Coding for Combined Block–Symbol Error Correction

Pascal O. Vontobel

Talk at WCI 2013, Hong Kong, December 13, 2013

Joint work with Ron M. Roth

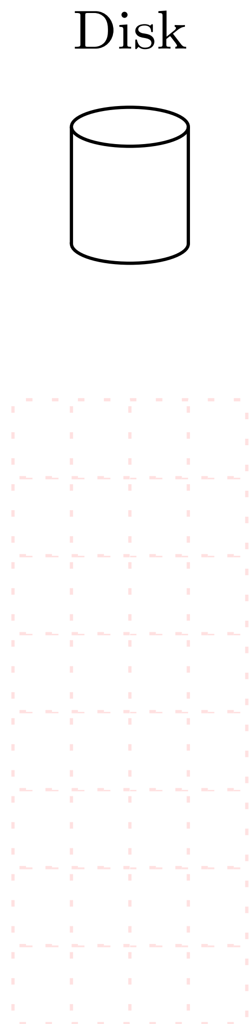
http://arxiv.org/abs/1302.1931

Overview

- Motivation
- Proposed coding scheme
- Decoding
- Conclusions / open problems

Motivation

Storing Data



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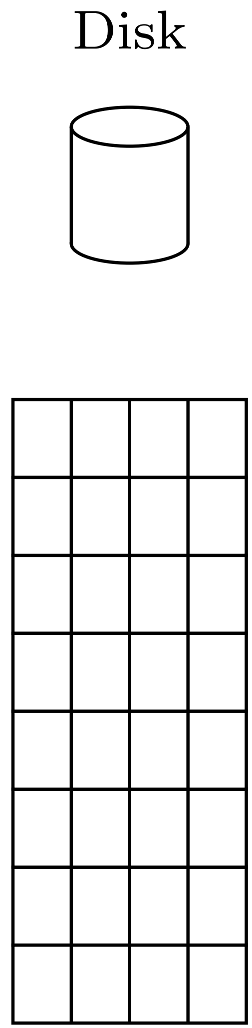

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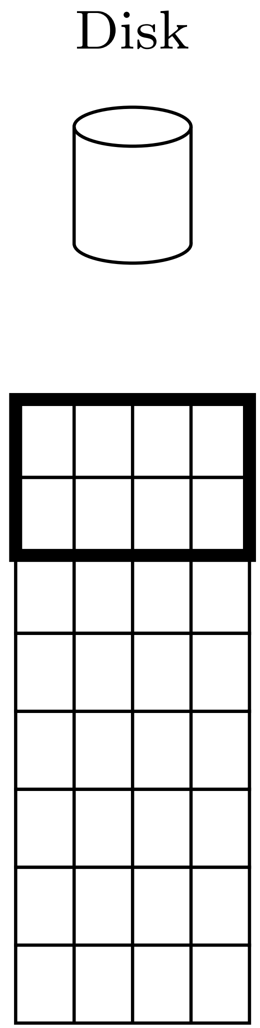

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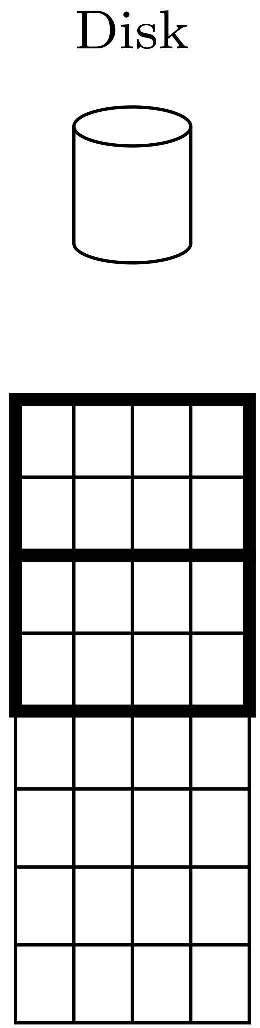

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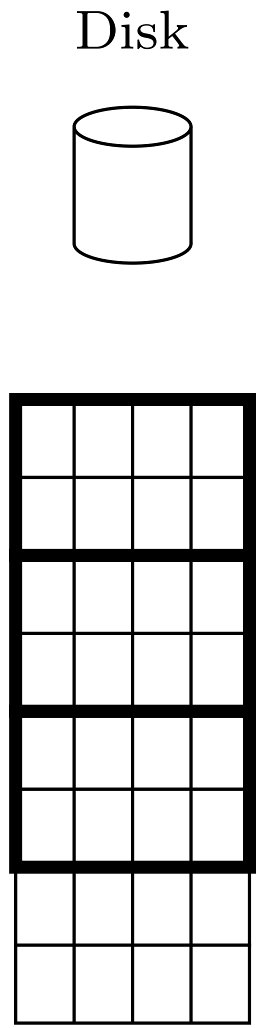

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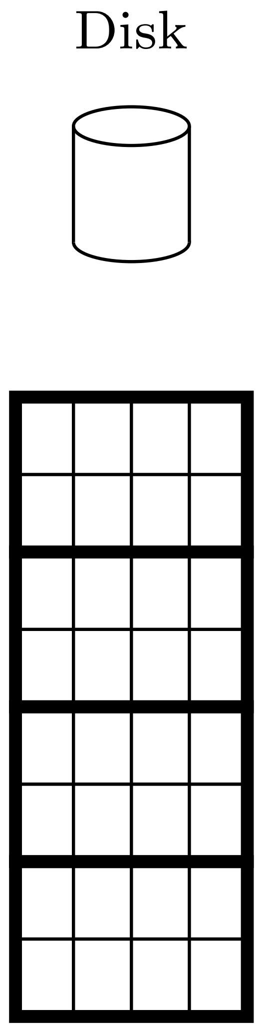

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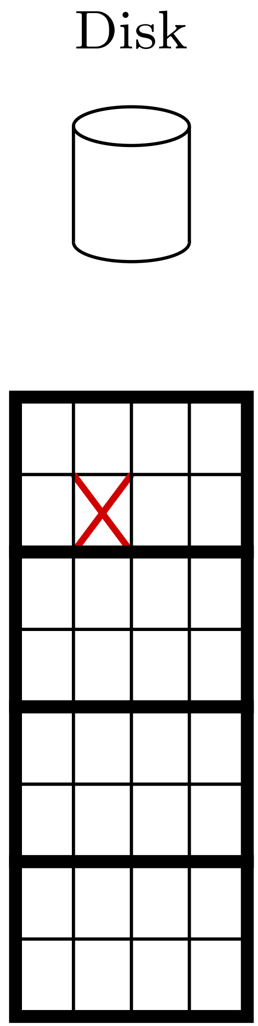

1 symbol error

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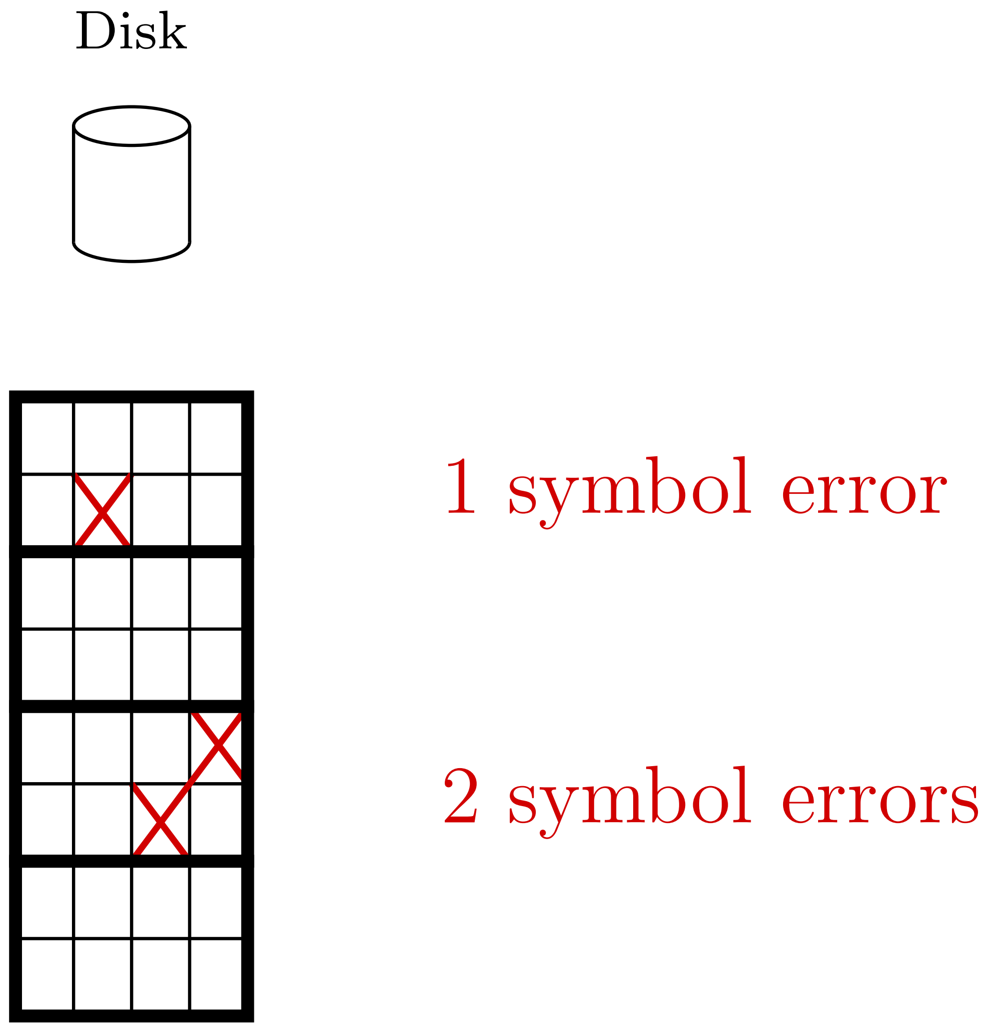

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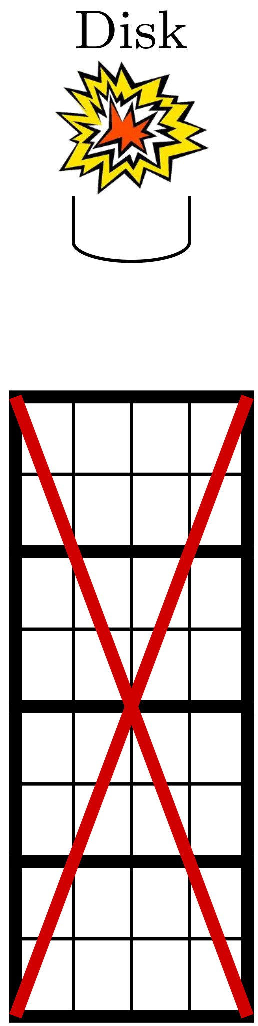
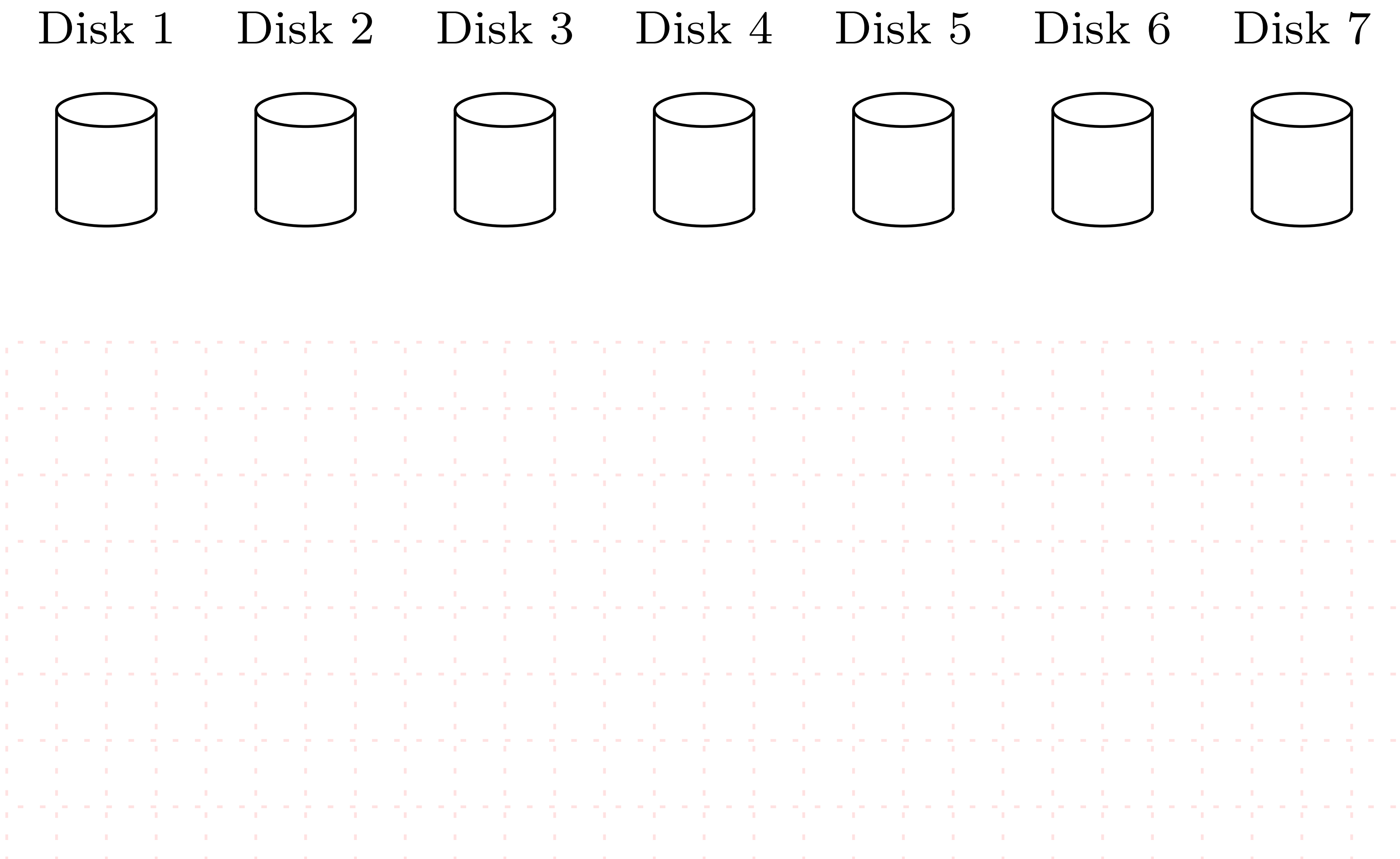


1 burst error

RAID:
Redundant Array of Inexpensive Disks



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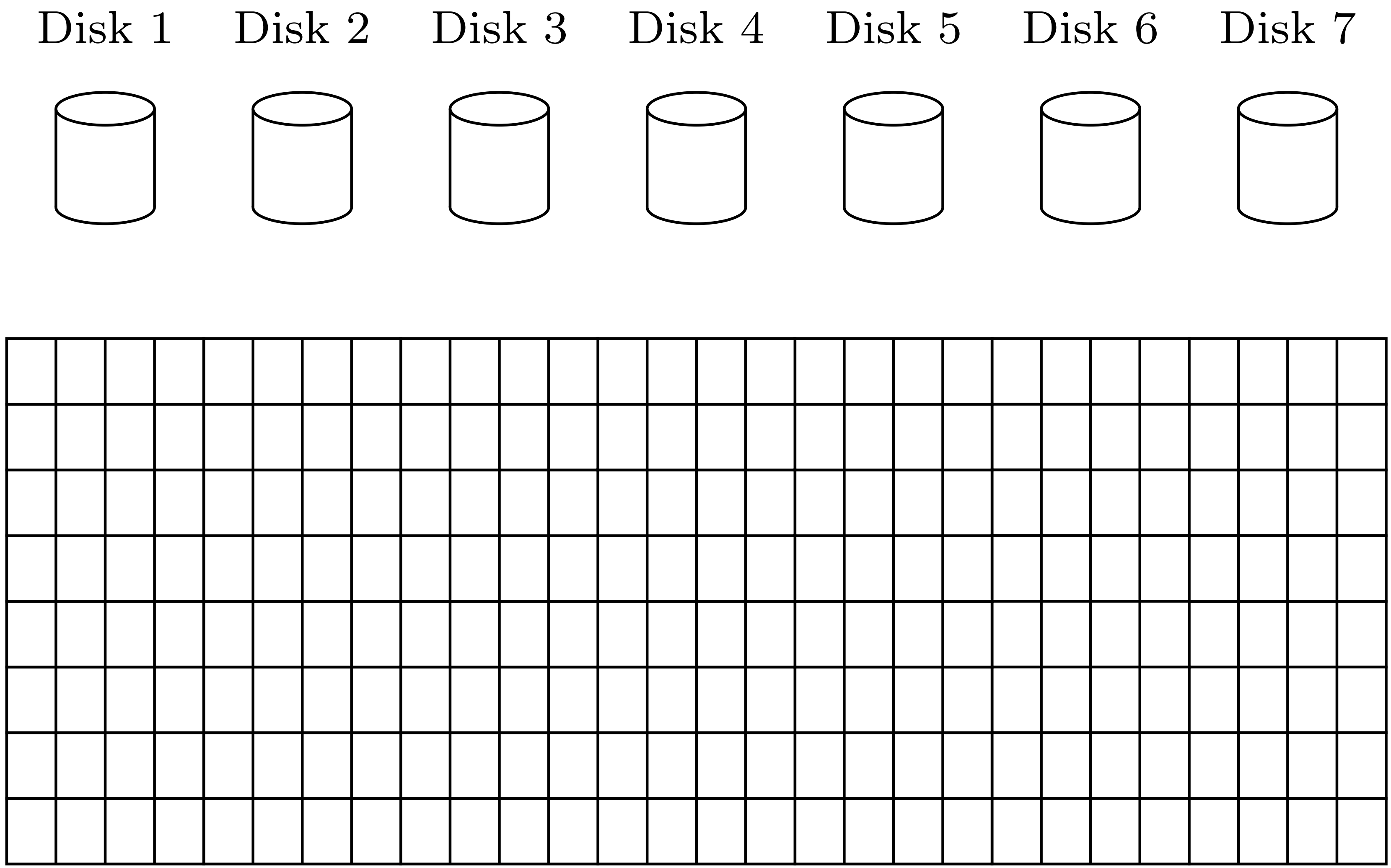

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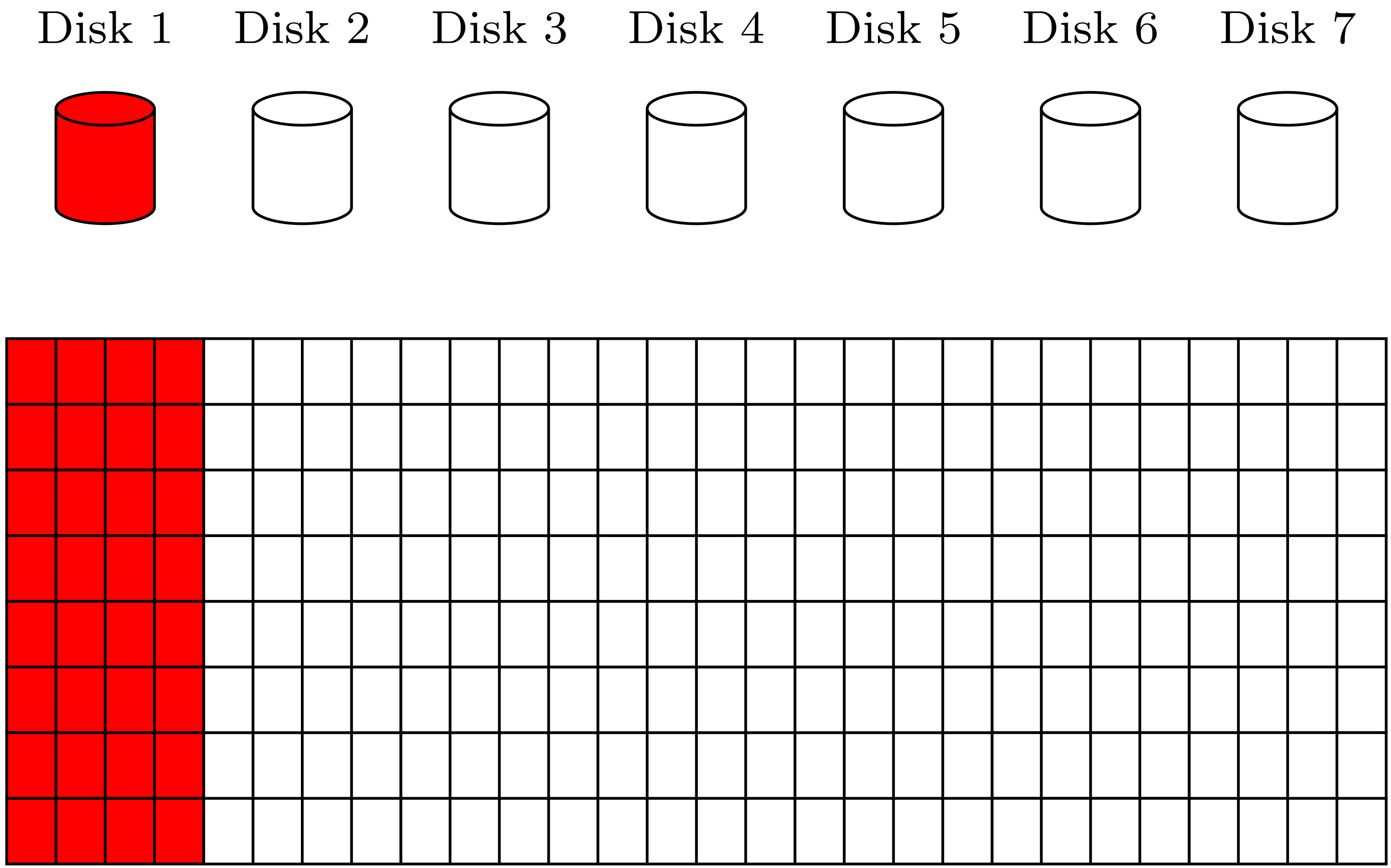

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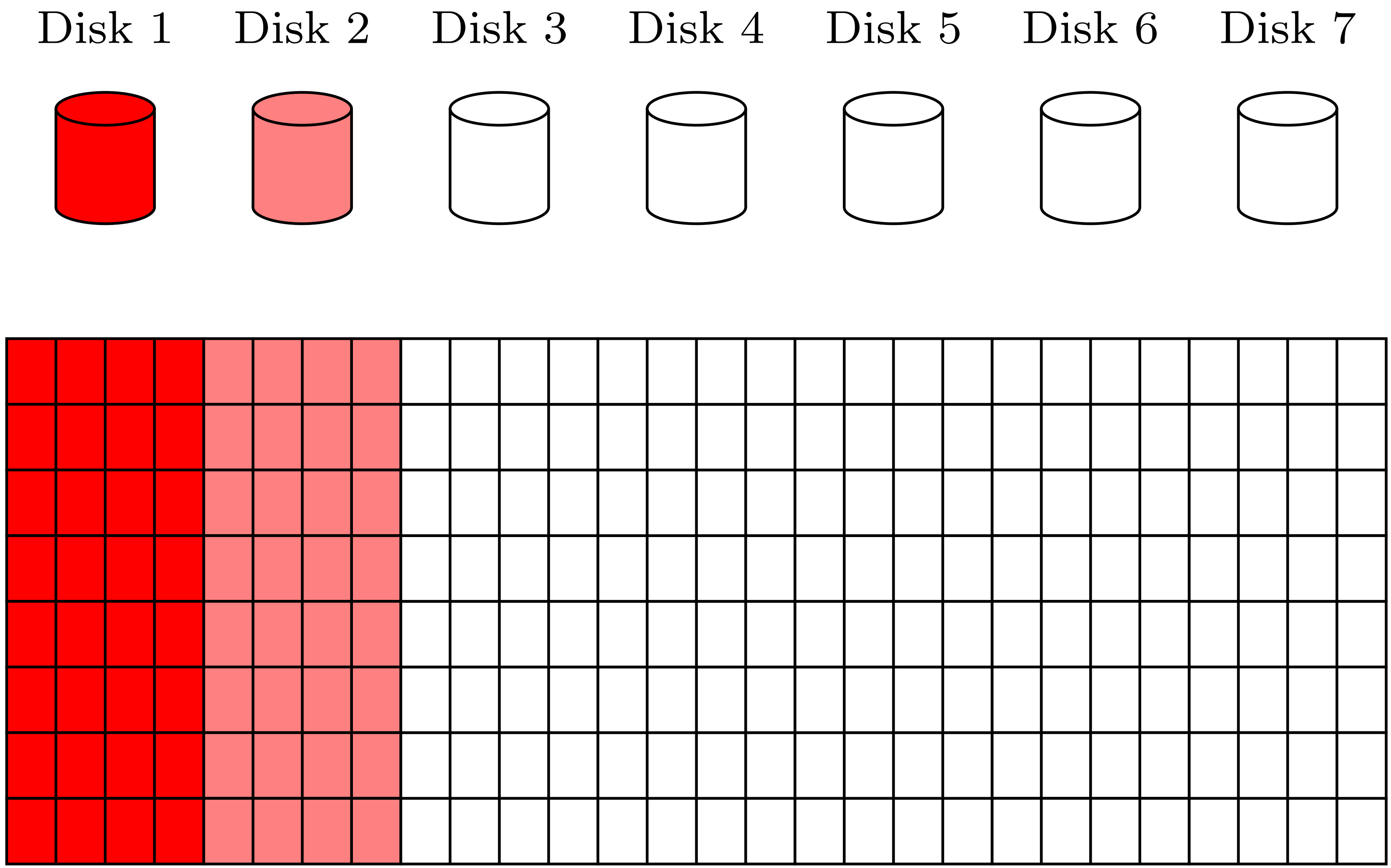

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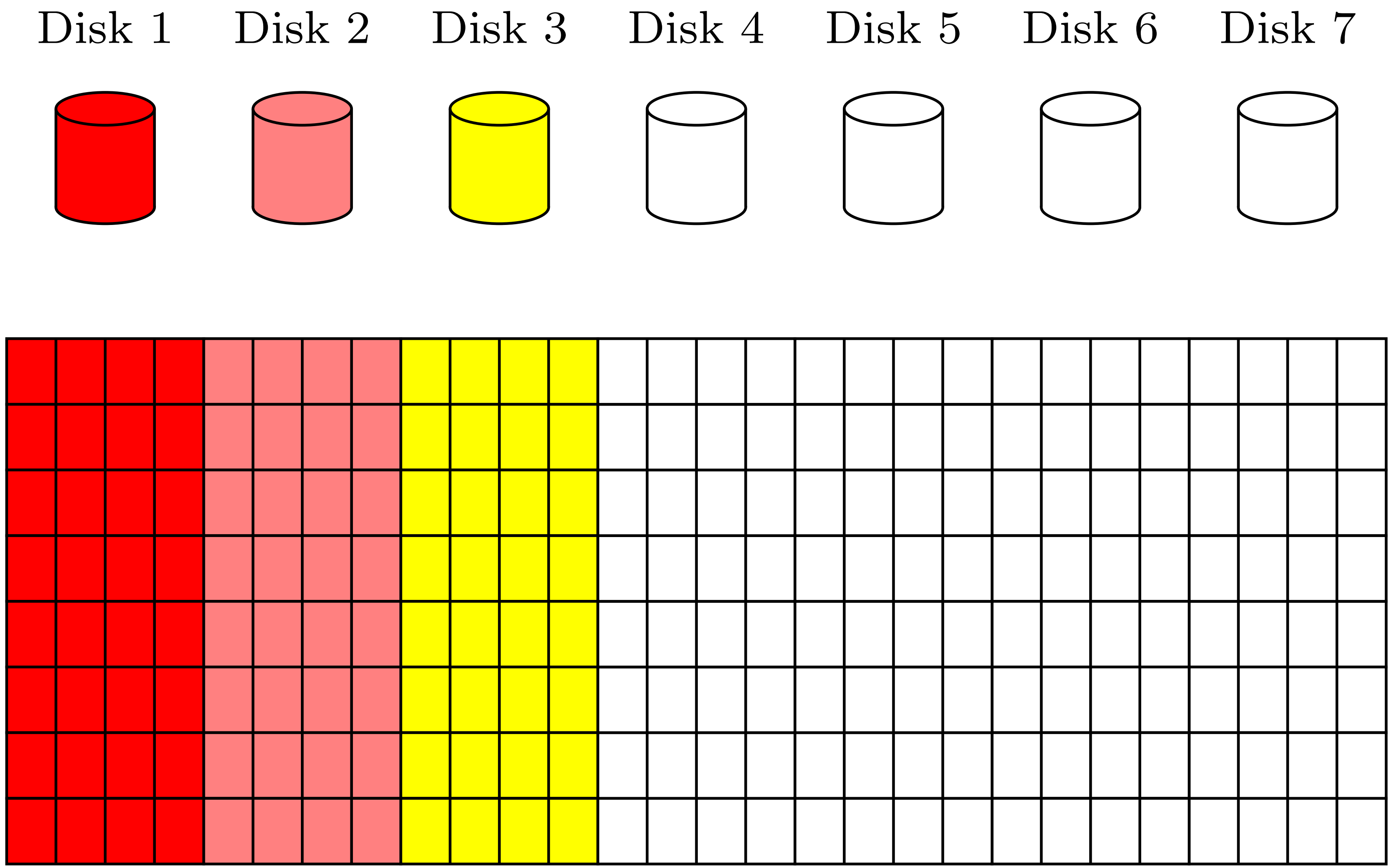

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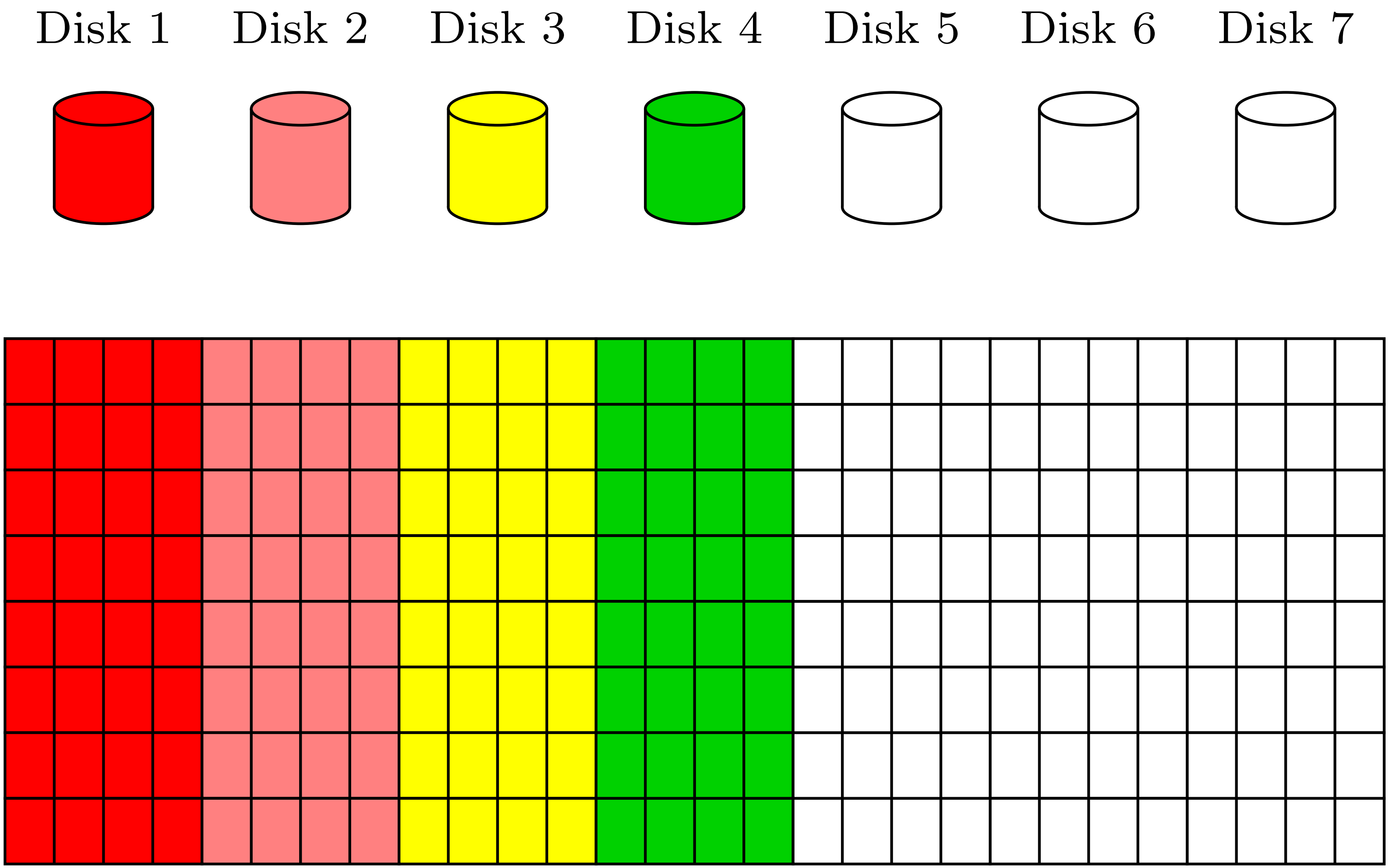

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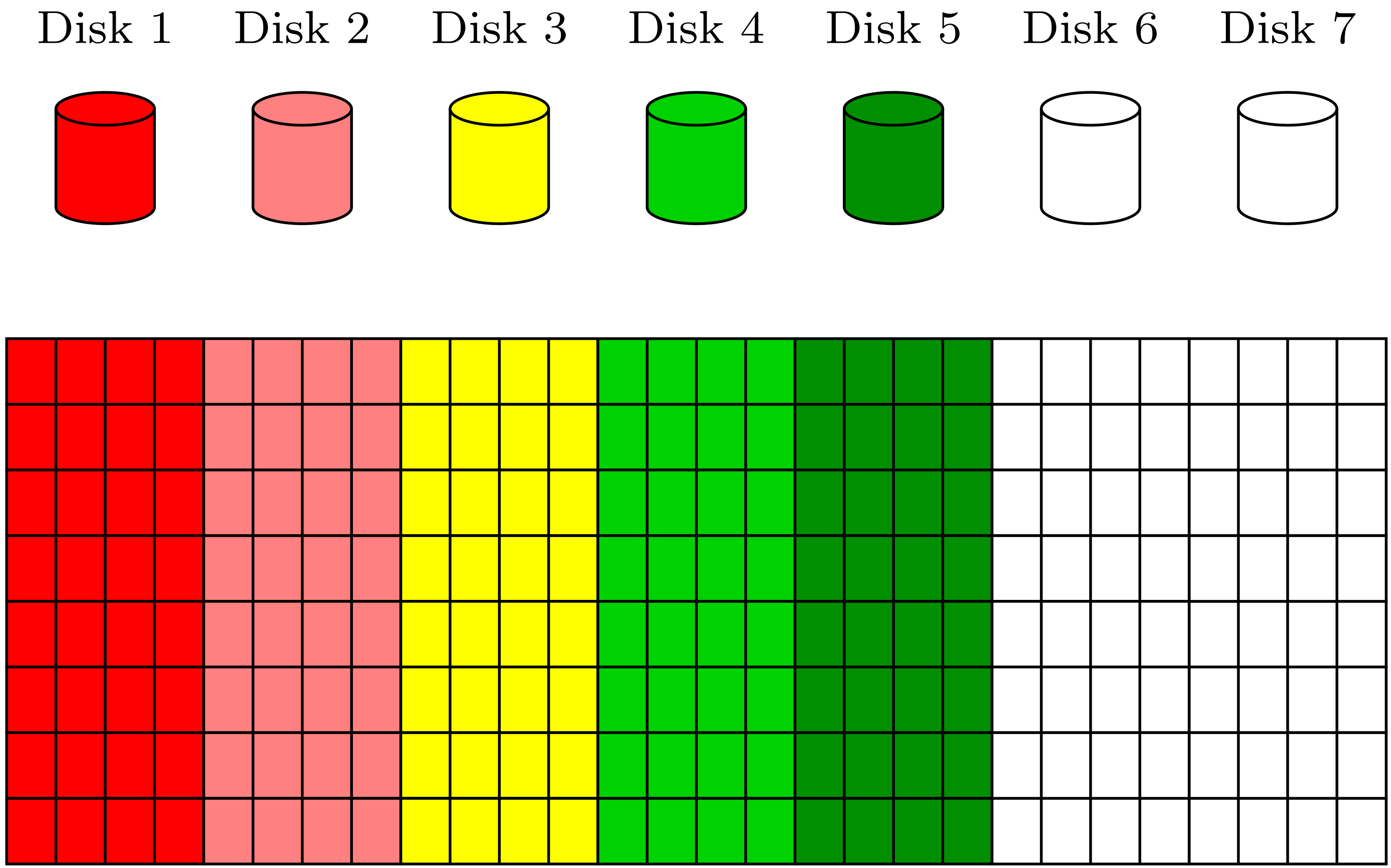

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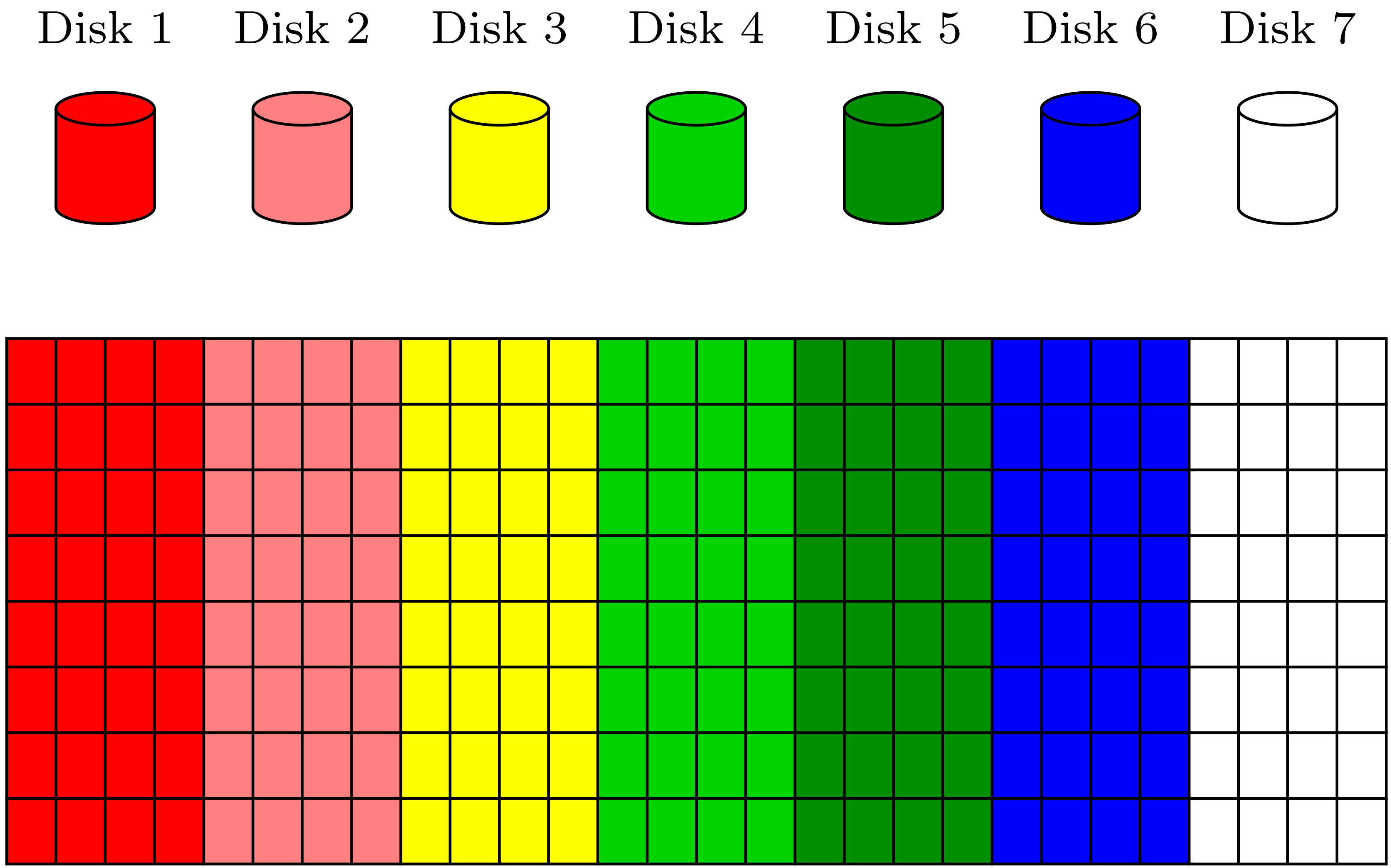

