A Generalization of the I-MMSE Formula

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The mutual information between two random elements X and Y, a measure of their mutual dependence, can be defined as

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \\ &= \mathbb{E}\left[\log \frac{d\mu_{XY}}{d\mu_X \times \mu_Y}(X,Y)\right] \\ &= \int\int \log \frac{f_{XY}(x,y)}{f_X(x)f_Y(y)} f_{XY}(x,y)dxdy. \end{aligned}$$

- Consider the discrete-time Gaussian channel

$$Y^\rho = \rho X + Z,$$

where Y^ρ: output, X: input, Z: Gaussian r.v., and ρ^2: signal-to-noise ratio.

- In the paper Mutual information and minimum mean-square error in Gaussian channels by *Guo, Shamai, and Verdu,* the I-MMSE formula was obtained:

$$\frac{d}{d\rho} I(X; Y^\rho) = \rho \mathbb{E}\left[(X - \mathbb{E}[X|Y^\rho])^2 \right].$$

For the continuous-time channel

$$Y_t^\rho = \rho \int_0^t X_s ds + W_t, \; t \in [0, T],$$

where W is a Brownian motion, the I-MMSE formula is

$$\frac{d}{d\rho} I(X; Y^\rho) = \rho \mathbb{E} \int_0^T \left(X_s - \mathbb{E}[X_s | Y^\rho]\right)^2 ds.$$

We refer to the Shannon Lecture From Constrained Signaling to Network Interference Via An Information-Estimation Perspective given by *Shamai* for an overview of the applications of I-MMSE.

- Consider the channel

$$Y^{\rho} = \rho X + Z.$$

-

$$I(X; Y^{\rho}) = H(Y^{\rho}) - H(Y^{\rho}|X) = H(Y^{\rho}) - H(Z).$$

-

$$\frac{d}{d\rho} I(X; Y^{\rho}) = \frac{d}{d\rho} H(Y^{\rho}) = -\frac{d}{d\rho} \mathbb{E}[\log f_{Y^{\rho}}(Y^{\rho})].$$

Recall that $Y^\rho = \rho X + Z$.

Computation of

$$\frac{d}{d\rho}\mathbb{E}[\log f_{Y^\rho}(Y^\rho)].$$

- The approach by Guo, Shamai and Verdu,

$$\frac{d}{d\rho}\int f_{Y^\rho}(y)\log f_{Y^\rho}(y)dy = \int \frac{\partial}{\partial\rho} f_{Y^\rho}(y)\log f_{Y^\rho}(y)dy.$$

- Our approach

$$\frac{d}{d\rho}\int\int f_X(x)f_Z(z)\log f_{XZ}(\rho x + z)dxdz = \mathbb{E}\left(\frac{d}{d\rho}\log f_{Y^\rho}(Y^\rho)\right).$$

- Suppose Z now is a general random variable with density function $e^{T(z)}$, and $\mathbb{E}(X) = 0$. Consider the channel

$$Y^{\rho} = \rho X + Z.$$

- It has been studied in Additive non-Gaussian noise channels: mutual information and conditional mean estimation by Guo, Shamai and Verdu.
- Note that $f_{Y^{\rho}|X=x}(y;x) = e^{T(y-\rho x)}$, $f_Y(y) = \int_{\mathbb{R}} e^{T(y-\rho x)} d\mu_X(x)$.

