Capacity of a POST Channel with and without Feedback

Workshop On Coding and Information Theory

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directed information

$$I(X^n \to Y^n) \triangleq \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1})$$

directed information

$$I(X^n \to Y^n) \triangleq \sum_{i=1}^{n} I(X^{i}; Y_i | Y^{i-1})$$

compare to:

$$I(X^n; Y^n) = \sum_{i=1}^{n} I(X^n; Y_i | Y^{i-1})$$

Directed Information [Massey90] inspired by [Marko 73]

$$I(X^n \to Y^n) \triangleq H(Y^n) - H(Y^n || X^n)$$

$$I(X^n; Y^n) \triangleq H(Y^n) - H(Y^n | X^n)$$

Causal Conditioning [Kramer98]

$$H(Y^n || X^n) \triangleq E[-\log P(Y^n || X^n)]$$

$$H(Y^n | X^n) \triangleq E[-\log P(Y^n | X^n)]$$

$$P(y^n || x^n) \triangleq \prod_{i=1}^{n} P(y_i | x^i, y^{i-1})$$

$$P(y^n || x^{n-1}) \triangleq \prod_{i=1}^{n} P(y_i | x^{i-1}, y^{i-1})$$

causal conditioning

$$P(y^n||x^n) \triangleq \prod_{i=1}^{n} P(y_i|x^i, y^{i-1})$$

$$P(y^n||x^{n-1}) \triangleq \prod_{i=1}^{n} P(y_i|x^{i-1}, y^{i-1})$$

$$p(x^n, y^n) = p(x^n||y^{n-1})p(y^n||x^n)$$

why directed information?

consider, e.g.:

$$\mathrm{BSC}(1/2)$$

$$X_i = Y_{i-1}$$

$$I(X^n; Y^n) = ?$$

$$I(X^n \to Y^n) = ?$$

optimization

$$\max_{p(x^n||y^{n-1})} I(X^n \to Y^n)$$

note:

- concavity of the function
- convexity of the set

on capacity

(under conditions)

$$C = \lim_{n \to \infty} \max \frac{1}{n} I(X^n \to Y^n)$$

[Massey 1990], [Kramer 1998], [Chen and Berger 2005], [Tatikonda and Mitter 2010], [Kim 2010], [Permuter, Goldsmith and W. 2010]

Finite State Channels

$$P(y_i, s_i | x^i, s^{i-1}, y^{i-1}) = P(y_i, s_i | x_i, s_{i-1})$$

explicit computations

- memoryless channels
- mod-additive channels
- Gaussian with and without FB
- trapdoor with FB
- Ising with FB
- some more

POST channel
Previous Output is the STate

$$S_i = Y_{i-1}$$

$$\Leftrightarrow$$

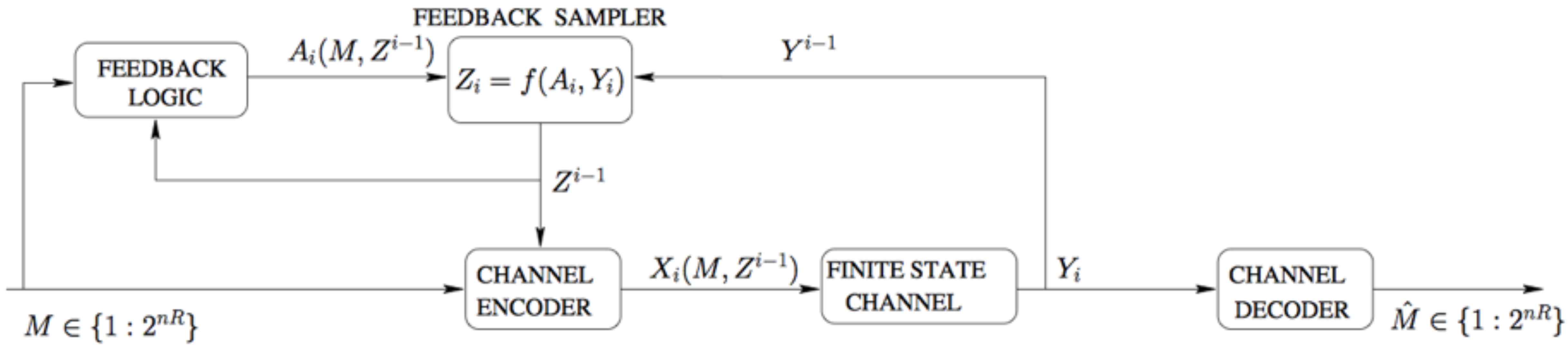
$$p(y_i|x^i, y^{i-1}) = p(y_i|x_i, y_{i-1})$$

[T. Berger, 2002 Shannon lecture
"living information theory"]

motivation

- simple
- good model
- to feed or not

“To Feed or Not to Feed Back”



questions for the POST channel

- C_{FB}
- C_{NFB}
- $C_{FB} > C_{NFB}$?

