Subspace Codes and Orbit Codes

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Hong Kong University, December 12, 2013

joint work with
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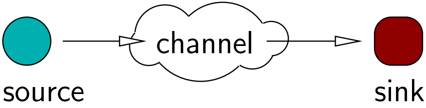
Outline

Traditional Communication Channel



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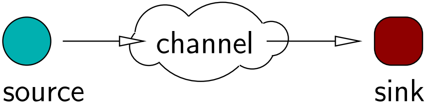

Setting:

- Communication between single source and sink.
- In the channel messages are forwarded.

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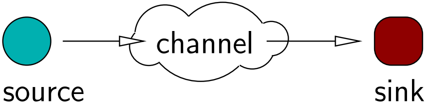

Setting:

- Communication between single source and sink.
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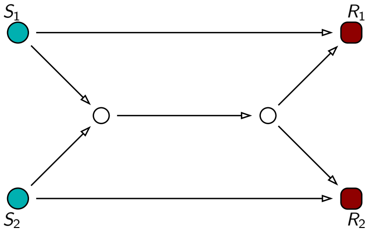
Question

Why do we consider only communications between single entities? Is it natural?

Example - Butterfly Network

Question

Is it possible that both S_1 and S_2 communicate their messages to both R_1 and R_2 in only one "round time"?



Channel setting

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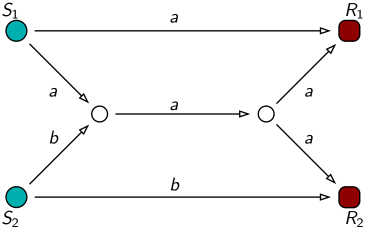

Traditional communication channel approach:
Throughput is limited by the Max-Flow, Min-Cut Theorem.

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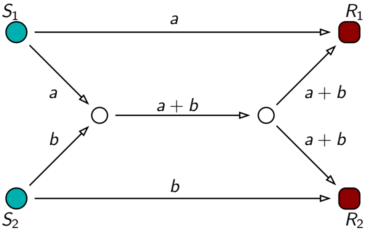
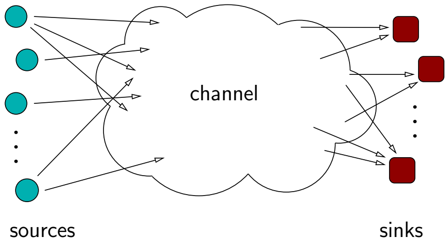


Linear Network coding approach increases Throughput!

Linear Network Coding



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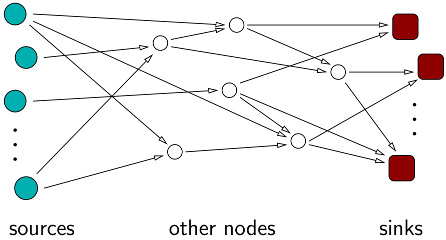
Linear Network Coding



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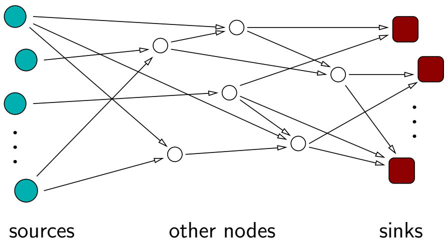
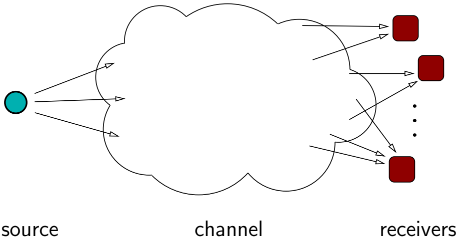


Setting:

- digraph $\mathfrak{G} = (V, E)$ with capacities on the edges.
- the output messages of a channel nodes are linear combinations of input ones.

Single User many Receivers



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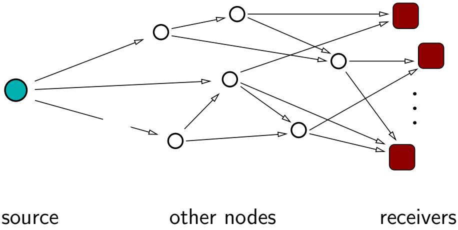

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- others.