$$\begin{aligned} \frac{d}{d\rho} f_{Y^{\rho}}(Y^{\rho}) &= \frac{d}{d\rho} \int_{\mathbb{R}} e^{T(Y^{\rho}-\rho x)} \mu_X(dx) \\ &= \frac{d}{d\rho} \int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} \mu_X(dx) \end{aligned}$$

$$= \int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$= \int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$= \int_{\mathbb{R}} f_{Y^\rho|X=x}(Y^\rho; x) T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$=\int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$=\int_{\mathbb{R}} f_{Y^\rho|X=x}(Y^\rho;x) T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$=f_{Y^\rho}(Y^\rho)\int_{\mathbb{R}} T'(Y^\rho-\rho x)(X-x))\mu_{X|Y^\rho}(dx;Y^\rho) \text{ (explain later)}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} T'(\rho(X-x)+Z)(X-x)\mu_X(dx) \\
&= \int_{\mathbb{R}} f_{Y^\rho|X=x}(Y^\rho; x) T'(\rho(X-x)+Z)(X-x)\mu_X(dx) \\
&= f_{Y^\rho}(Y^\rho) \int_{\mathbb{R}} T'(Y^\rho - \rho x)(X-x))\mu_{X|Y^\rho}(dx; Y^\rho) \text{ (explain later)} \\
&= f_{Y^\rho}(Y^\rho) \left(X\mathbb{E}[T'(Y^\rho - \rho X)|Y^\rho] - \mathbb{E}[T'(Y^\rho - \rho X)X|Y^\rho] \right)
\end{aligned}
$$

$$= \int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$= \int_{\mathbb{R}} f_{Y^\rho|X=x}(Y^\rho; x) T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$= f_{Y^\rho}(Y^\rho) \int_{\mathbb{R}} T'(Y^\rho - \rho x)(X - x))\mu_{X|Y^\rho}(dx; Y^\rho) \text{ (explain later)}$$

$$= f_{Y^\rho}(Y^\rho)\left(X\mathbb{E}[T'(Y^\rho - \rho X)|Y^\rho] - \mathbb{E}[T'(Y^\rho - \rho X)X|Y^\rho]\right)$$

$$= f_{Y^\rho}(Y^\rho)\left(X\mathbb{E}[T'(Z)|Y^\rho] - \mathbb{E}[T'(Z)X|Y^\rho]\right).$$

$$= \int_{\mathbb{R}} e^{T(\rho(X-x)+Z)} T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$= \int_{\mathbb{R}} f_{Y^\rho|X=x}(Y^\rho; x) T'(\rho(X-x)+Z)(X-x)\mu_X(dx)$$

$$= f_{Y^\rho}(Y^\rho) \int_{\mathbb{R}} T'(Y^\rho - \rho x)(X - x))\mu_{X|Y^\rho}(dx; Y^\rho) \text{ (explain later)}$$

$$= f_{Y^\rho}(Y^\rho)\left(X\mathbb{E}[T'(Y^\rho - \rho X)|Y^\rho] - \mathbb{E}[T'(Y^\rho - \rho X)X|Y^\rho]\right)$$

$$= f_{Y^\rho}(Y^\rho)\left(X\mathbb{E}[T'(Z)|Y^\rho] - \mathbb{E}[T'(Z)X|Y^\rho]\right).$$

- $$f_{Y^\rho|X=x}(y; x)\mu_X(dx) = f_{Y^\rho|X=x}(y; x)f_X(x)dx = f_{XY^\rho}(x, y)dx$$ $$= f_{Y^\rho}(y)f_{X|Y^\rho=y}(x; y)dx = f_{Y^\rho}(y)\mu_{X|Y^\rho=y}(dx; y).$$

- Hence we have

$$\begin{aligned}
&\frac{d}{d\rho} I(X;Y^\rho) \\
&= -\frac{d}{d\rho}\mathbb{E}[\log f_{Y^\rho}(Y^\rho)] = -\mathbb{E}\left[\frac{1}{f_{Y^\rho}(Y^\rho)}\frac{d}{d\rho}f_{Y_\rho}(Y^\rho)\right] \\
&= -\mathbb{E}\left(X\mathbb{E}[T'(Z)|Y^\rho]\right) + \mathbb{E}\left(\mathbb{E}[T'(Z)X|Y^\rho]\right) \\
&= -\mathbb{E}\left(X\mathbb{E}[T'(Z)|Y^\rho]\right) + \mathbb{E}[T'(Z)X] \\
&= -\mathbb{E}\left(X\mathbb{E}[T'(Z)|Y^\rho]\right).
\end{aligned}$$

