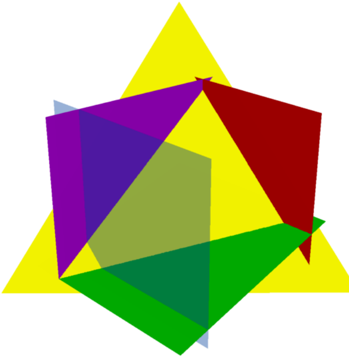
A point of a tropical polyhedron is basic if it saturates n inequalities. A tropically extreme point (member of a minimal generating family) is basic, but not vice versa.





Picture from [Allamigeon, Benchimol, Gaubert, and Joswig, 2015a].





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Tropical sign genericity and optimal assignments

Let $M \in \mathbb{R}_{\max}^{n\times n}$ and $\epsilon \in \{\pm 1, 0\}^{n\times n}$. $\mathbf{M} \in \mathbb{K}^{n\times n}$ is a lift of (ϵ, M) if $\operatorname{val} \mathbf{M} = M$ and $\operatorname{sgn} \mathbf{M} = \epsilon$. We have

$$\operatorname{val} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{1\leqslant i\leqslant n} \mathbf{M}_{i\sigma(i)} \leqslant \max_{\sigma\in S_n} \sum_{1\leqslant i\leqslant n} M_{i\sigma(i)}$$

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with $=$ if (ϵ, M) is tropically sign generic, meaning that all optimal permutations yield the same $\operatorname{sgn}(\sigma)\prod_{1\leqslant i\leqslant n} \epsilon_{i\sigma(i)}$.

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with $=$ if (ϵ, M) is tropically sign generic, meaning that all optimal permutations yield the same $\operatorname{sgn}(\sigma)\prod_{1\leqslant i\leqslant n} \epsilon_{i\sigma(i)}$.

It is generic if there is only one optimal permutation.

Tropical sign genericity and optimal assignments

Let $M \in \mathbb{R}_{\max}^{n\times n}$ and $\epsilon \in \{\pm 1, 0\}^{n\times n}$. $\mathbf{M} \in \mathbb{K}^{n\times n}$ is a lift of (ϵ, M) if $\operatorname{val}\mathbf{M} = M$ and $\operatorname{sgn}\mathbf{M} = \epsilon$. We have

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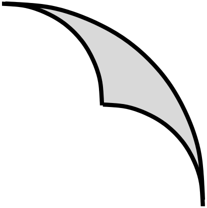
It is generic if there is only one optimal permutation.

Sign-genericity is related to even cycle problem and Polya's permanent problem. Checkable in polynomial time, as well as genericity (simpler).

LP tropically sign generic means that every minor of "$(A^+ - A^-, b^+ - b^-)$" is sign generic.

sign generic condition not satisfied, valuation does not commute with the external representation.



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$$X_i = \log x_i / \log t, \qquad t \to 0.$$

$$\max(X_1, X_2) \leqslant 0, \ \max(1 + X_1, X_2) \geqslant 0, \ \max(X_1, 1 + X_2) \geqslant 0 \ .$$

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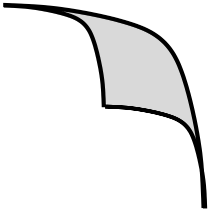

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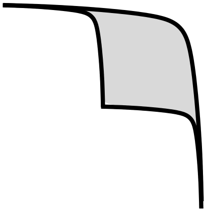

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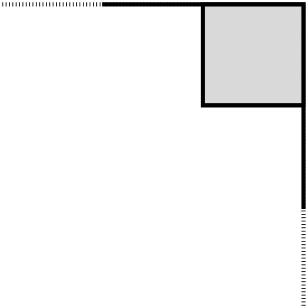

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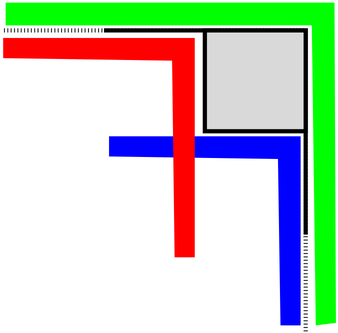

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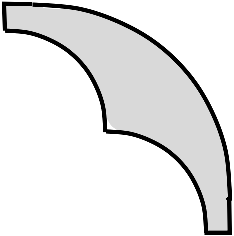

$$x_1 + x_2 \leqslant t^{\epsilon},\ t^1 x_1 + x_2 \geqslant 1,\ x_1 + t^1 x_2 \geqslant 1,\ t^2 x_1 \geqslant 1,\ t^2 x_2 \geqslant 1$$