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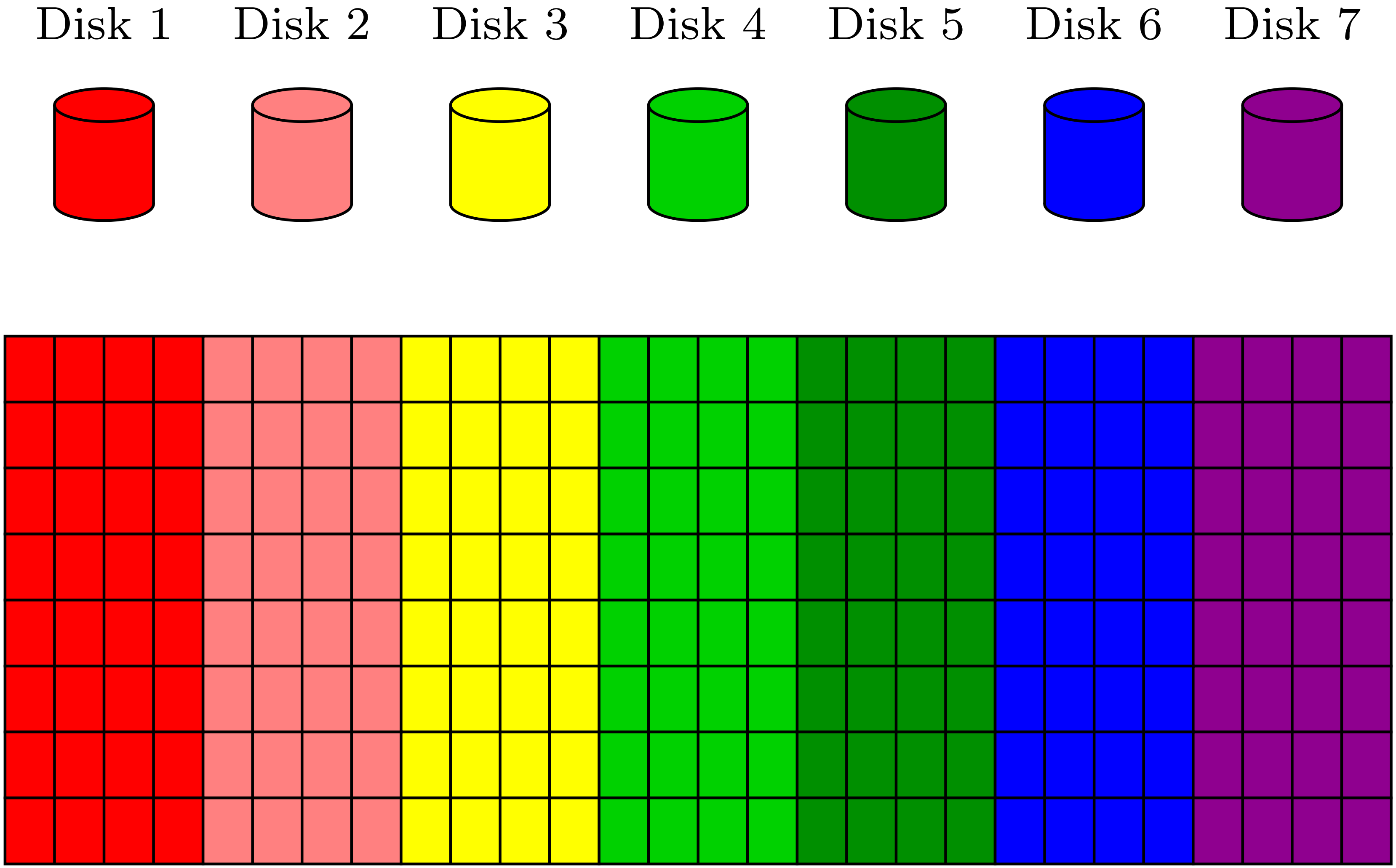

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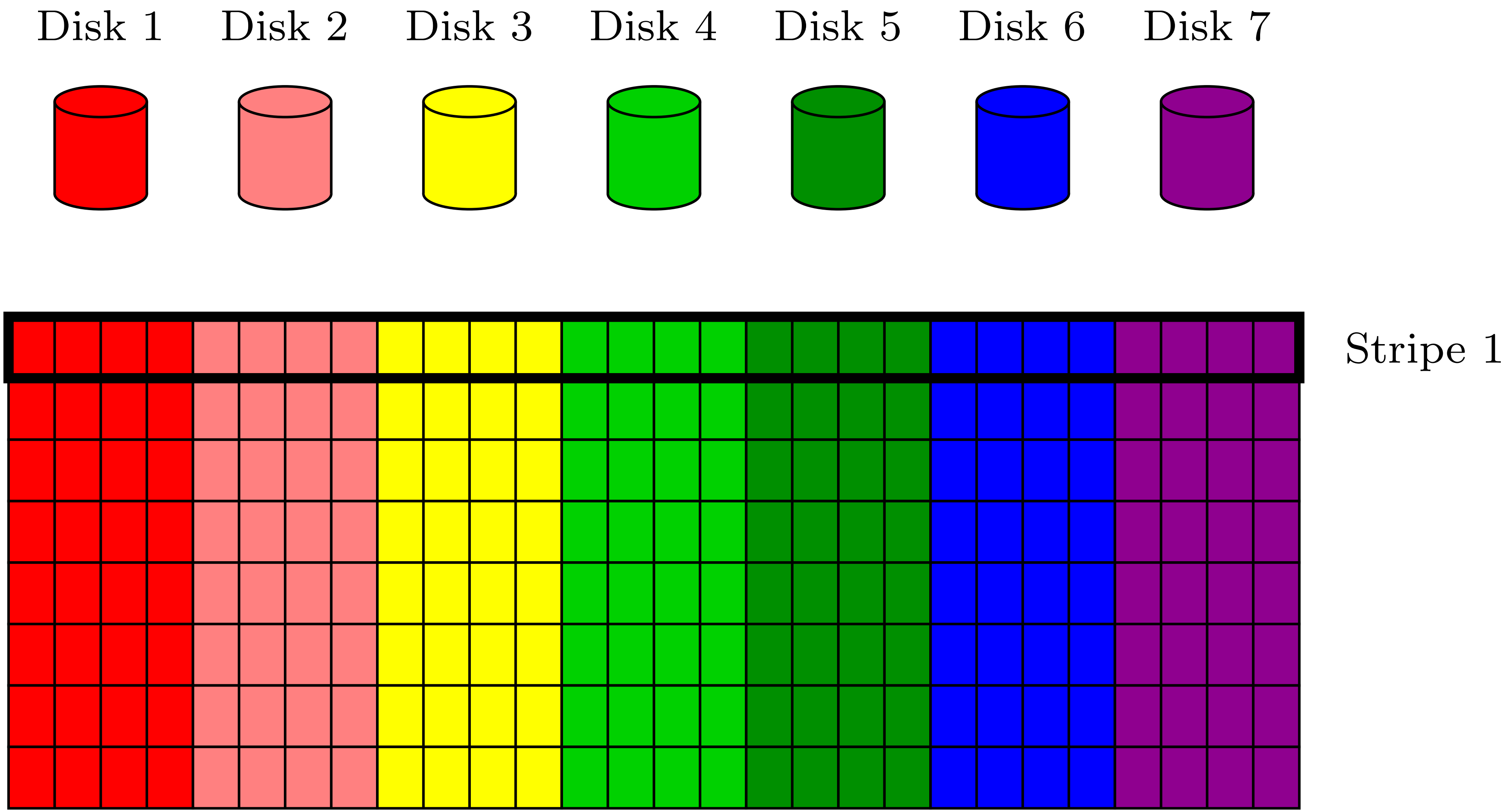

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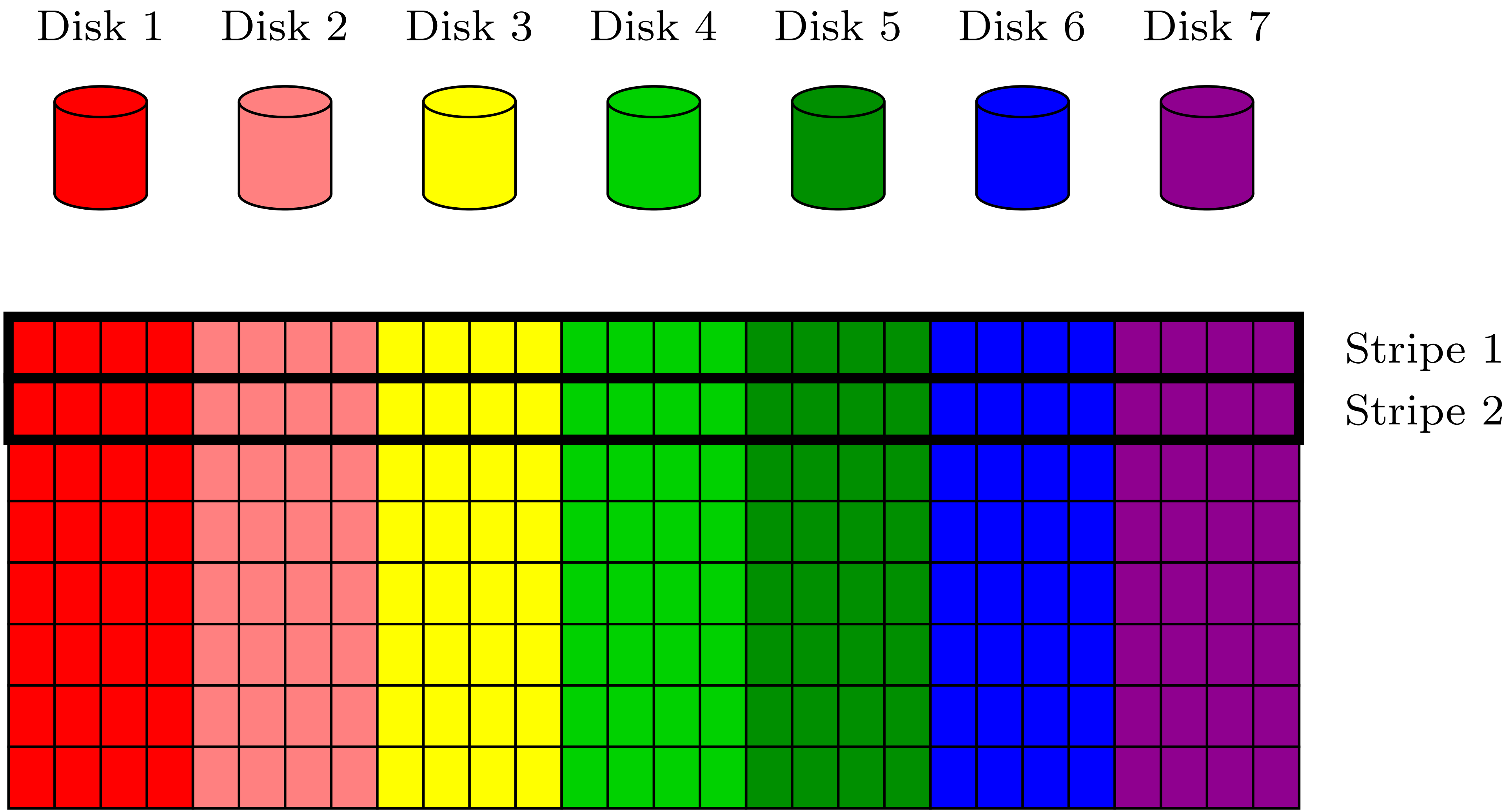

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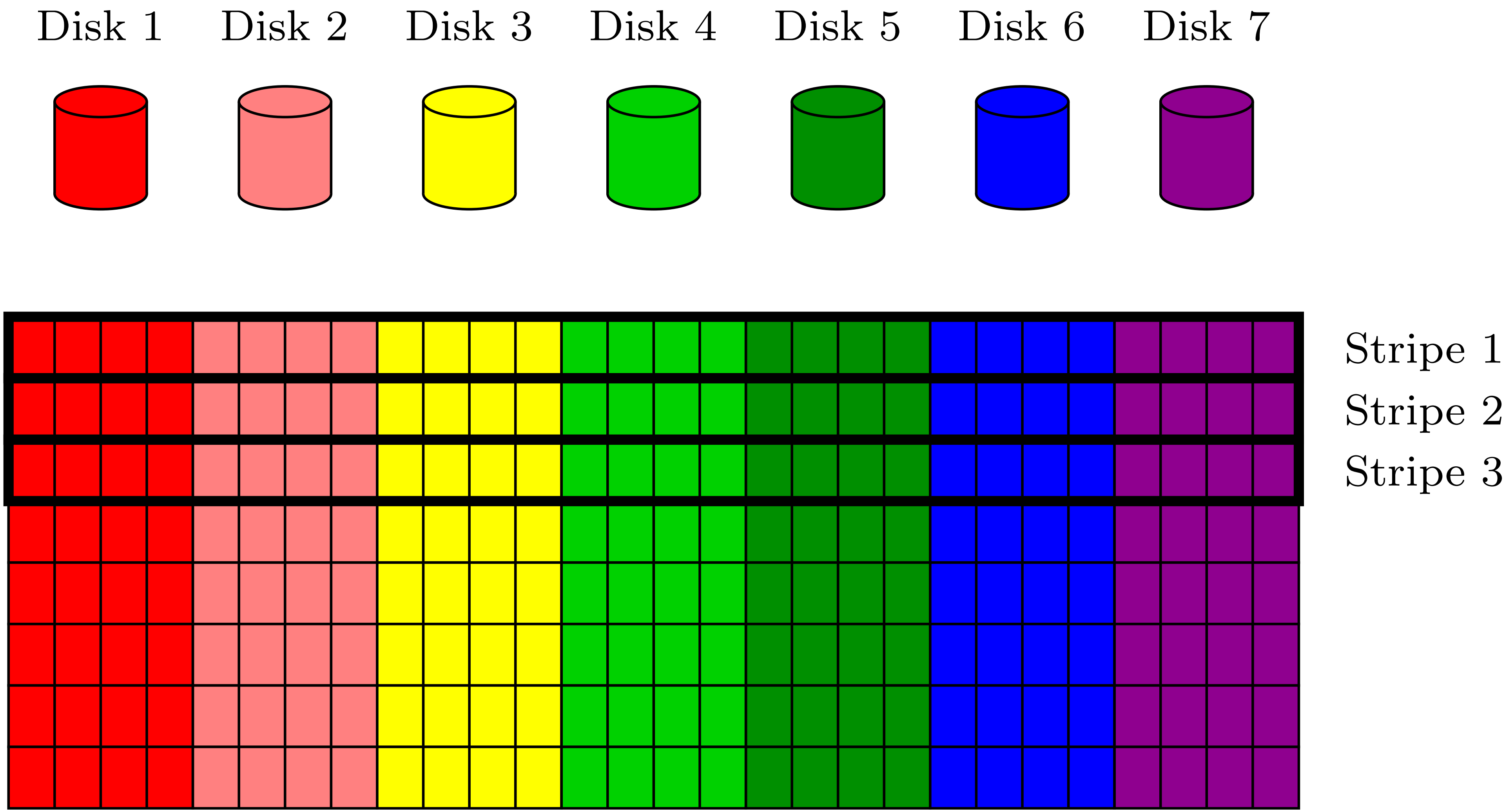

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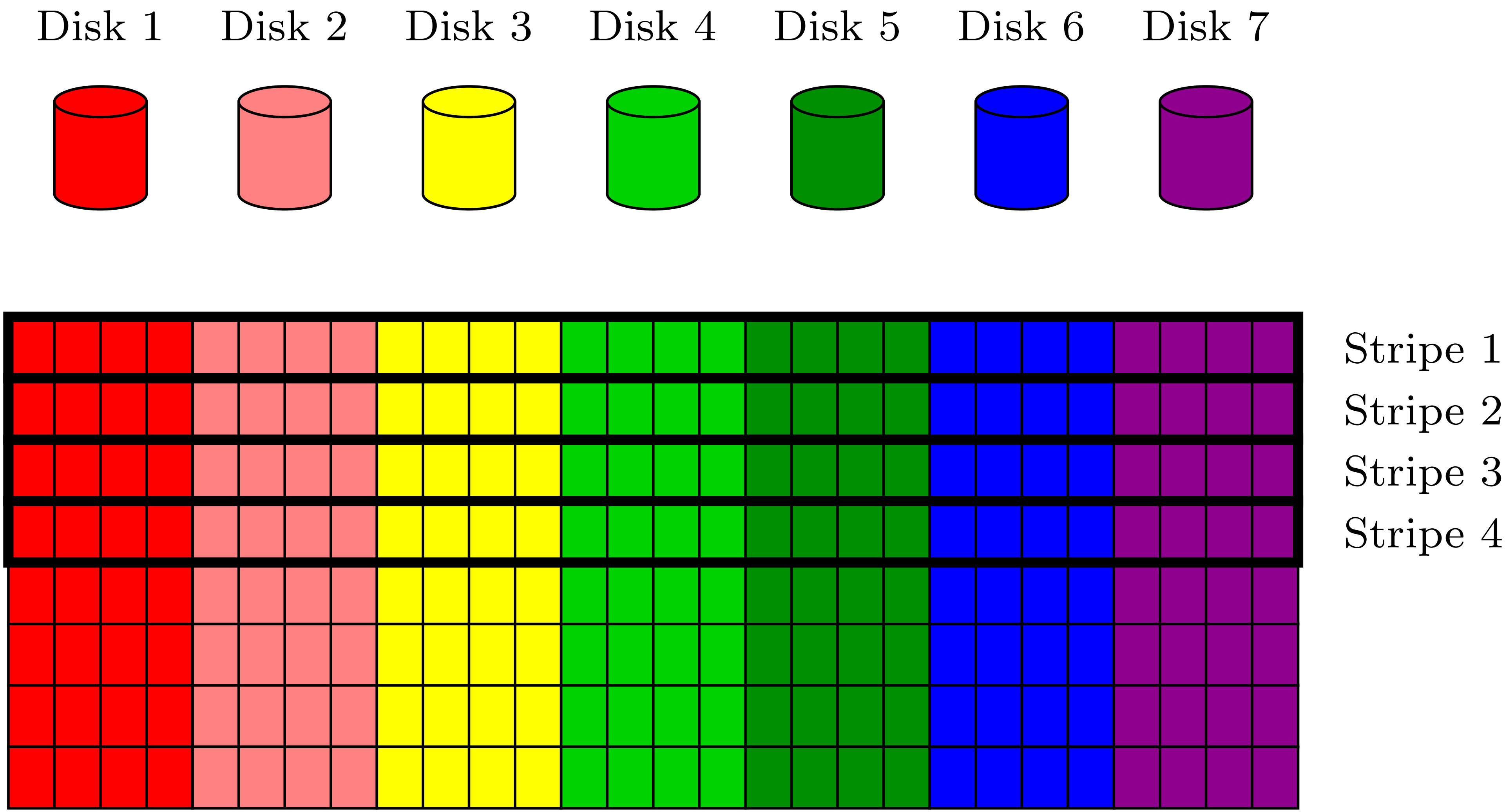

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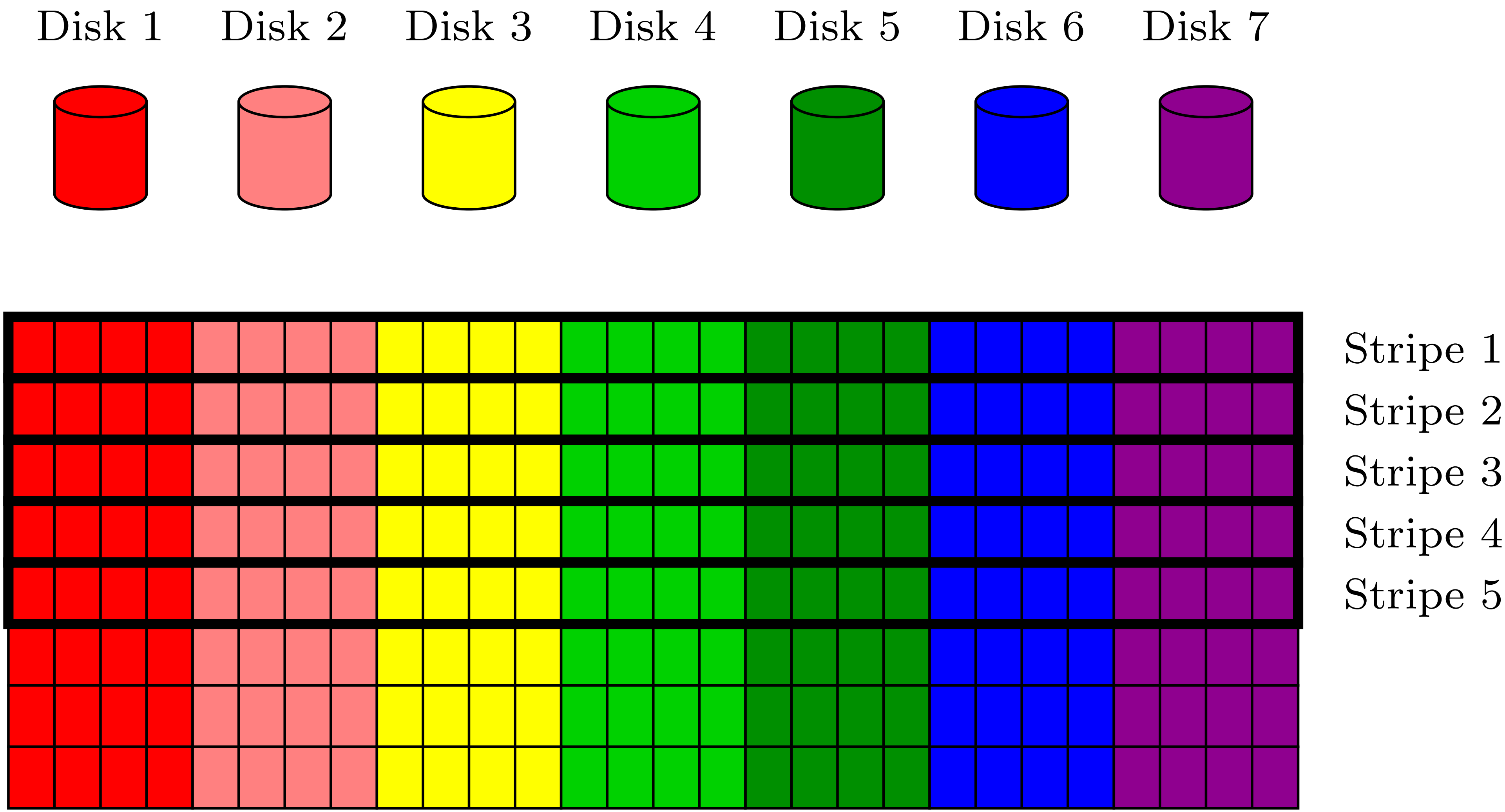

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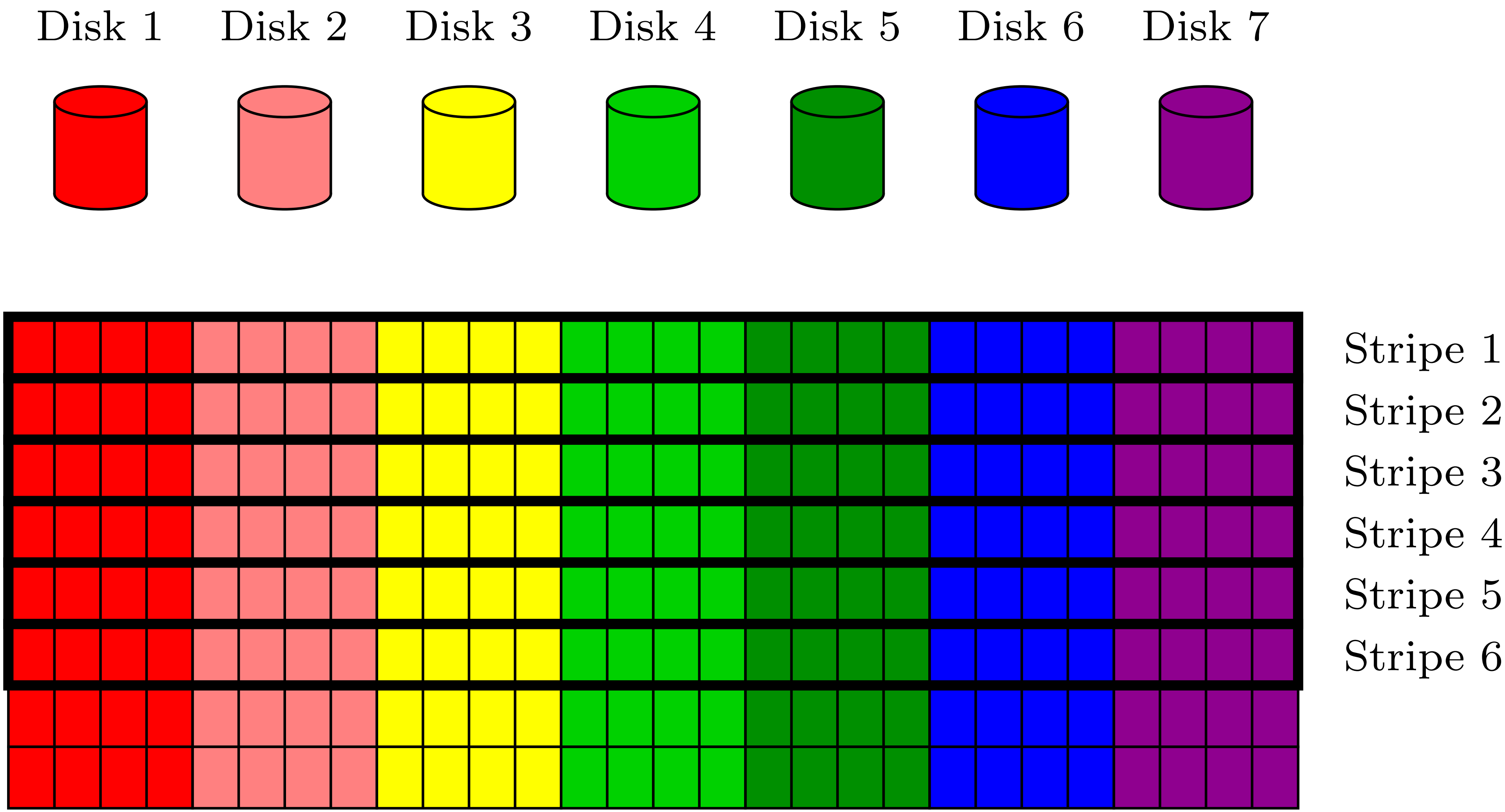

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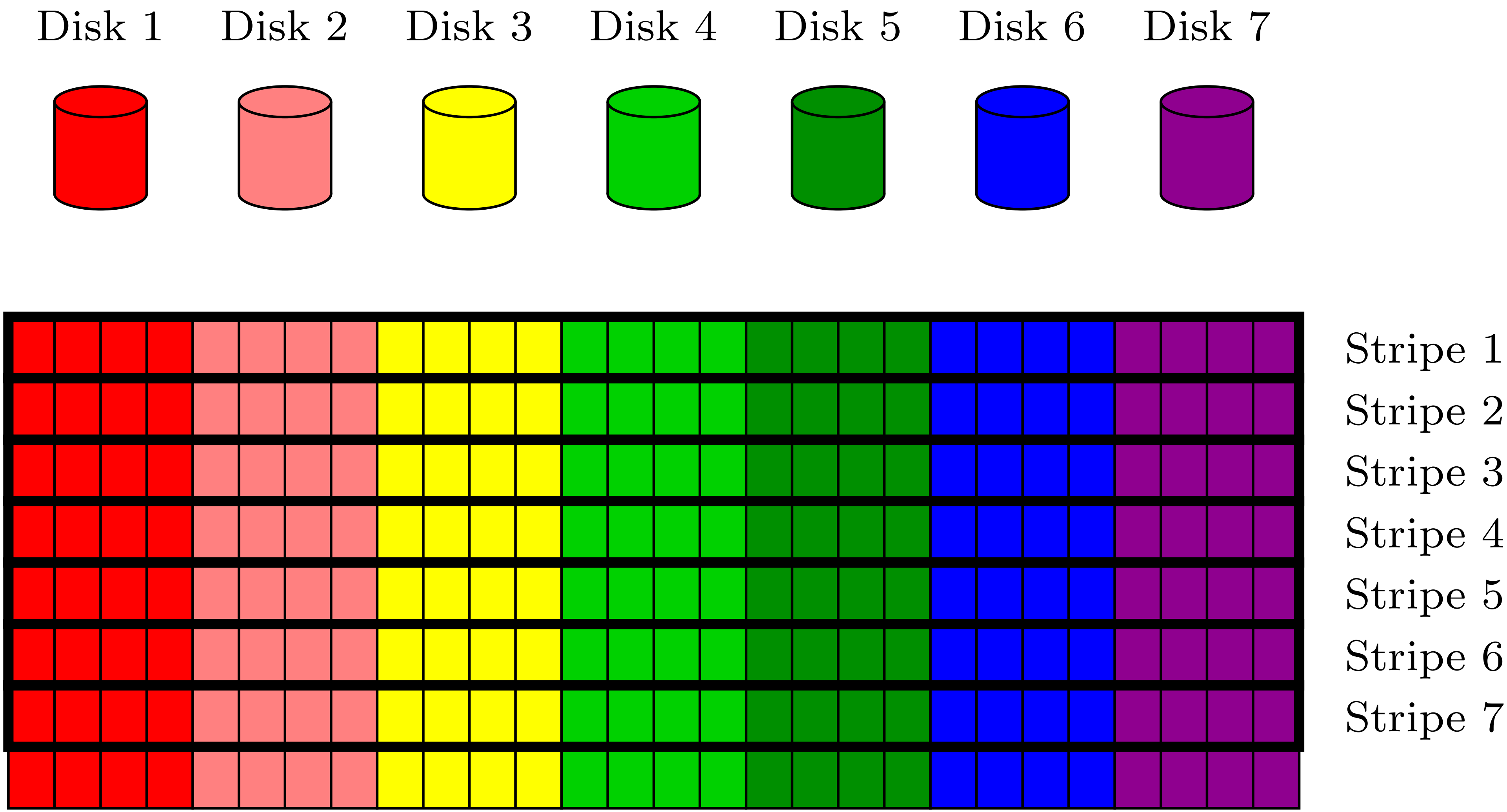

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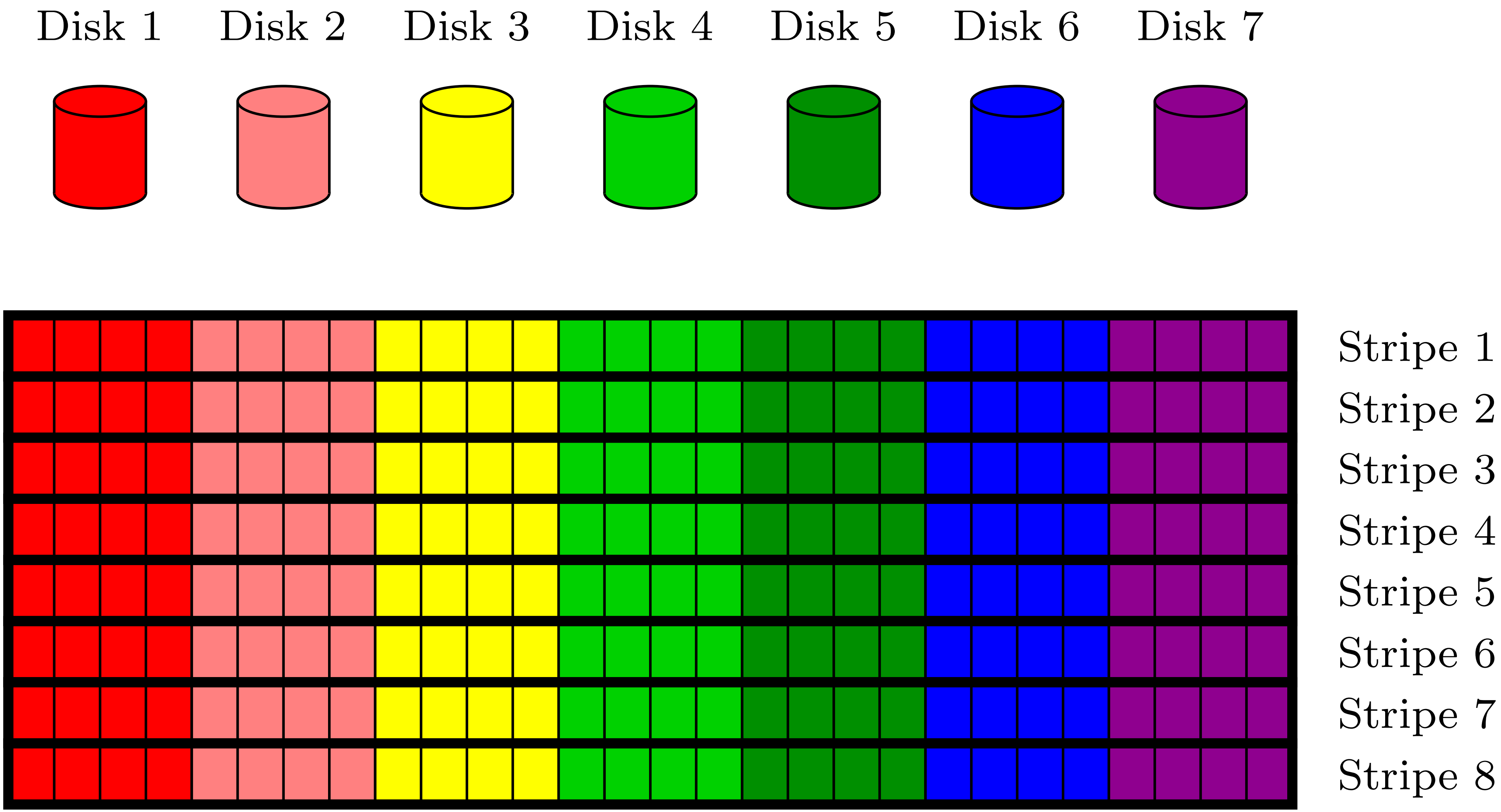

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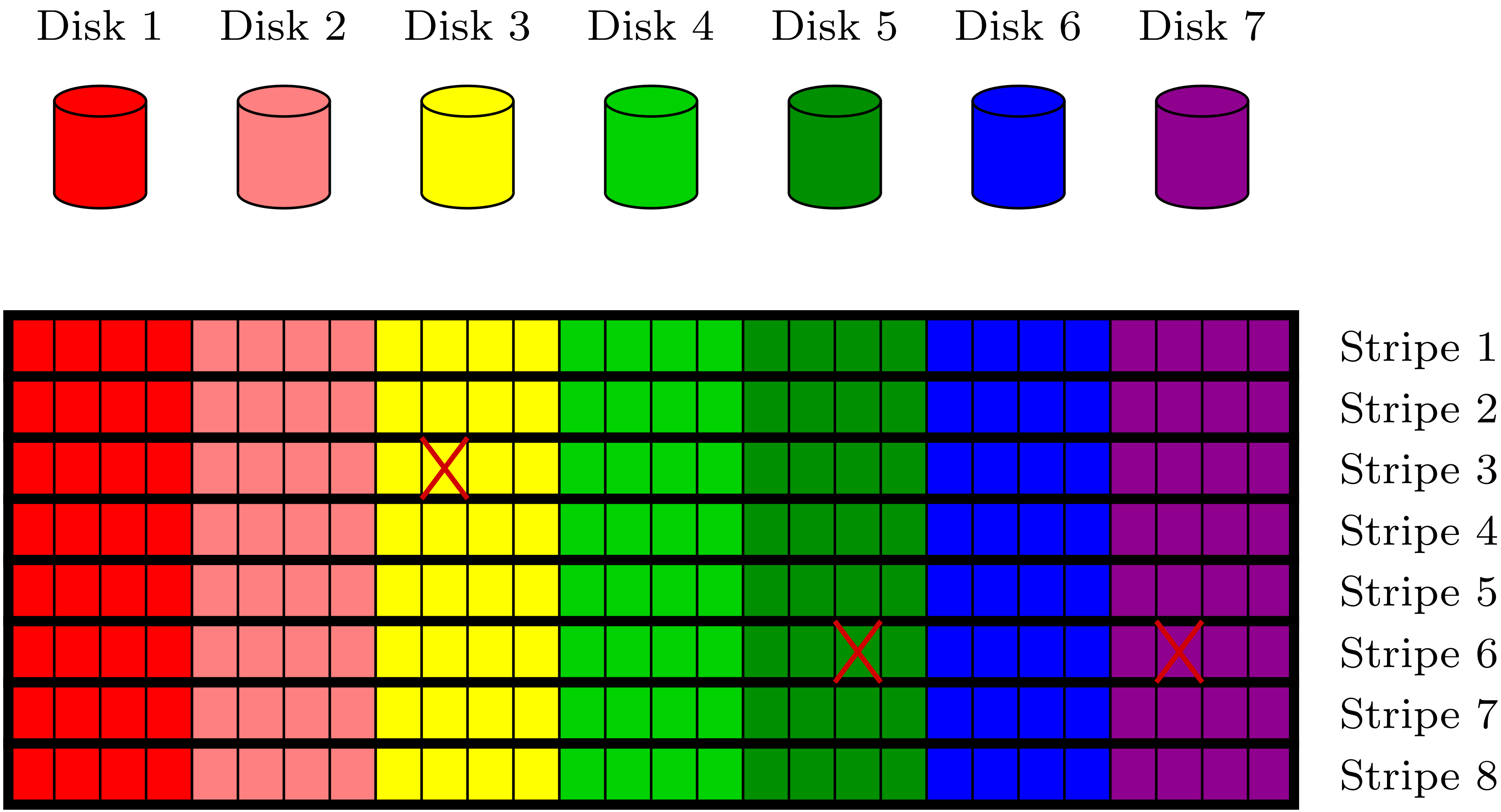

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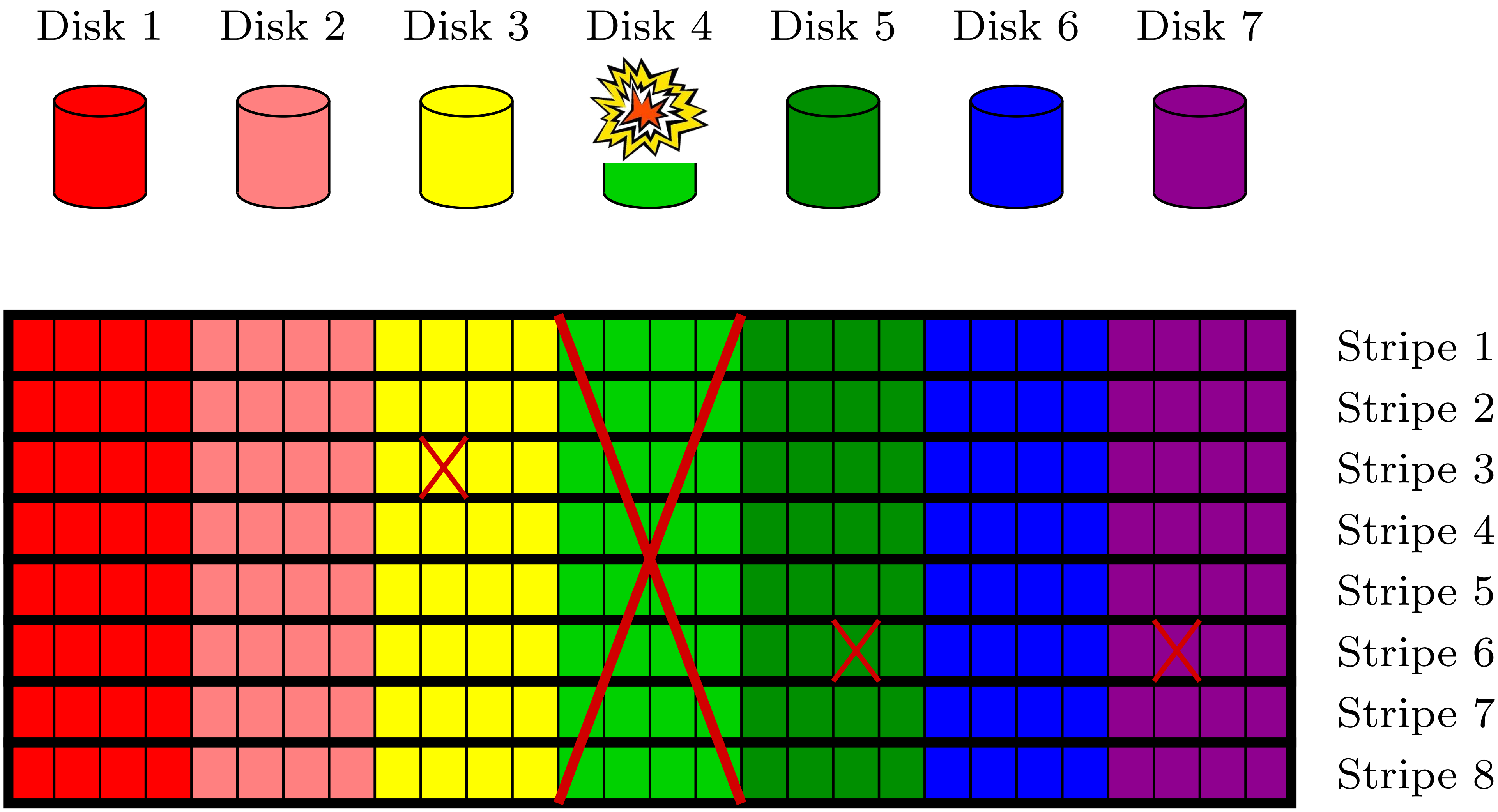