- When $T(x) = -x^2/2$,

$$-\mathbb{E}(X\mathbb{E}[T'(Z)|Y^\rho]) = \mathbb{E}(X\mathbb{E}[Z|Y^\rho]) = \mathbb{E}(X\mathbb{E}[Y^\rho - \rho X|Y^\rho])$$

- When $T(x) = -x^2/2$,

$$
\begin{aligned}
&-\mathbb{E}(X\mathbb{E}[T'(Z)|Y^\rho]) = \mathbb{E}(X\mathbb{E}[Z|Y^\rho]) = \mathbb{E}(X\mathbb{E}[Y^\rho - \rho X|Y^\rho]) \\
=&\mathbb{E}(XY^\rho) - \rho\mathbb{E}(X\mathbb{E}[X|Y^\rho]) = \rho\mathbb{E}(X^2) + \mathbb{E}(XZ) - \rho\mathbb{E}(X\mathbb{E}[X|Y^\rho])
\end{aligned}
$$

- When $T(x) = -x^2/2$,

$$
\begin{aligned}
&-\mathbb{E}(X\mathbb{E}[T'(Z)|Y^\rho]) = \mathbb{E}(X\mathbb{E}[Z|Y^\rho]) = \mathbb{E}(X\mathbb{E}[Y^\rho - \rho X|Y^\rho]) \\
=&\mathbb{E}(XY^\rho) - \rho\mathbb{E}(X\mathbb{E}[X|Y^\rho]) = \rho\mathbb{E}(X^2) + \mathbb{E}(XZ) - \rho\mathbb{E}(X\mathbb{E}[X|Y^\rho]) \\
=&\rho\mathbb{E}(X^2) - \rho\mathbb{E}(X\mathbb{E}[X|Y^\rho]) = \rho\mathbb{E}(X - \mathbb{E}[X|Y^\rho])^2.
\end{aligned}
$$

Theorem

Let X be any random variable with a finite variance with density $f_X(x)$. Let Z be an independent standard normally distributed random variable. Then

$$\frac{d}{d\rho}H(X+\sqrt{\rho}Z)=\frac{1}{2}J(X+\sqrt{\rho}Z),$$

where $J(\cdot)$ is the Fisher information.

- Let $Y^\rho = X+\sqrt{\rho}Z$.
- Density function of Y^ρ :

$$f_{Y^\rho}(y)=\int_{\mathbb{R}}\frac{f_X(x)}{\sqrt{2\pi\rho}}\exp\left(-\frac{(y-x)^2}{2\rho}\right)dx.$$

- $$f_{Y^\rho}(Y^\rho) = f_{Y^\rho}(X+\sqrt{\rho}Z) = \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi\rho}} \exp\left(-\frac{(X+\sqrt{\rho}Z-x)^2}{2\rho}\right) dx.$$

- $$\begin{aligned} &\frac{d}{d\rho} f_{Y^\rho}(Y^\rho) \\ =& \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi\rho}} \exp\left(-\frac{(X+\sqrt{\rho}Z-x)^2}{2\rho}\right)\left(\frac{(X-x)(X+\sqrt{\rho}Z-x)}{2\rho^2} - \frac{1}{2\rho}\right) dx \\ =& f_{Y^\rho}(Y^\rho) \int_{\mathbb{R}} f_{X|Y^\rho}(x|Y^\rho) \left(\frac{(X-x)(Y^\rho - x)}{2\rho^2} - \frac{1}{2\rho}\right) dx. \end{aligned}$$

- The Blue terms $= f_{XY^\rho}(x, Y^\rho)$.