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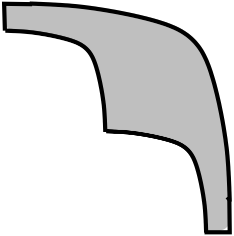

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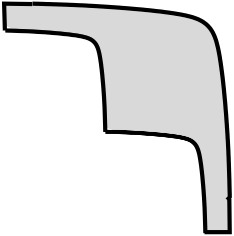

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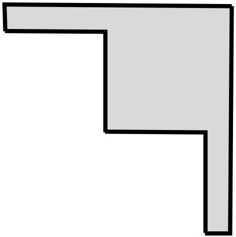

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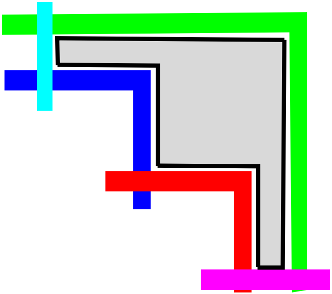

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Assume that the data are tropically in general position.

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1308.0454, SIAM J. Disc. Math)

The valuation of the path of the simplex algorithm over $\mathbb{K}$ can be computed tropically (with a compatible pivoting rule). One iteration takes $O(n(m+n))$ time.

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Pivoting is more subttle tropically. Tropical general position assumption (only one optimal permutation) stronger than sign general position (all optimal permutations yield monomials with the same sign). The $O(n(n+m))$ bound arises by tracking the deformations of a hypergraph along a tropical edge. We can still pivot tropically if the data are only in sign general position, but then we (currently) loose a factor n in time.

Example of compatible pivoting rule. A rule is **combinatorial** if any entering/leaving inequalities are functions of the history (sequence of bases) and of the signs of the minors of the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^\top & 0 \end{pmatrix} .$$

(eg signs of reduced costs).
Most known pivoting rules are combinatorial.

Mean payoff games reduce to tropical LP

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1309.5925, SIAM J. Opt, + refinement in Benchimol's PhD)

If any combinatorial (or even "semialgebraic") rule in classical linear programming would run in strongly polynomial time, then, mean payoff games could be solved in strongly polynomial time.

The "semialgebraic" rule must satisfy a mild technical assumption (polynomial time solvability of LP's over Newton polytopes).

Sketch of Proof

1. Mean payoff games are equivalent to feasibility problems in tropical linear programming (Akian, SG, Guterman)
2. Tropical linear programs can be lifted to a subclass of classical linear programs over $\mathbb{K}$.
3. The set of runs (sequences of bases) of the classical simplex algorithm equipped with a combinatorial (or even semialgebraic) pivoting rule is independent of the real closed field. Being a run is a first order property, apply Tarski's theorem. So, number of iterations of classical simplex over $\mathbb{K}$ is the same as over $\mathbb{R}$.
4. Can simulate the classical simplex on $\mathbb{K}$ tropically, every pivot being strongly polynomial.

Technicalities were hidden here.
Instead of $\mathbb{K}$, we eventually use a field of formal Hahn series

$$\mathbb{R}[[t^{\mathbb{R}^N}]], \quad \text{the value group } \mathbb{R}^N \text{ is equipped with lex order}$$

to encode a symbolic perturbation scheme (needs to encode a MPG by a LP in general position).

Part V
Tropicalization of the central path

Primal-dual central path

$$\begin{aligned} &\text{minimize} \quad \frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \\ &\text{subject to} \quad Ax + w = b,\ x > 0, w > 0. \end{aligned} \tag{1}$$

$$\begin{aligned} Ax + w &= b \\ -A^\top y + s &= c \\ w_i y_i &= \mu \quad \text{for all } i \in [m] \\ x_j s_j &= \mu \quad \text{for all } j \in [n] \\ x, w, y, s &> 0\ . \end{aligned} \tag{2}$$

For any $\mu > 0$, $\exists!$ $(x^\mu, w^\mu, y^\mu, s^\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$. The central path is the image of the map $\mathcal{C} : \mathbb{R}_{>0} \to \mathbb{R}^{2m+2n}$ which sends $\mu > 0$ to the vector $(x^\mu, w^\mu, y^\mu, s^\mu)$.

The tropical central path

Assume now that $\mathbf{A}(t), \mathbf{b}(t), \mathbf{c}(t)$ have entries in $\mathbb{K}$ (absolutely converging Puiseux series with real exponents, $t \to \infty$).

The tropical central path is the log-limit:

$$\mathcal{C}^{\mathrm{trop}} : \lambda \mapsto \lim_{t \to \infty} \frac{\log \mathcal{C}(t^{\lambda})}{\log t} \ . \tag{3}$$

The pointwise limit does exist since $\mathcal{C}(\cdot)$ is definable in a polynomially bounded o-minimal structure.