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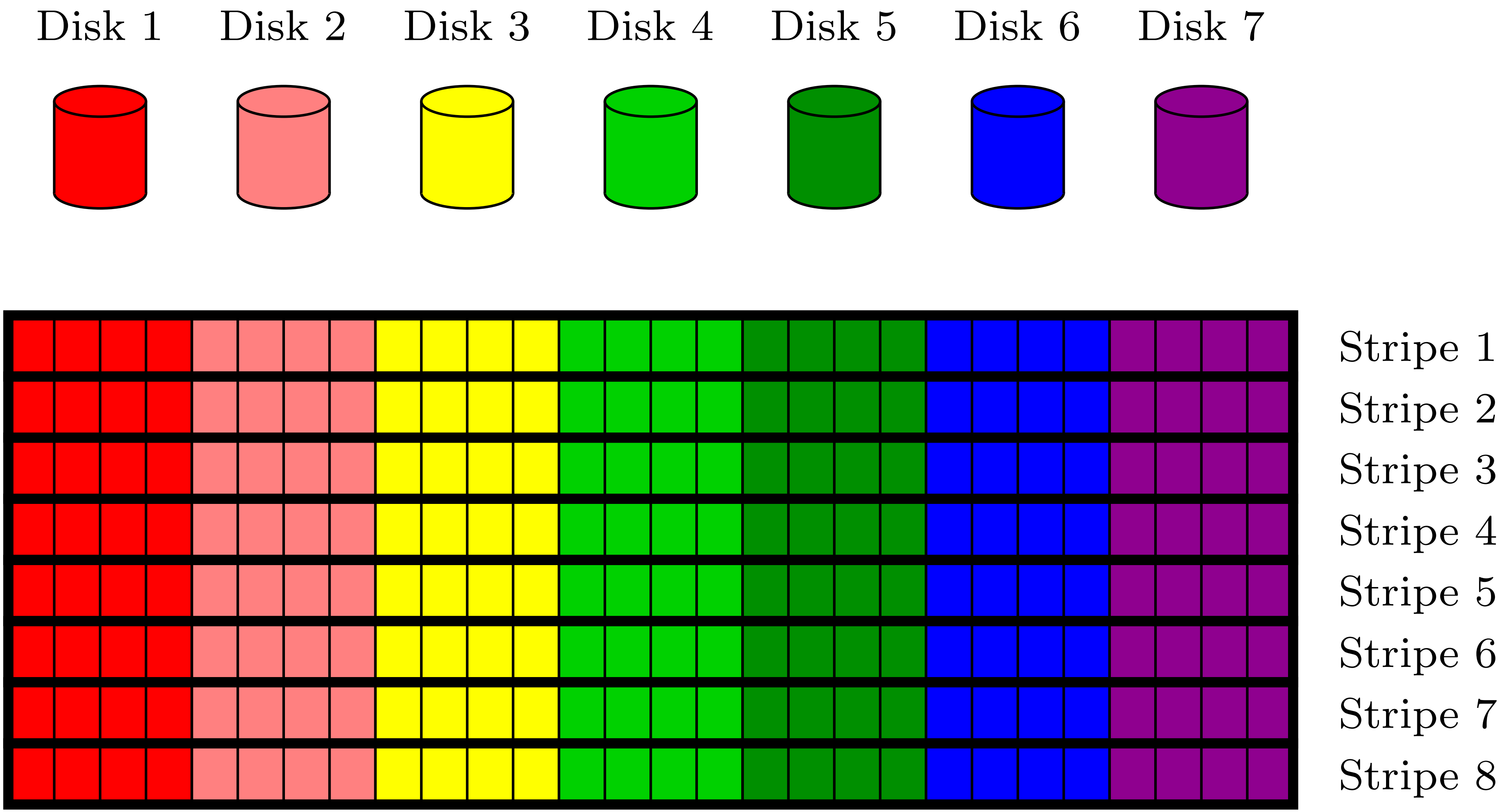
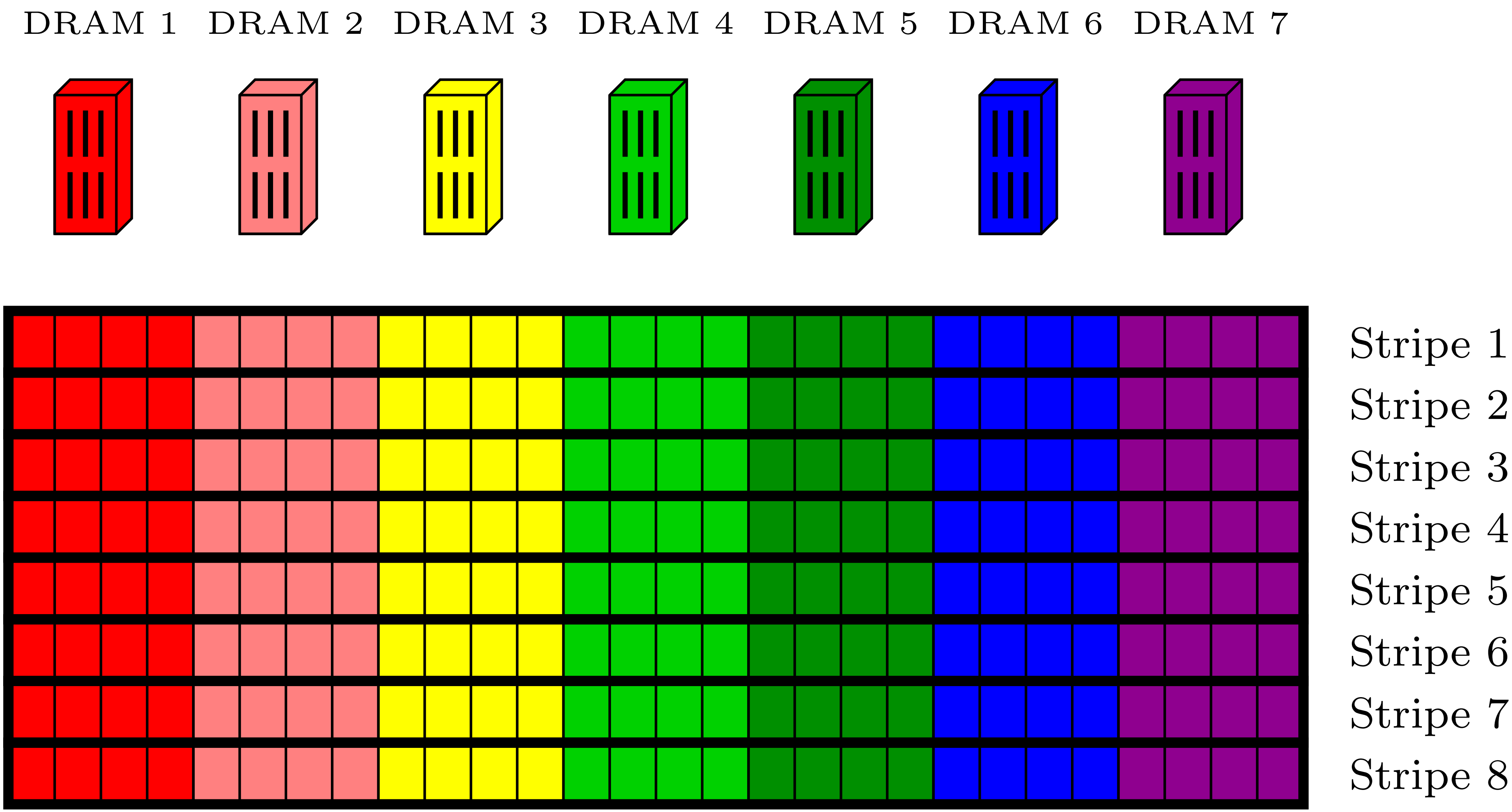


Similar Principle with DRAMs



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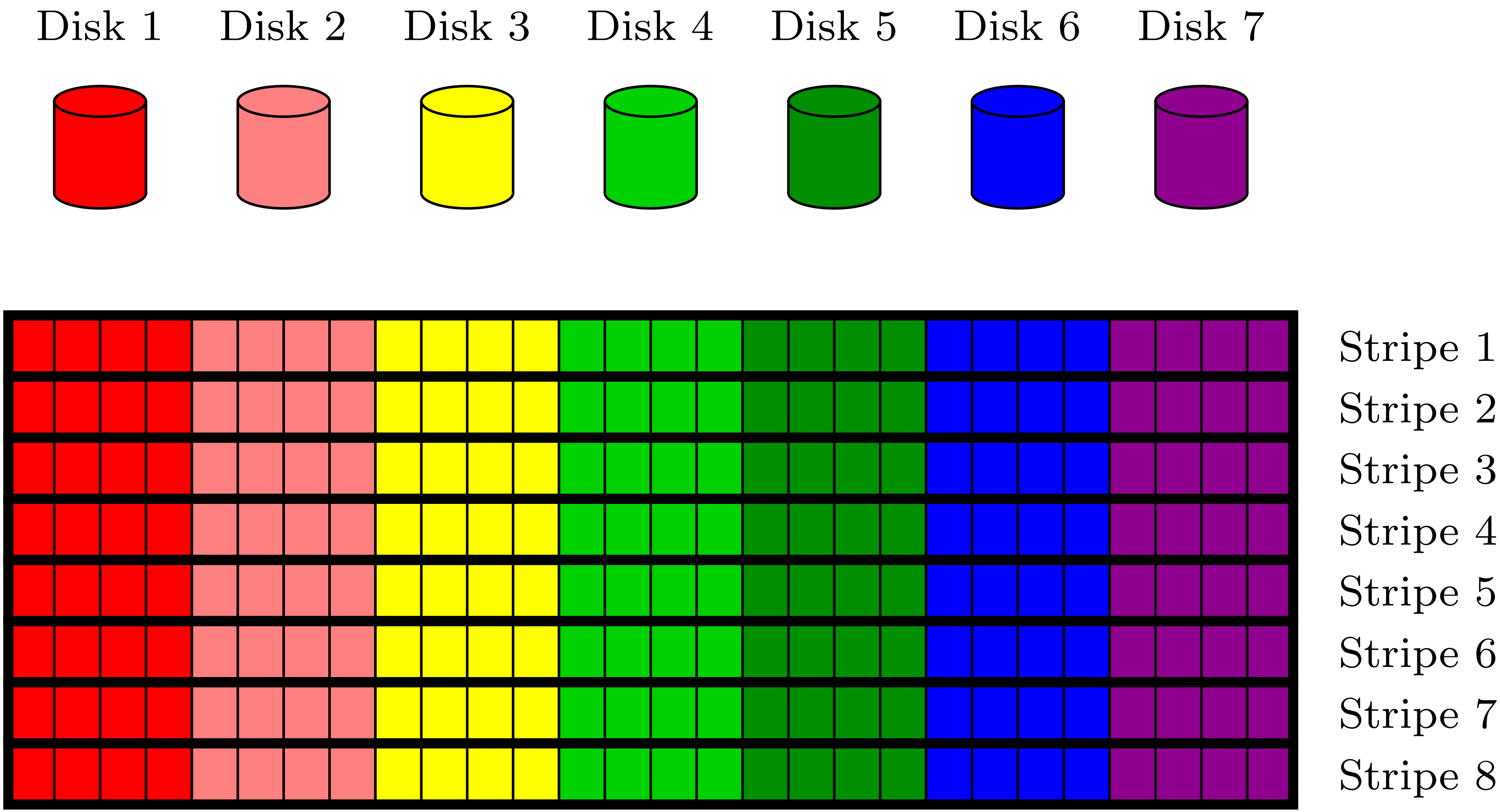
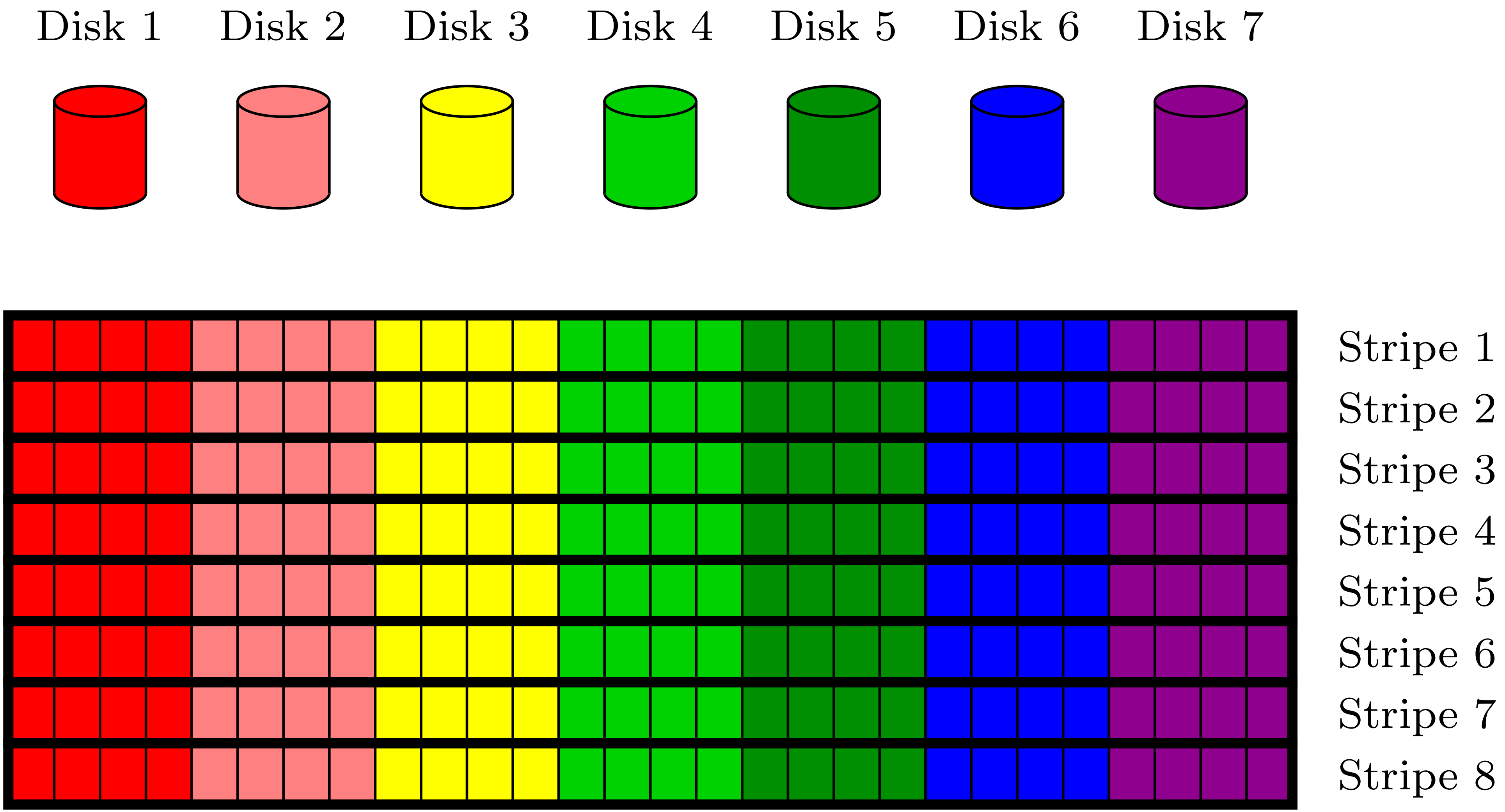


Error / erasure model

Error / Erasure Model



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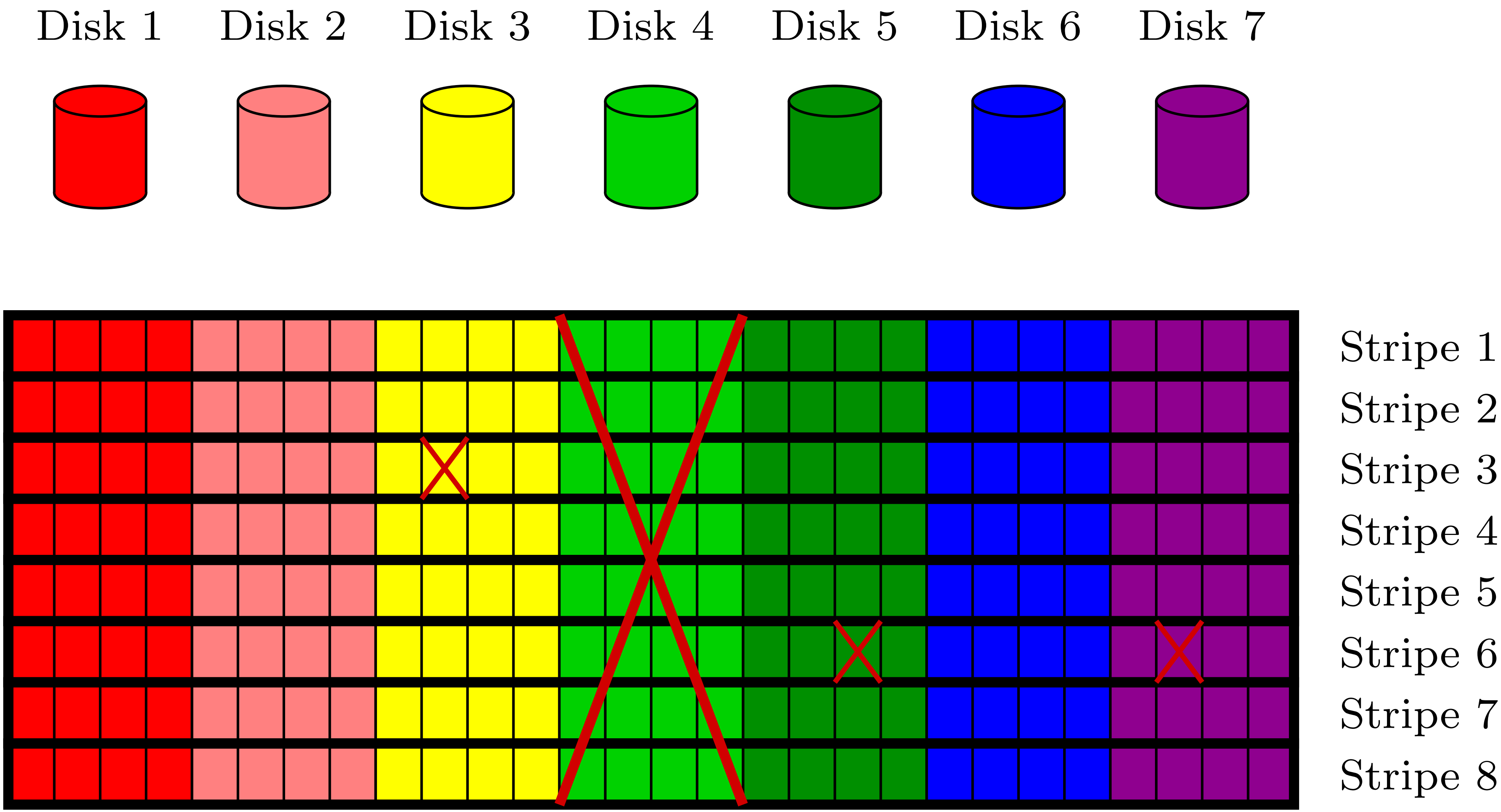

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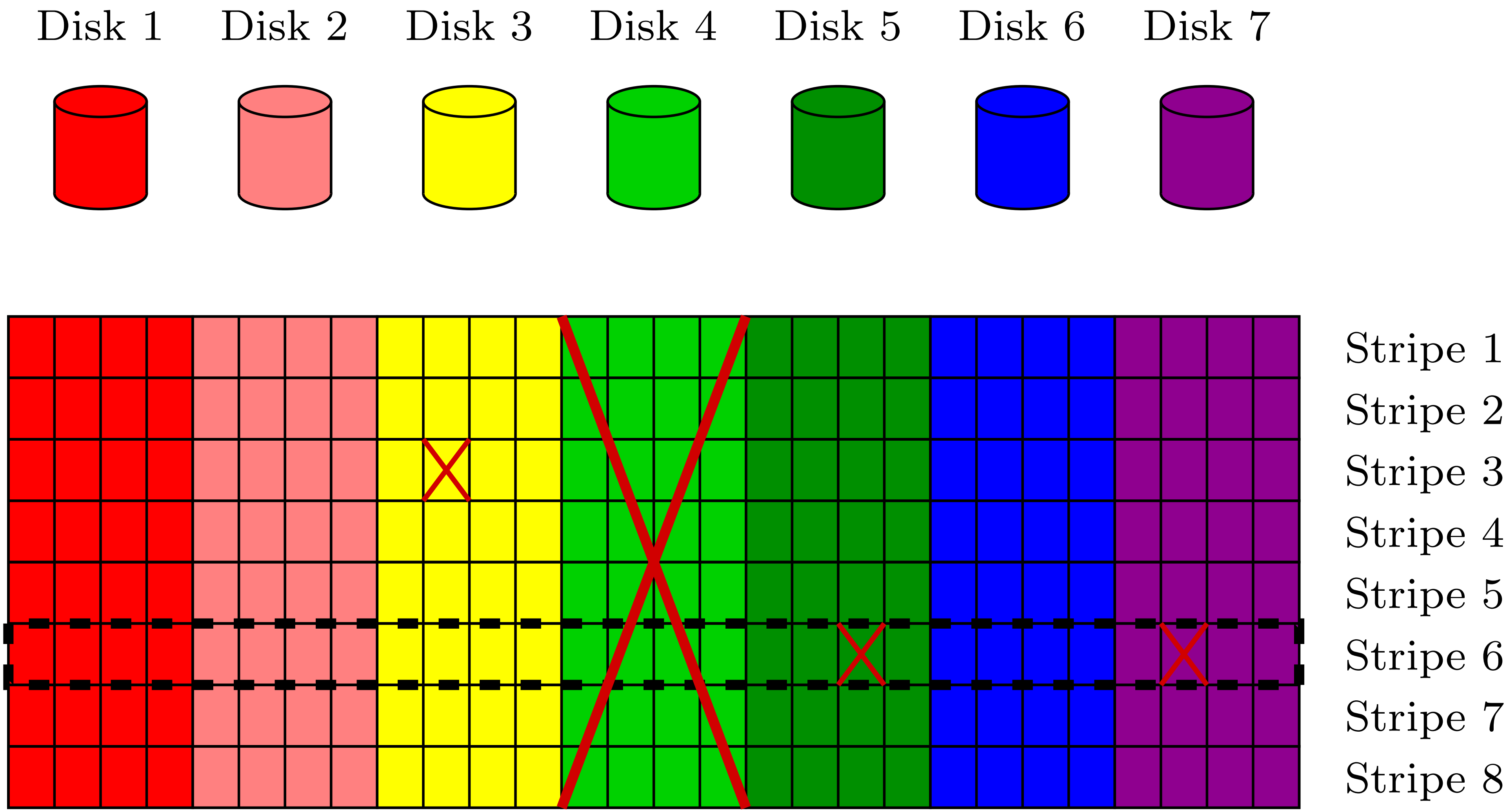

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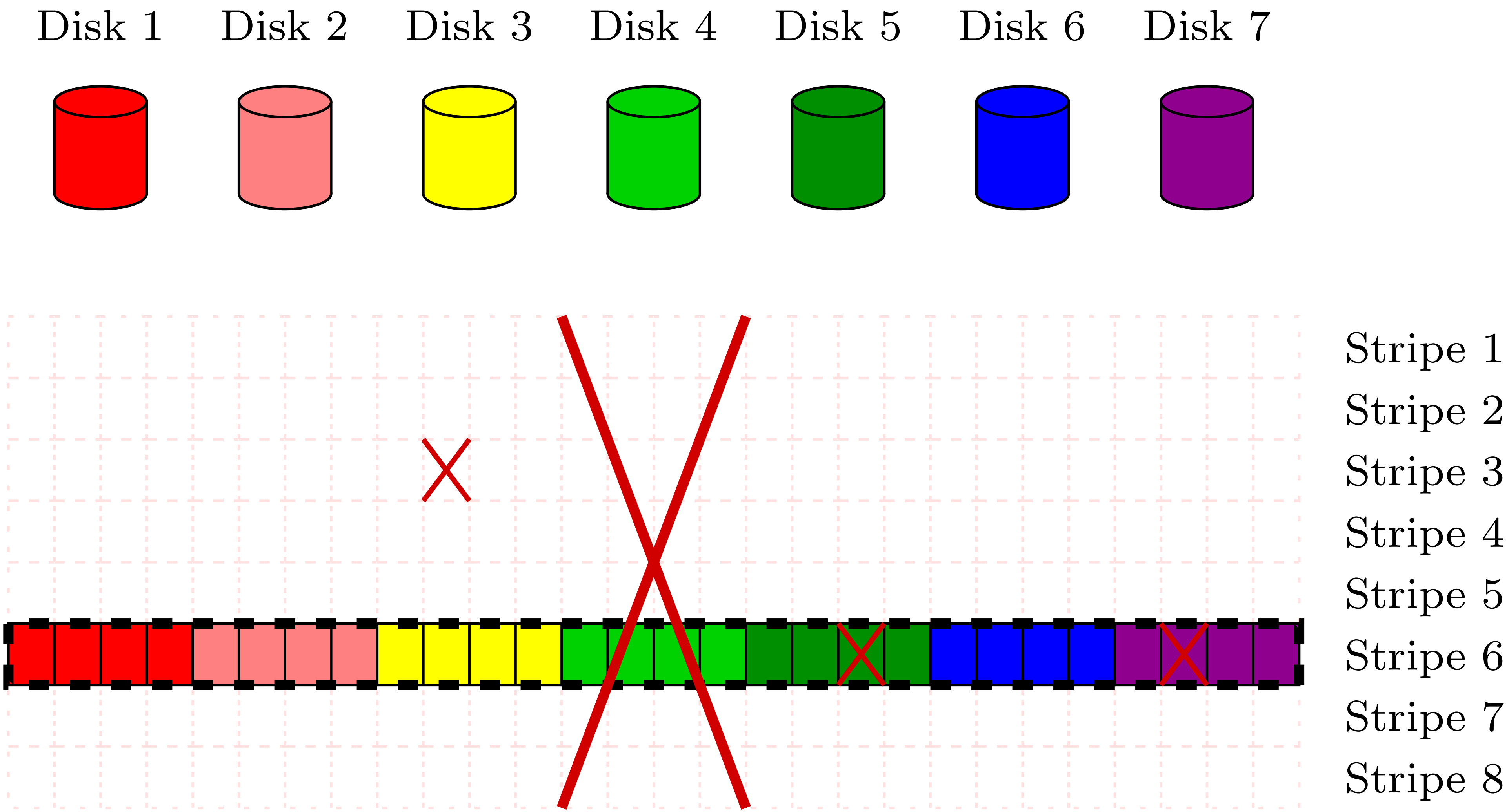

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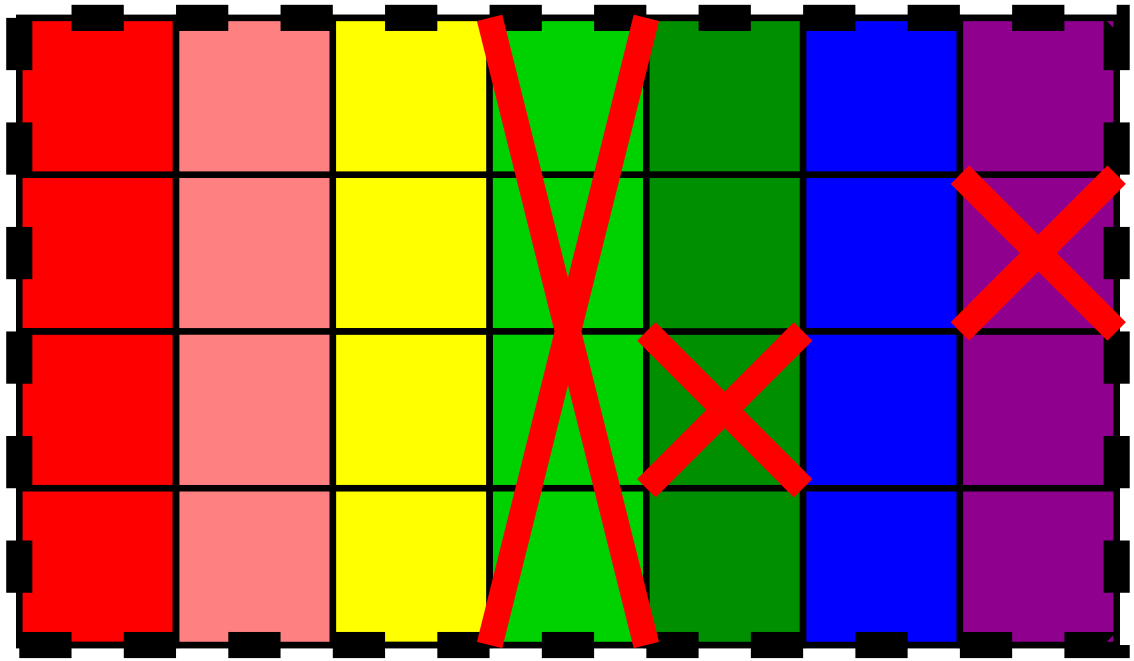

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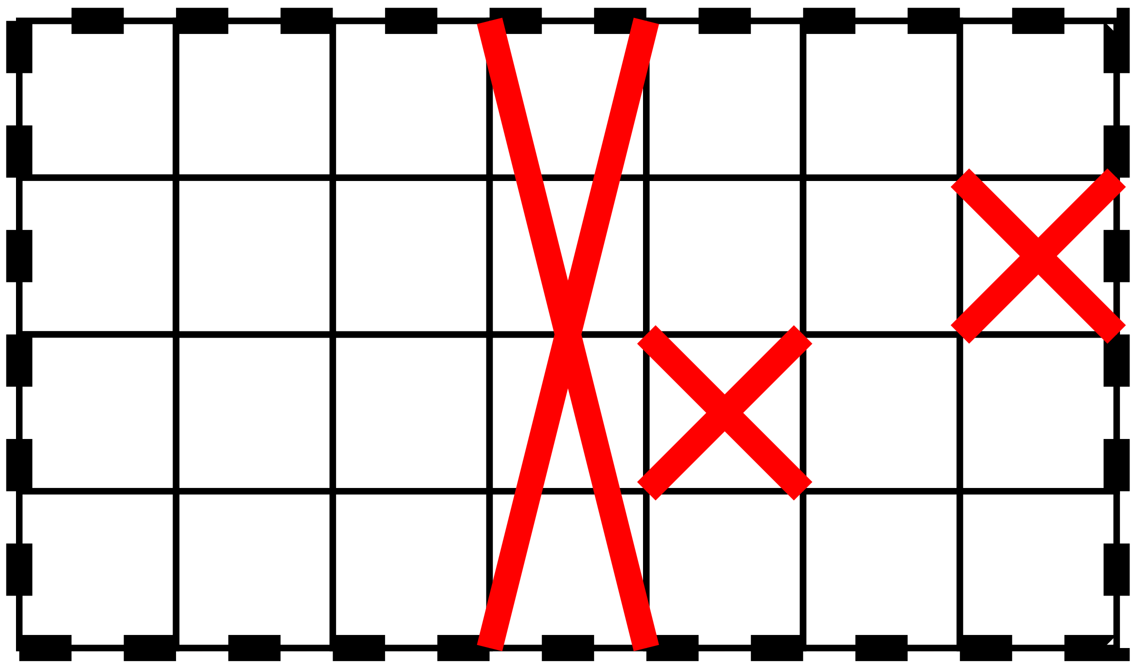

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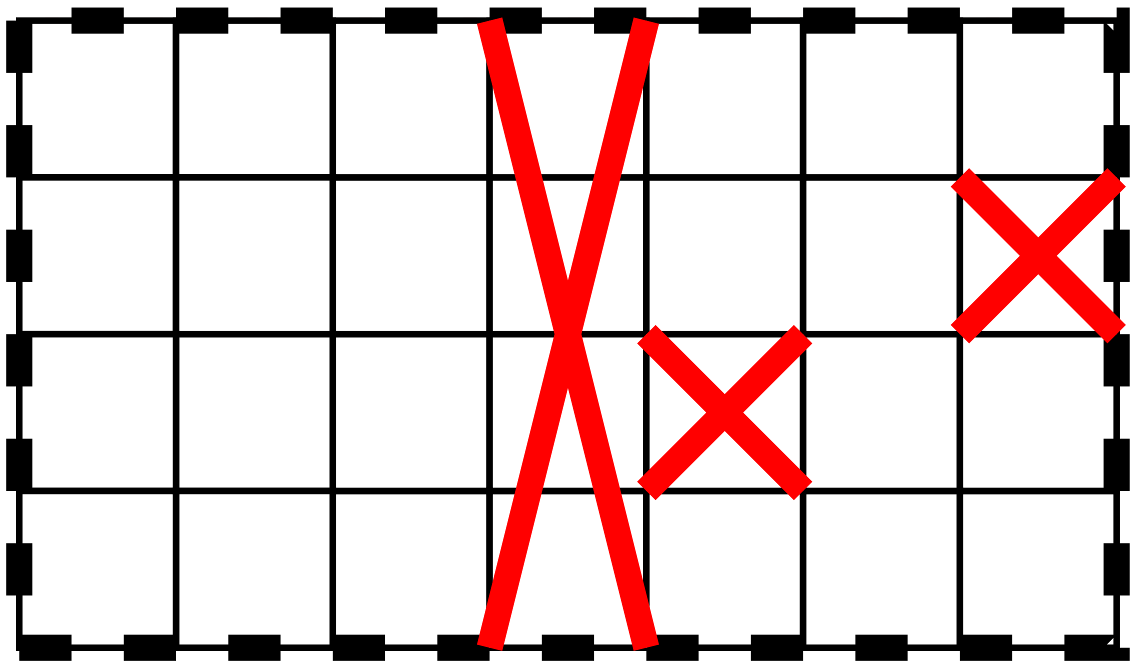

$m \times n$

Wish List for ECC Scheme

We want ECC schemes that can jointly handle

- burst errors,
- symbol errors.

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Moreover, the ECC scheme schould also be able to handle

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Finally, the ECC scheme should have

- low encoding complexity,
- low decoding complexity.

Overview

Example of previous coding schemes for related setups.

[Blokh, Zyablov, 1974]
[Kasahara, Hirasawa, Sugiyama, Namekawa, 1976]
[Zinov'ev, Zyablov, 1979]
[Zinov'ev, 1981]
[Abdel-Ghaffar, Hassner, 1991]
[Feng, Tzeng, 1991]
[Dumer, 1998]
[Metzner, Kapturowski, 1990]
[Sakata, 1991]
[Krachkovsky, Lee, 1998]
[Roth, Seroussi, 1998]
[Haslach, Vinck, 1999, 2000]
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Our proposed coding scheme:

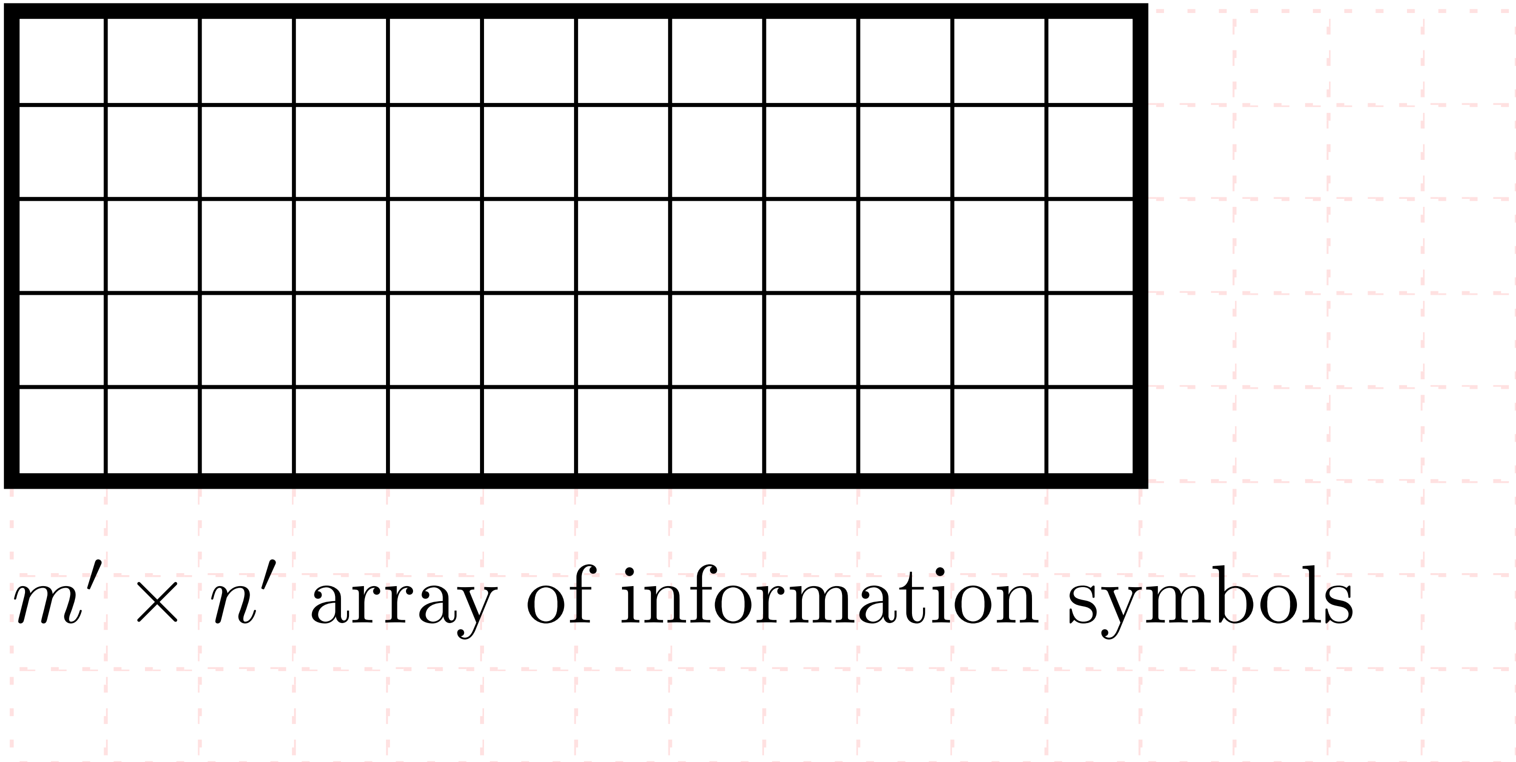
- Code construction
- Code properties
- Decoding algorithms

Example of a more “traditional” coding scheme

Concatenated Coding Scheme

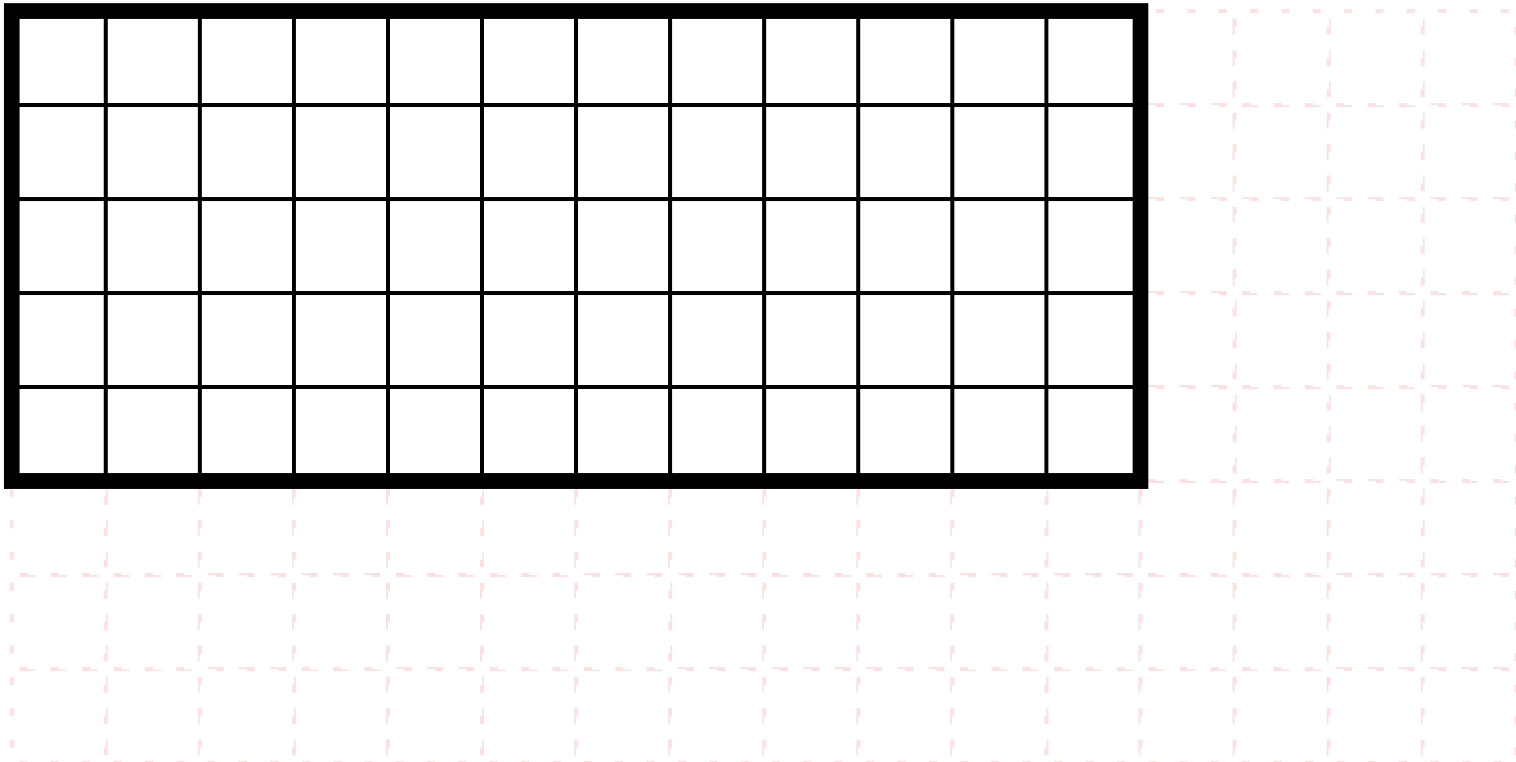
In Particular: Product Coding Scheme

Concatenated Coding Scheme
In Particular: Product Coding Scheme



$m' \times n'$ array of information symbols

Concatenated Coding Scheme
In Particular: Product Coding Scheme



Concatenated Coding Scheme

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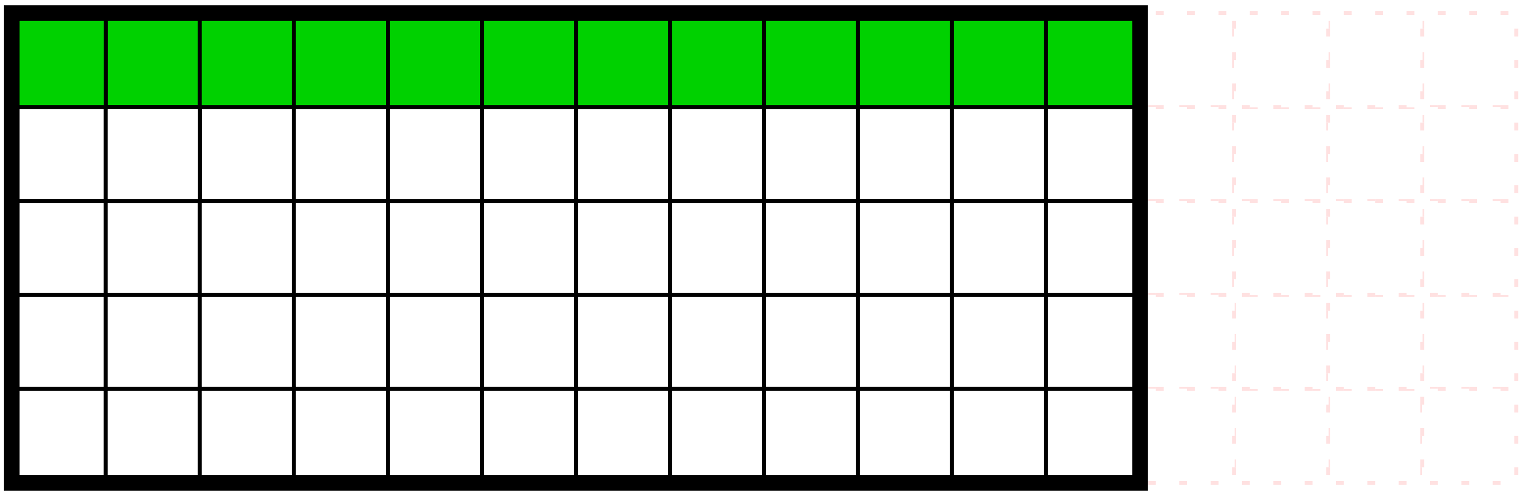