The Role of Vector Spaces

Let $\mathbb{F}_q$ be a finite field and n, k two nonzero natural numbers. Denote by $m_1, \ldots, m_k \in \mathbb{F}_q^n$ the messages trasmitted by k different sources.
Assume the messages to be linear independent.

$$m_1, \ldots, m_k \to M = \begin{pmatrix} m_1^t \\ m_2^t \\ \vdots \\ m_k^t \end{pmatrix} \in Mat_{k\times n}(\mathbb{F}_q) \to \mathrm{rowsp}(M) \in \mathrm{G}(k, \mathbb{F}_q^n)$$

where $\mathrm{G}(k, \mathbb{F}_q^n)$ is the Grassmannian of all k-dimensional vector subspaces of $\mathbb{F}_q^n$.

Metric on $\mathcal{P}(n)$

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Remark

Check that the map: $d_S : \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{N}_+$ defines a metric on $\mathcal{P}(n)$.

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In the usual way one defines the distance of the subspace code $\mathcal{C} \subset \mathcal{P}(n)$ through:

$$\mathrm{dist}(\mathcal{C}) := \min \{ d_S(V, W) \mid V, W \in \mathcal{C},\ V \neq W \}$$

and the size of $\mathcal{C}$ as $M := |\mathcal{C}|$.

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Remark

In the usual way one has the goal to construct for any natural numbers n, M and any finite field $\mathbb{F}_q$ codes having maximal distance d and efficient decoding algorithms.

Induced Metric on the the Grassmannian $\mathrm{G}(k, \mathbb{F}_q^n)$

Definition

In the sequel we will assume that a subspace code is a subset of the Grassmannian $\mathrm{G}(k, \mathbb{F}_q^n)$. We call such codes also constant-dimension codes.

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Remark

The main constant-dimension subspace coding problem is: For every size M construct codes $\mathcal{C} \subset \mathrm{G}(k, \mathbb{F}_q^n)$ having maximal possible distance.

Errors and Erasures

Decoder: Minimum Distance Decoder (closest codeword given a received vector space).

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Question

How do we expect errors and erasures to be?

- *Error* $\leftrightarrow$ *Increase in dimension.*
- *Erasure* $\leftrightarrow$ *Decrease in dimension.*

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Fundamental Research Questions

- For every finite field and positive intgers d, k, n find the maximum number of subspaces in the Grassmannian $G(k, \mathbb{F}_q^n)$ such that this code has distance d.

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- For every finite field and positive intgers d, k, n find the maximum number of subspaces in the Grassmannian $G(k, \mathbb{F}_q^n)$ such that this code has distance d.
- Find constructions of codes together with efficient decoding algorithms.

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Nota Bene: It will turn out that the problem of list decoding is an intersection problem between the *Schubert variety* S_W and the subspace code $\mathcal{C} \subset \mathrm{Grass}(k, V)$.

Geometric Questions of Schubert

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Answer Schubert: By Poncelet's principle of conservation of numbers we can assume lines 1 and 2 intersect and lines 3 and 4 intersect. So there are 2 solutions in general.

A Result of Schubert

Theorem (Schubert [2])

Given $N := k(n-k)$ linear subspace $U_i, i = 1, \dots, N$ in V having dimension k each. If the base field $\mathbb{F}$ is algebraically closed and the subspaces are in general position then there exist exactly

$$\frac{1!2!\cdots(k-1)!(N)!}{(n-k)!(n-k+1)!\cdots(n-1)!} \tag{1}$$

subspaces W of dimension $(n-k)$ intersecting each of the subspaces U_i nontrivially.

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Hermann Cäsar Hannibal Schubert (1848-1911)

Schubert Varieties

Definition

A flag $\mathcal{F}$ is a sequence of nested subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_n = V \tag{2}$$

where we assume that $\dim V_i = i$ *for* $i = 1, \ldots, n$.