Theorem

The family of maps $(\log_t \mathcal{C}(t,\cdot))_t$ *converges uniformly on any closed interval* $[a,b] \subset \mathbb{R}$ *to the tropical central path* $\mathcal{C}^{\mathrm{trop}}$*.*

Proof of uniformity uses

$$\max(a,b) \leqslant \log_t(t^a + t^b) \leqslant \log_t 2 + \max(a,b) \enspace .$$

Computing the tropical central path

$$\mathcal{P} := \{(\mathbf{x}, \mathbf{w}) \in \mathbb{K}^{n+m} \mid \mathbf{A}\mathbf{x} + \mathbf{w} = \mathbf{b},\ \mathbf{x} \geqslant 0,\ \mathbf{w} \geqslant 0\}$$

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

Assume that $\mathbf{b}, \mathbf{c} \geqslant 0$. *Then, for* $\boldsymbol{\mu} = t^{\lambda}$,

$$\mathrm{val}(\mathbf{x}^{\boldsymbol{\mu}}, \mathbf{w}^{\boldsymbol{\mu}}) = \max\left(\mathrm{val}\,\mathcal{P} \cap \{(x, w) \in \mathbb{R}_{\max}^{m+n} \mid c^{\top} x \leqslant \lambda\}\right) .$$

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In particular, the valuation of the analytic center (take $\lambda = \infty$*) is the maximal element of* $\operatorname{val}\boldsymbol{\mathcal{P}}$*.*

$$
\begin{aligned}
\mathbf{x}_1 + \mathbf{x}_2 &\leqslant 2 \\
t^1\mathbf{x}_1 &\leqslant 1 + t^2\mathbf{x}_2 \\
t^1\mathbf{x}_2 &\leqslant 1 + t^3\mathbf{x}_1 \\
\mathbf{x}_1 &\leqslant t^2\mathbf{x}_2 \\
\mathbf{x}_1, \mathbf{x}_2 &\geqslant 0 \ .
\end{aligned}
\tag{4}
$$

Its value $\mathrm{val}(\mathcal{P}_t)$ is the tropical set described by the inequalities:

$$
\begin{aligned}
\max(x_1, x_2) &\leqslant 0 \\
1 + x_1 &\leqslant \max(0, 2 + x_2) \\
1 + x_2 &\leqslant \max(0, 3 + x_1) \\
x_1 &\leqslant 2 + x_2 \ .
\end{aligned}
\tag{5}
$$

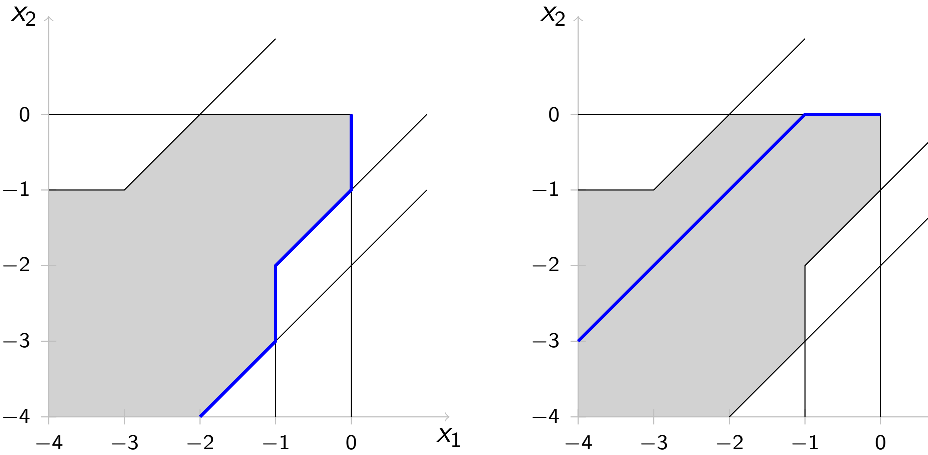


Figure : Tropical central paths on the Puiseux polyhedron (4) for the objective function $\min \mathbf{x}_2$ (left) and $\min t^1\mathbf{x}_1 + \mathbf{x}_2$ (right).