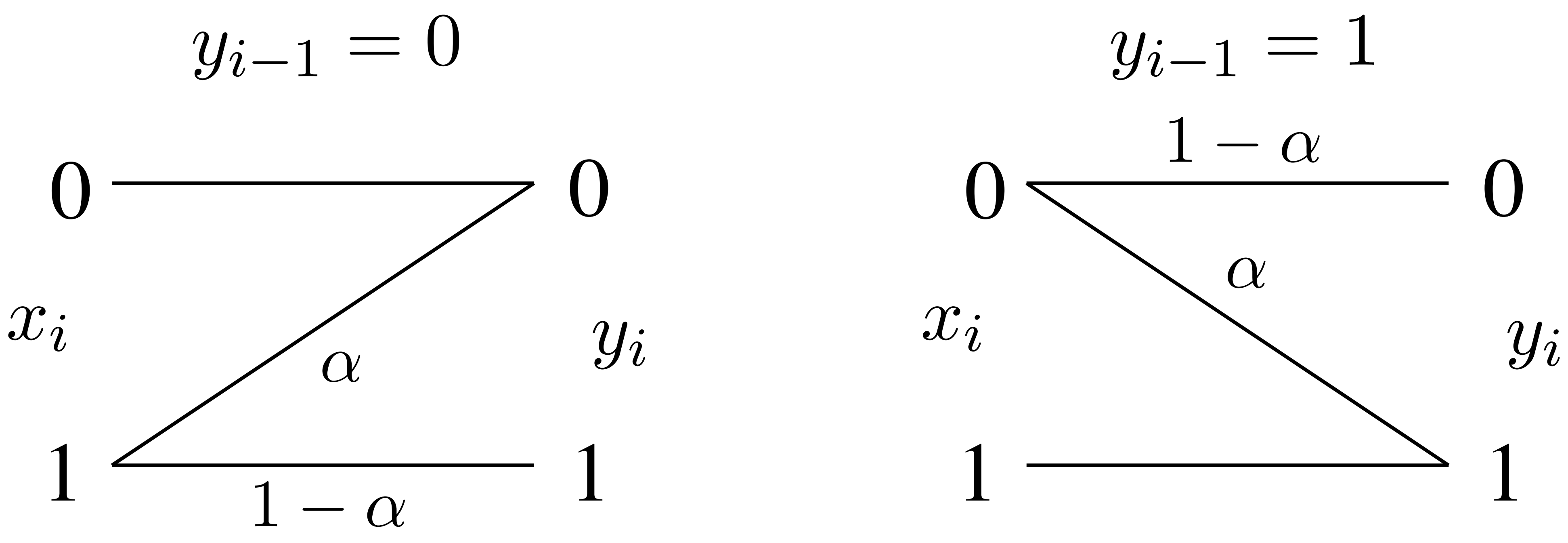
feedback capacity of the POST channel

[Chen and Berger 2005]:

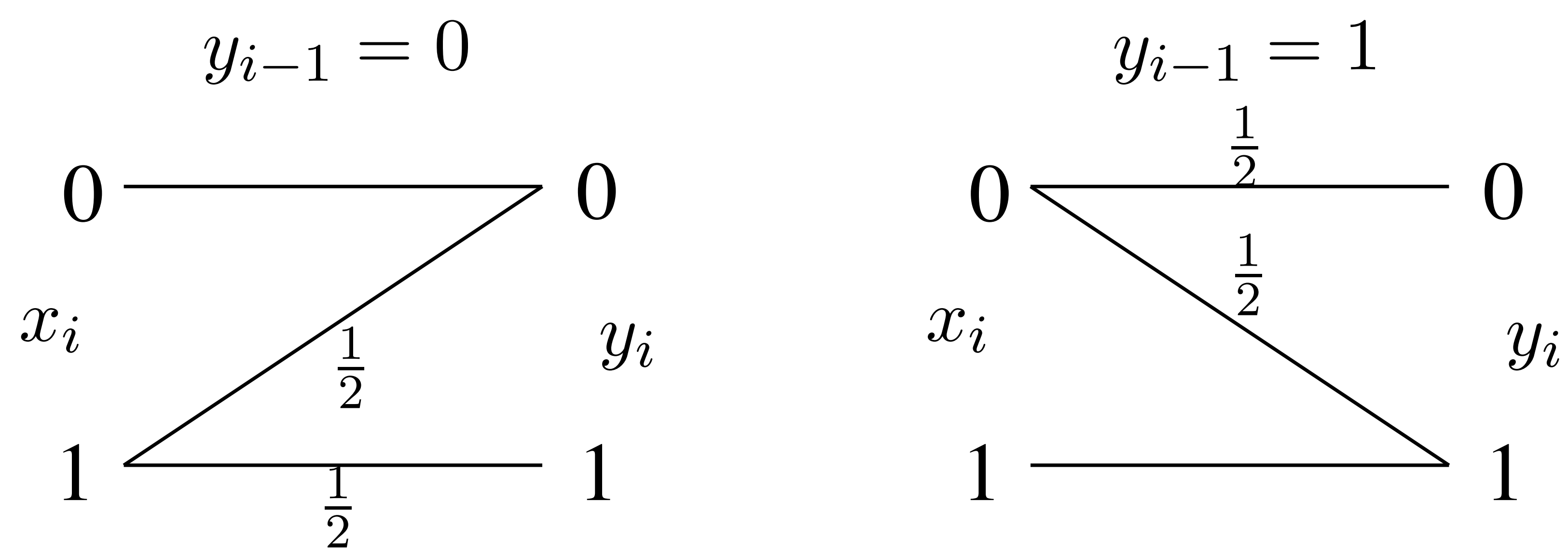
$$C_{FB} = \max_{p(x_1|y_0)} I(X_1; Y_1|Y_0)$$