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Definition

For each flag $\mathcal{F}$ and each multiindex $\underline{i}$

$$S(\underline{i}; \mathcal{F}) := \{W \in \mathrm{Grass}(k, V) \mid \dim(W \bigcap V_{i_s}) \geq s\}$$

Central Question of Schubert Calculus

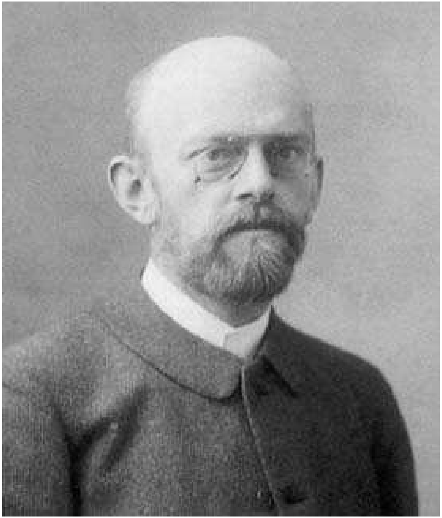
Problem

Given two Schubert varieties $S(\nu; \mathcal{F})$ and $S(\tilde{\nu}; \tilde{\mathcal{F}})$. Describe as explicitly as possible the intersection variety

$$S(\nu; \mathcal{F}) \cap S(\tilde{\nu}; \tilde{\mathcal{F}}).$$

Hilbert Problem Number 15, Paris 1900
Rigorous foundation of Schubert's enumerative calculus

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him. Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.



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David Hilbert (1862-1943)

Plücker Embedding

Consider the vector space of alternating k–tensors $\wedge^k V$. Let $\mathbb{P}(\wedge^k V)$ be the projective space consisting of all lines in $\wedge^k V$.

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$$\begin{array}{rrcl} \varphi : & \mathrm{Grass}(k, V) & \longrightarrow & \mathbb{P}(\wedge^k V) \\ & \mathrm{span}(v_1, \ldots, v_k) & \longmapsto & \mathbb{F} v_1 \wedge \cdots \wedge v_k . \end{array} \tag{4}$$

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Plücker Coordinates

Assume

$$v_i = \sum_{j=1}^{n} a_{ij} e_j,\ \ i = 1, \ldots, k.$$

Let A be the $k \times n$ matrix $(a_{i,j})$. The Plücker embedding writes:

$$\begin{array}{cccc} \varphi \ : & Mat_{k\times n} & \longrightarrow & \mathbb{P}(\wedge^k V) \\ & \mathrm{rowspace}(A) & \longmapsto & \displaystyle\sum_{1\le i_1<\cdots<i_k\le n} x_{i_1,\ldots,i_k} \cdot e_{i_1} \wedge \ldots \wedge e_{i_k}. \end{array} \quad (5)$$

The coordinates $x_{\underline{i}} := x_{i_1,\ldots,i_k}$ are called the Plücker coordinates of $\mathrm{rowspace}(A)$.

Shuffle Relations

Theorem

$$\sum_{\lambda=1}^{k+1}(-1)^{\lambda}\cdot x_{i_1,\ldots,i_{k-1},j_\lambda}\cdot x_{j_1,\ldots,\hat{j_\lambda},\ldots,j_{k+1}}=0 \tag{6}$$

describes the image of the Grassmannian in the projective space $\mathbb{P}(\wedge^k V)$

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Example

$\text{Grass}(2, \mathbb{F}^4)$ is embedded in $\mathbb{P}^5$ and $\varphi(Grass(2,4))$ is described by a single relation

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0 \qquad (7)$$

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Shuffle Relations

Example

$\text{Grass}(2, \mathbb{F}^5)$ is embedded in $\mathbb{P}^9$ and the defining relations are:

$$
\begin{aligned}
x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} &= 0 \\
x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23} &= 0 \\
x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{14} &= 0 \\
x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34} &= 0 \\
x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34} &= 0
\end{aligned}
$$

Defining Equations of Schubert Varieties

Bruhat order:
Let $\underline{i} := (i_1, \ldots, i_k)$ and $\underline{j} := (j_1, \ldots, j_k)$ be two set of indices satisfying

$$1 \leq i_1 < \ldots < i_k \leq n$$

respectively

$$1 \leq j_1 < \ldots < j_k \leq n.$$

Then one defines:

$$\underline{i} \leq \underline{j}$$

if and only if $i_t \leq j_t$ for $t = 1, \ldots, k$.