- $$\begin{aligned}\frac{d}{d\rho}H(Y^\rho) &= -\mathbb{E}\left[\frac{1}{f_{Y^\rho}(Y^\rho)}\frac{d}{d\rho}f_{Y^\rho}(Y^\rho)\right]\\ &= \mathbb{E}\left[\int_{\mathbb{R}} f_{X|Y^\rho}(x|Y^\rho)\left(-\frac{(X-x)(Y^\rho-x)}{2\rho^2}+\frac{1}{2\rho}\right)dx\right]\\ &= \frac{1}{2\rho^2}\mathbb{E}\left(-XY^\rho+(X+Y^\rho)\mathbb{E}[X|Y^\rho]-\mathbb{E}[X^2|Y^\rho]\right)+\frac{1}{2\rho}\\ &= \frac{1}{2\rho^2}\left(-\mathbb{E}[X^2]+\mathbb{E}[(\mathbb{E}[X|Y^\rho])^2]\right)+\frac{1}{2\rho}\end{aligned}$$

- For the Fisher information $J(Y^\rho) = \mathbb{E}\left[\left(\frac{f'_{Y^\rho}(Y^\rho)}{f_{Y^\rho}(Y^\rho)}\right)^2\right]$.

$$\begin{aligned} f'_{Y^\rho}(y) &= \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi\rho}} \exp\left(\frac{-(y-x)^2}{2\rho}\right) \frac{x-y}{\rho} dx \\ &= f_{Y^\rho}(y) \int_{\mathbb{R}} f_{X|Y^\rho}(x|y) \frac{x-y}{\rho} dx. \end{aligned}$$

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$$f'_{Y^\rho}(Y^\rho) = f_{Y^\rho}(Y^\rho) \int_{\mathbb{R}} f_{X|Y^\rho}(x|Y^\rho) \frac{x - Y^\rho}{\rho} dx = \frac{1}{\rho} f_{Y^\rho}(Y^\rho) \mathbb{E}\left[X - Y^\rho | Y^\rho\right].$$

- $Y^\rho = X + \sqrt{\rho} Z.$

$$J(Y^\rho) = \mathbb{E}\left[\left(\frac{f'_{Y^\rho}(Y^\rho)}{f_{Y^\rho}(Y^\rho)}\right)^2\right]$$
$$= \frac{1}{\rho^2}\mathbb{E}\left(\mathbb{E}\left[X - Y^\rho | Y^\rho\right]\right)^2 = \frac{1}{\rho}\mathbb{E}\left(\mathbb{E}\left[Z | Y^\rho\right]\right)^2$$
$$= \frac{1}{\rho^2}\left(\mathbb{E}[(\mathbb{E}[X|Y^\rho])^2 + E[(Y^\rho)^2] - 2\mathbb{E}[XY^\rho]\right),$$

- $$\frac{d}{d\rho}H(Y^\rho) = \frac{1}{2}J(Y^\rho),$$

  since

$$\rho = \mathbb{E}[(X - Y^\rho)^2].$$

Theorem (Han-Song 2013)

Consider a discrete-time channel with feedback

$$Y_i = \rho g_i(X_i, Y_1^{i-1}) + Z_i,\ i = 1, \cdots, n,$$

where the density function of Z_i's is $e^{T(x)}$. Then we have

$$\frac{dI(X_1^n; Y_1^n)}{d\rho} = \sum_{i=1}^{n} \left(-\mathbb{E}\left((g_i + \rho \frac{d}{d\rho} g_i) \mathbb{E}\left[T'(Z_i) \middle| Y_1^n \right] \right) + \mathbb{E}\left[(g_i + \rho \frac{d}{d\rho} g_i)) T'(Z_i) \right] \right)$$

where $g_i = g_i(X_i, Y_1^{i-1})$.