(under benign conditions)

POST(α) channel



(simple) POST($\alpha = \frac{1}{2}$)

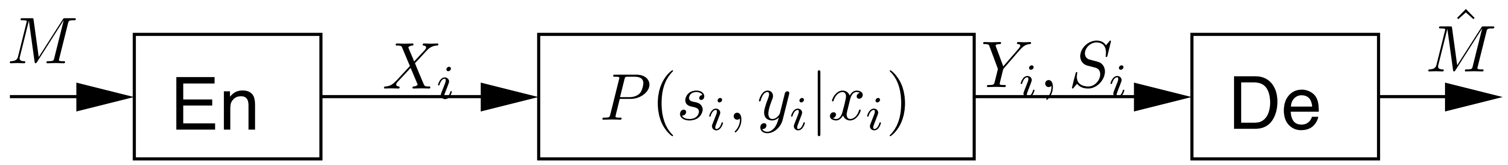


alternatively:

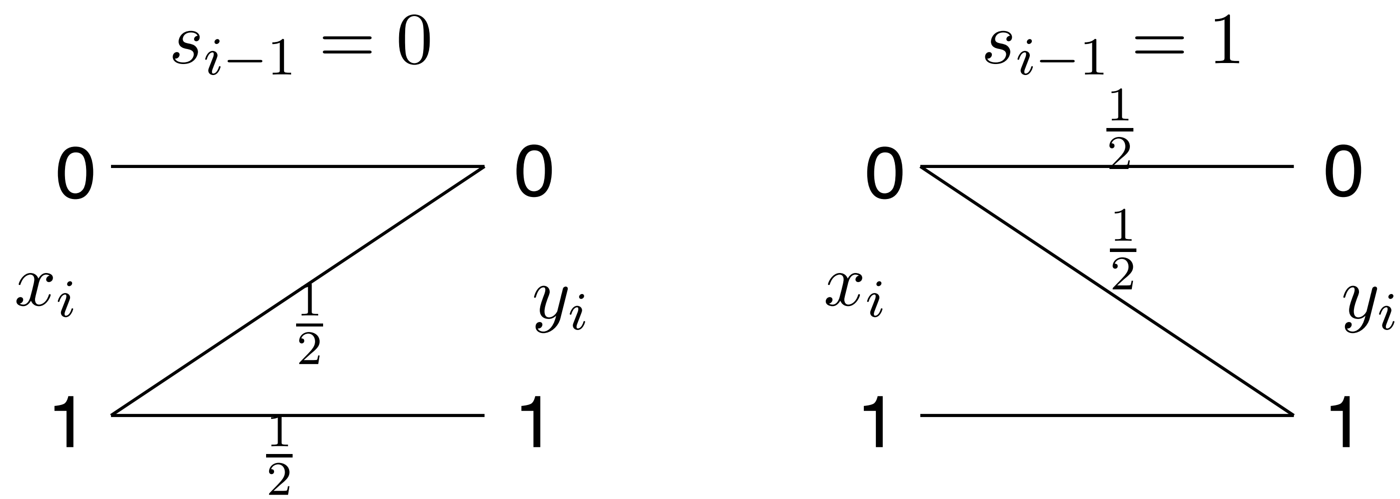
if $X_i = Y_{i-1}$, $\quad Y_i = X_i$

otherwise, $\quad Y_i \sim Bernouli(\frac{1}{2})$

intuition?