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Theorem

The defining equations in terms of Plücker coordinates of the Schubert variety $S(\underline{i}; \mathcal{F})$ are given by the quadratic shuffle relations together with the linear equations $x_{\underline{j}} = 0$ for all $\underline{j} \not\leq \underline{i}$.

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We could show how to efficiently describe the equations for the variety S_W.

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Gabidulin provided several constructions and decoding algorithms of rank metric codes with good distances.

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Remark

The map

$$\phi : \mathbb{F}^{k\times m} \longrightarrow \mathrm{G}(k, \mathbb{F}_q^{k+m}),\ X \longmapsto \mathrm{rowsp}[I_k\ X]$$

defines an embedding and one sometimes calls the image the thick open cell of the Grassmannian.

Spread of $\mathbb{F}_q^n$

Definition

$S \subset \mathrm{G}(k, \mathbb{F}_q^n)$ *is a spread of* $\mathbb{F}_q^n$ *if:*

- $V \cap W = \{0\}$ *for all* $V, W \in S$, *and*
- *for any* $v \in \mathbb{F}_q^n$, $v \neq 0$, *exists unique* $V \in S$ *such that* $v \in V$.

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There exists a spread $S \subset \mathrm{G}(k, \mathbb{F}_q^n)$ if and only if $k \mid n$.

Spreads in Projective Geometry [Hirschfeld 98]

Remark

k-dim subspaces in $\mathbb{F}_q^n \overset{1-1}{\longleftrightarrow} (k-1)$*-dim subspaces in* $\mathbb{P}_{\mathbb{F}_q}^{n-1}$.
It follows $\mathrm{G}(k, \mathbb{F}_q^n) \cong \mathrm{G}(k-1, \mathbb{P}_{\mathbb{F}_q}^{n-1})$.

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There exists a spread $S \subset \mathrm{G}(k-1, \mathbb{P}_{\mathbb{F}_q}^{n-1})$ if and only if $k \mid n$.

Spread Codes

Setting:

- $n, k, r \in \mathbb{N}_+$ such that $n = kr$;
- $p \in \mathbb{F}_q[x]$ irreducible of degree k and $P \in Mat_{k\times k}(\mathbb{F}_q)$ its companion matrix;
- $\mathbb{F}_q[P] \subset GL_k(\mathbb{F}_q)$, $\mathbb{F}_q[P] \cong \mathbb{F}_{q^k}$.

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Theorem

The collection of subspaces

$$\mathcal{S} := \bigcup_{i=1}^{r} \{\mathrm{rowsp}\,[0_k \;\cdots\; 0_k \;\; I_k \;\; A_{i+1} \;\cdots\; A_r] \mid A_{i+1}, \ldots, A_r \in \mathbb{F}_q[P]\}$$

is a subset of $\mathrm{G}(k, \mathbb{F}_q^n)$ *and a spread of* $\mathbb{F}_q^n$*.*

Definition and Properties

Definition

The set $\mathcal{S}$ constructed as in the previous slide will be called a Spread Codes of $\mathrm{G}(k, \mathbb{F}_q^n)$.

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Properties:

- MDS-like for the distance $d = 2k$.
- every nonzero vector of $\mathbb{F}_q^n$ belong to one and only one codeword.

Orbit codes

$GL_n(\mathbb{F}_q)$ (right) action on Grassmannians:

$$\begin{array}{ccc} \mathcal{G}(k,n) \times GL_n(\mathbb{F}_q) & \to & \mathcal{G}(k,n) \\ (\mathcal{U}, A) & \mapsto & \mathcal{U} \cdot A := \mathrm{rowsp}(U \cdot A) \end{array}$$

Proposition

Let $\mathcal{U}, \mathcal{V} \in \mathcal{G}(k,n)$. Then

$$d(\mathcal{U}, \mathcal{V}) = d(\mathcal{U} \cdot A, \mathcal{V} \cdot A) \quad \forall A \in GL_n(\mathbb{F}_q).$$

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Definition (orbit codes)

Let $\mathcal{U} \in \mathcal{G}(k,n)$ and $\mathfrak{G} < GL_n(\mathbb{F}_q)$. An orbit code is

$$\mathcal{C} = \{\mathcal{U}\cdot A \mid A \in \mathfrak{G}\}.$$

Representation of Grassmannian via $GL_n(\mathbb{F}_q)$

Definition

- *Let $\mathcal{U} \in \mathcal{G}(k,n)$. The stabilizer of $\mathcal{U}$ is*

$$Stab(\mathcal{U}) := \{A \in GL_n(\mathbb{F}_q) \mid \mathcal{U} = \mathcal{U} \cdot A\}.$$