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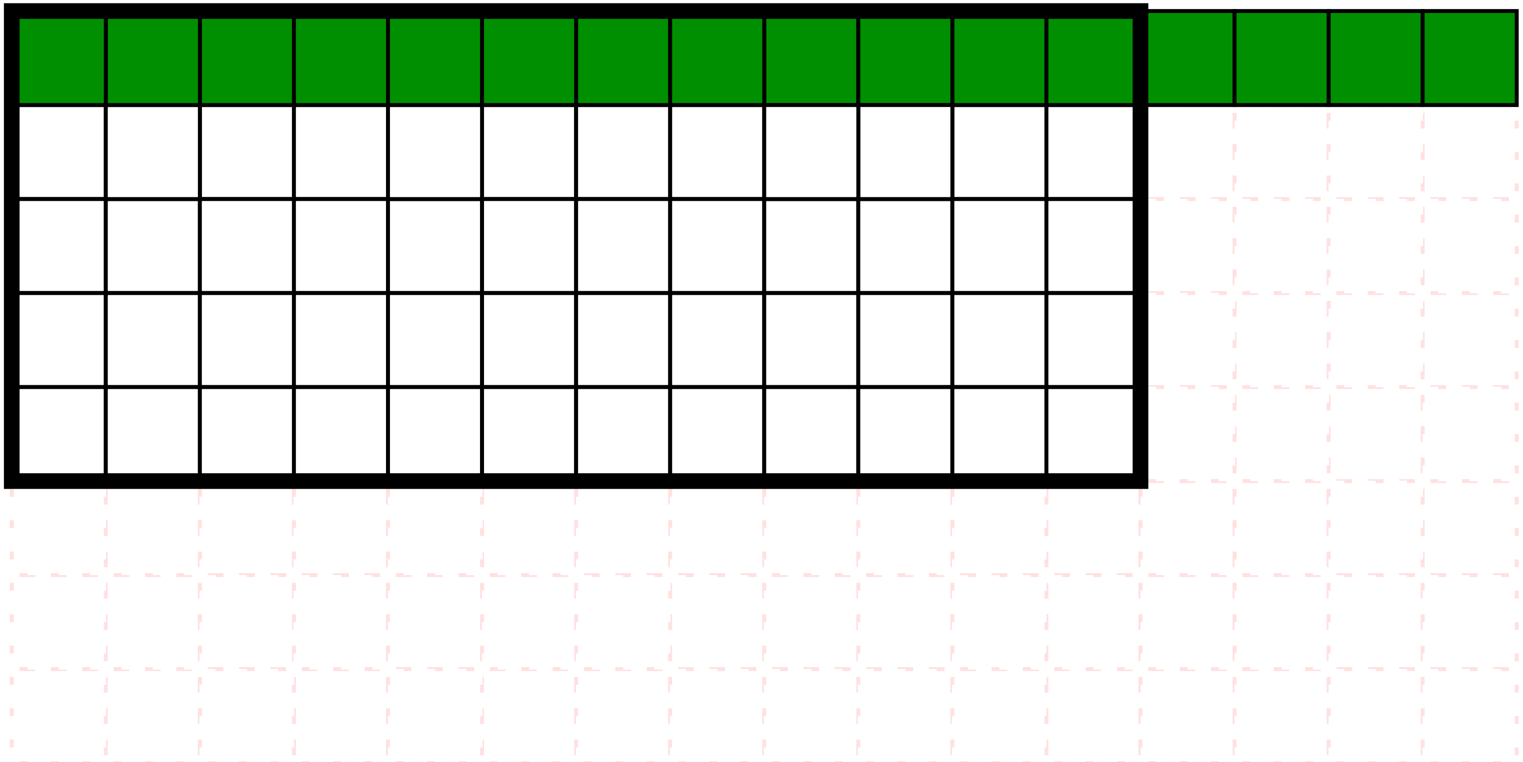

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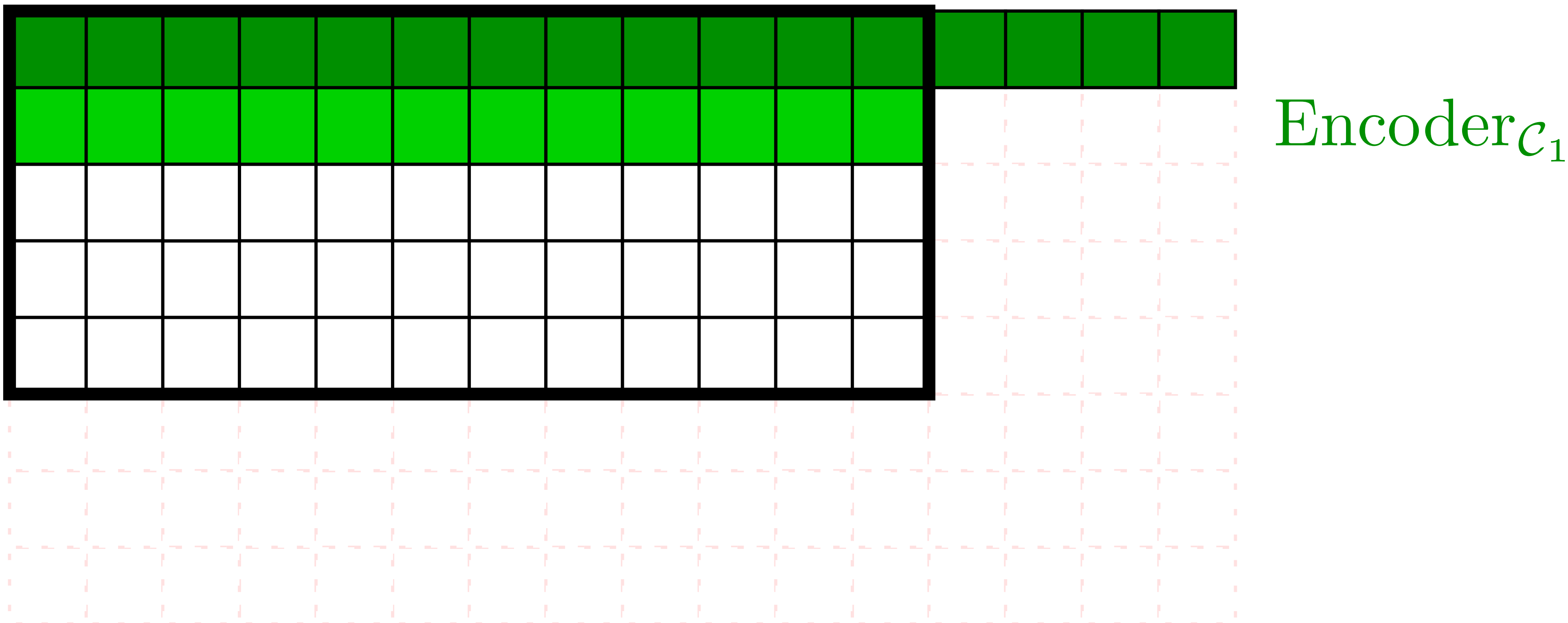

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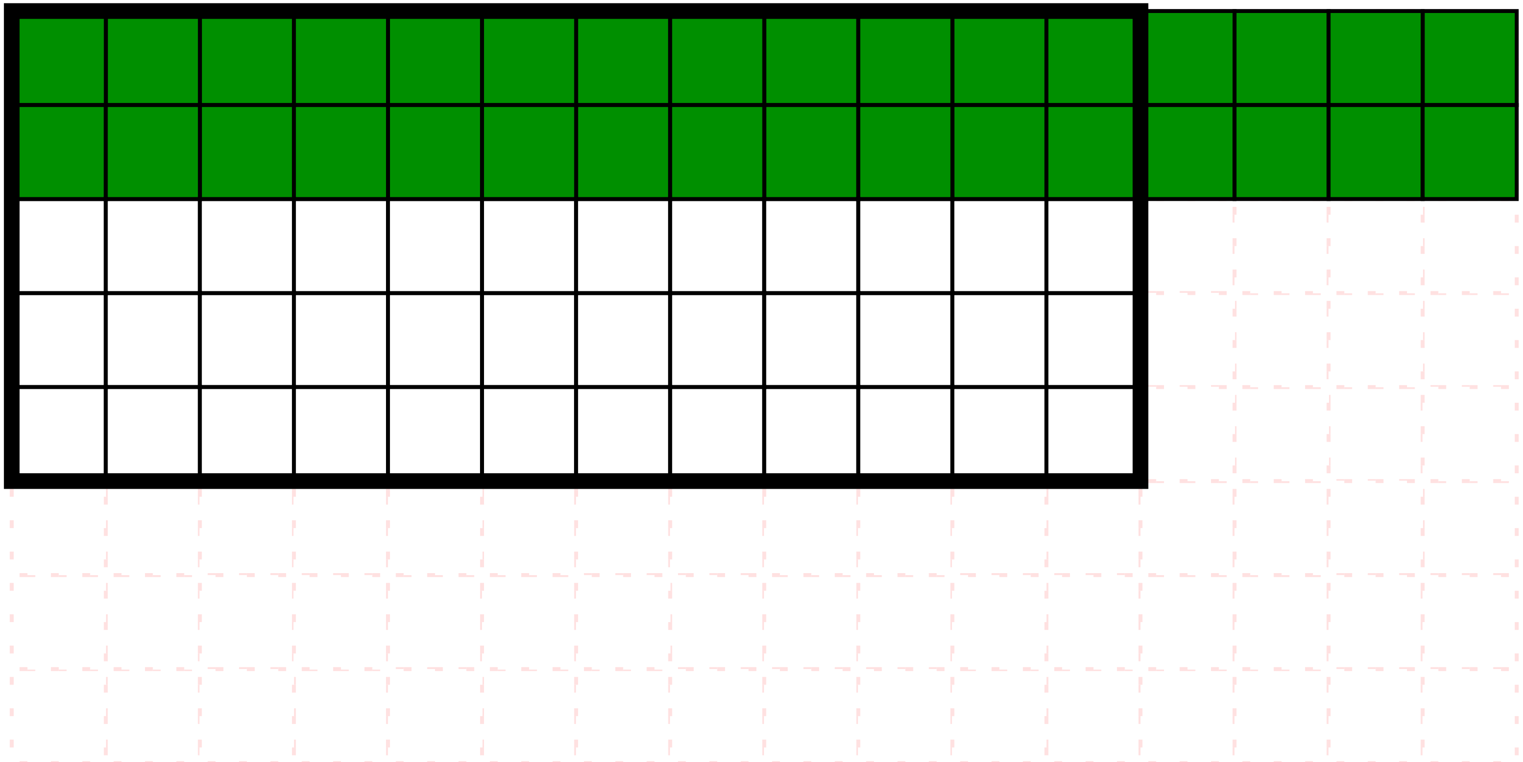

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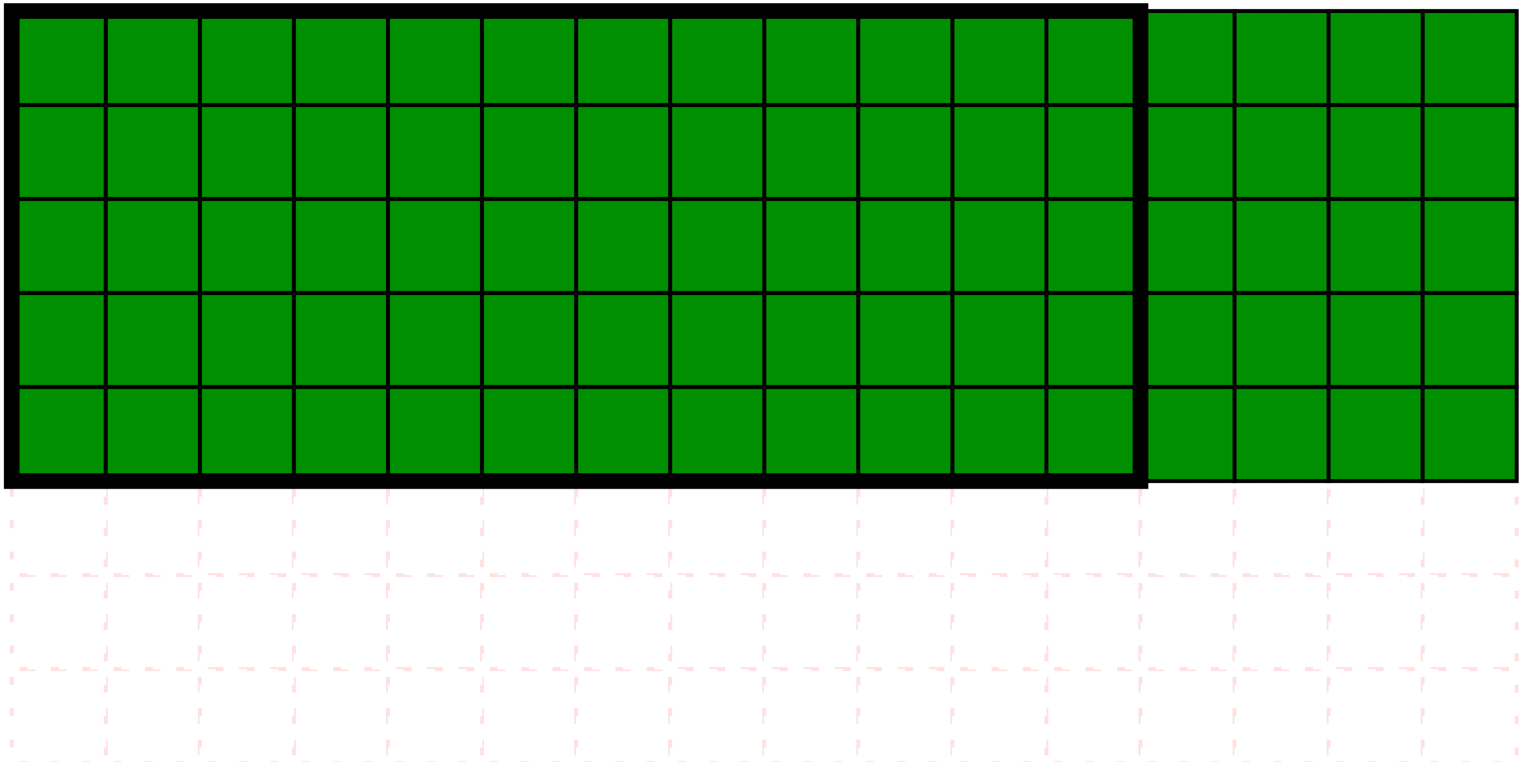

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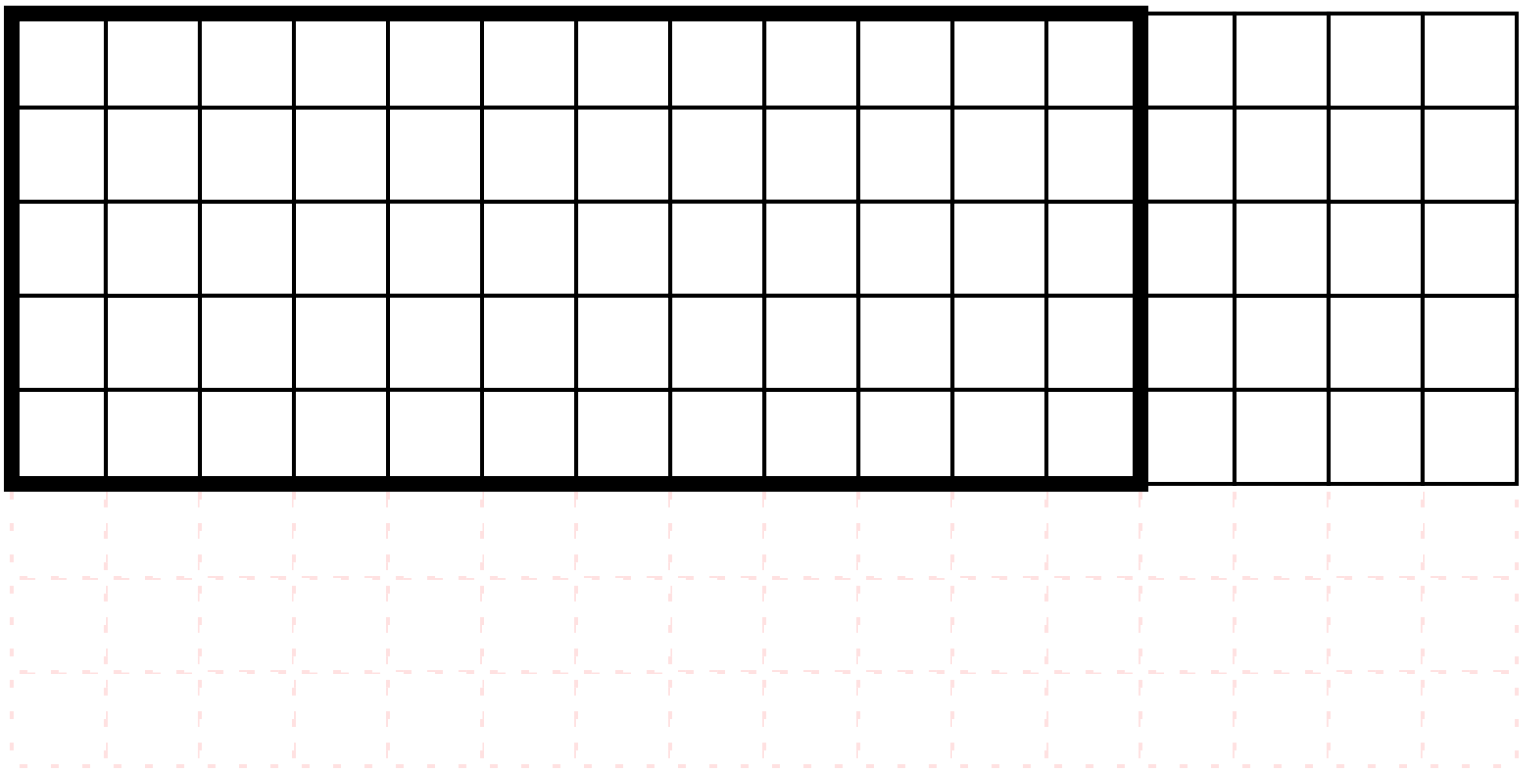

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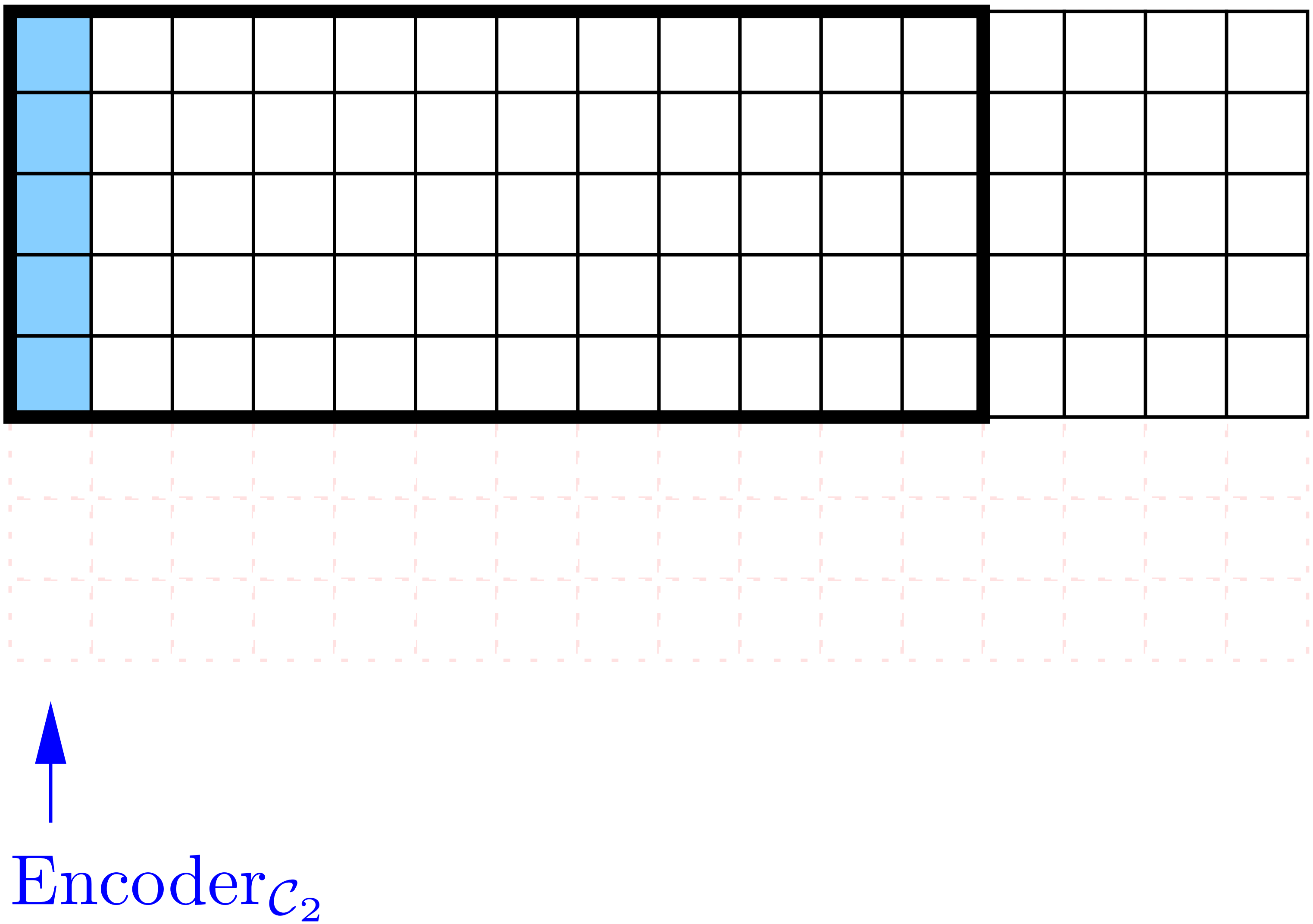

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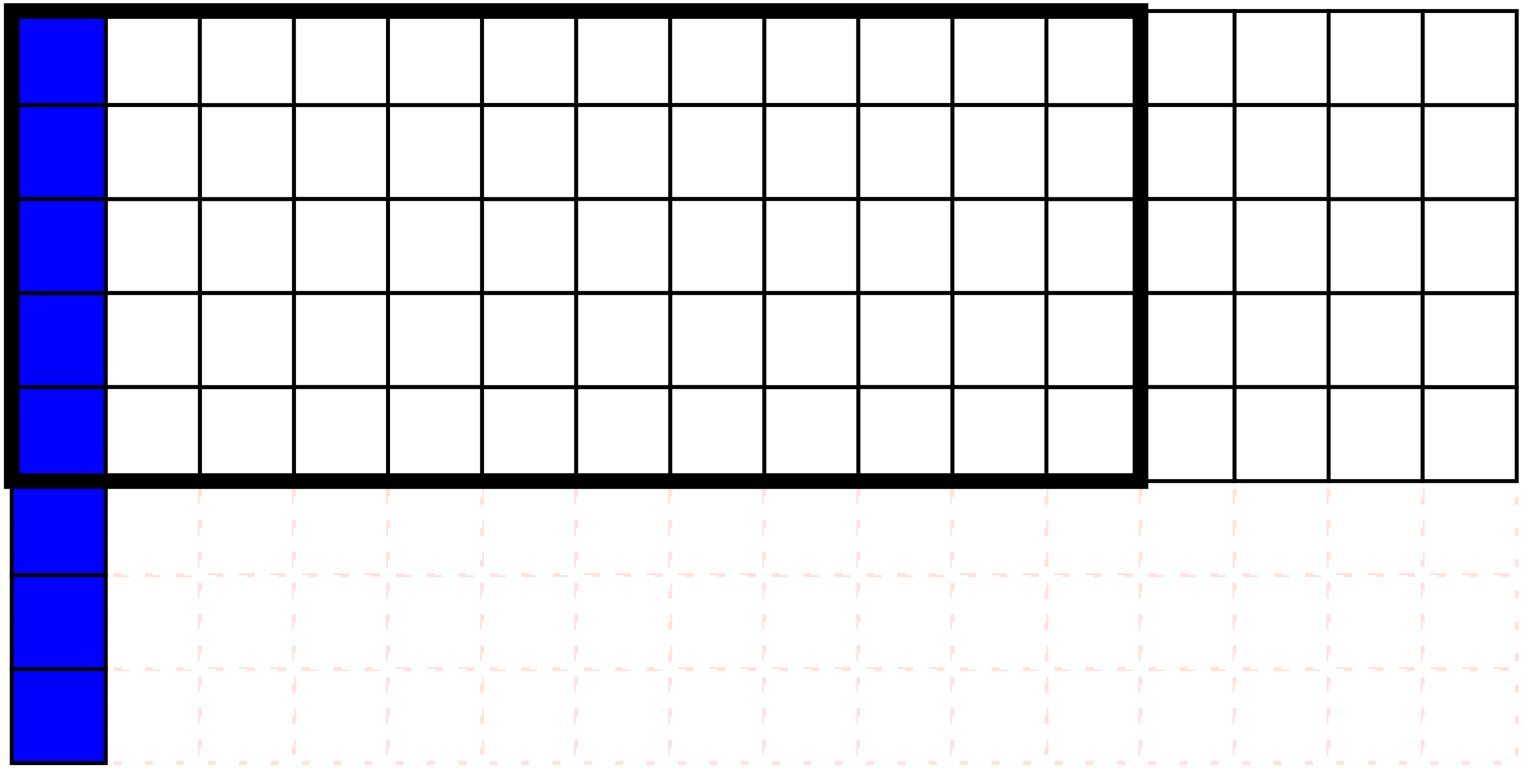

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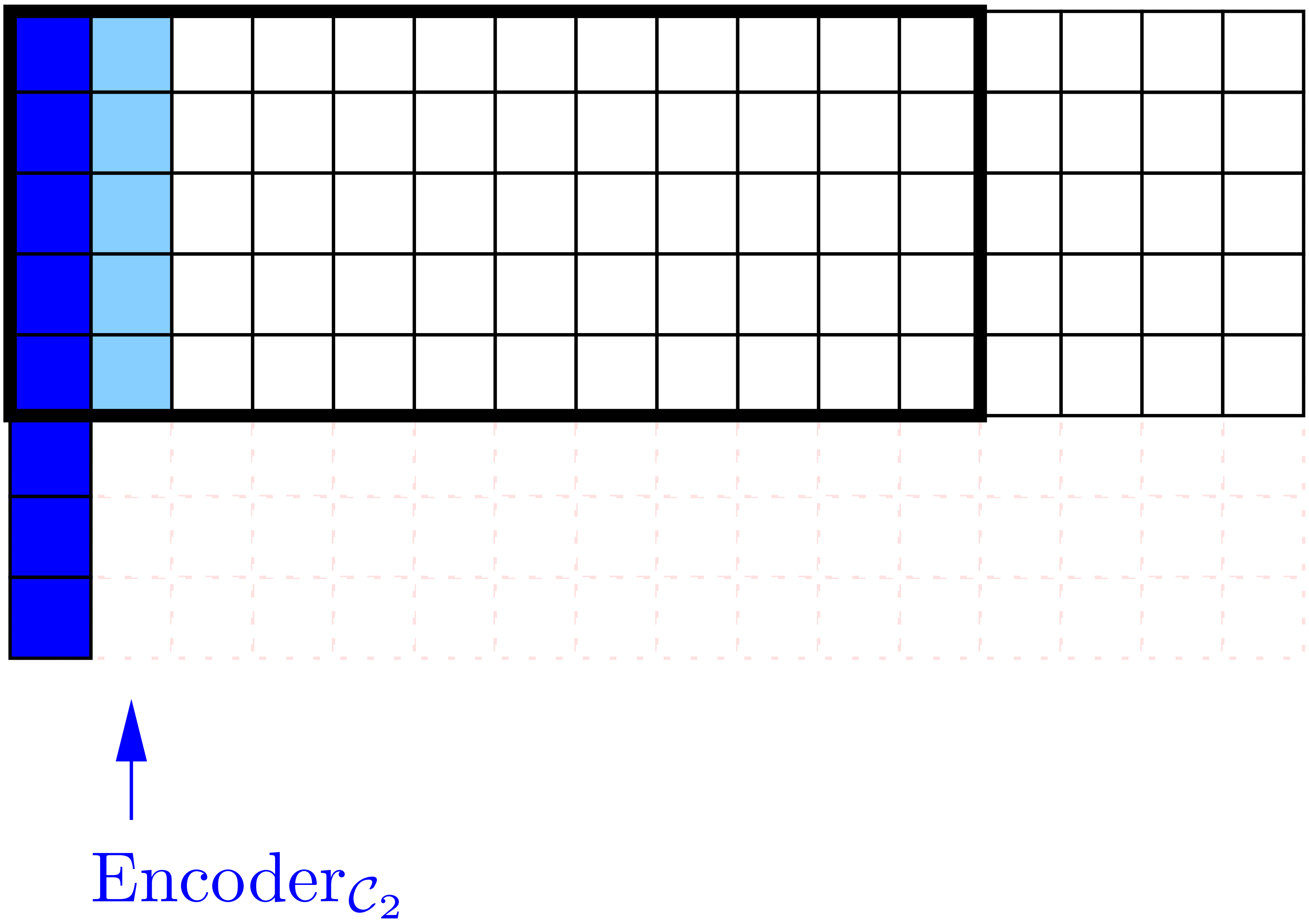

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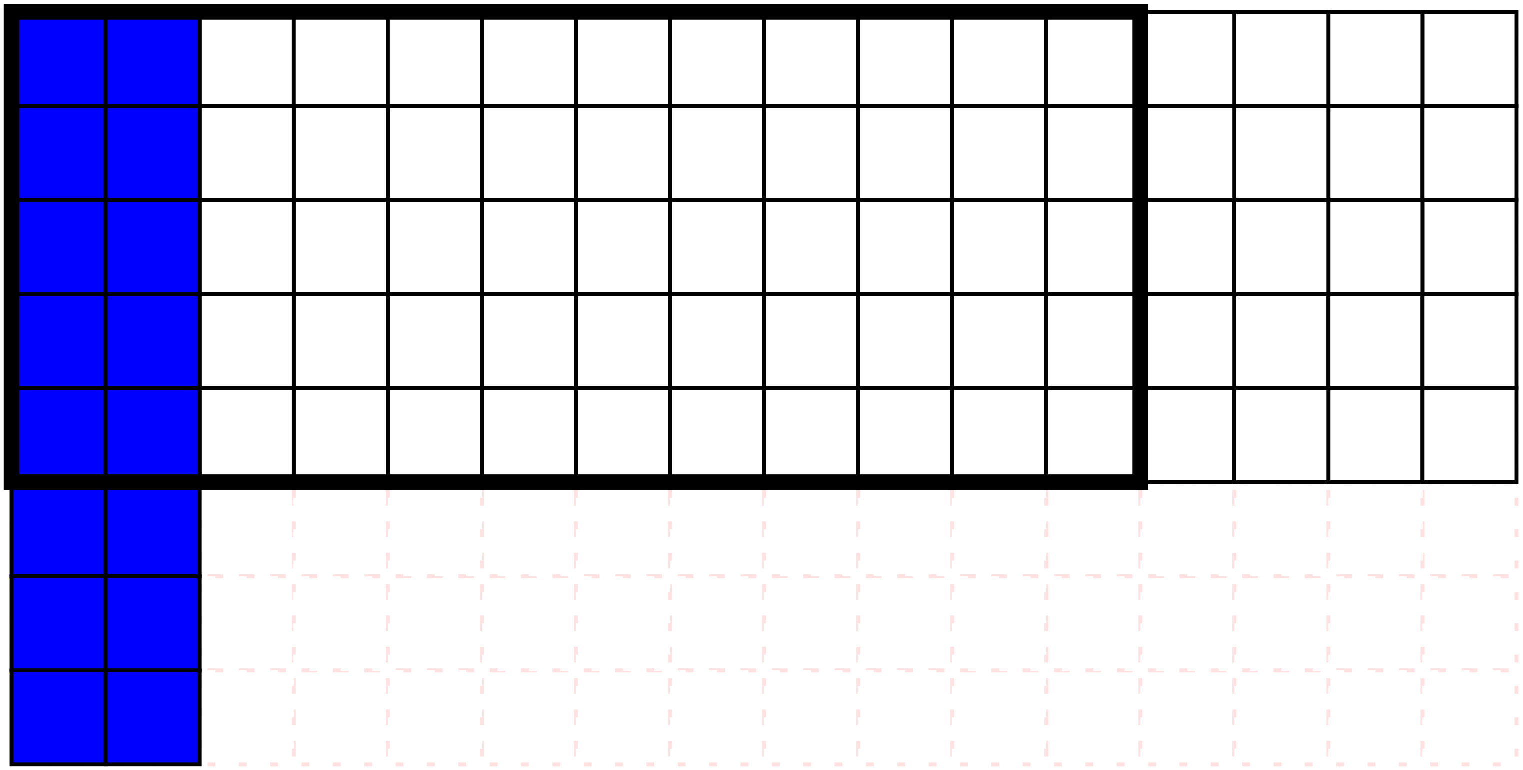

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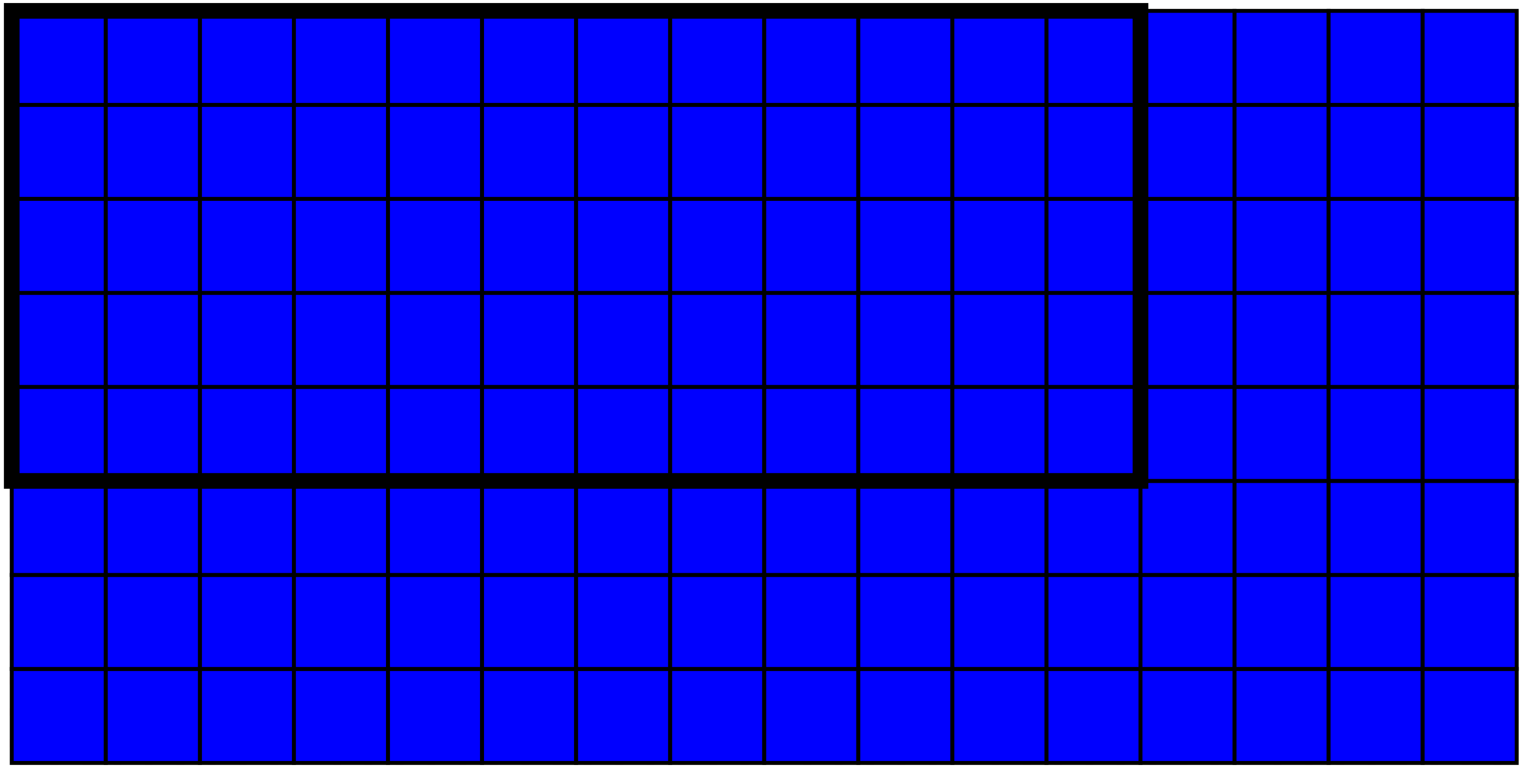

$m \times n$ array of codeword symbols

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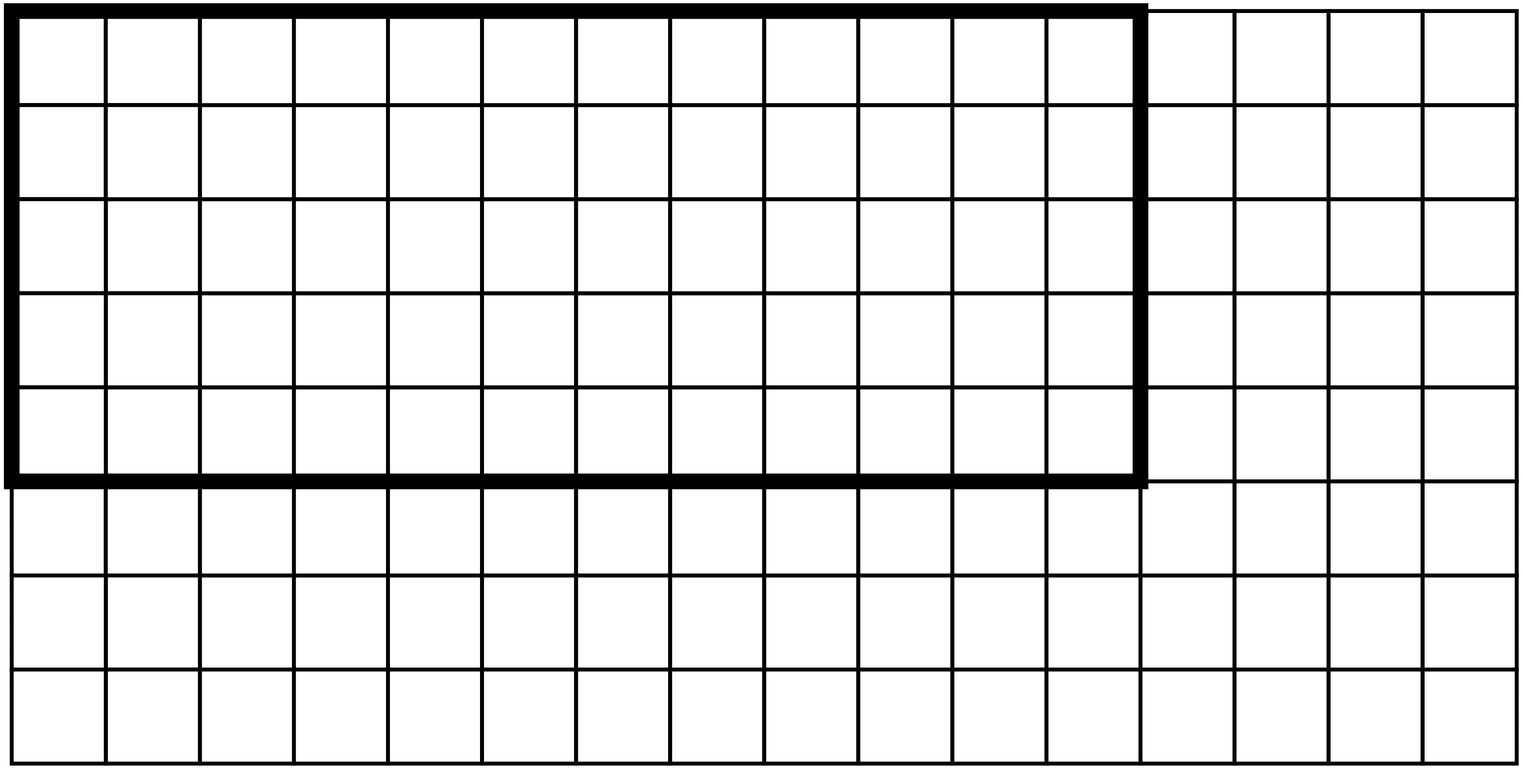