S_i i.i.d. $\sim$Bernoulli($\frac{1}{2}$)



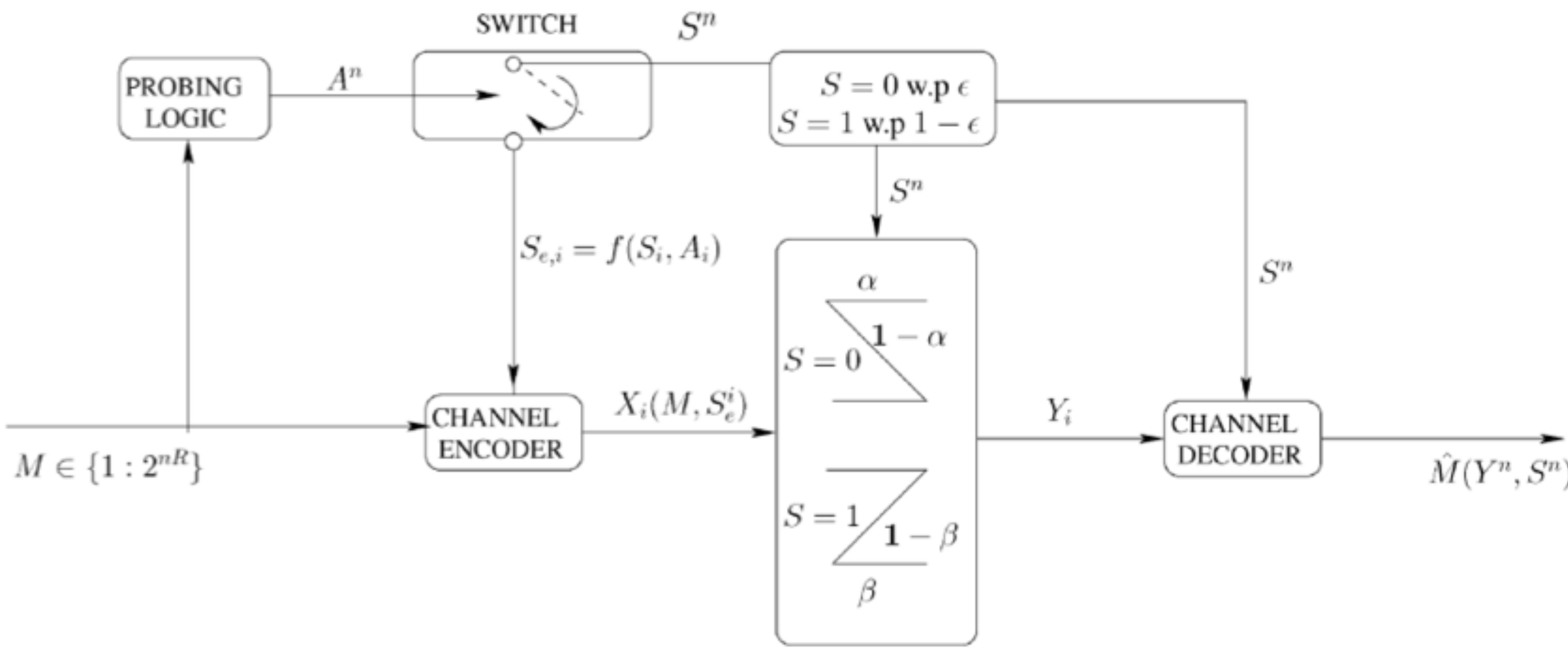
- Regular capacity

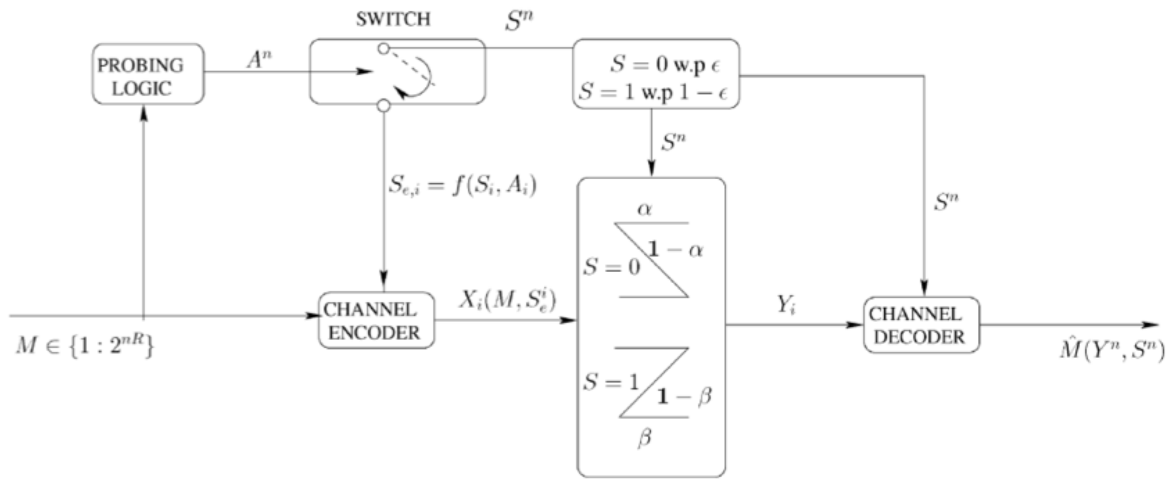
$$C = \max_{P(x)} I(X;Y,S) = H_b(\frac{1}{4}) - \frac{1}{2} = 0.3111$$