- $f(y_1^n)$ means the density function of Y_1^n, $f(Y_1^n)$ means the density function of Y_1^n evaluated at Y_1^n, $f(y_1^n|x_1^n)$ means the conditional density function of Y_1^n given $X_1^n = x_1^n$, and so on.
- $$I(X_1^n; Y_1^n) = H(Y_1^n) - H(Y_1^n|X_1^n) = -\mathbb{E} \log f(Y_1^n) - H(Y_1^n|X_1^n).$$
- $$H(Y_1^n|X_1^n) = \sum_{i=1}^{n} H(Y_i|X_1^n, Y_1^{i-1}) = nH(Z_1).$$
- Hence
$$\frac{dI(X;Y)}{d\rho} = -\mathbb{E}\left[\frac{d}{d\rho} \log f(Y_1^n)\right] = -\mathbb{E}\left[\frac{1}{f(Y_1^n)} \frac{d}{d\rho} f(Y_1^n)\right].$$

- $$f(y_1^n|x_1^n) = f(y_1|x_1^n)f(y_2|y_1,x_1^n)\cdots f(y_n|y_1^{n-1},x_1^n)$$
$$= \prod_{i=1}^{n} \exp\left(T(y_i - \rho g_i(x_i, y_1^{i-1}))\right).$$

- $$\frac{d}{d\rho} f(Y_1^n)$$
$$= \frac{d}{d\rho}\int_{\mathbb{R}^n} f(Y_1^n|x_1^n)f(x_1^n)dx = \int_{\mathbb{R}^n} \frac{d}{d\rho} f(Y_1^n|x_1^n)f(x_1^n)dx$$
$$= \int_{\mathbb{R}^n} \frac{d}{d\rho} \prod_{i=1}^{n} \exp\left(T\left(Y_i - \rho g_i(x_i, Y_1^{i-1})\right)\right) f(x_1^n)dx$$
$$= \int_{\mathbb{R}^n} \frac{d}{d\rho} \prod_{i=1}^{n} \exp\left(T\left(\rho g_i(X_i, Y_1^{i-1}) - \rho g_i(x_i, Y_1^{i-1}) + Z_i\right)\right) f(x_1^n)dx$$

- $$= \int_{\mathbb{R}^n} f(Y_1^n|x_1^n) \sum_{i=1}^n T'(Y_i - \rho g_i(x_i, Y_1^{i-1}) \Big[g_i(X_i, Y_1^{i-1}) - g_i(x_i, Y_1^{i-1}) + \rho \frac{d}{d\rho} \left(g_i(X_i, Y_1^{i-1}) - g_i(x_i, Y_1^{i-1}) \right) \Big] f(x_1^n) dx$$
$$= f(Y_1^n) \int_{\mathbb{R}^n} \sum_{i=1}^n T'(Y_i - \rho g(x_i, Y_1^{i-1}) \Big[g_i(X_i, Y_1^{i-1}) - g_i(x_i, Y_1^{i-1}) + \rho \frac{d}{d\rho} \left(g_i(X_i, Y_1^{i-1}) - g_i(x_i, Y_1^{i-1}) \right) \Big] f(x_1^n | Y_1^n) dx.$$
- Denote $g_i := g_i(X_i, Y_1^{i-1})$ and $\tilde{g}_i := g_i(x_i, Y_1^{i-1})$.

- $$\begin{aligned}&\frac{d}{d\rho} f_X(Y_1^n)\\ =&f_X(Y_1^n)\sum_{i=1}^{n}\int_{\mathbb{R}^n} T'(Y_i-\rho\tilde{g}_i)\left[(g_i+\rho\frac{d}{d\rho}g_i)-(\tilde{g}_i+\rho\frac{d}{d\rho}\tilde{g}_i)\right] f_X(x_1^n|Y_1^n)dx\\ =&f_X(Y_1^n)\sum_{i=1}^{n}\left((g_i+\rho\frac{d}{d\rho}g_i)\mathbb{E}\left[T'(Y_i-\rho g_i)\middle| Y_1^n\right]-\mathbb{E}\left[(g_i+\rho\frac{d}{d\rho}g_i)T'(Y_i-\rho g_i)\middle| Y_1^n\right]\right)\\ =&f_X(Y_1^n)\sum_{i=1}^{n}\left((g_i+\rho\frac{d}{d\rho}g_i)\mathbb{E}\left[T'(Z_i)\middle| Y_1^n\right]-\mathbb{E}\left[(g_i+\rho\frac{d}{d\rho}g_i)T'(Z_i)\middle| Y_1^n\right]\right).\end{aligned}$$