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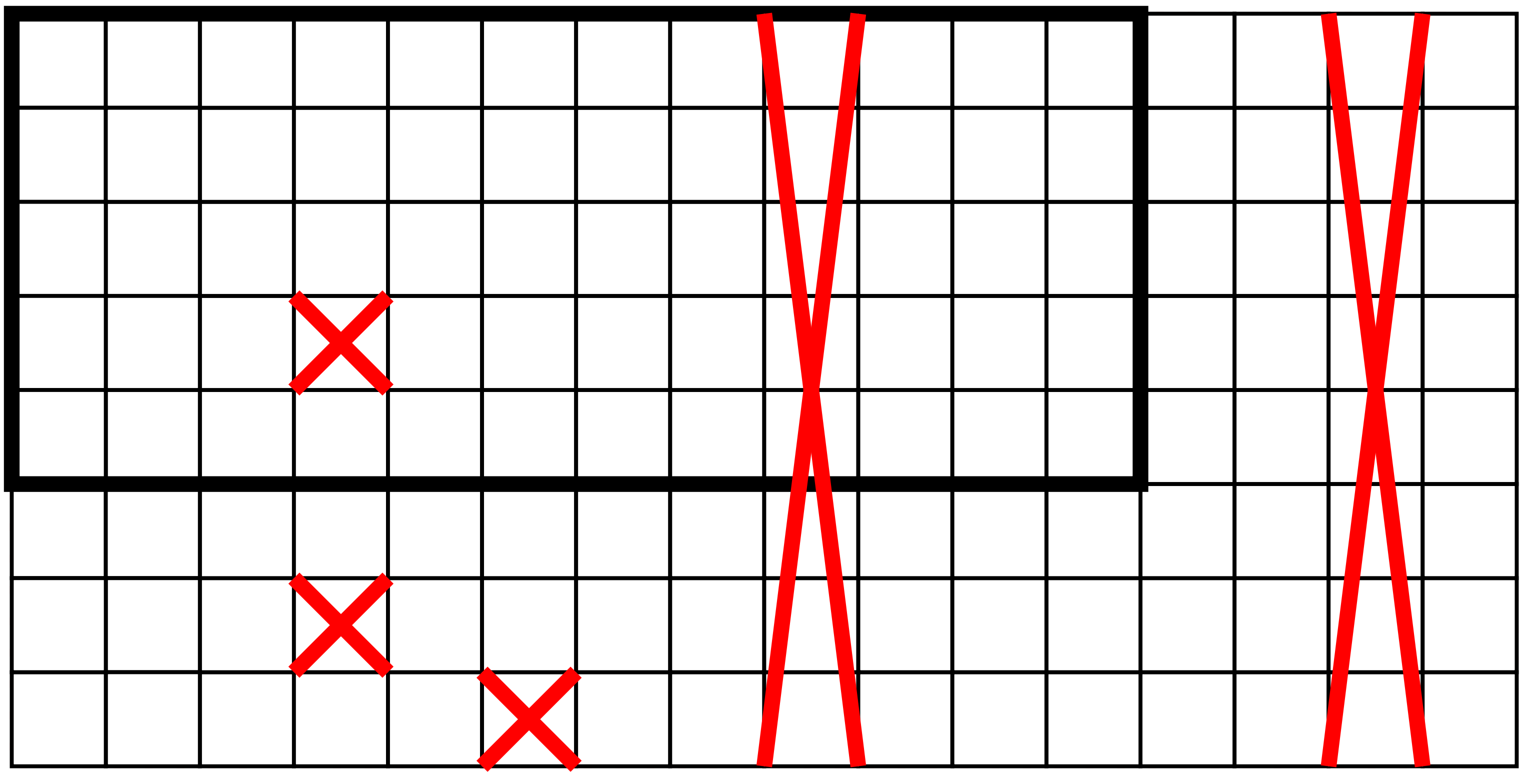

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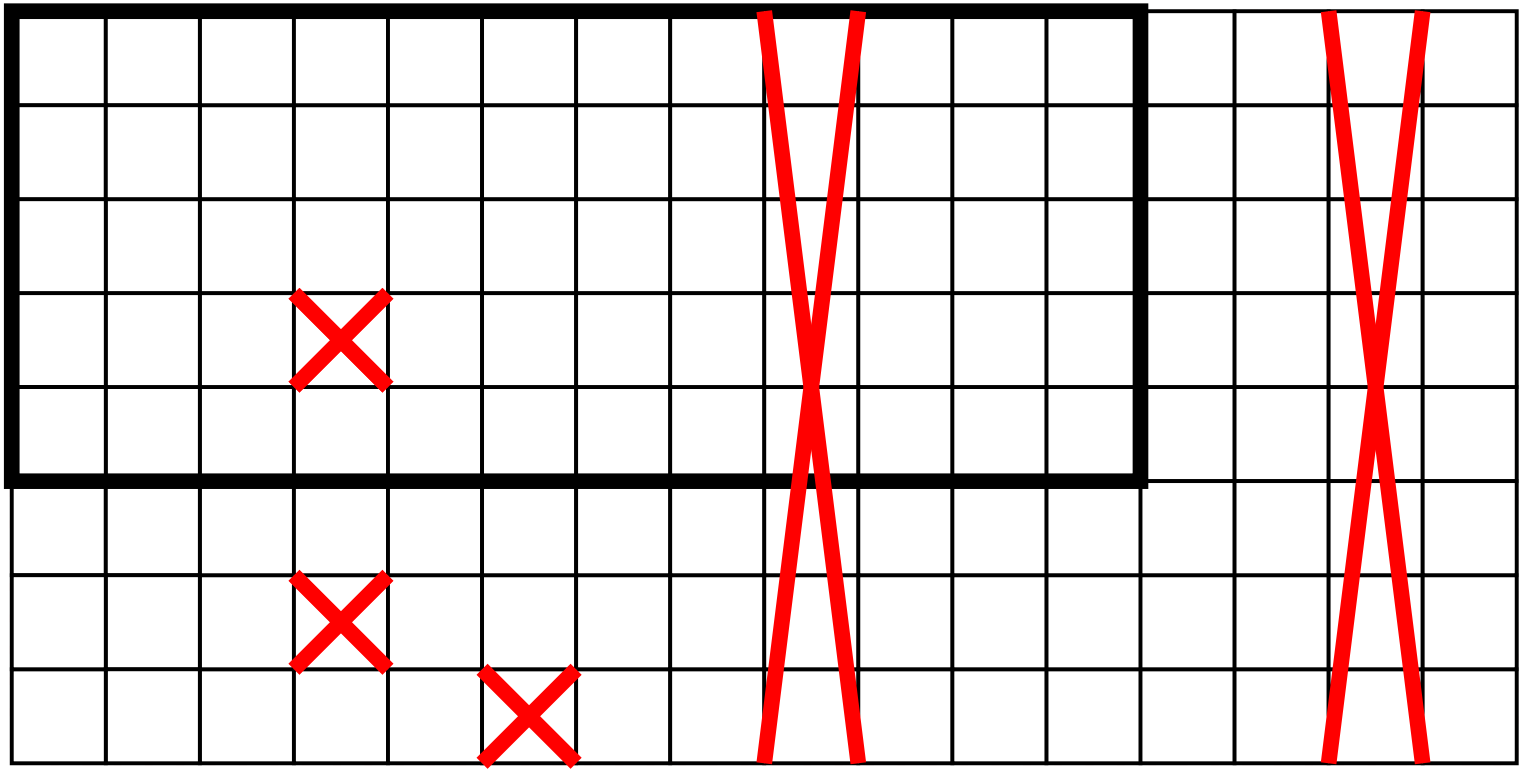
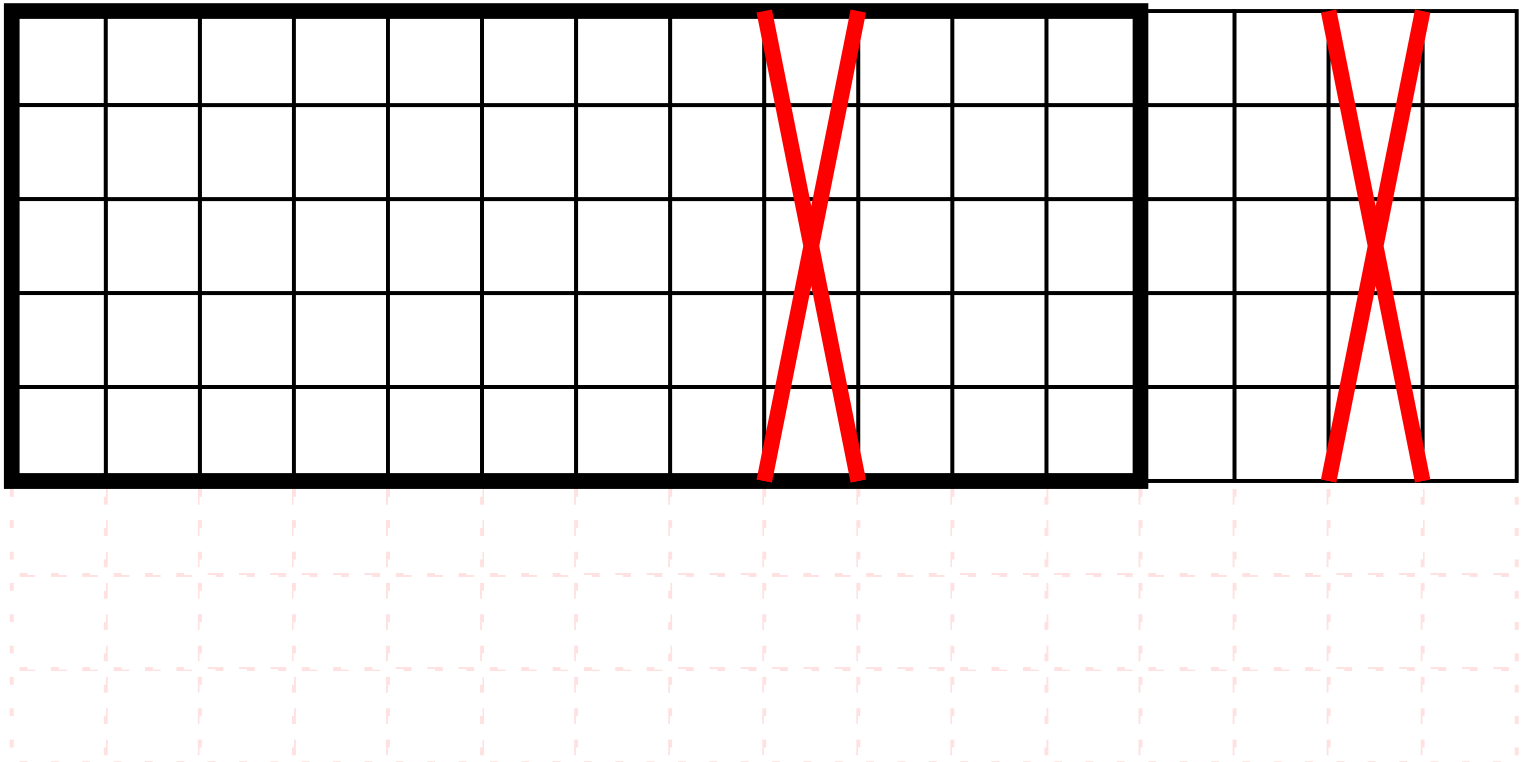


Decoding of columns based on $\mathcal{C}_2$:

- corrects symbol errors (as far as possible),
- leaves or modifies block errors.

Concatenated Coding Scheme
In Particular: Product Coding Scheme



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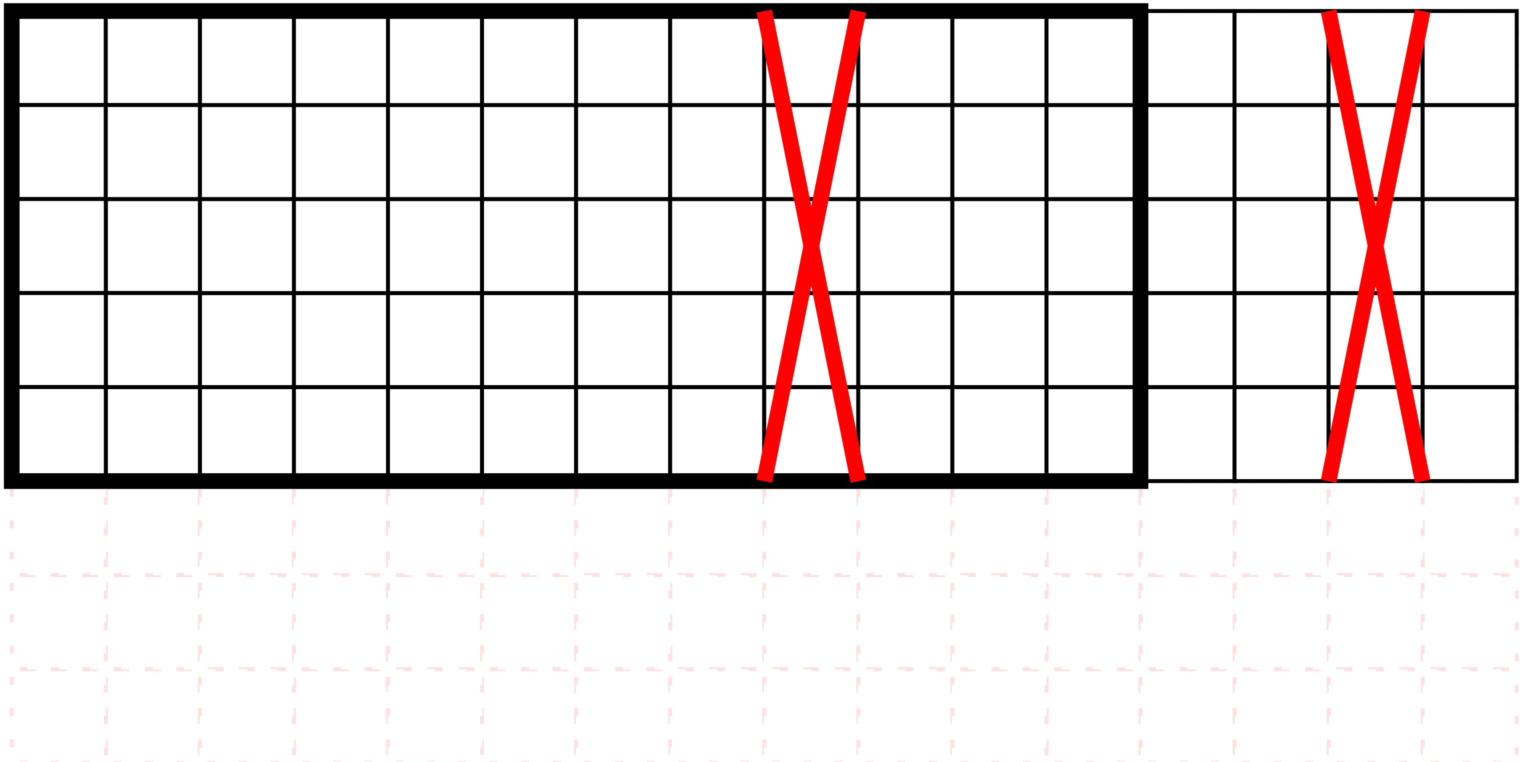

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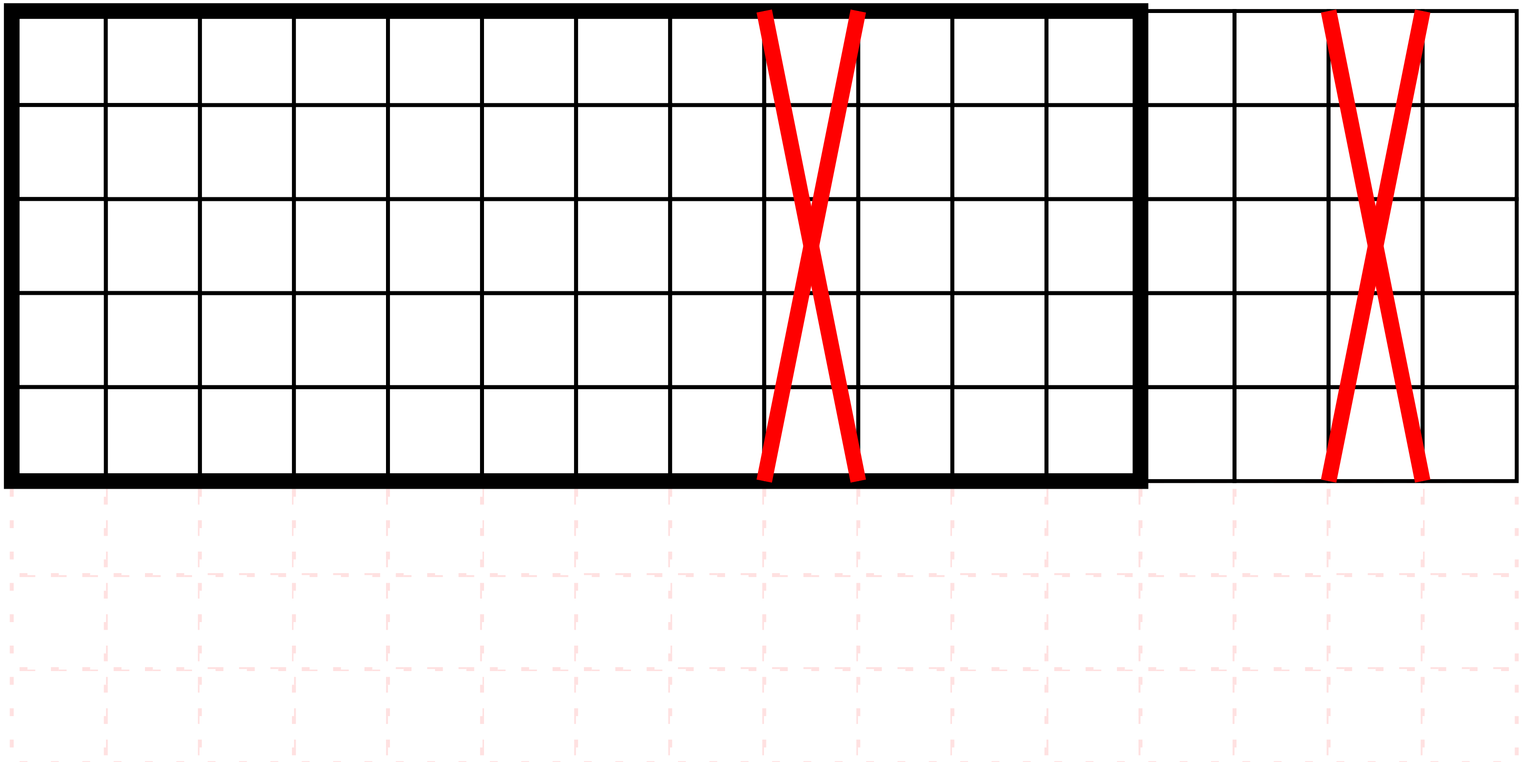

Decoding of rows based on $\mathcal{C}_1$:

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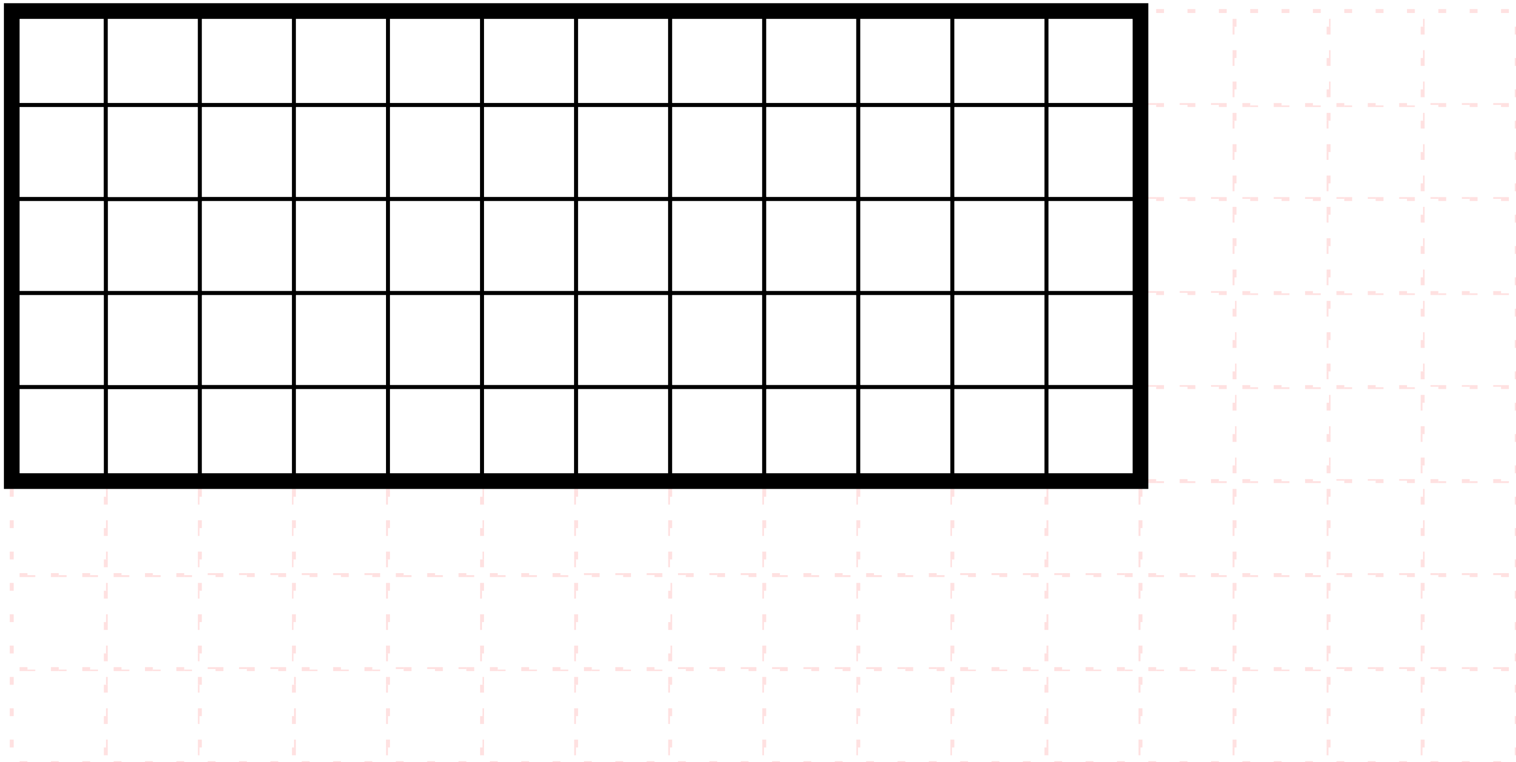

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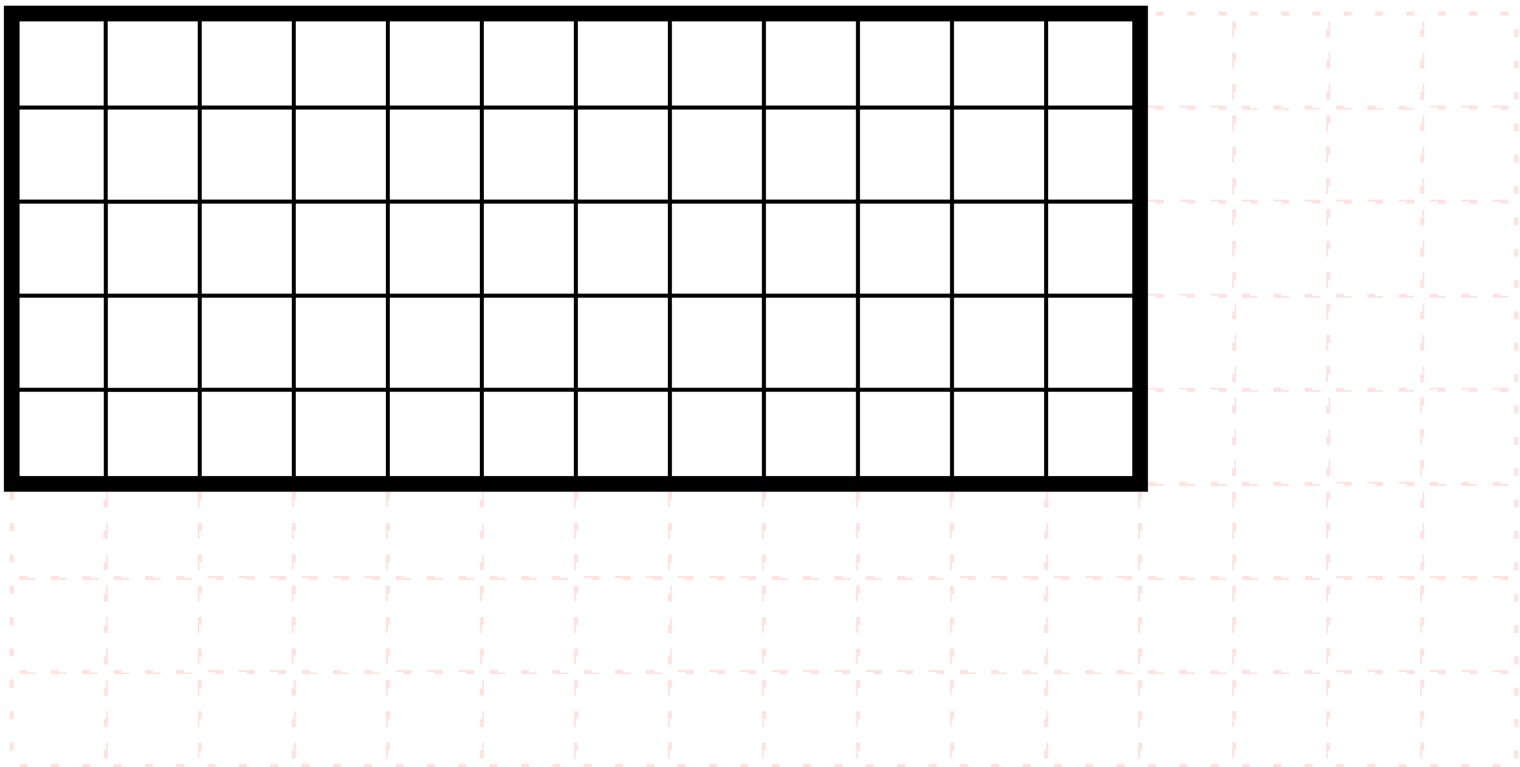
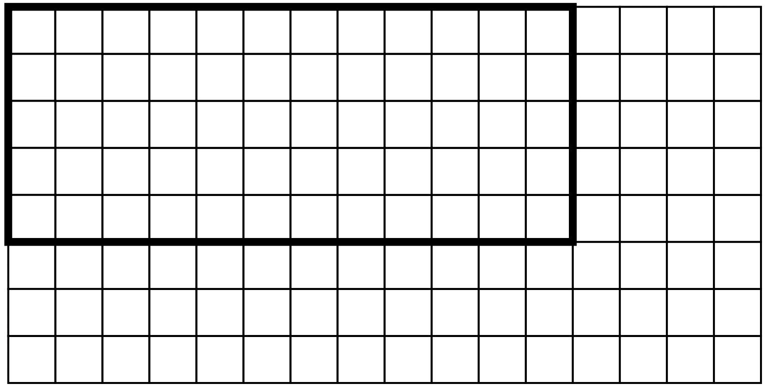
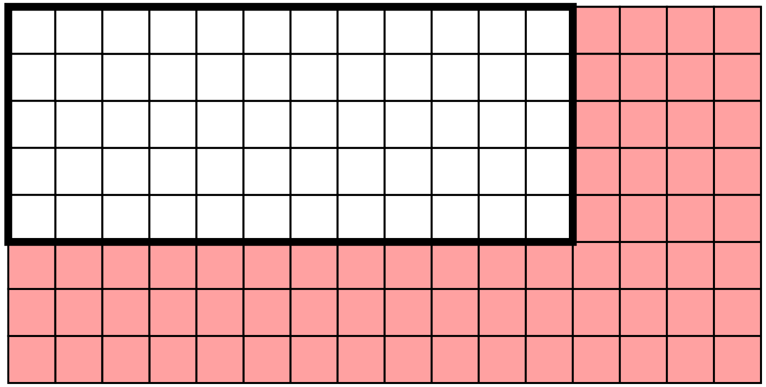


Disadvantages of Product Coding Scheme



Product coding schemes have many favorable aspects.

Disadvantages of Product Coding Scheme



Product coding schemes have many favorable aspects.

However:

- $\mathrm{Rate}(\mathcal{C}_1) < 1$.

  $\Rightarrow$ The redundancy of the coding scheme is at least linear in m.

- $\mathrm{Rate}(\mathcal{C}_2) < 1$.

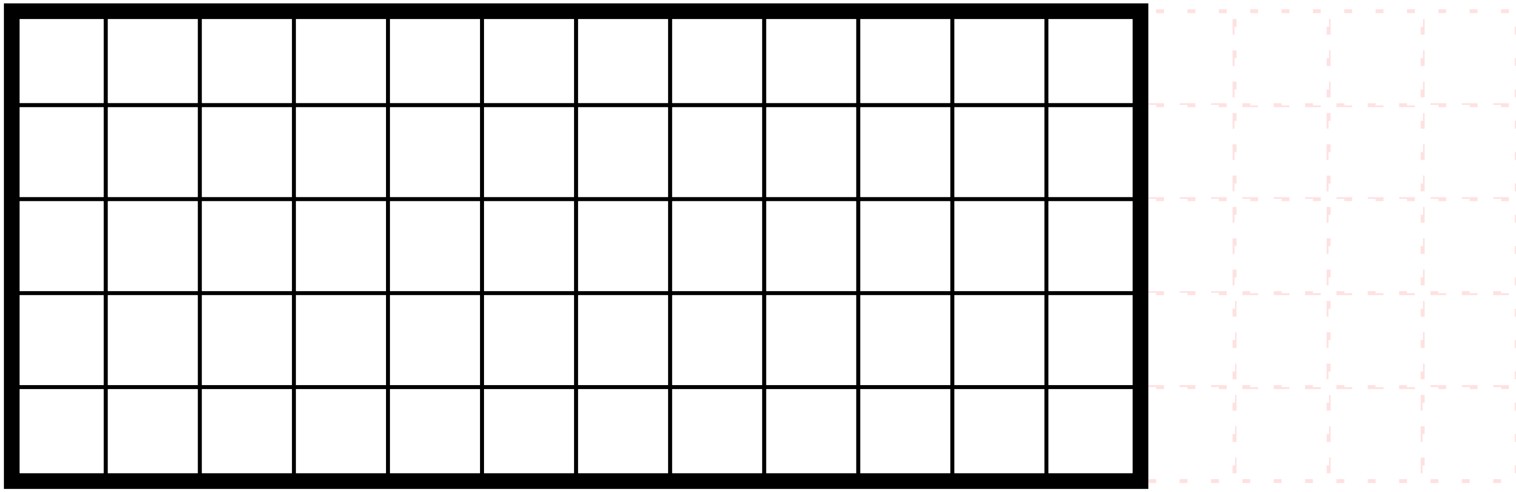
  $\Rightarrow$ The redundancy of the coding scheme is at least linear in n.

Proposed coding scheme:

Overview

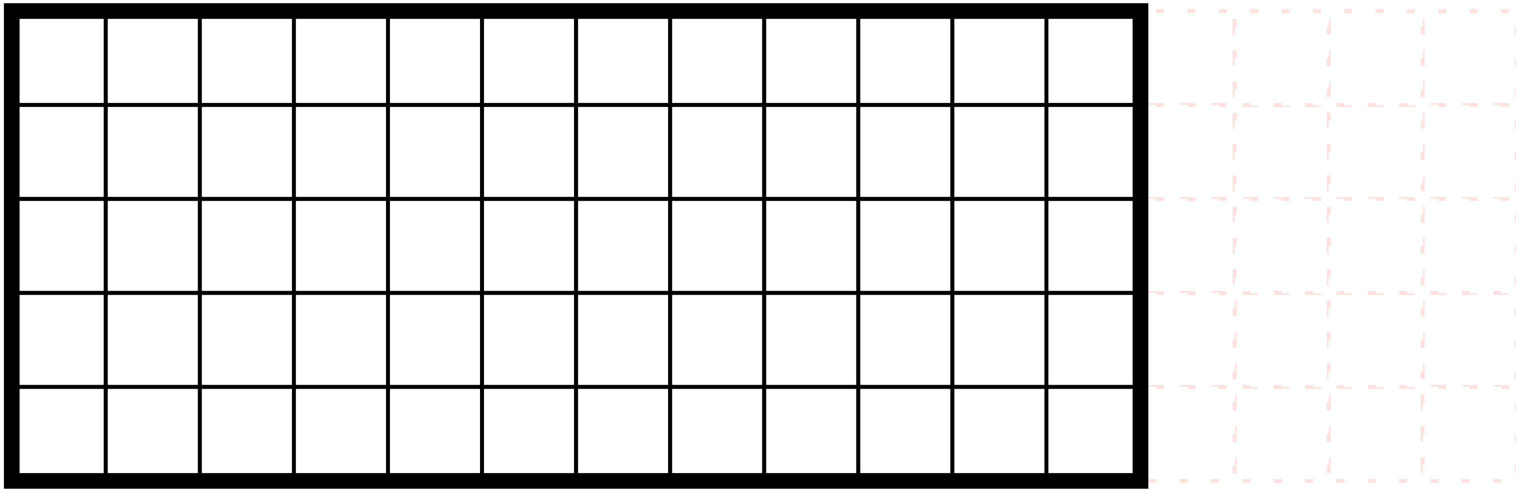
Overview of Proposed Coding Scheme

Overview of Proposed Coding Scheme



$m \times n'$ array of information symbols

Overview of Proposed Coding Scheme



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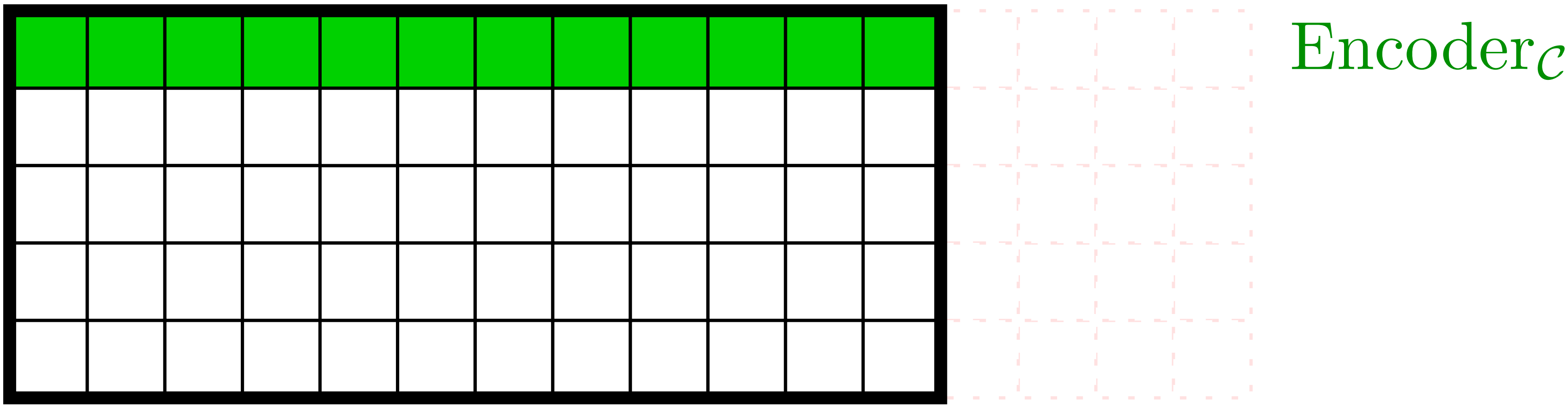

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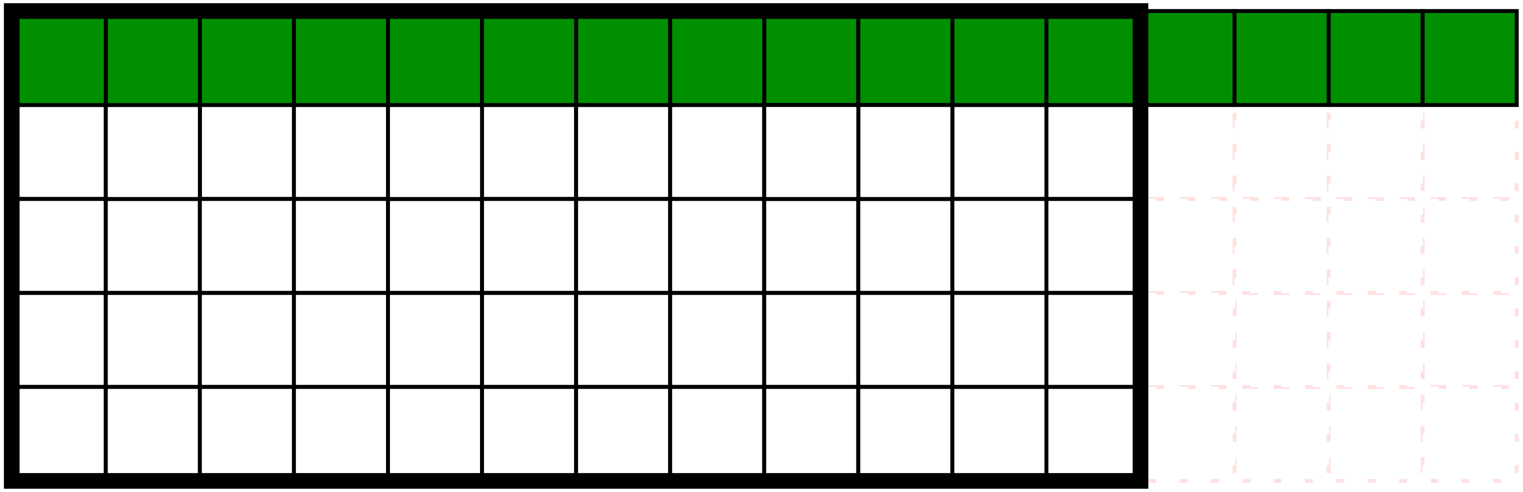

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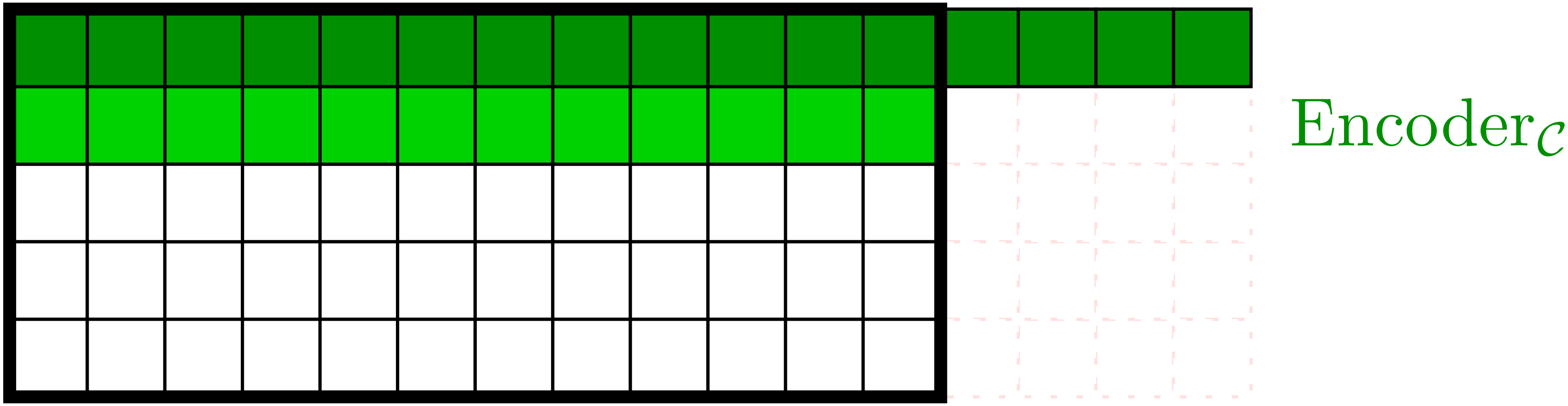