Theorem

Let $\mathcal{U} \in \mathcal{G}(k,n)$. Then

$$\mathcal{G}(k,n) \cong GL_n(\mathbb{F}_q)/Stab(\mathcal{U}).$$

Cyclic orbit codes

$$GL_n(\mathbb{F}_q) \xrightarrow{\pi} GL_n(\mathbb{F}_q)/Stab(\mathcal{U}) \longleftrightarrow \mathcal{G}(k,n)$$

Proposition

Let $\mathfrak{G}_1, \mathfrak{G}_2 < GL_n$. Then

$$\pi(\mathfrak{G}_1) = \pi(\mathfrak{G}_2) \iff \mathcal{C}_{\mathfrak{G}_1} = \mathcal{C}_{\mathfrak{G}_2}.$$

Definition

An orbit code $\mathcal{C}$ is cyclic if there exists $\mathfrak{G} < GL_n(\mathbb{F}_q)$ cyclic defining it.

"Linearity" of orbit codes

Properties

Let $\mathfrak{G} < GL_n(\mathbb{F}_q)$. *Then*

- $|C| = \dfrac{|\mathfrak{G}|}{|\mathfrak{G} \cap Stab(\mathcal{U})|}$.
- $d_{\min} = \min_{A \in \mathfrak{G} \setminus Stab(\mathcal{U})} d(\mathcal{U}, \mathcal{U} \cdot A)$.
- $\mathcal{C}^{\perp} := \{\mathcal{U}^{\perp} \in \mathcal{G}(n-k, n) \mid \mathcal{U} \in \mathcal{C}\}$ *is an orbit code.*

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Spread codes as cyclic orbit codes

Lemma

If $k|n, c := \frac{q^n-1}{q^k-1}$ *and* α *a primitive element of* $\mathbb{F}_{q^n}$*, then the vector space generated by* $1, \alpha^c, ..., \alpha^{(k-1)c}$ *is equal to* $\{\alpha^{ic} | i = 0, ..., q^k - 2\} \cup \{0\} = \mathbb{F}_{q^k}$.

Spread codes as cyclic orbit codes

Lemma

If $k|n, c := \frac{q^n-1}{q^k-1}$ *and* α *a primitive element of* $\mathbb{F}_{q^n}$, *then the vector space generated by* $1, \alpha^c, ..., \alpha^{(k-1)c}$ *is equal to* $\{\alpha^{ic} | i = 0, ..., q^k - 2\} \cup \{0\} = \mathbb{F}_{q^k}$.

Lemma

For every $\beta \in \mathbb{F}_{q^n}$ *the set*

$$\beta \cdot \mathbb{F}_{q^k} = \{\beta\alpha^{ic} | i = 0, ..., q^k - 2\} \cup \{0\}$$

defines an $\mathbb{F}_q$*-subspace of dimension* k.

Spread codes as cyclic orbit codes

Theorem

The set

$$\mathcal{S} = \left\{ \alpha^i \cdot \mathbb{F}_{q^k} \mid i = 0, \ldots, c-1 \right\}$$

defines a spread code.

Spread codes as cyclic orbit codes

Theorem

The set

$$\mathcal{S} = \left\{ \alpha^i \cdot \mathbb{F}_{q^k} \mid i = 0, \ldots, c-1 \right\}$$

defines a spread code.

Proof.