- Feedback capacity is the capacity of the Z channel

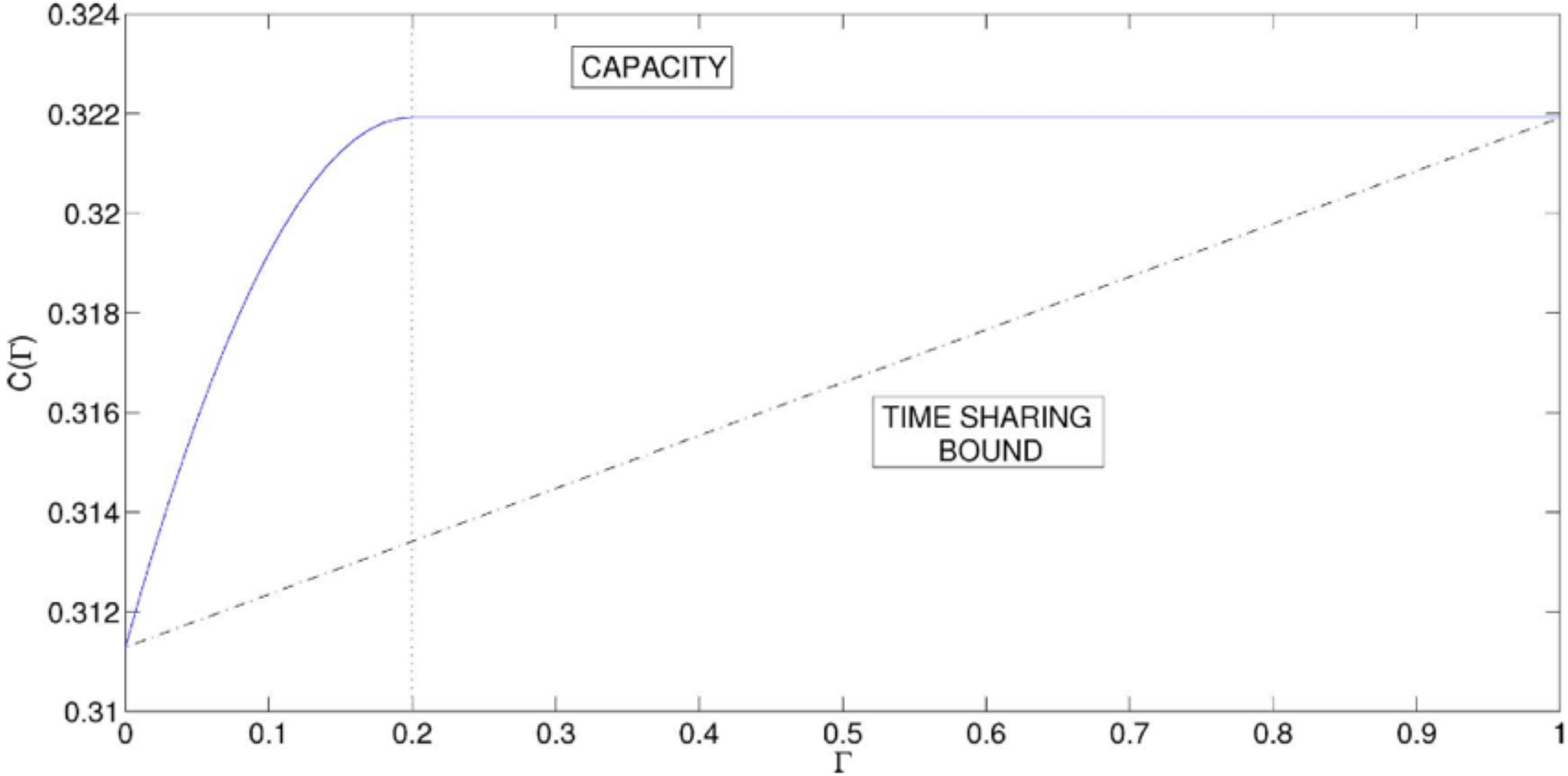
$$C_{fb} = -\log_2 0.8 = 0.3219$$

channel probing ([Asnani, Permuter and W. 2011])