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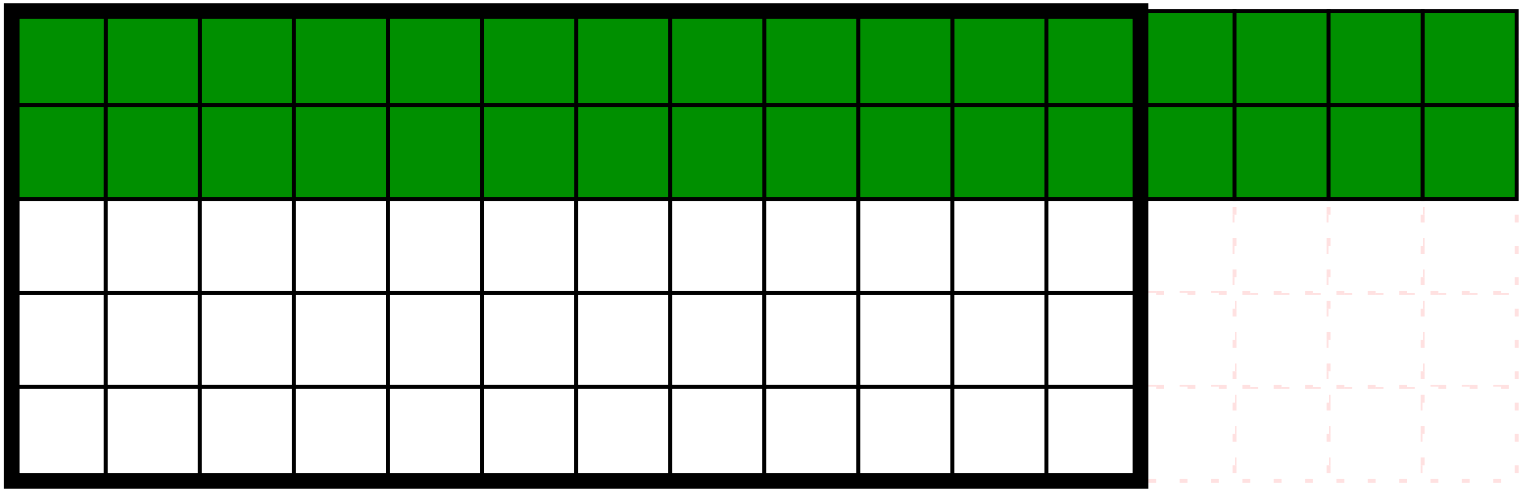

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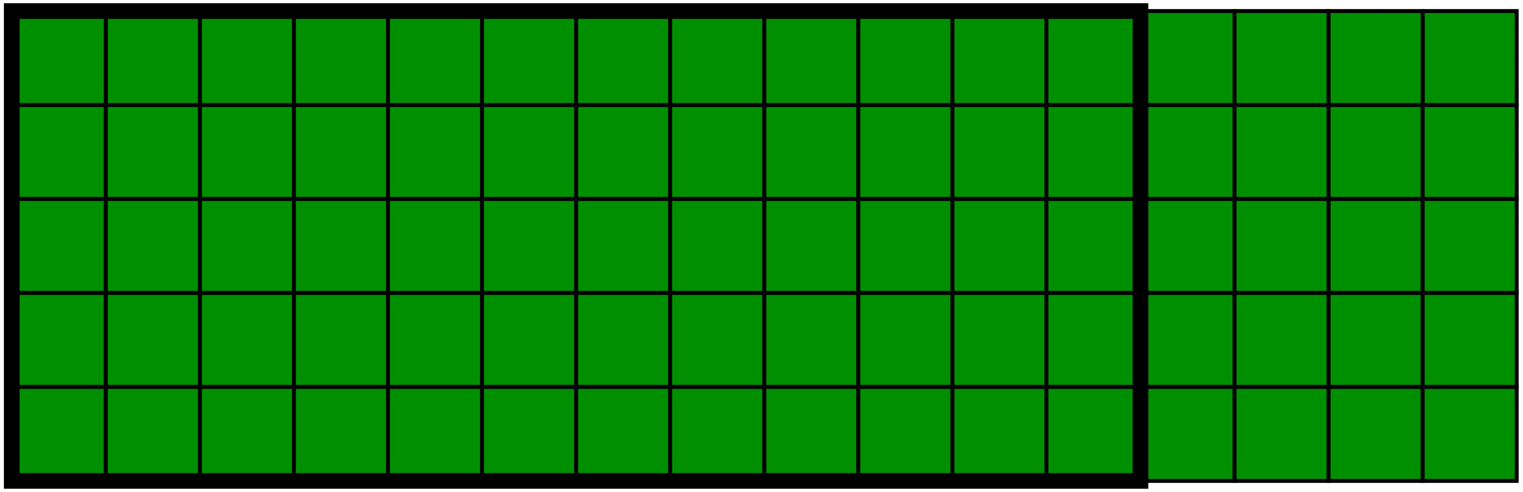

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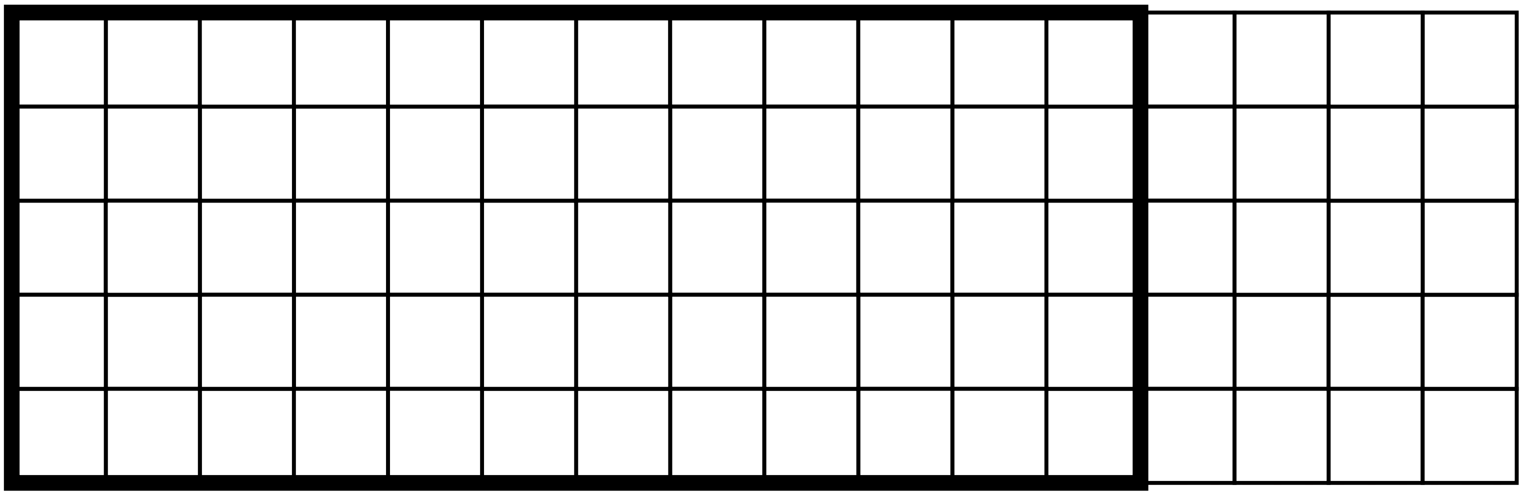

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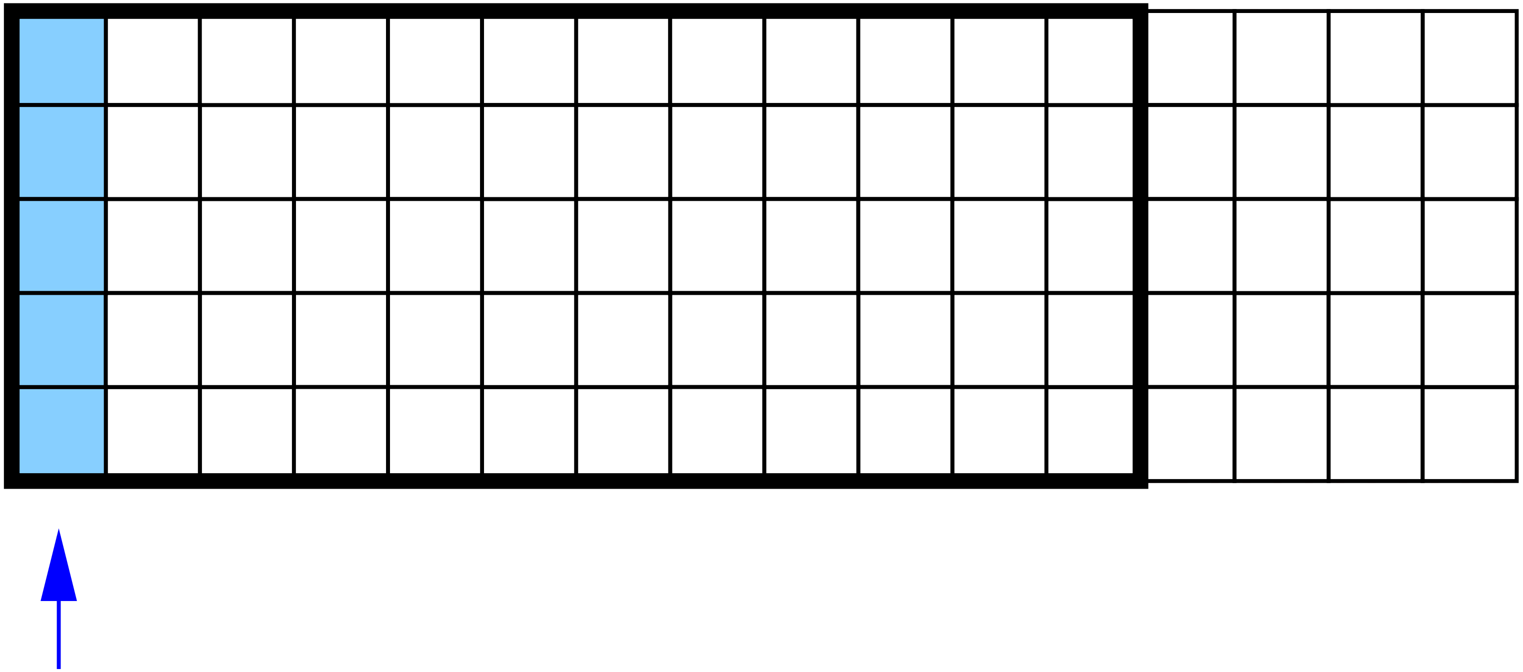

multiply by square matrix H_0^{-1}

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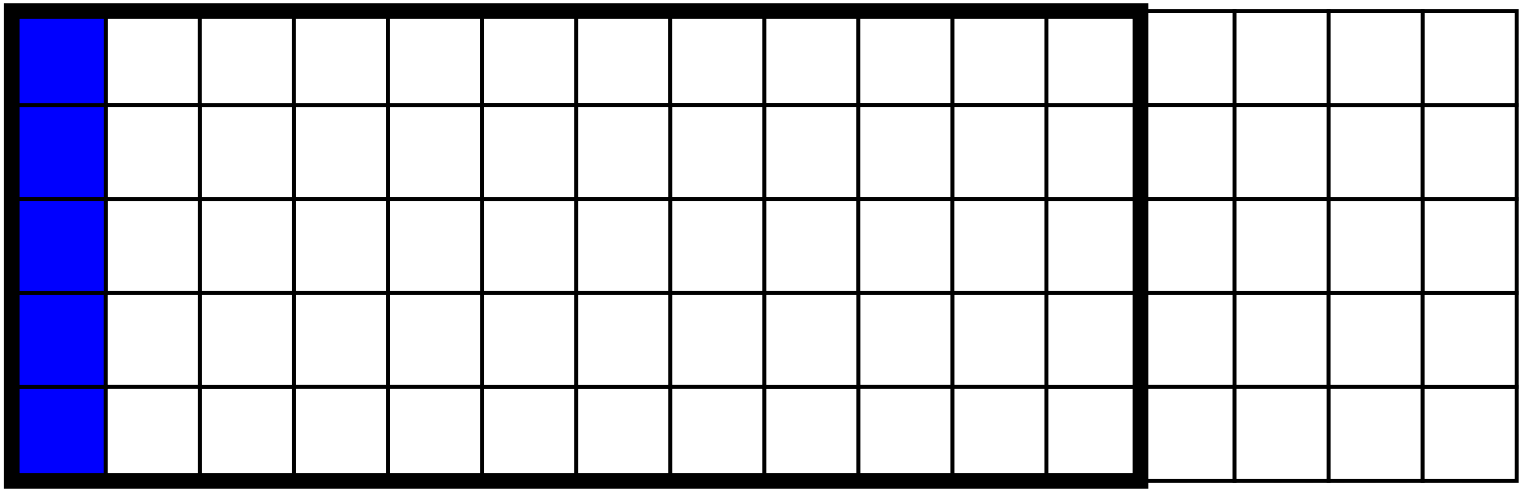

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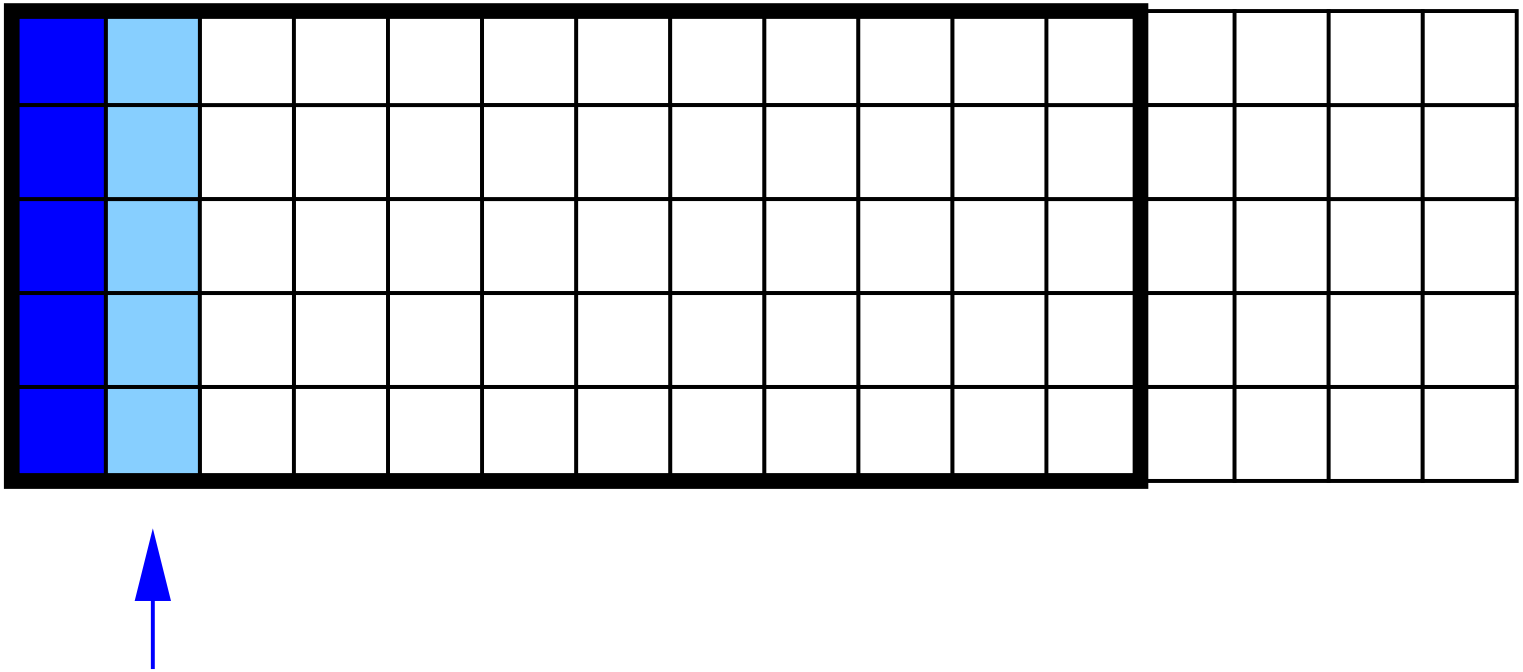

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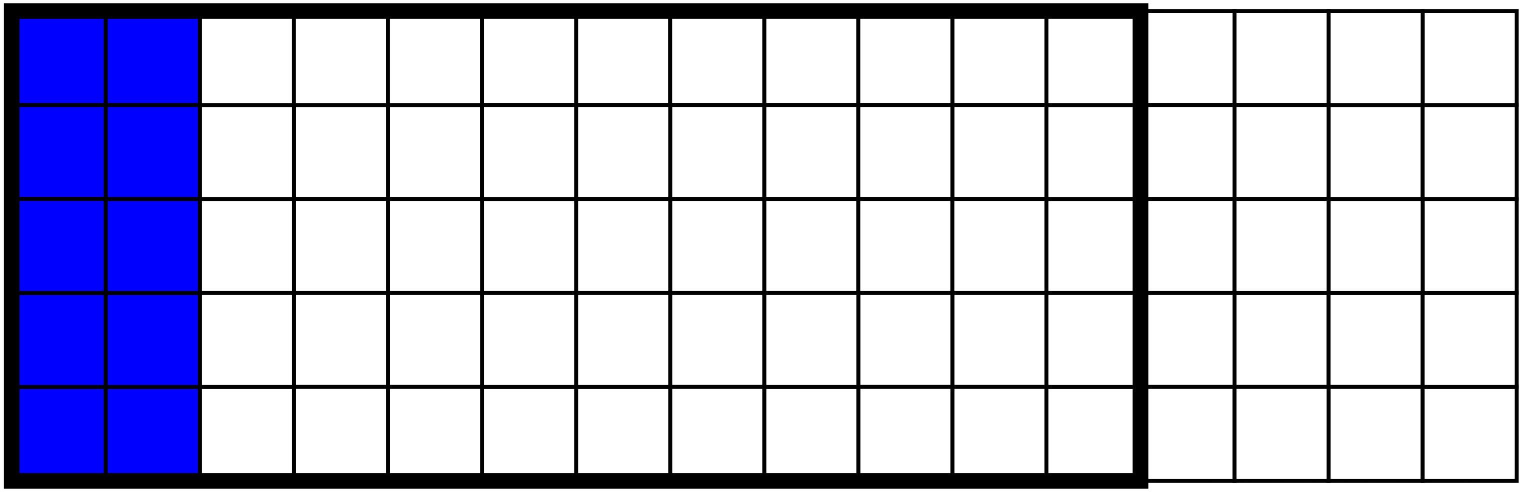

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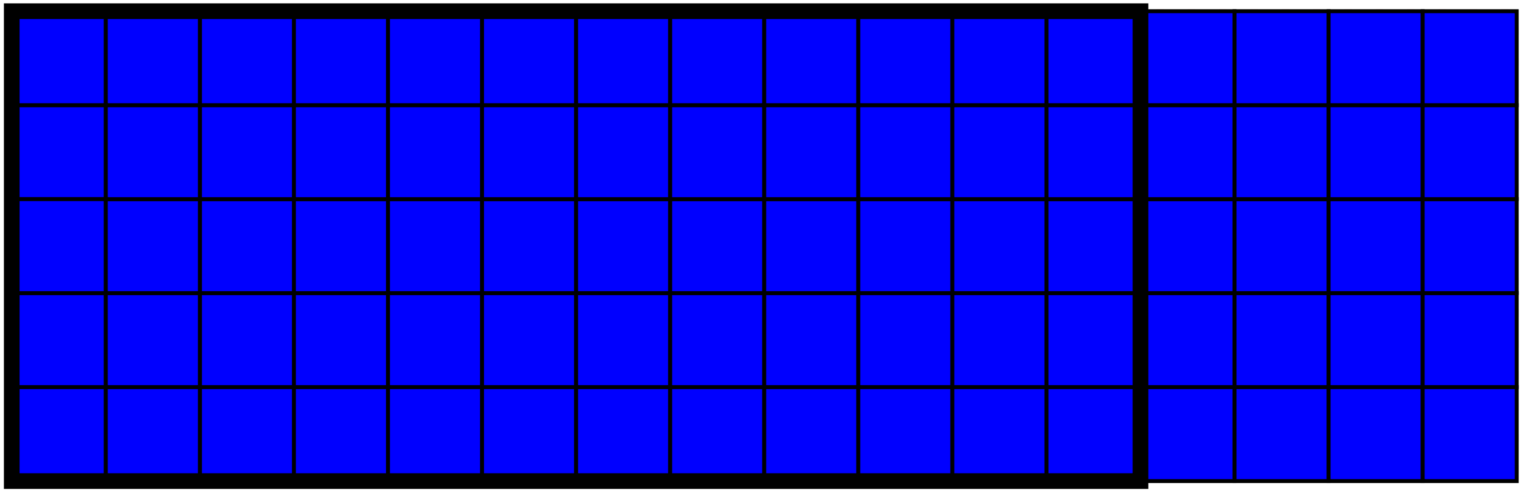

$m \times n$ array of codeword symbols

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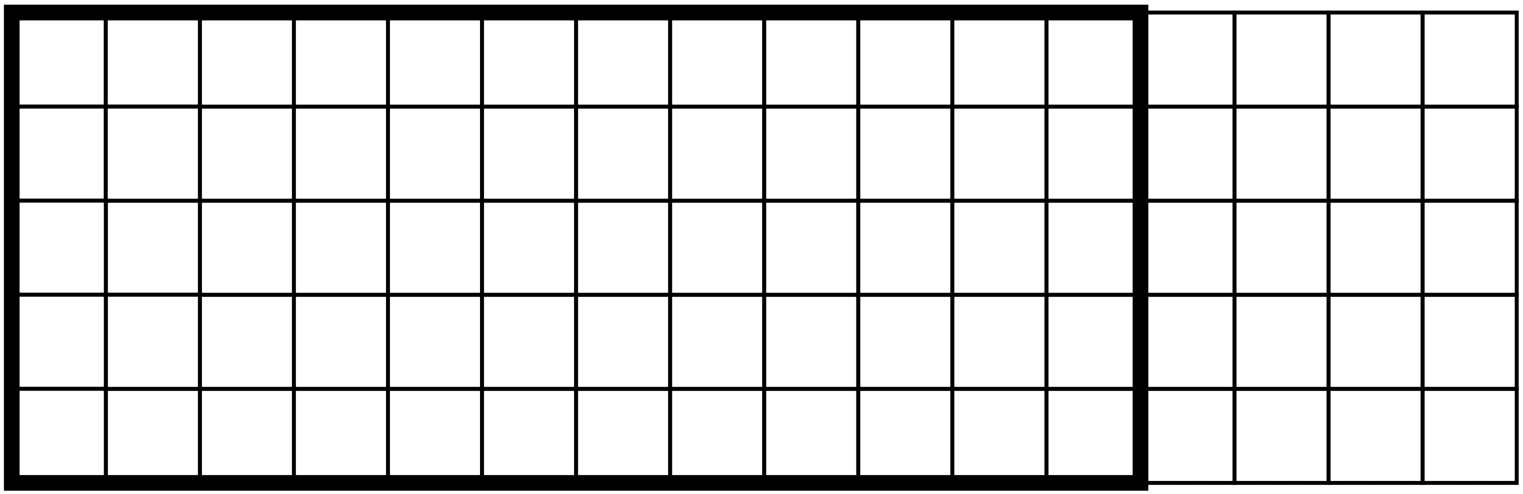

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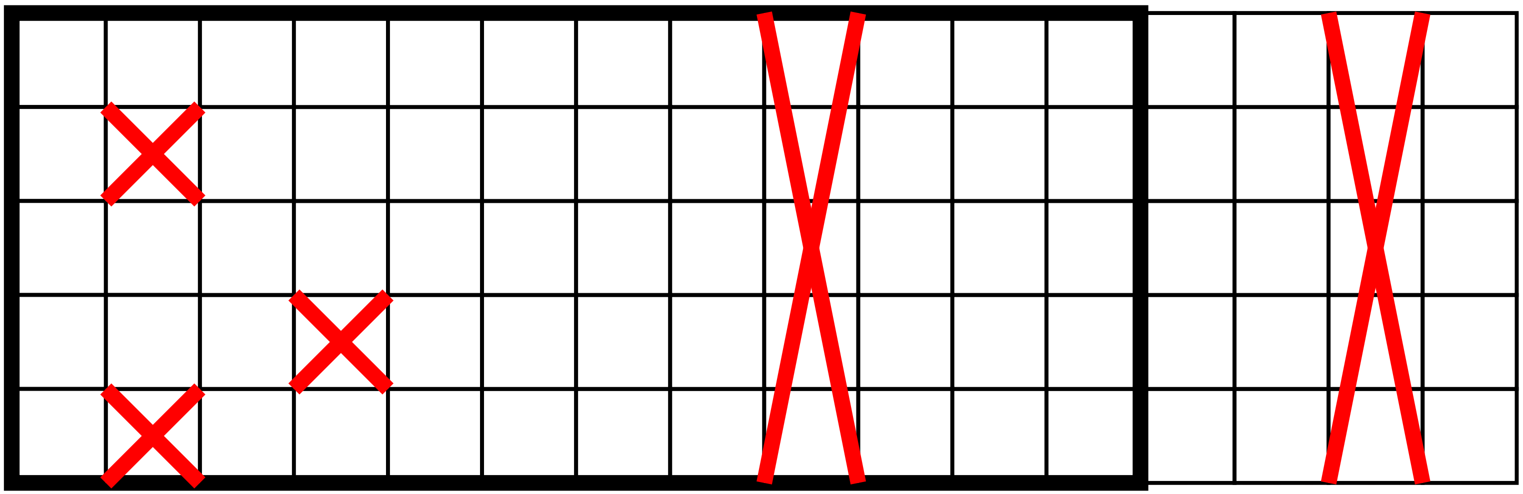

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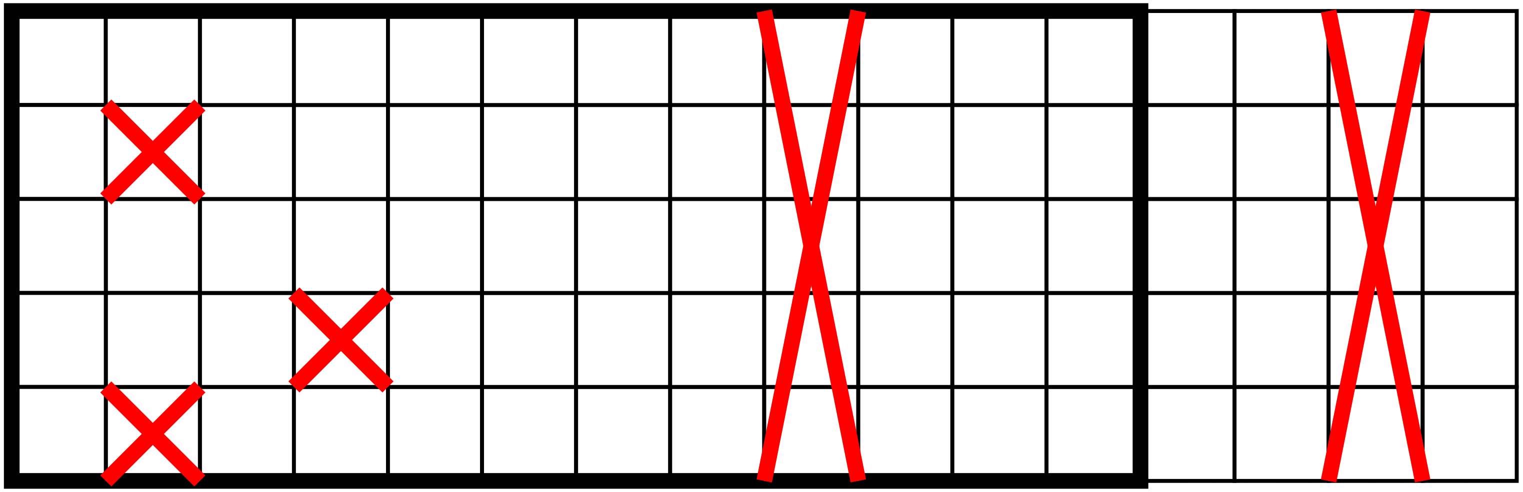
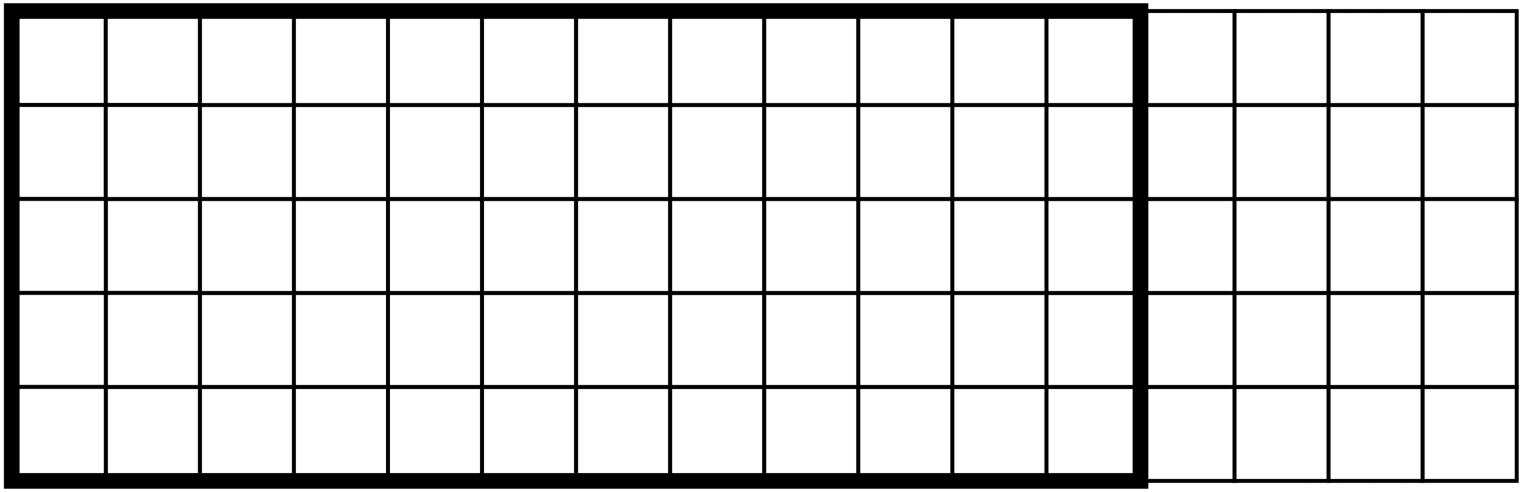


Decoding based on code $\mathcal{C}$ and matrices H_0, H_1, ...

Overview of Proposed Coding Scheme



Decoding based on code $\mathcal{C}$ and matrices $H_0, H_1, \ldots$.

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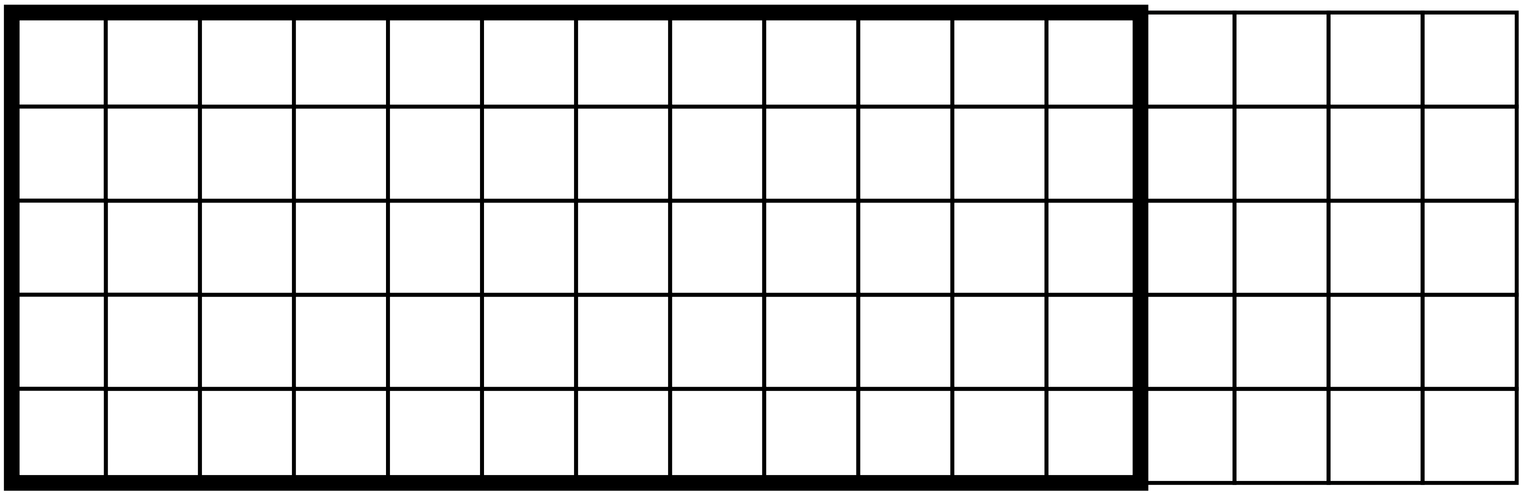

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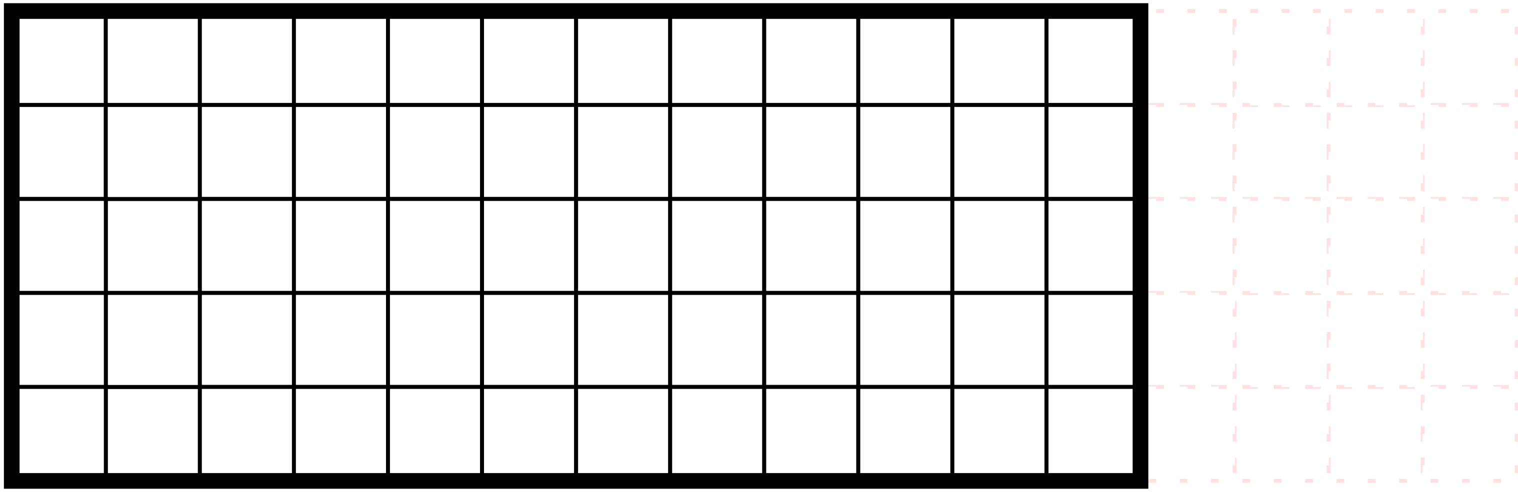

Proposed coding scheme:

Code definition

Definition of Proposed Code

- Let $\mathcal{C}$ be a linear $[n, k, d]$ code over F.

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    every submatrix H_j is invertible over F,
  - every subset of $\delta - 1$ columns in H_{in} is linearly independent.

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such that each row in

$$Z = \left(\underset{m\times 1}{H_0\Gamma_0} \quad \underset{m\times 1}{H_1\Gamma_1} \quad \underset{m\times 1}{\ldots} \quad \underset{m\times 1}{\ldots} \quad \underset{m\times 1}{H_{n-1}\Gamma_{n-1}} \right)$$

is a codeword of $\mathcal{C}$.

Proposed coding scheme:

Correction capabilities

Correction Capability of Proposed Code

	error	erasure
block	τ columns in error	ρ columns erased
symbol	ϑ symbols in error	ϱ symbols erased

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Theorem: There exists a decoder for the code $\mathbb{C}$ that correctly recovers the transmitted array in the presence of the above error and erasure types (which may occur simultaneously), whenever

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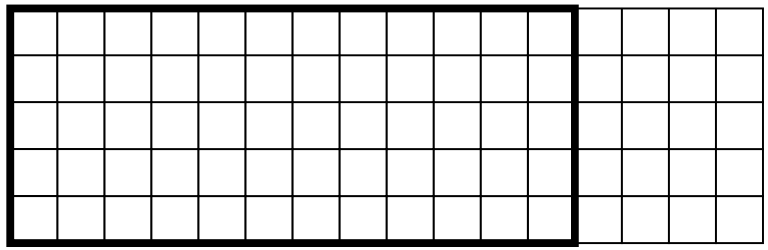
☺ **Surprisingly simple conditions!**

Proposed coding scheme:

Advantages

Advantages of Proposed Coding Scheme



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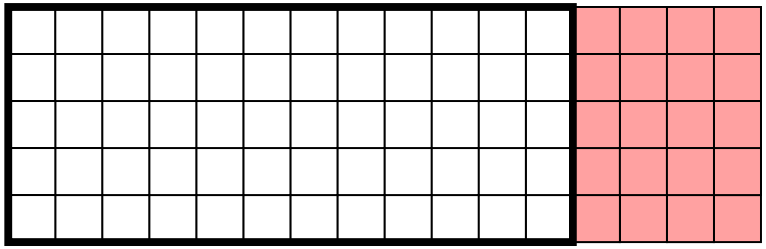

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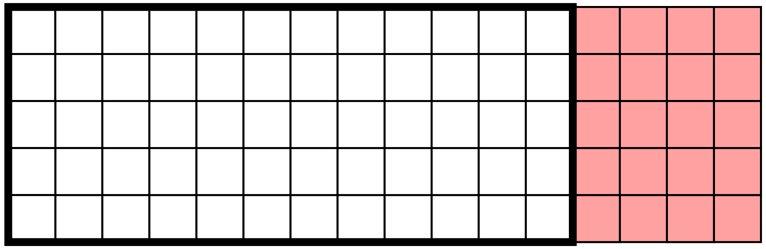
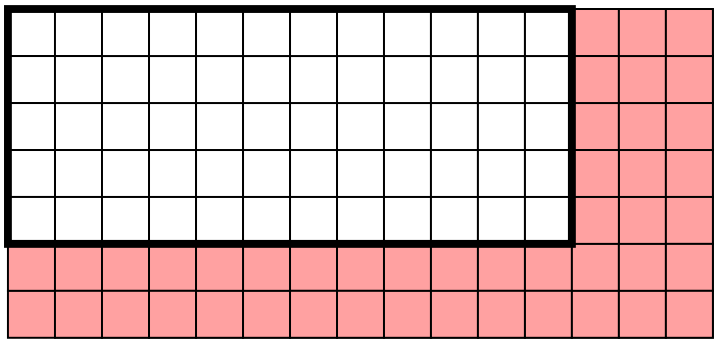
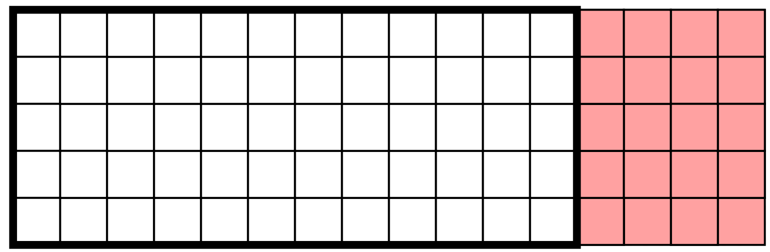


- Multiplication by H_j^{-1} is like a rate-1 inner code.

  $\Rightarrow$ The redundancy of the coding scheme can be independent of n.

Advantages of Proposed Coding Scheme





Overall, we want to handle ϑ symbol errors and ϱ symbol erasures.

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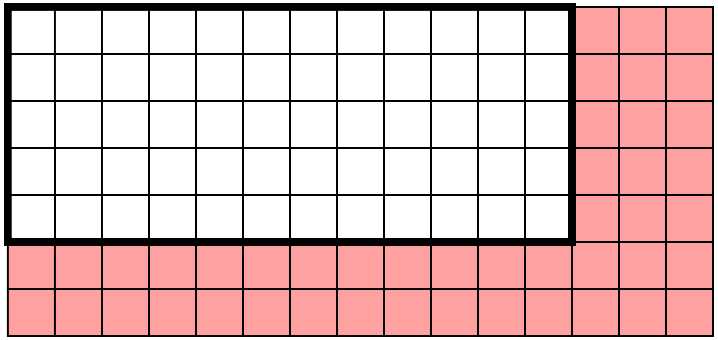

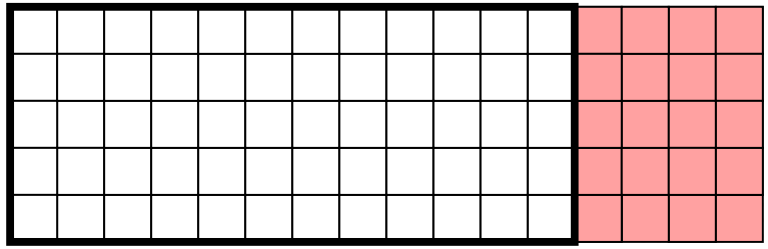

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code $\mathcal{C}_2$ needs to be designed such that it can handle the worst case where ϑ symbol errors and ϱ symbol erasures appear all in the same column.