It is enough to show that the subspace $\alpha^i \cdot \mathbb{F}_{q^k}$ and $\alpha^i \cdot \mathbb{F}_{q^k}$ are pairwise disjoint whenever $0 \leq i < j \leq c-1$. For this assume that there are field elements $c_i, c_j \in \mathbb{F}_{q^k}$, such that

$$v = \alpha^i c_i = \alpha^j c_j \in \alpha^i \cdot \mathbb{F}_{q^k} \cap \alpha^j \cdot \mathbb{F}_{q^k}.$$

If $v \neq 0$ then $\alpha^{i-j} = c_j c_i^{-1} \in \mathbb{F}_{q^k}$. But this means $i - j \equiv 0 \mod c$ and $\alpha^i \cdot \mathbb{F}_{q^k} = \alpha^j \cdot \mathbb{F}_{q^k}$. It follows that $\mathcal{S}$ is a spread. $\square$

Translation into matrix setting

Theorem

Let $p(x)$ be an irreducible polynomial over $\mathbb{F}_q$ of degree n and P its companion matrix. Furthermore let $\alpha \in \mathbb{F}_{q^n}$ be a root of $p(x)$ and ϕ be the canonical homomorphism

$$\phi : \mathbb{F}_q^n \to \mathbb{F}_{q^n}, \quad (v_1, \ldots, v_n) \mapsto \sum_{i=1}^{n} v_i \alpha^{i-1}$$

Then the following diagram commutes (for $v \in \mathbb{F}_q^n$):

$$\begin{array}{ccc} v & \xrightarrow{\cdot P} & vP \\ \phi \downarrow & & \downarrow \phi \\ v' & \xrightarrow[\cdot \alpha]{} & v'\alpha \end{array}$$

Example 1

Over the binary field let $p(x) := x^6 + x + 1$ primitive, α a root of $p(x)$ and P its companion matrix. For the 3-dimensional spread compute $c = \frac{63}{7} = 9$ and construct a basis for the starting point of the orbit:

$$u_1 = \phi^{-1}(1) = (100000)$$
$$u_2 = \phi^{-1}(\alpha^9) = \phi^{-1}(\alpha^4 + \alpha^3) = (000110)$$
$$u_3 = \phi^{-1}(\alpha^{18}) = \phi^{-1}(\alpha^3 + \alpha^2 + \alpha + 1) = (111100)$$

The starting point is

$$\mathcal{U} = \mathrm{rowsp} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} = \mathrm{rowsp} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and the orbit of the group generated by P on $\mathcal{U}$ is a spread code.

Example 2

For the 2-dimensional spread compute $c = \frac{63}{3} = 21$ and construct the starting point

$$u_1 = \phi^{-1}(1) = (100000)$$
$$u_2 = \phi^{-1}(\alpha^{21}) = \phi^{-1}(\alpha^2 + \alpha + 1) = (111000)$$

The starting point is

$$\mathcal{U} = \mathrm{rowsp} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rowsp} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and the orbit of the group generated by P is a spread code.

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Thank you for your attention.

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T. Etzion and N. Silberstein.
Error-correcting codes in projective spaces via rank-metric codes and Ferrers diagrams.
IEEE Trans. Inform. Theory, 55(7):2909–2919, March 2009.

R. Kötter and F.R. Kschischang.
Coding for errors and erasures in random network coding.
IEEE Transactions on Information Theory, 54(8):3579–3591, August 2008.

F. Manganiello, E. Gorla, and J. Rosenthal.
Spread codes and spread decoding in network coding.
In *Proceedings of the 2008 IEEE International Symposium on Information Theory*, pages 851–855, Toronto, Canada, 2008.

J. Rosenthal and A.-L. Trautmann.
A complete characterization of irreducible cyclic orbit codes and their Plücker embedding.
Des. Codes Cryptogr., 66(1–3):275–289, 2013.

H. Schubert.
Kalkühl der abzählenden Geometrie.
Teubner, Leipzig, 1879.

D. Silva, F.R. Kschischang, and R. Kötter.
A rank-metric approach to error control in random network coding.
Proceedings of the 2008 IEEE International Symposium on Information Theory, 54(9):3951–3967, Sept. 2008.

A.-L. Trautmann, F. Manganiello, M. Braun, and J. Rosenthal.
Cyclic orbit codes.
IEEE Transactions on Information Theory, 59(11):7386–7404, November 2013.

A.-L. Trautmann, F. Manganiello, and J. Rosenthal.
Orbit codes - a new concept in the area of network coding.
In *Information Theory Workshop (ITW), 2010 IEEE*, pages 1 –4, Dublin, Ireland, August 2010.

A.-L. Trautmann and J. Rosenthal.
New improvements on the echelon-ferrers construction.
In *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS*, pages 405–408, Budapest, Hungary, 2010.