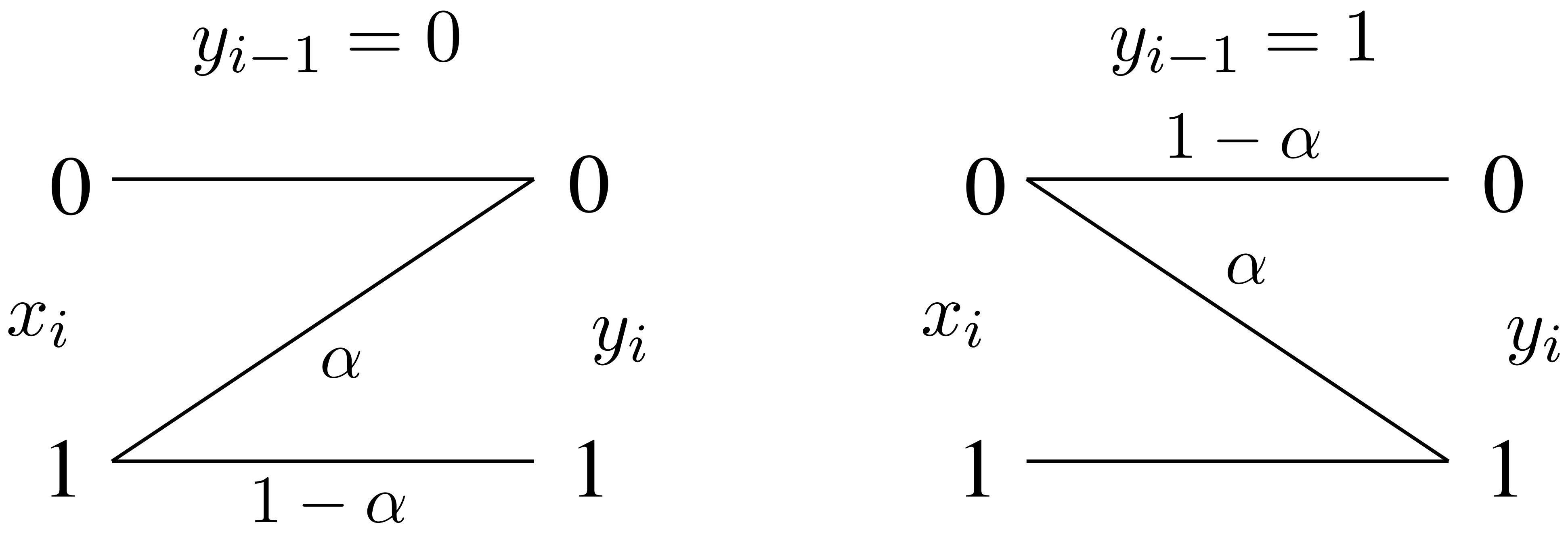


PROBING LOGIC
A^n
SWITCH
S^n
$S = 0$ w.p ϵ
$S = 1$ w.p $1-\epsilon$
S^n
$S_{e,i} = f(S_i, A_i)$
S^n
α
$S=0$
$1-\alpha$
$S=1$
$1-\beta$
β
CHANNEL ENCODER
$X_i(M, S_e^i)$
Y_i
CHANNEL DECODER
$M \in \{1 : 2^{nR}\}$
$\hat{M}(Y^n, S^n)$



CAPACITY
TIME SHARING BOUND
C(Γ)
Γ
0.324
0.322
0.32
0.318
0.316
0.314
0.312
0.31
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

POST(α) channel



- C_{FB}
- C_{NFB}
- $C_{FB} > C_{NFB}$?

Theorem

Feedback does not increase the capacity of the POST(α) channel.

main idea

show:

$$\max_{P(x^n||y^{n-1})} I(X^n \to Y^n) = \max_{P(x^n)} I(X^n \to Y^n)$$

Necessary and sufficient for $\max I(X^n \to Y^n)$

Theorem

A set of necessary and sufficient conditions for an input probability $P(x^n||y^{n-1})$ to maximize $I(X^n \to Y^n)$ is that for some numbers $\beta_{y^{n-1}}$

$$\sum_{y_n} p(y^n||x^n) \log \frac{p(y^n||x^n)}{ep(y^n)} = \beta_{y^{n-1}}, \ \forall x^n, y^{n-1}, \text{ if } p(x^n||y^{n-1}) > 0$$

$$\sum_{y_n} p(y^n||x^n) \log \frac{p(y^n||x^n)}{ep(y^n)} \leq \beta_{y^{n-1}}, \ \forall x^n, y^{n-1}, \text{ if } p(x^n||y^{n-1}) = 0$$

where $p(y^n) = \sum_{x^n} p(y^n||x^n)p(x^n||y^{n-1})$. The solution of the optimization is

$$\max_{P(x^n||y^{n-1})} I(X^n \to Y^n) = \sum_{y^{n-1}} \beta_{y^{n-1}} + 1.$$

the case n=1

[Gallager 1968]