Proposed code:

"same redundancy symbols" can be used to handle overall ϑ symbol errors and ϱ symbol erasures.

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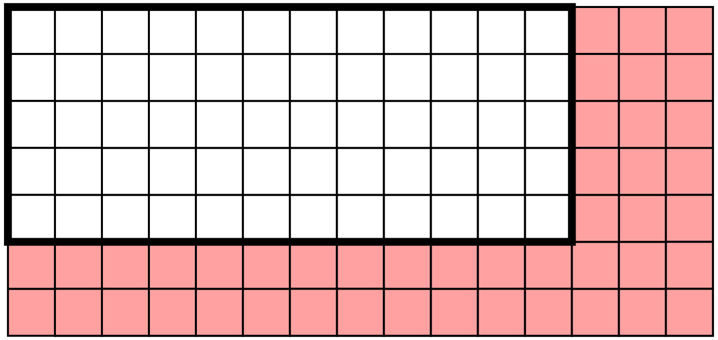

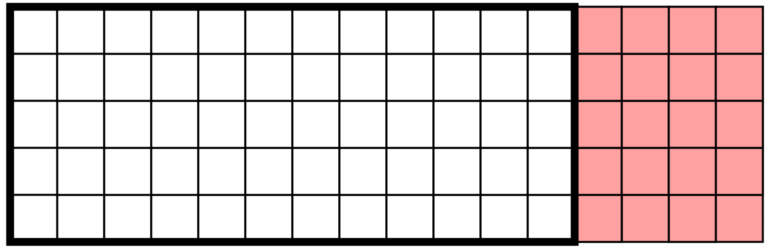

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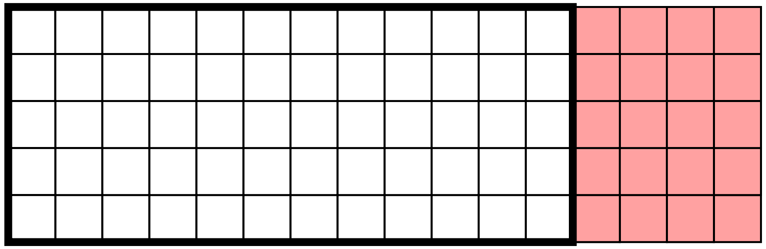
$\Rightarrow$ "Price" that is paid for this:

$$2\tau + \rho \leq d - 2$$

vs.

$$2\tau + \rho \leq d - 1$$

Advantages of Proposed Coding Scheme



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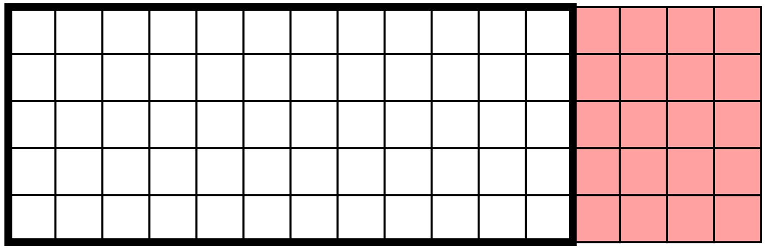
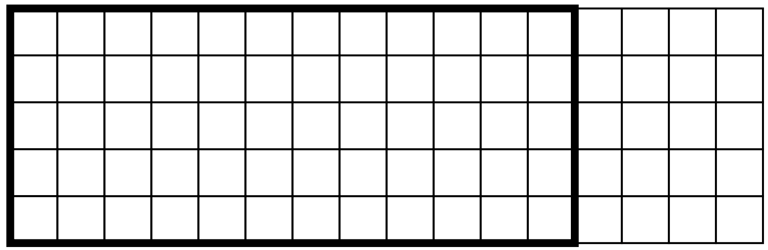


- One can identify a range of code parameters for $\mathbb{C}$ for which the resulting redundancy improves upon the best known. (To the best of our knowledge.)

Advantages of Proposed Coding Scheme



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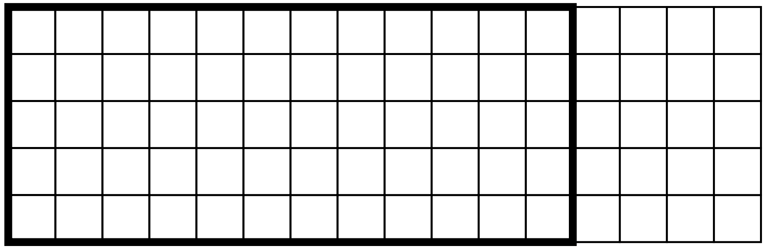
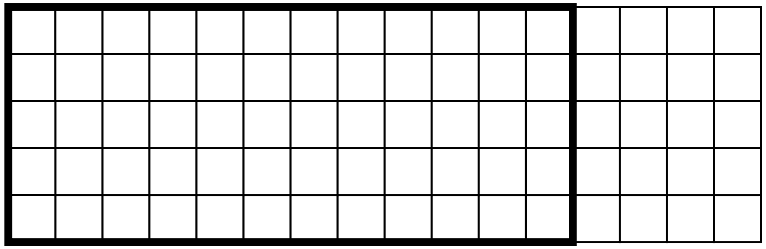


- One can devise efficient decoders for combinations of block and symbol errors and erasures most relevant in practical applications.

Advantages of Proposed Coding Scheme



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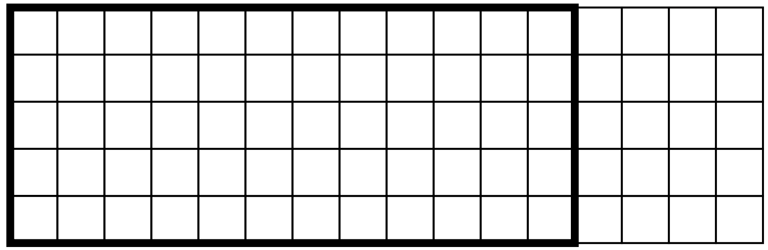

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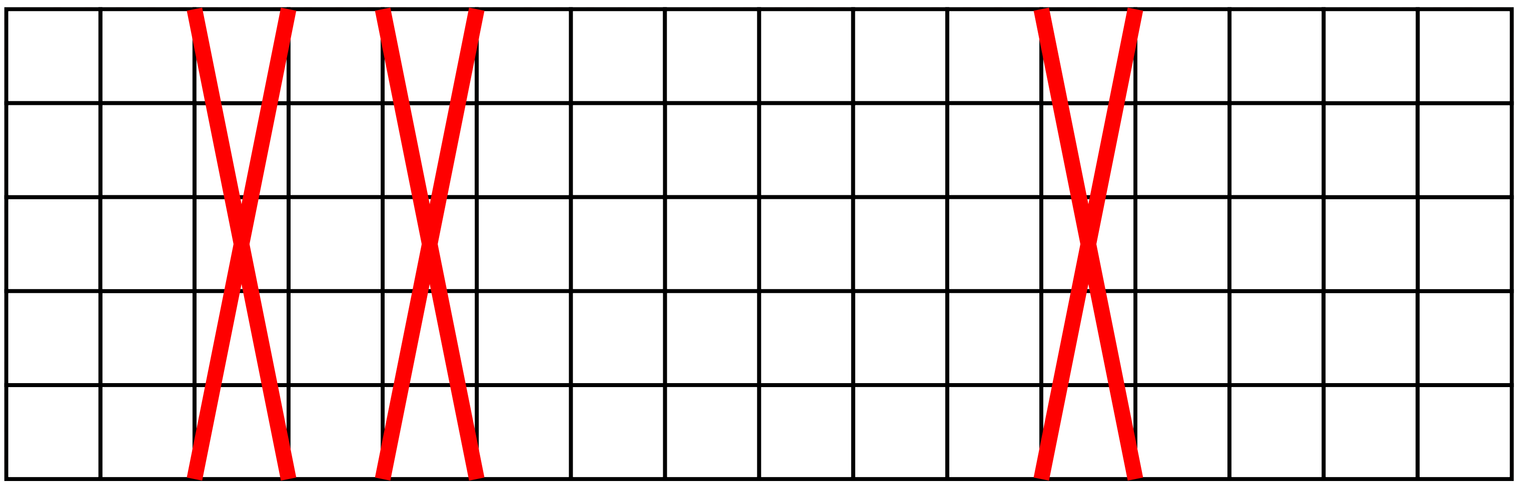
- In particular, these decoders are more efficient than a corresponding decoder for a suitably chosen Reed–Solomon code of length mn over F, assuming such a Reed–Solomon code exists in the first place.
- Finding efficient decoders for the general case is still an open problem.

Decoding

A Simplified Setup

A Simplified Setup: Error Model



- Encoded array has size $m \times n$.
- Up to t burst errors happen.

A Simplified Setup: Encoding

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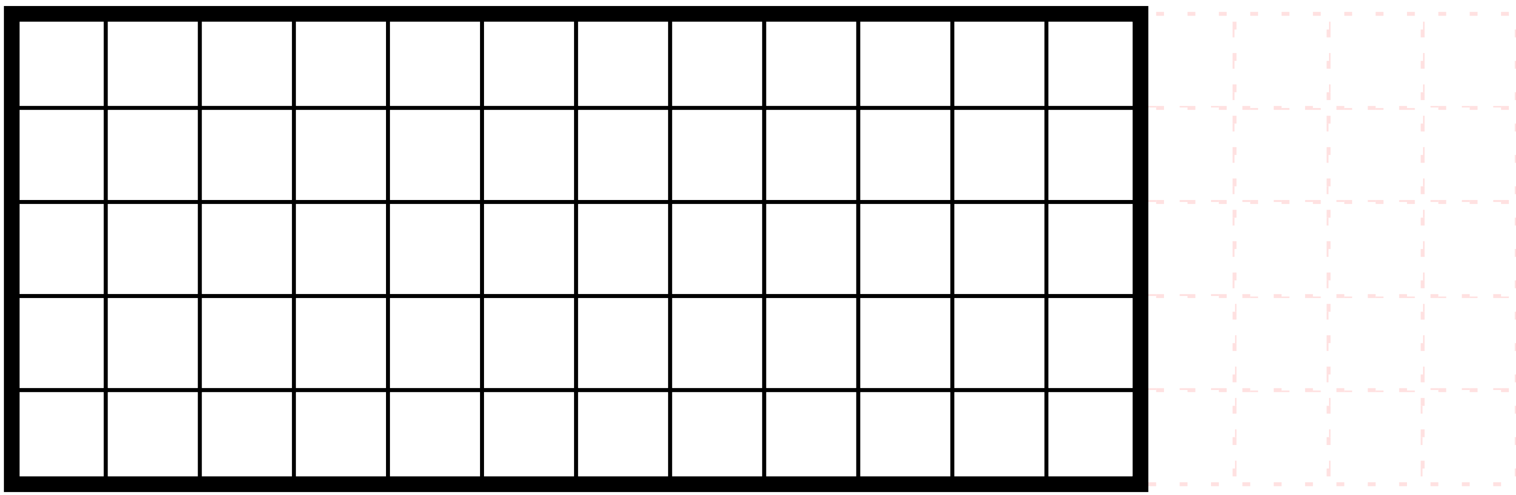

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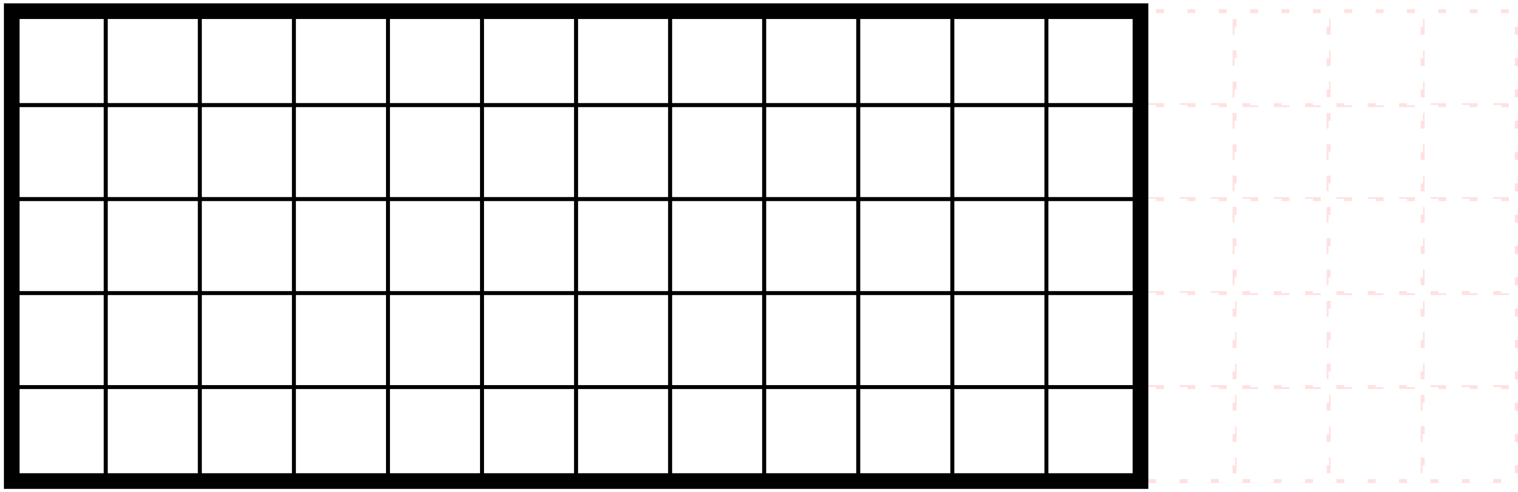

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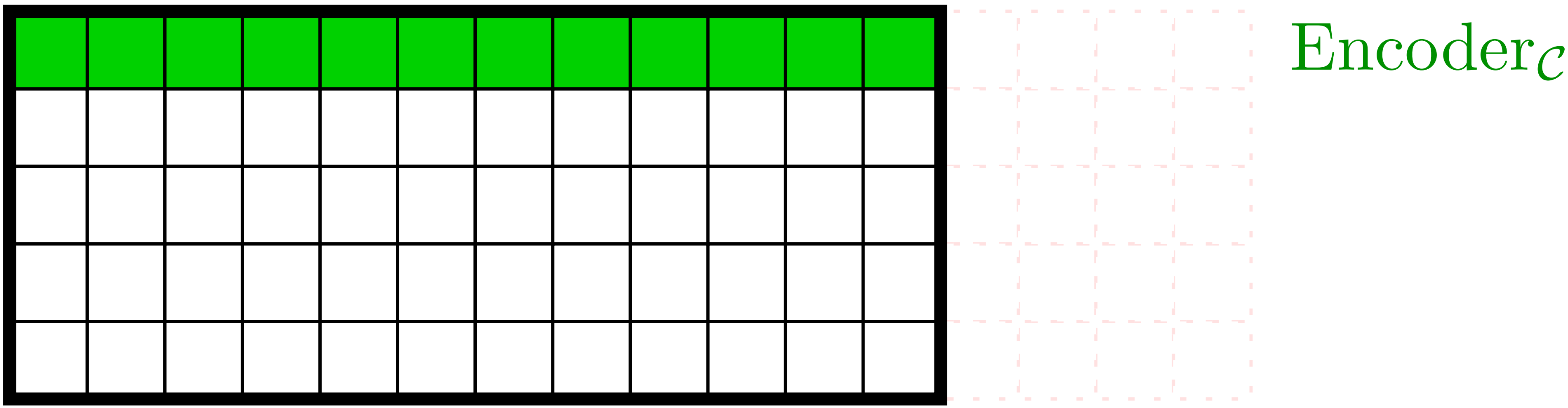

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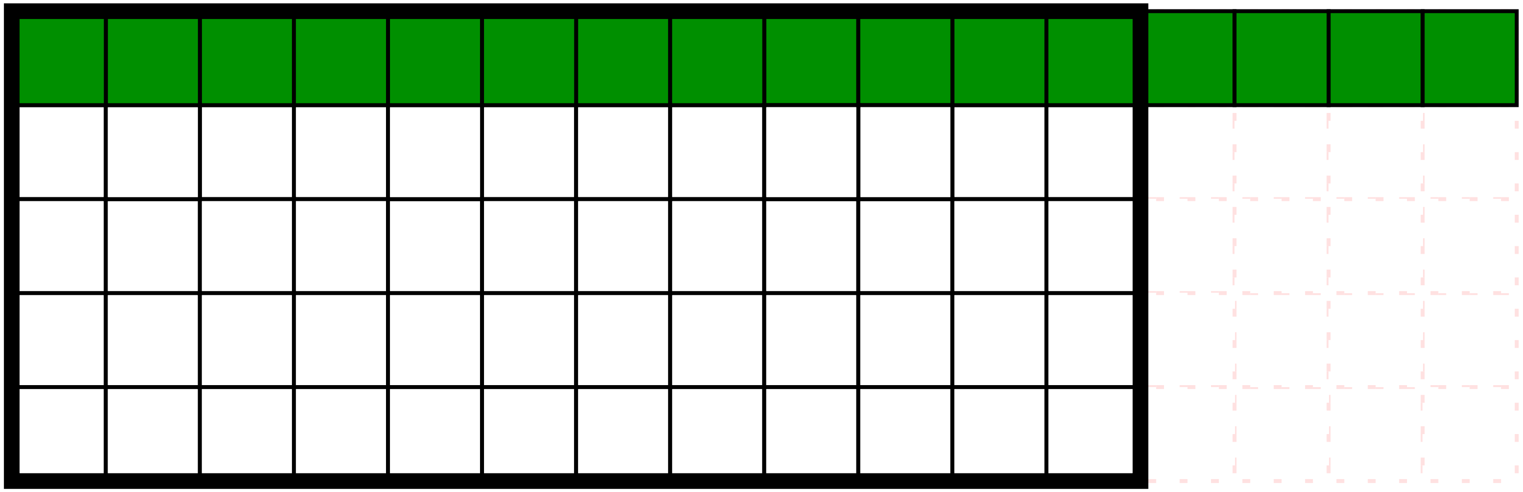

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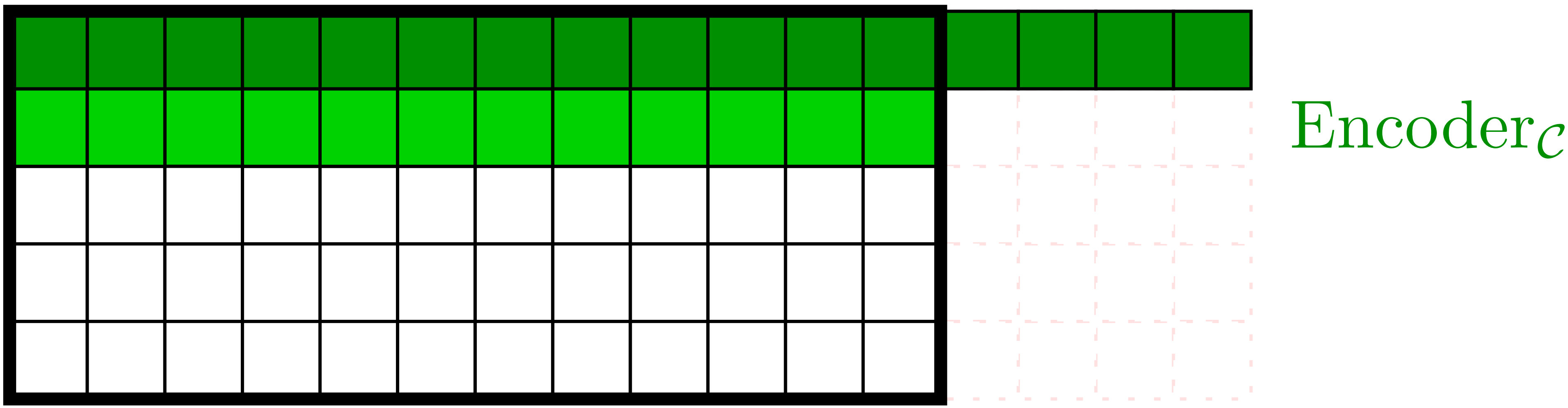

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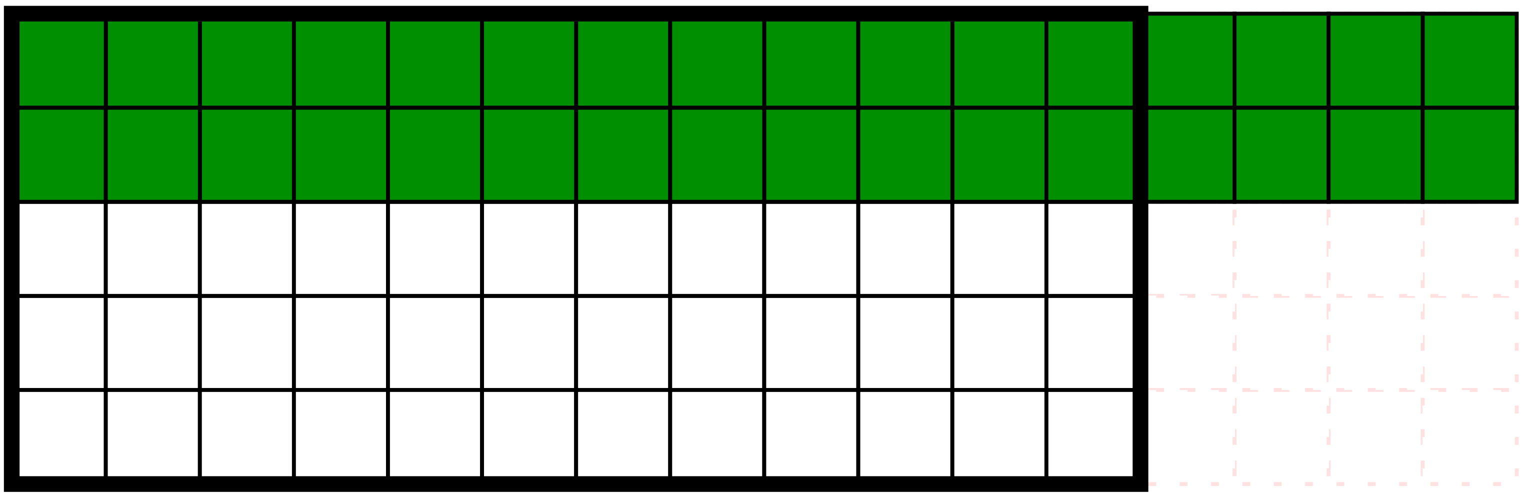

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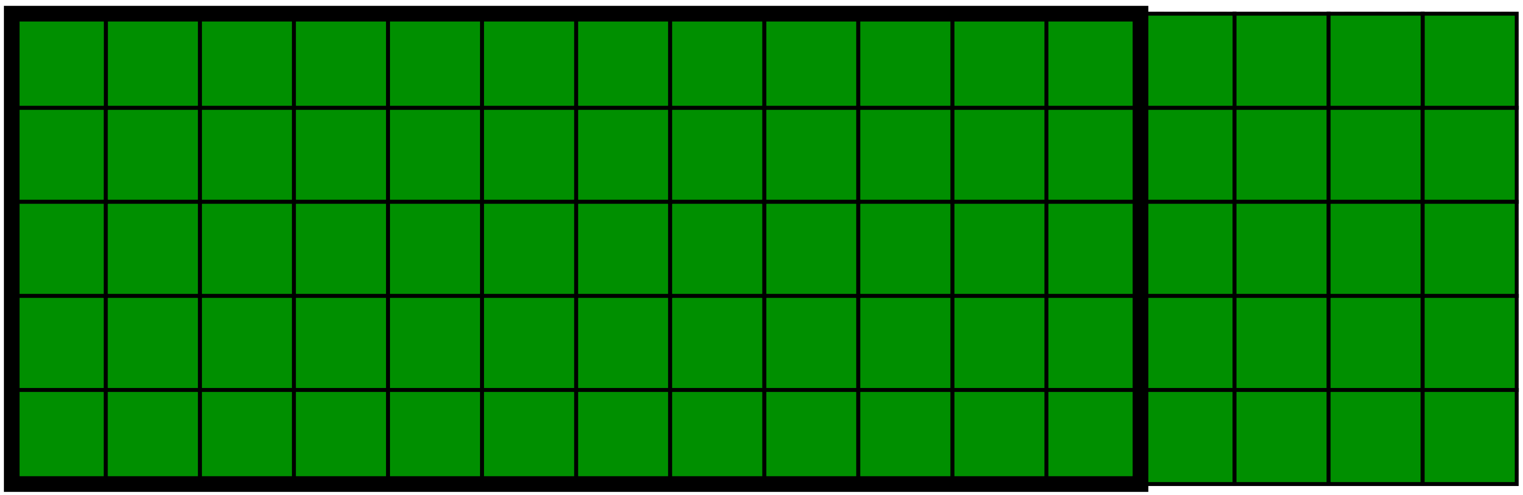

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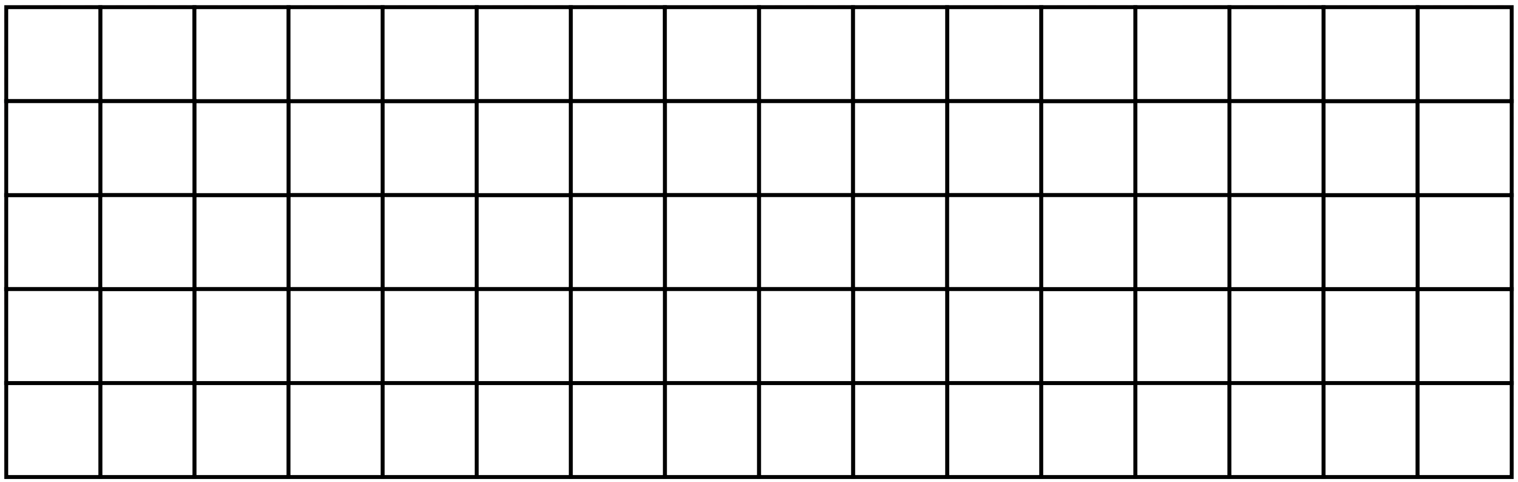

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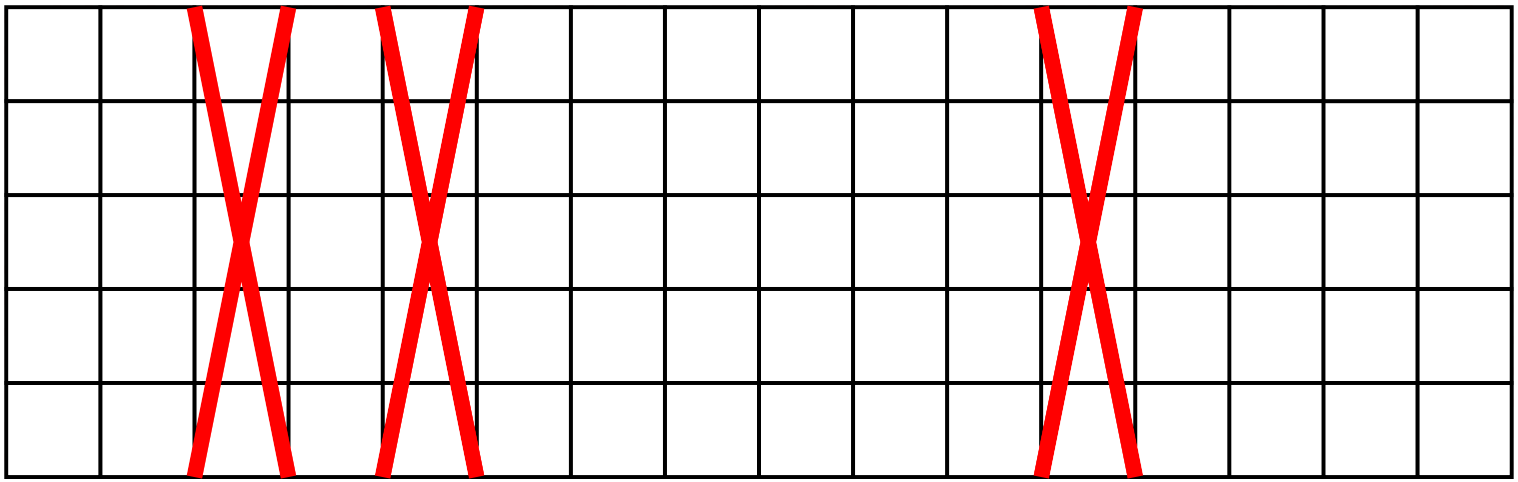

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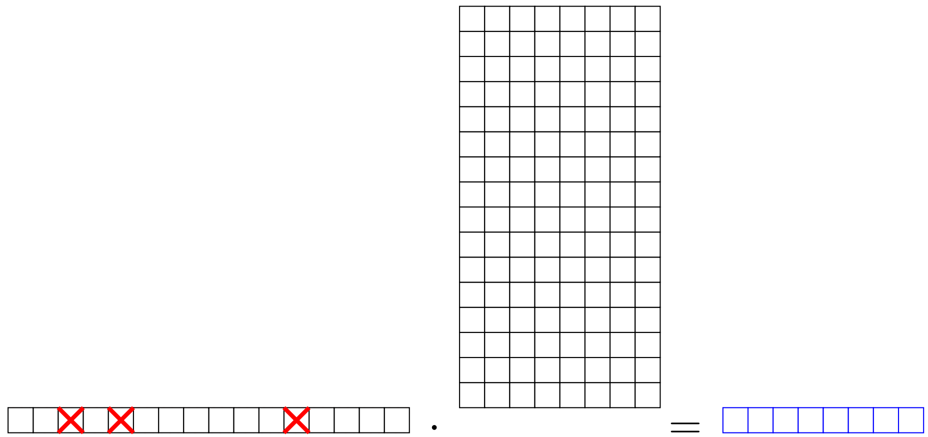
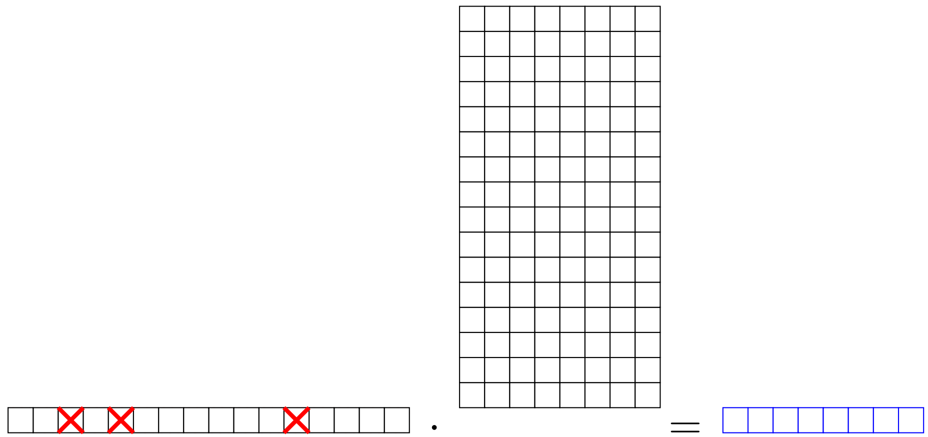


$$E \cdot H^\top = S$$

We consider first the case $m = 1$.

A Simplified Setup: Decoding



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A necessary condition for being able to correct (up to) t errors:

$\#$error patterns that we want to correct $\leq$ $\#$different syndromes.

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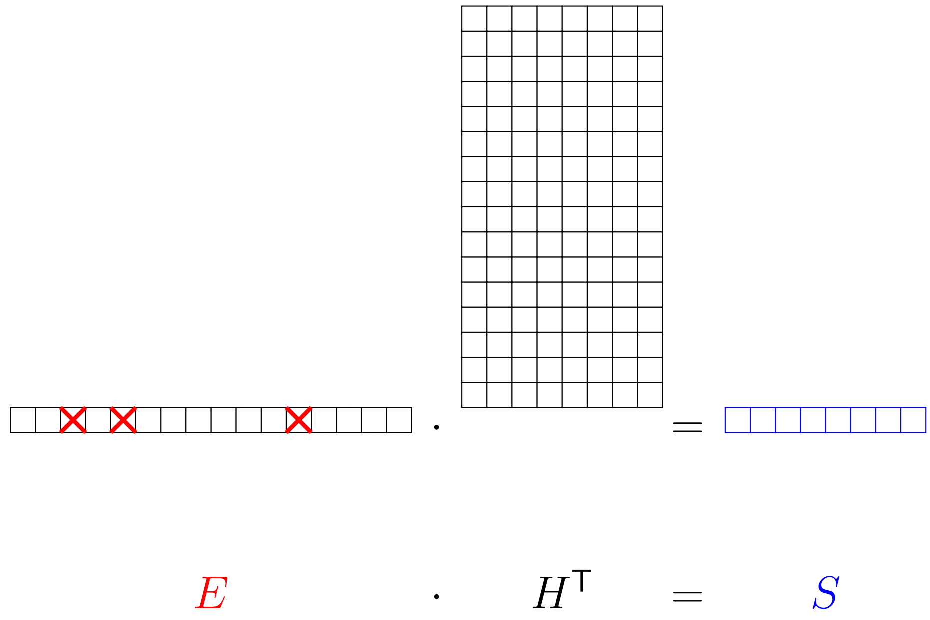

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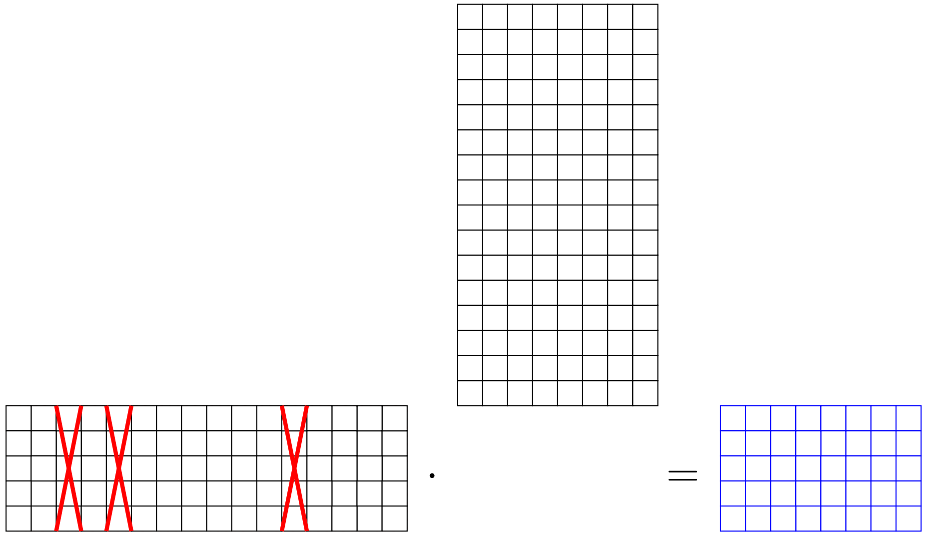
A necessary condition for being able to correct (up to) t errors:

$$\binom{n}{t} \cdot q^t \quad \lesssim \quad q^{n-k}.$$

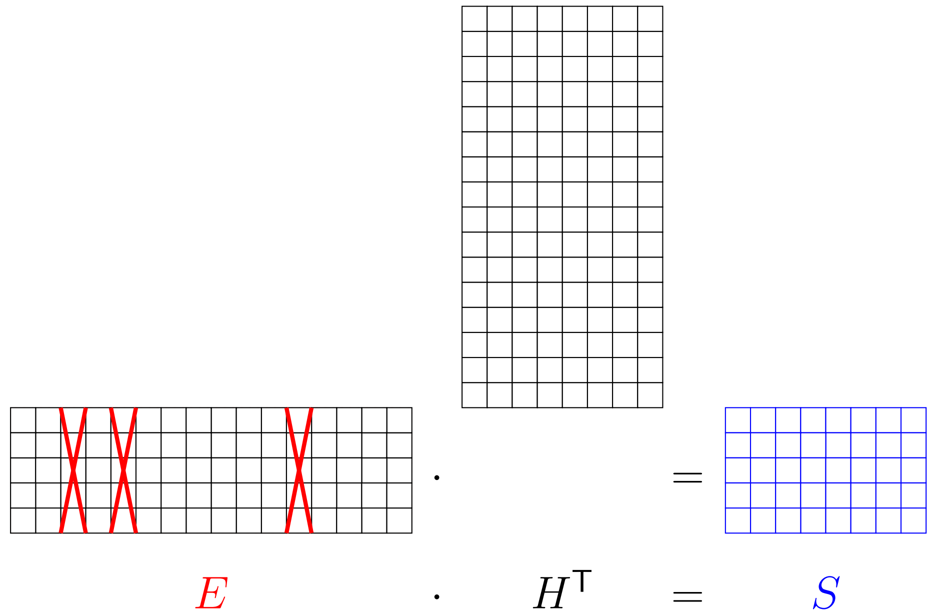
A Simplified Setup: Decoding



$$E \cdot H^{\mathsf{T}} = S$$

Now we consider the case of general m.

A Simplified Setup: Decoding

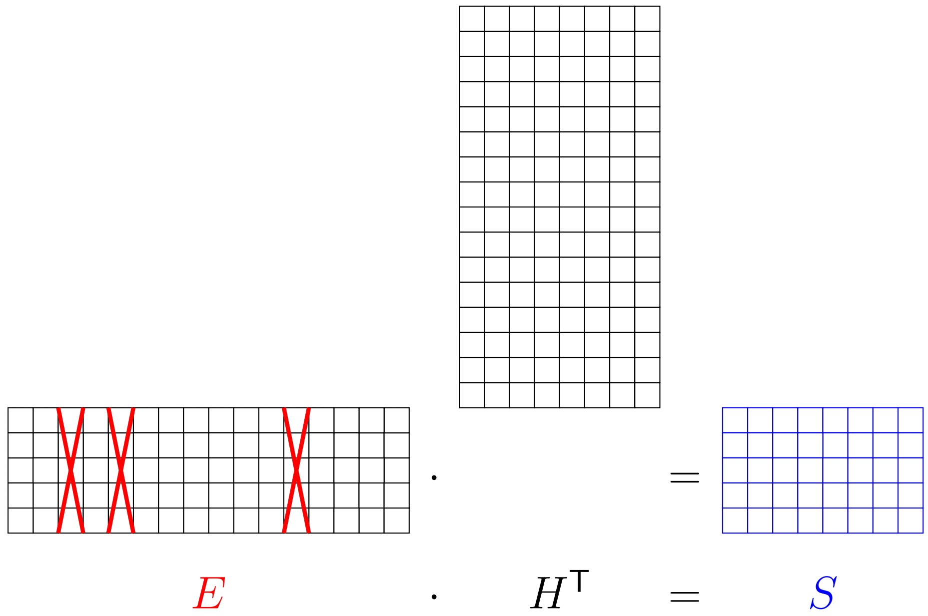


Now we consider the case of general m.

A necessary condition for being able to correct (up to) t burst errors:

$$\#\text{error patterns that we want to correct} \leq \#\text{different syndromes}.$$

A Simplified Setup: Decoding

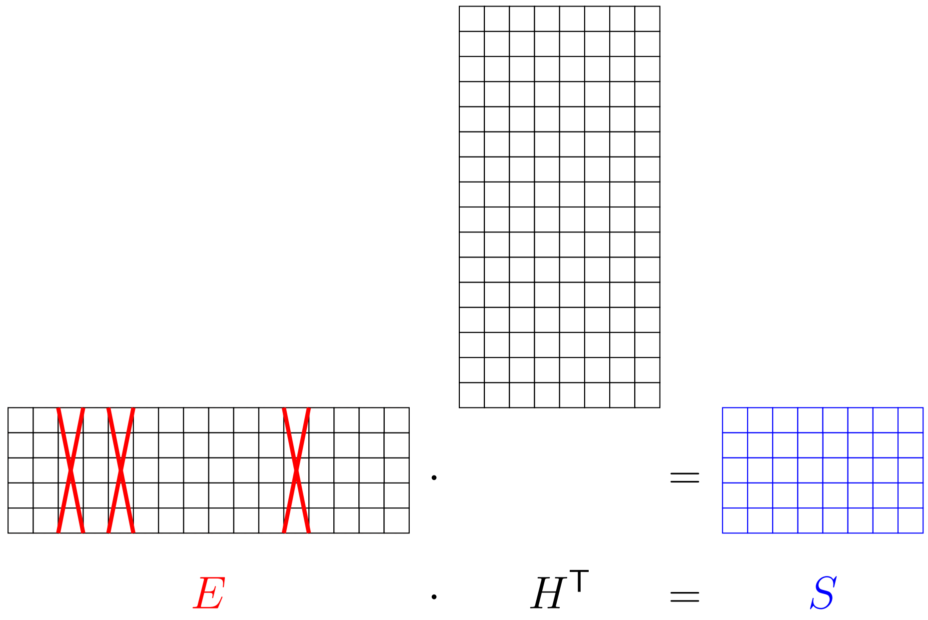


Now we consider the case of general m.