- $$\begin{aligned}\frac{dI(X;Y^\rho)}{d\rho}&=-\mathbb{E}\left[\frac{d}{d\rho}\log f_X(Y_1^n)\right]\\ &=\sum_{i=1}^{n}\left(-\mathbb{E}\left((g_i+\rho\frac{d}{d\rho}g_i)\mathbb{E}\left[T'(Z_i)\middle| Y_1^n\right]\right)+\mathbb{E}\left[(g_i+\rho\frac{d}{d\rho}g_i)T'(Z_i)\right]\right).\end{aligned}$$

- For the Gaussian channel where $T(x) = -x^2/2$, we have

$$\frac{dI(X;Y^\rho)}{d\rho}$$

$$= \sum_{i=1}^{n} \left(\mathbb{E}\left((g_i + \rho \frac{d}{d\rho} g_i) \mathbb{E}[Z_i | Y_1^n] \right) - \mathbb{E}\left[(g_i + \rho \frac{d}{d\rho} g_i) Z_i \right] \right)$$

$$= \sum_{i=1}^{n} \left(\mathbb{E}\left((g_i + \rho \frac{d}{d\rho} g_i)(Y_i - \rho \mathbb{E}[g_i | Y_1^n]) \right) - \mathbb{E}\left[(g_i + \rho \frac{d}{d\rho} g_i)(Y_i - \rho g_i) \right] \right)$$

$$= \sum_{i=1}^{n} \left(\rho \mathbb{E}\left[(g_i - \mathbb{E}(g_i | Y_1^n))^2 \right] + \rho^2 \mathbb{E}\left[(g_i - \mathbb{E}(g_i | Y_1^n)) \frac{d}{d\rho} g_i \right] \right).$$

Theorem (Han-Song 2013)

Consider the continuous-time channel with feedback

$$Y_t^\rho = \rho \int_0^t g(s, X_s, Y^\rho) ds + W_t,\ t \in [0, T], \tag{1}$$

where $g : [0, T] \times \mathbb{R} \times C([0, T]; \mathbb{R}) \to \mathbb{R}$ *is a bounded progressively measurable function. We have*

$$\frac{dI(X; Y^\rho)}{d\rho} = \rho \mathbb{E} \int_0^T (g_s - \mathbb{E}[g_s | Y^\rho])^2 \, ds + \rho^2 \int_0^T \mathbb{E}\left[(g_s - \mathbb{E}[g_s | Y^\rho]) \frac{\partial g_s}{\partial \rho} \right] ds,$$

where $g_s = g(s, X_s, Y^\rho)$.

- For simplicity, consider the case $g(s, X_s, Y) = X_s$.

Lemma (Cameron-Martin)

$Y_t = \rho \int_0^t X_s ds + W_t$, where W is a Wiener process, and X is independent of W. Then we have

$$\frac{d\mu_{Y|X}}{d\mu_W}(y;\omega) = \exp\left(\rho \int_0^T \omega_s dy_s - \frac{\rho^2}{2}\int_0^t \omega_s^2 ds\right), y \in C[0,T],$$

and

$$\frac{d\mu_Y}{d\mu_W}(y) = \int_{C[0,T]} \frac{d\mu_{Y|X}}{d\mu_W}(y;\omega)\mu_X(d\omega).$$

- $$
\begin{aligned}
& I(X; Y^\rho) \\
=& \mathbb{E}\left[\log \frac{d\mu_{XY^\rho}}{d(\mu_X \times \mu_{Y^\rho})}(X, Y^\rho)\right] \\
=& \mathbb{E}\left[\log \frac{d\mu_{Y^\rho|X}}{d\mu_W}(Y^\rho; X)\right] - \mathbb{E}\left[\log \frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\right] \\
=& \frac{\rho^2}{2}\int_0^T \mathbb{E}[X_s^2]ds - \mathbb{E}\left[\log \frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\right]
\end{aligned}
$$