4.5 Finding Channel Capacity for a Discrete Memoryless Channel

Theorem 4.5.1. A set of necessary and sufficient conditions on an input probability vector $\mathbf{Q} = [Q(0), \ldots, Q(K-1)]$ to achieve capacity on a discrete memoryless channel with transition probabilities $P(j \mid k)$ is that for some number C,

$$I(x = k; Y) = C; \qquad \text{all } k \text{ with } Q(k) > 0 \tag{4.5.1}$$

$$I(x = k; Y) \leq C; \qquad \text{all } k \text{ with } Q(k) = 0 \tag{4.5.2}$$

in which $I(x = k; Y)$ is the mutual information for input k averaged over the outputs,

$$I(x = k; Y) = \sum_j P(j \mid k) \log \frac{P(j \mid k)}{\sum_i Q(i) P(j \mid i)} \tag{4.5.3}$$

Necessary and sufficient for $\max I(X^n \to Y^n)$

Theorem

A set of necessary and sufficient conditions for an input probability $P(x^n||y^{n-1})$ to maximize $I(X^n \to Y^n)$ is that for some numbers $\beta_{y^{n-1}}$

$$\sum_{y_n} p(y^n||x^n) \log \frac{p(y^n||x^n)}{ep(y^n)} = \beta_{y^{n-1}}, \ \forall x^n, y^{n-1}, \text{ if } p(x^n||y^{n-1}) > 0$$

$$\sum_{y_n} p(y^n||x^n) \log \frac{p(y^n||x^n)}{ep(y^n)} \leq \beta_{y^{n-1}}, \ \forall x^n, y^{n-1}, \text{ if } p(x^n||y^{n-1}) = 0$$

where $p(y^n) = \sum_{x^n} p(y^n||x^n)p(x^n||y^{n-1})$. The solution of the optimization is

$$\max_{P(x^n||y^{n-1})} I(X^n \to Y^n) = \sum_{y^{n-1}} \beta_{y^{n-1}} + 1.$$

key tool

Corollary

Let $P^*(x^n||y^{n-1})$ achieve the maximum of $\max_{P(x^n||y^{n-1})} I(X^n \to Y^n)$ and let $P^*(y^n)$ be the induced output pmf. If there exists an input probability distribution $P(x^n)$ such that

$$p^*(y^n) = \sum_{x^n} p(y^n||x^n)p(x^n),$$

then

$$\max_{P(x^n||y^{n-1})} I(X^n \to Y^n) = \max_{P(x^n)} I(X^n \to Y^n)$$

route for showing $C_{NFB} = C_{FB}$

Find:

1. $p^*(y^n)$
2. P_n^{-1} (where $P_n(y^n, x^n) = P(y^n | x^n)$)
3. $P_n^{-1} \cdot p^*(y^n)$

is $P_n^{-1} \cdot p^*(y^n) \geq 0$?

if yes $\forall n \Rightarrow C_{NFB} = C_{FB}$

specifically, for POST

- simple structure and evolution of:
  - optimal output distribution
  - channel matrix
  - its inverse

Simple POST channel

Binary symmetric Markov $\{Y\}_{i\geq 1}$ with transition probability 0.2 can be described recursively

$$P_0(y^n) = \begin{bmatrix} 0.8P_0(y^{n-1}) \\ 0.2P_1(y^{n-1}) \end{bmatrix} \qquad P_1(y^n) = \begin{bmatrix} 0.2P_0(y^{n-1}) \\ 0.8P_1(y^{n-1}) \end{bmatrix},$$

where $P_0(y^0) = P_1(y^0) = 1.$

(matrix of the) Simple POST channel

$P(Y_1|X_1, s_0 = 0)$

$Y_1 \backslash X_1$	0	1
0	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$