A necessary condition for being able to correct (up to) t burst errors:

$$\binom{n}{t} \cdot q^{mt} \quad \lesssim \quad q^{m(n-k)}.$$

A Simplified Setup: Decoding



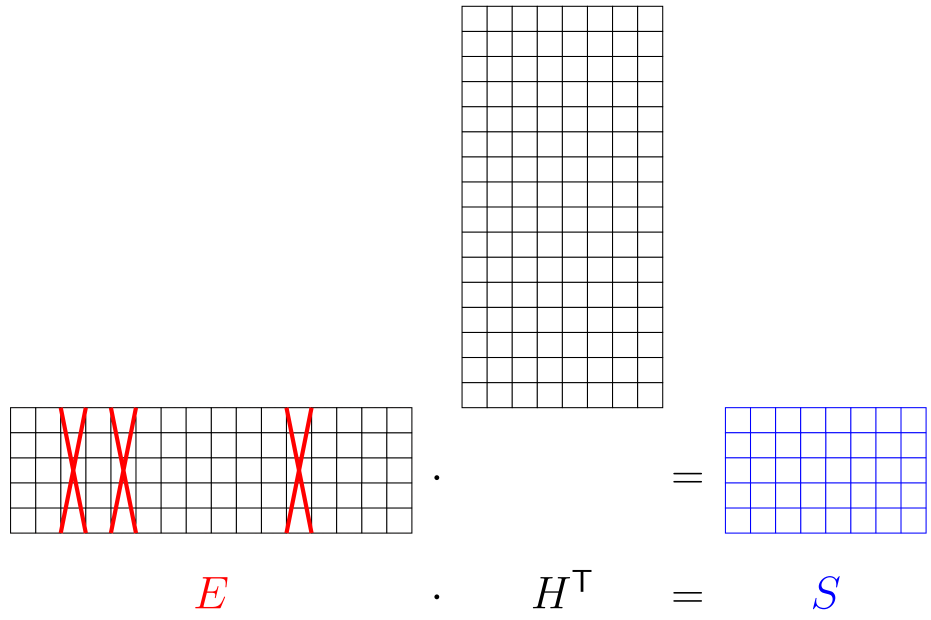
Now we consider the case of general m.

A necessary condition for being able to correct (up to) t burst errors:

$$t \lesssim \frac{m}{m+1} \cdot (n-k).$$

(Assumption: $n \approx q$ and t small.)

A Simplified Setup: Decoding



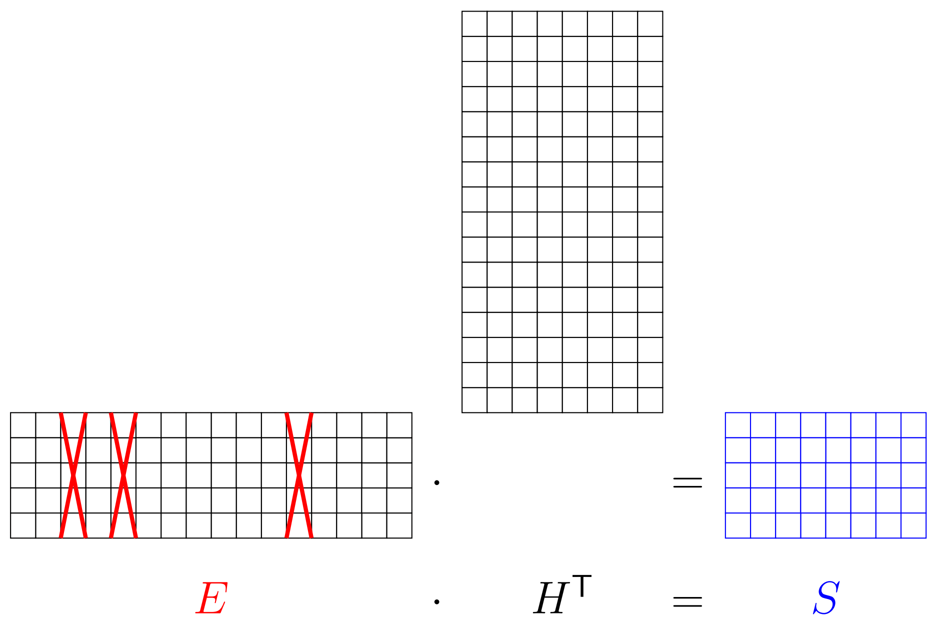
Now we consider the case of general m.

A necessary condition for being able to correct (up to) t burst errors:

$$t \leq \frac{1}{2} \cdot \big(n - k + \mathrm{rank}(E) - 1\big)$$

(Assumption: MDS code.)

A Simplified Setup: Decoding



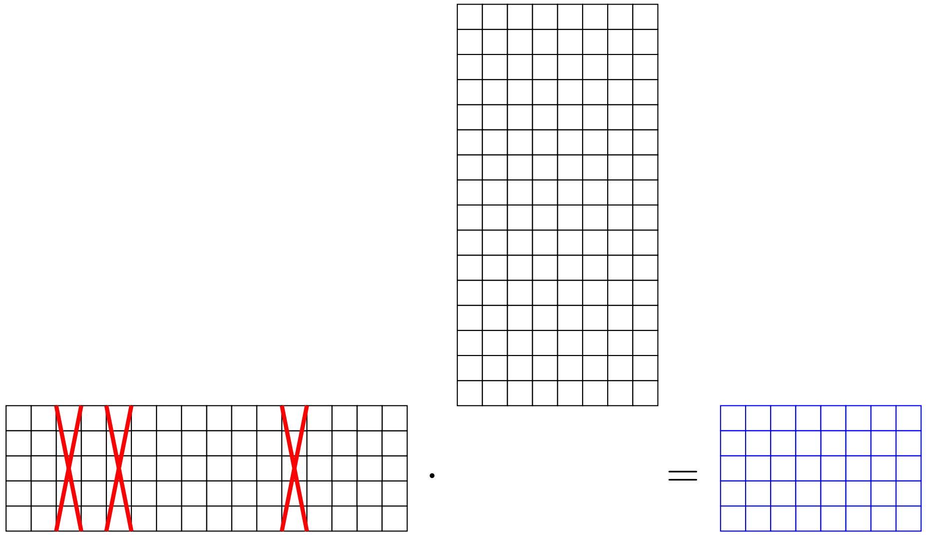
Now we consider the case of general m.

A necessary condition for being able to correct (up to) t burst errors:

$$t \leq \frac{1}{2} \cdot \left(n - k + \mathsf{rank}(E) - 1\right) = \begin{cases} \frac{1}{2} \cdot (n-k) & (\mathsf{rank}(E) = 1) \\ n - k - 1 & (\mathsf{rank}(E) = t) \end{cases}.$$

(Assumption: $\mathcal{C}$ is an MDS code.)

A Simplified Setup: Decoding



$$E \cdot H^{\top} = S$$

Now we consider the case of general m.

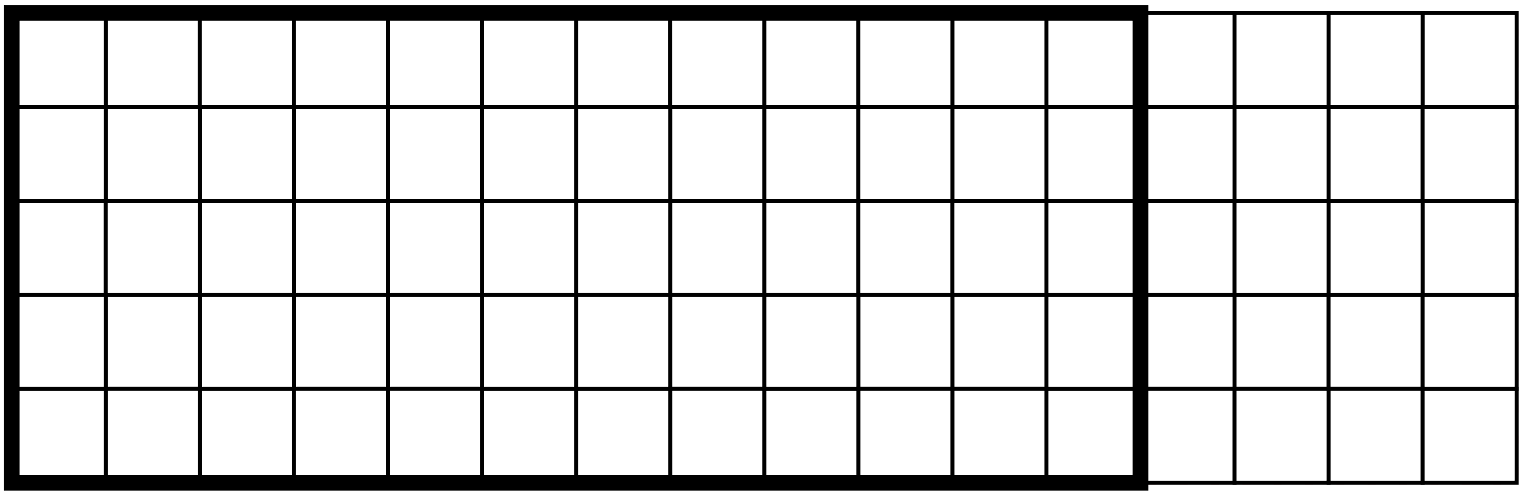
Remarkable:

Decoding can be done by Gaussian elimination,
independently of the chosen code $\mathcal{C}$.

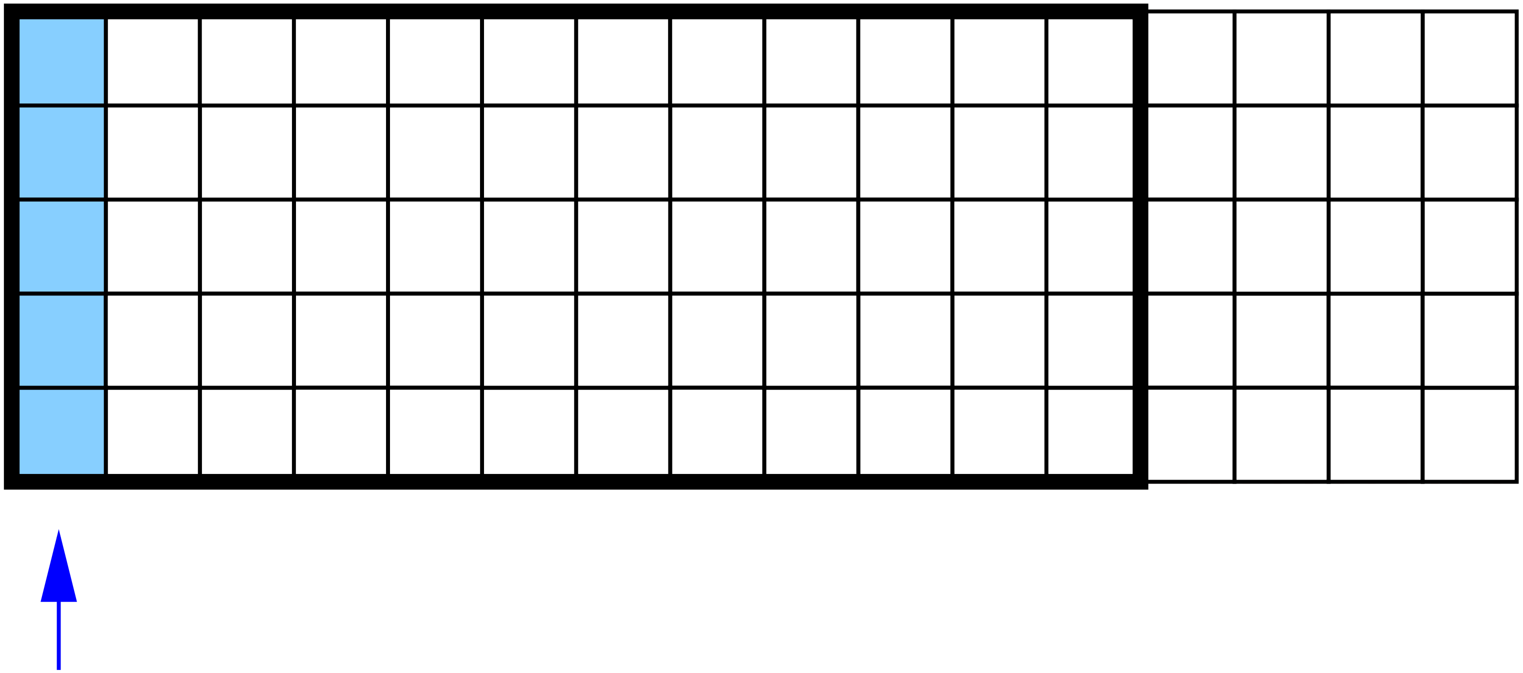
See [Metzner, Kapturowski, 1990] and [Haslach, Vinck, 2000/2001].

Proposed Coding Scheme: Encoding and Decoding

Assume that the rows have already been encoded.

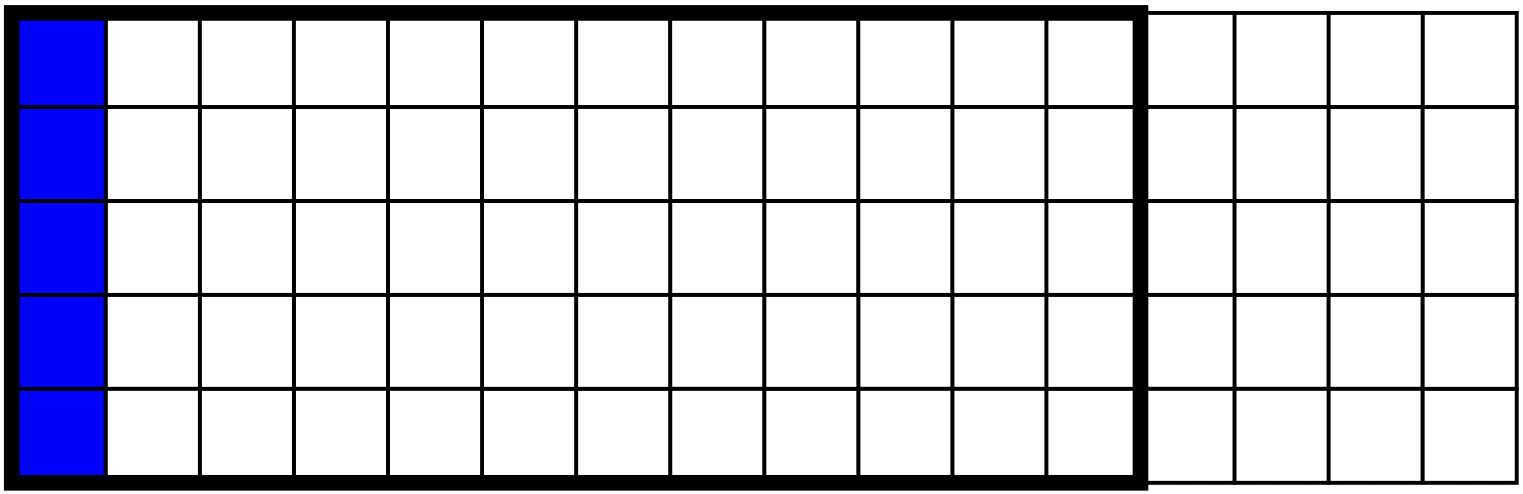


Proposed Coding Scheme: Encoding and Decoding

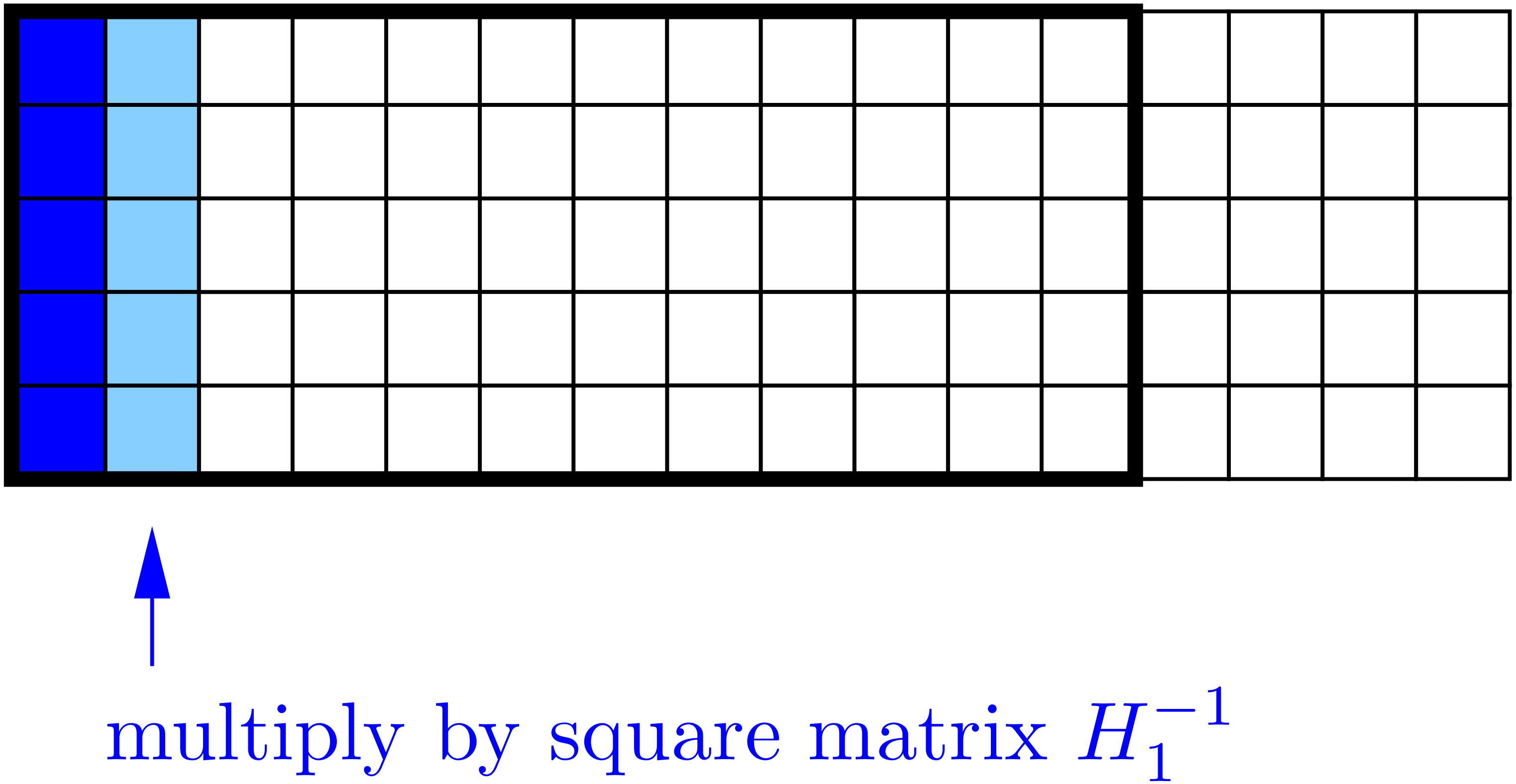


multiply by square matrix H_0^{-1}

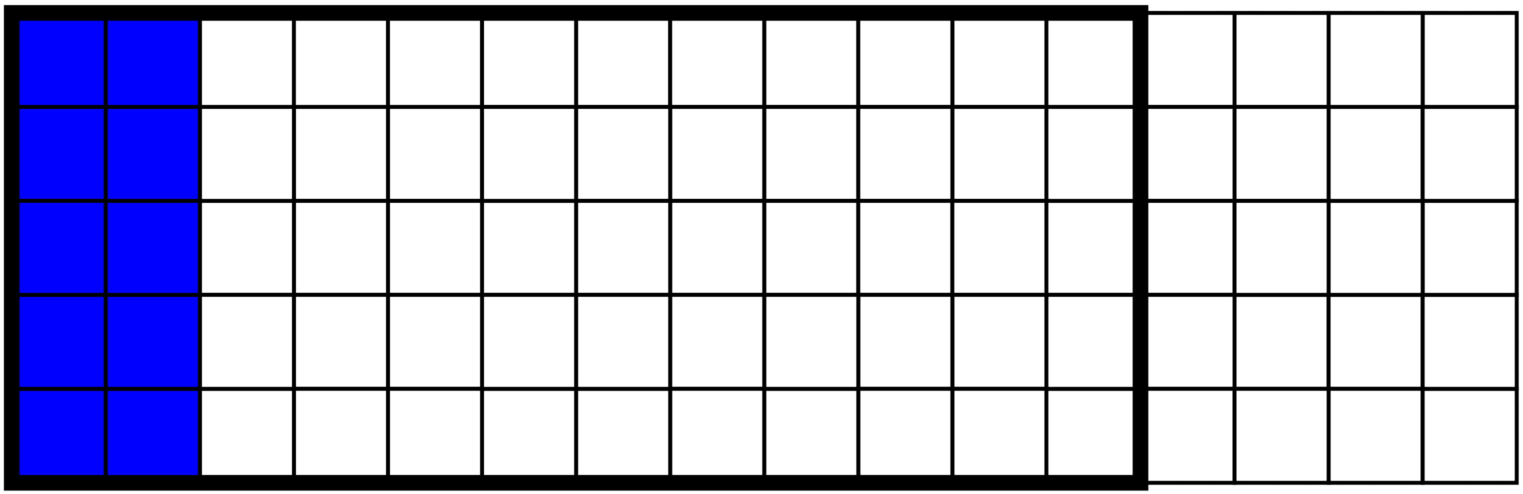
Proposed Coding Scheme: Encoding and Decoding



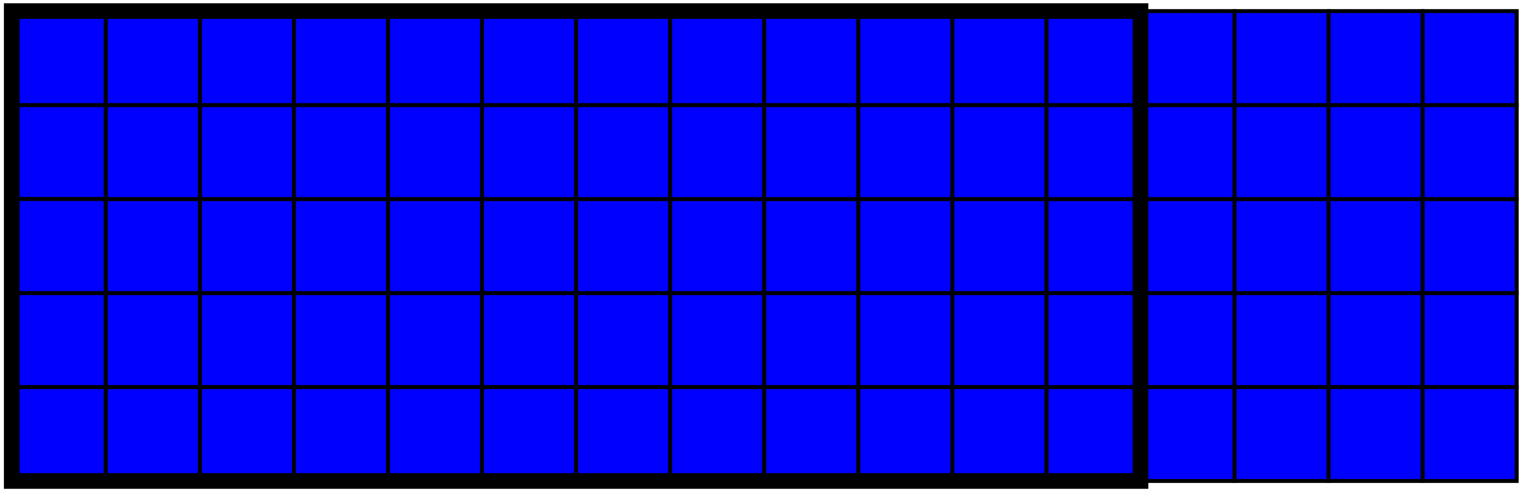
Proposed Coding Scheme: Encoding and Decoding



Proposed Coding Scheme: Encoding and Decoding

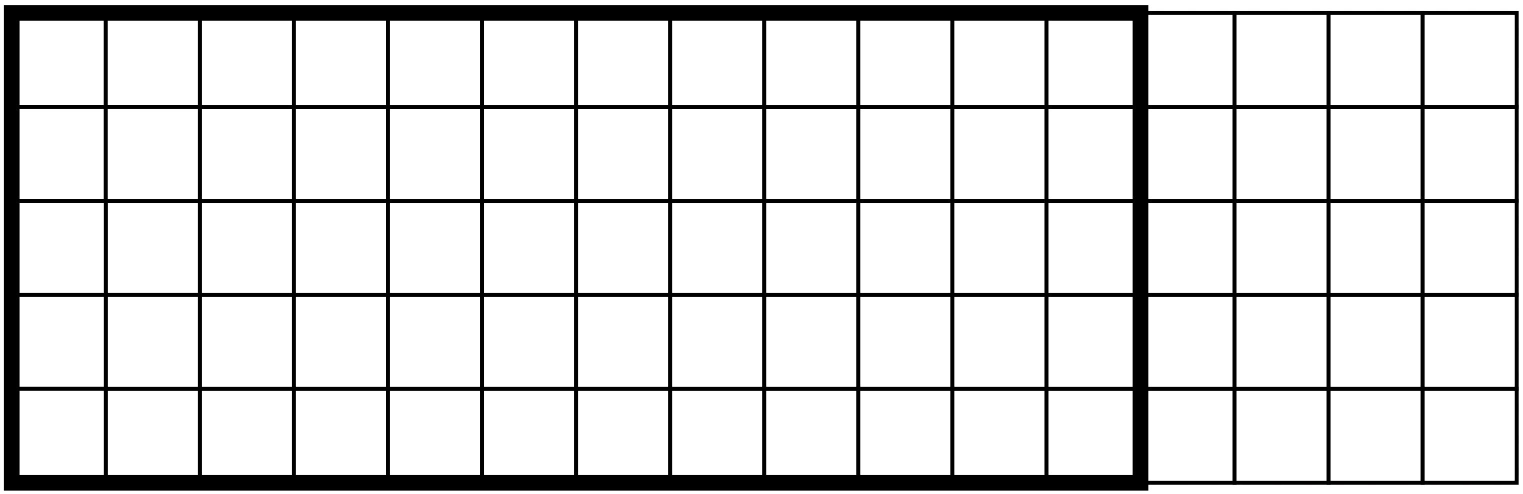


Proposed Coding Scheme: Encoding and Decoding

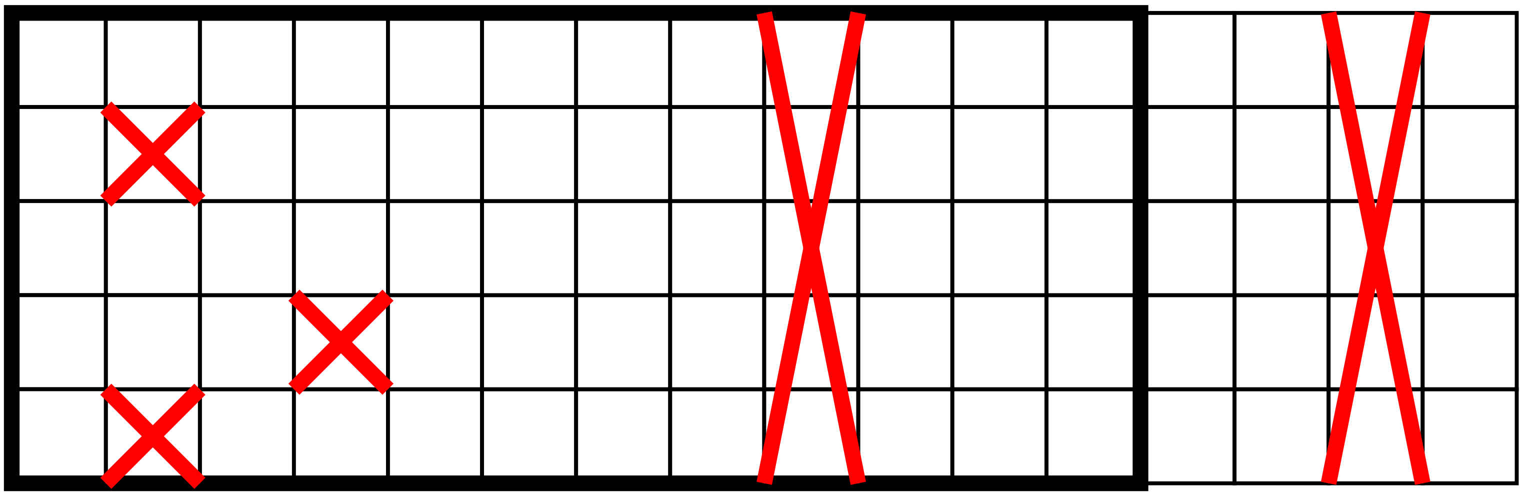


$m \times n$ array of codeword symbols

Proposed Coding Scheme: Encoding and Decoding

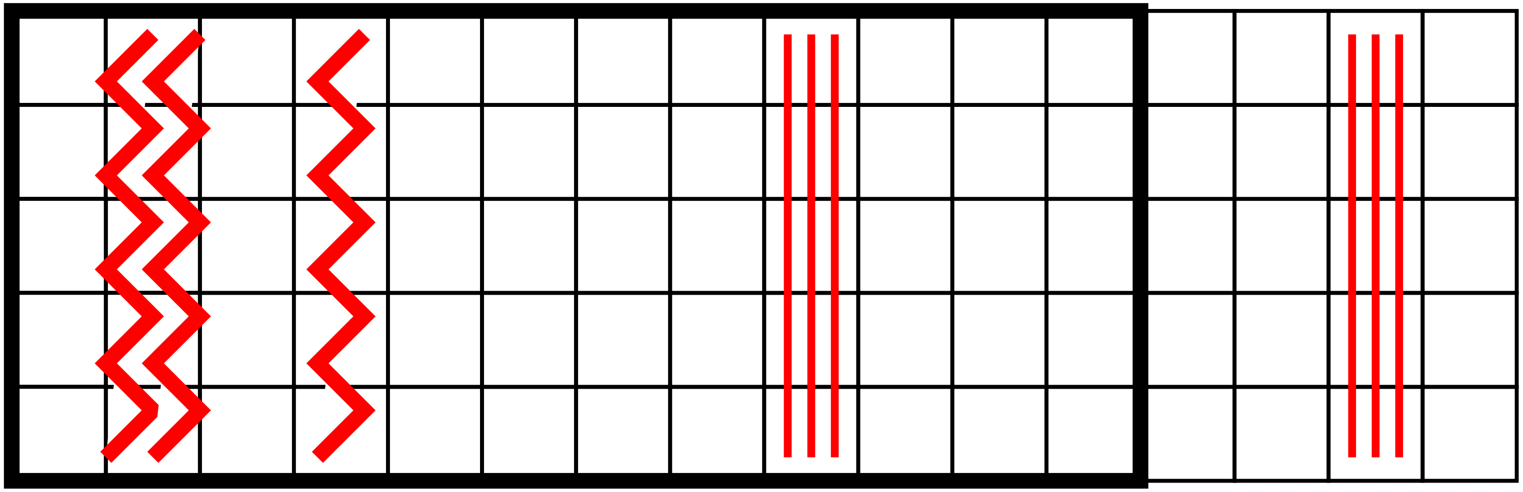


Proposed Coding Scheme: Encoding and Decoding



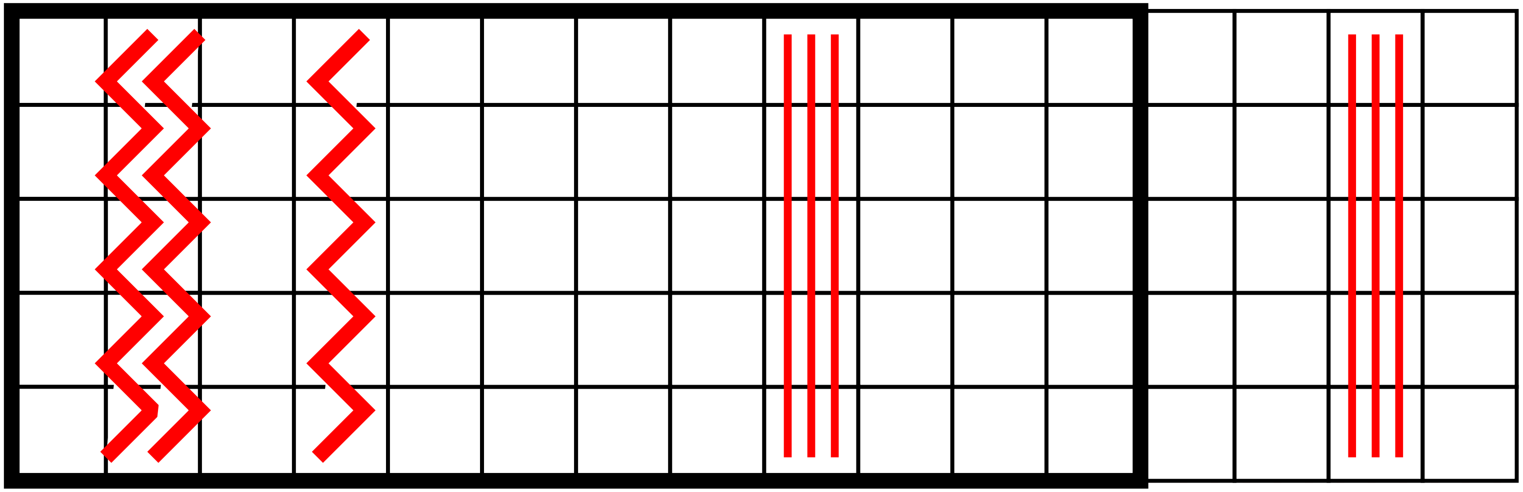
Proposed Coding Scheme: Encoding and Decoding

First decoding step: multiply the i-th column by H_i.



Proposed Coding Scheme: Encoding and Decoding

First decoding step: multiply the i-th column by H_i.



Decoder then takes advantage of special structure of burst errors in this modified array.

Conclusions

Conclusions

- Motivation for block and symbol errors / erasures.
- Discussed traditional and novel ECC schemes
  that can handle such errors / erasures with low complexity
  and compared their advantages / disadvantages.
- Finding efficient decoders for the general case is still an open problem.
- More details, in particular decoding schemes and comparison with other coding schemes, can be found in the paper available at **http://arxiv.org/abs/1302.1931**

Original Talk Topic

Connections between
Techniques for Analyzing Finite State Channels and
Techniques for Analyzing Graphical Models

- Theorems 26 and Corollary 27 in

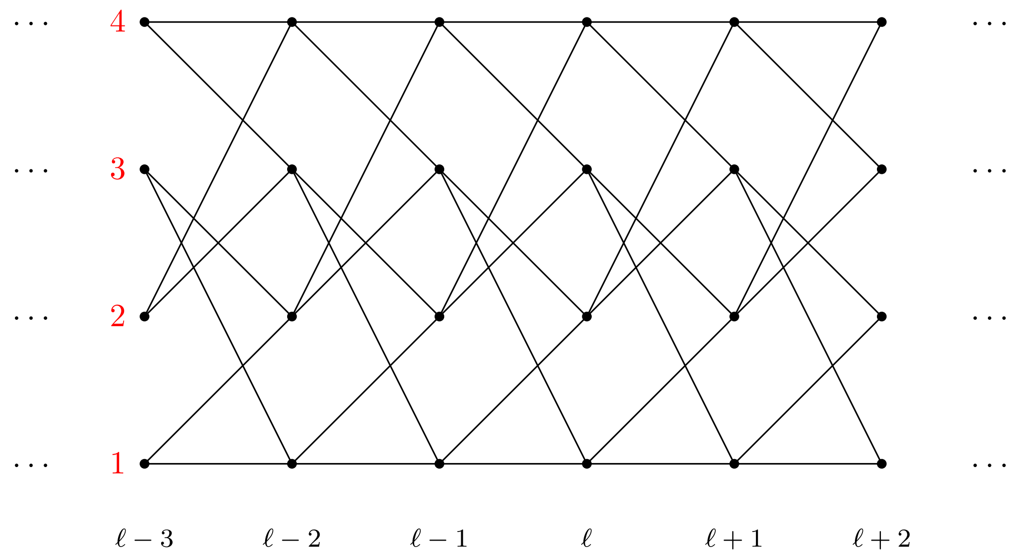
  P. O. Vontobel, "The Bethe permanent of a non-negative matrix," IEEE Trans. Inf. Theory, vol. 59, no. 3, pp. 1866–1901, Mar. 2013.

- Theorem 8 and Corollary 9 in

  H. D. Pfister and P. O. Vontobel, "On the relevance of graph covers and zeta functions for the analysis of SPA decoding of cycle codes," Proc. ISIT 2013.

Optimization Problem

$$\max_{\{\mu_i, p_{ij}\}} f\big(\{\mu_i, p_{ij}\}\big)$$



$$f\big(\{\mu_i, p_{ij}\}\big) = -\sum_{(i,j)\in\mathcal{B}} \mu_i \cdot p_{ij} \cdot \log(p_{ij})$$

$\mathcal{S}$: state alphabet

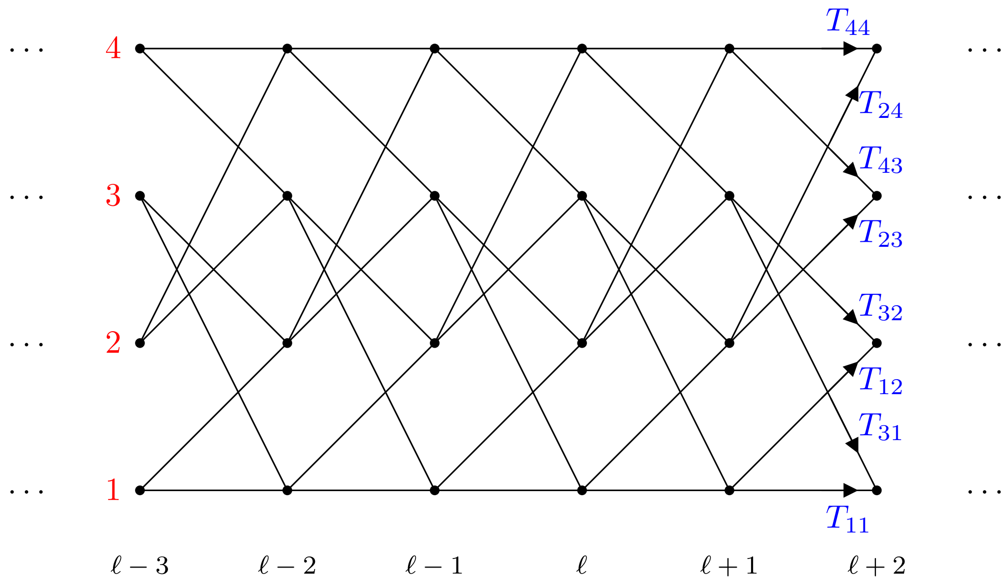
$\mathcal{B}$: branch alphabet

$\{\mu_i\}$: stationary state probabilities

$\{p_{ij}\}$: transition probabilities

Optimization Problem

$$\max_{\{\mu_i, p_{ij}\}} f\big(\{\mu_i, p_{ij}\}\big)$$



$$f\big(\{\mu_i, p_{ij}\}\big) = -\sum_{(i,j)\in\mathcal{B}} \mu_i \cdot p_{ij} \cdot \log(p_{ij}) + \sum_{(i,j)\in\mathcal{B}} \mu_i \cdot p_{ij} \cdot T_{ij}$$

$\mathcal{S}$: state alphabet

$\mathcal{B}$: branch alphabet

$\{\mu_i\}$: stationary state probabilities

$\{p_{ij}\}$: transition probabilities

$\{T_{ij}\}$: branch weights

Solution of this Optimization Problem

Definitions:

- "Noisy adjacency matrix" $\mathbf{A}$ with $A_{ij} = \begin{cases} \mathrm{e}^{T_{ij}} & \text{if } (i,j) \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$.
- Let ρ be the maximal (real) eigenvalue of $\mathbf{A}$.
- Let $\boldsymbol{\beta}^{\mathsf{T}}$ and $\boldsymbol{\gamma}$ be the corresponding left and right eigenvectors.

Then:

$$p_{ij}^{*} = \frac{\gamma_j}{\gamma_i} \cdot \frac{A_{ij}}{\rho} \qquad (i,j) \in \mathcal{B},$$

$$\mu_i^{*} \propto \beta_i \cdot \gamma_i \qquad i \in \mathcal{S},$$

$$f\left(\{\mu_i^{*}, p_{ij}^{*}\}\right) = \log(\rho).$$

This and similar problems were solved in, e.g., [Justesen, Høholdt, 1984], [Khayrallah, Neuhoff, 1996], [V., Kavčić, Arnold, Loeliger, 2008].


Thank you!
http://arxiv.org/abs/1302.1931