- $$
\begin{aligned}
& \frac{d}{d\rho} I(X; Y^\rho) \\
=& \rho \int_0^T \mathbb{E}[X_s^2]ds - \frac{d}{d\rho}\mathbb{E}\left[\log \frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\right].
\end{aligned}
$$

- $$\begin{aligned}
&\frac{d}{d\rho}\left(\frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\right)\\
=&\frac{d}{d\rho}\int_{C[0,T]}\frac{d\mu_{Y^\rho|X=\omega}}{d\mu_W}(Y^\rho;\omega)\mu_X(d\omega)\\
=&\frac{d}{d\rho}\int_{C[0,T]}\exp\left\{\rho\int_0^T\omega_s dY_s^\rho-\frac{\rho^2}{2}\int_0^T\omega_s^2 ds\right\}\mu_X(d\omega)\\
=&\frac{d}{d\rho}\int_{C[0,T]}\exp\left\{\rho^2\int_0^T\omega_s X_s ds+\rho\int_0^T\omega_s dW_s-\frac{\rho^2}{2}\int_0^T\omega_s^2 ds\right\}\mu_X(d\omega)\\
=&\int_{C[0,T]}\left(\rho\int_0^T\omega_s(X_s-\omega_s)ds+\int_0^T\omega_s dY_s^\rho\right)\frac{d\mu_{Y^\rho|X=\omega}}{d\mu_W}(Y^\rho;\omega)\mu_X(d\omega)
\end{aligned}$$

- $$=\int_{C[0,T]}\left(\rho\int_0^T \omega_s(X_s-\omega_s)ds+\int_0^T \omega_s dY_s^\rho\right)\frac{d\mu_{XY^\rho}}{d\mu_W}(d\omega,Y^\rho)$$
$$=\frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\int_{C[0,T]}\left(\rho\int_0^T \omega_s(X_s-\omega_s)ds+\int_0^T \omega_s dY_s^\rho\right)\mu_{X|Y^\rho}(d\omega;Y^\rho)$$
$$=\frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\left(\rho\int_0^T \mathbb{E}\left[X_s|\,Y^\rho\right]X_s ds-\rho\int_0^T \mathbb{E}\left[X_s^2\middle|\,Y^\rho\right]ds+\mathbb{E}\left[\int_0^T X_s dY_s^\rho\middle|\,Y^\rho\right]\right)$$

- $$\mathbb{E}\left[\frac{d}{d\rho}\left(\frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\right)\frac{d\mu_W}{d\mu_{Y^\rho}}(Y^\rho)\right]$$
$$=\rho\int_0^T \mathbb{E}\left[\mathbb{E}\left[X_s|\, Y^\rho\right]X_s\right]ds-\rho\int_0^T\mathbb{E}\left[X_s^2\right]ds+\mathbb{E}\left[\int_0^T X_s dY_s^\rho\right]$$
$$=\rho\int_0^T\mathbb{E}\left[\left(\mathbb{E}\left[X_s|\, Y^\rho\right]\right)^2\right]ds.$$

- $$\frac{d}{d\rho}I(X;Y^\rho)$$
$$=\rho\int_0^T\mathbb{E}[X_s^2]ds-\frac{d}{d\rho}\mathbb{E}\left[\log\frac{d\mu_{Y^\rho}}{d\mu_W}(Y^\rho)\right]$$
$$=\rho\int_0^T\mathbb{E}\left[\left(X_s-\mathbb{E}\left[X_s|\, Y^\rho\right]\right)^2\right]ds.$$

Further study

- I-MMSE when W is a general Gaussian process.
- Applications of I-MMSE.

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THANK YOU!