The counter example . . .

$$
\begin{array}{lll}
\min & \mathbf{v}_0 & \\
\text{s.t.} & \mathbf{u}_0 \leqslant t^1 & \\
& \mathbf{v}_0 \leqslant t^2 & \\
& \mathbf{v}_i \leqslant t^{(1-\frac{1}{2^i})}(\mathbf{u}_{i-1} + \mathbf{v}_{i-1}) & \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_i \leqslant t^1 \mathbf{u}_{i-1} & \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_i \leqslant t^1 \mathbf{v}_{i-1} & \text{for } 1 \leqslant i \leqslant r \\
& \mathbf{u}_r \geqslant 0, \ \mathbf{v}_r \geqslant 0 &
\end{array}
\qquad \mathbf{LP}_r
$$

Theorem (Allamigeon, Benchimol, SG, Joswig arXiv:1405.4161)

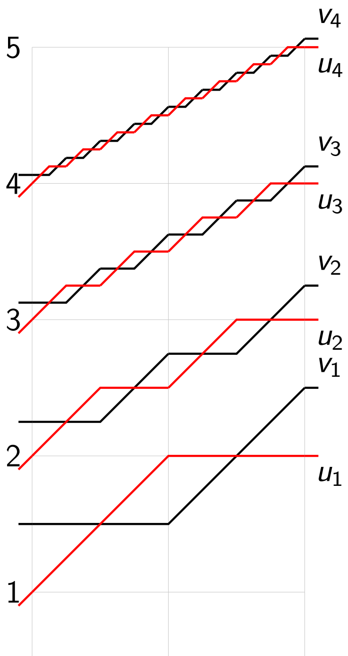
For t large enough, the total curvature of the central path is $\geqslant (2^{r-1} - 1)\pi/2$.

Large enough: $\log_2 t = \Omega(2^r)$. Need an exponential number of bits.

$$
\begin{aligned}
&\mathbf{u}_0 \leqslant t^1 && u_0 \leqslant 1 \\
&\mathbf{v}_0 \leqslant t^2 && v_0 \leqslant 2 \\
&\mathbf{v}_i \leqslant t^{(1-\frac{1}{2^i})}(u_{i-1} + v_{i-1}) && v_i \leqslant 1 - \frac{1}{2^i} + \max(u_{i-1}, v_{i-1}) \\
&\mathbf{u}_i \leqslant t^1 \mathbf{u}_{i-1} && u_i \leqslant 1 + u_{i-1} \\
&\mathbf{u}_i \leqslant t^1 \mathbf{v}_{i-1} && u_i \leqslant 1 + v_{i-1} \\
&\mathbf{u}_r \geqslant 0,\ \mathbf{v}_r \geqslant 0 && c^\top x = v_0 \leqslant \lambda
\end{aligned}
$$

The tropical central path is given by

$$
\begin{aligned}
u_0 &= 1 \\
v_0 &= \min(2, \lambda) \\
v_i &= 1 - \frac{1}{2^i} + \max(u_{i-1}, v_{i-1}) \\
u_i &= 1 + \min(u_{i-1}, v_{i-1})
\end{aligned}
$$



v_4
5
u_4
v_3
4
u_3
v_2
3
u_2
v_1
2
u_1
1

How the counter example was found

Bezem, Nieuwenhuis and Rodríguez-Carbonell (2008) constructed a class of tropical linear programs for which an algorithm of Butkovič and Zimmermann (2006) exhibits an exponential running time.

Their algorithm is, loosely speaking, in the family of tropical simplex algorithms.

There is a class of tropical LP for which the tropical central path degenerates to a simplex path (moves only on the edges).

This is the case on this example. The tropical central path passes through an exponential number of basic points.

The central path of $\mathcal{P}_t$ converges to the tropical central path as $t \to 0$ (dequantization).

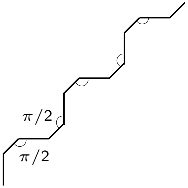
How we bound the classical curvature

Can define tropical angle $\angle^t \in \{0, \pi/2, \pi\}$,

$\angle^t PQR := \inf \angle \mathbf{PQR},\ \mathbf{P}, \mathbf{Q}, \mathbf{R} \in \mathbb{K}^n,\ \operatorname{val} \mathbf{P} = P, \operatorname{val} \mathbf{Q} = Q, \operatorname{val} \mathbf{R} = R$

Can define tropical total curvature

$$\kappa^t := \sup \sum_{i=0}^{k-1} \angle^t P_{i-1} P_i P_{i+1}, \qquad P_0, \dots, P_k \text{ points on the path } .$$



Total curvature of the classical path is $\geqslant$ than the tropical total curvature of its valuation, which is $\Omega(2^r)$ here. □

Concluding remarks

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- Can we make "an archimedean" version of the game algorithms which apply to generic nonarchimedean LP/SDP ?

The tentative conclusion of the story is that “detropicalization” yields unusual instances, combinatorially tractable: tropicalization is a formidable machine to find counter-examples.

Thank you !

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