$P(Y_1|X_1, s_0 = 1)$

$Y_1 \backslash X_1$	0	1
0	$\frac{1}{2}$	0
1	$\frac{1}{2}$	1

$$P_{n,0} = \begin{bmatrix} 1 \cdot P_{n-1,0} & \frac{1}{2} \cdot P_{n-1,0} \\ 0 \cdot P_{n-1,1} & \frac{1}{2} \cdot P_{n-1,1} \end{bmatrix} \quad P_{n,1} = \begin{bmatrix} \frac{1}{2} \cdot P_{n-1,0} & 0 \cdot P_{n-1,0} \\ \frac{1}{2} \cdot P_{n-1,1} & 1 \cdot P_{n-1,1} \end{bmatrix}$$

$$P_{n,0}^{-1} = \begin{bmatrix} 1 \cdot P_{n-1,0} & \alpha \cdot P_{n-1,0} \\ 0 \cdot P_{n-1,1} & \bar{\alpha} \cdot P_{n-1,1} \end{bmatrix}^{-1} = \begin{bmatrix} P_{n-1,0}^{-1} & -\frac{\alpha}{\bar{\alpha}} P_{n-1,1}^{-1} \\ 0 & \frac{1}{\bar{\alpha}} P_{n-1,1}^{-1} \end{bmatrix}$$

$$P_{n,1}^{-1} = \begin{bmatrix} \bar{\alpha} \cdot P_{n-1,0} & 0 \cdot P_{n-1,0} \\ \alpha \cdot P_{n-1,1} & 1 \cdot P_{n-1,1} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\bar{\alpha}} P_{n-1,0}^{-1} & 0 \\ -\frac{\alpha}{\bar{\alpha}} P_{n-1,0}^{-1} & P_{n-1,1}^{-1} \end{bmatrix}$$

using:

$$P_0(x^n) = P_{n,0}^{-1} P_0(y^n), \quad P_1(x^n) = P_{n,1}^{-1} P_1(y^n)$$

we obtain:

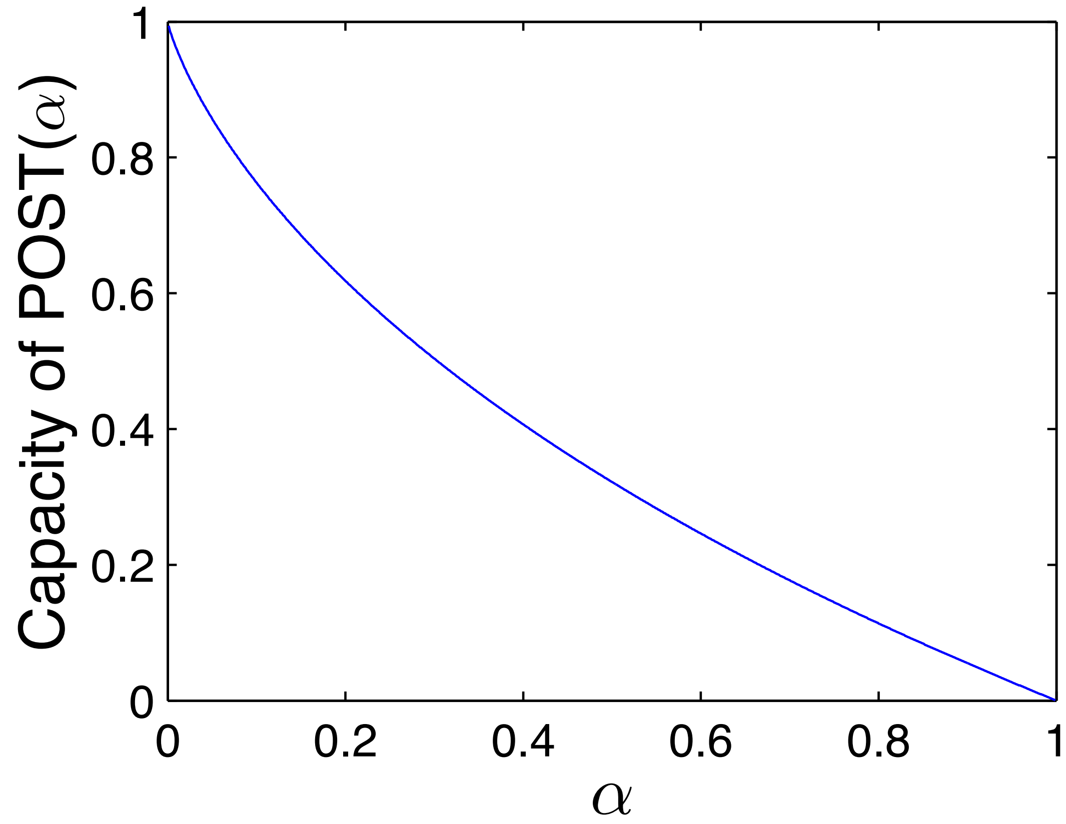
$$P_0(x^n) = \left[\begin{array}{c} 0.8P_0(x^{n-1}) - 0.2P_1(x^{n-1}) \\ 0.4P_1(x^{n-1}) \end{array} \right]$$

$$P_1(x^n) = \left[\begin{array}{c} 0.4P_0(x^{n-1}) \\ 0.8P_1(x^{n-1}) - 0.2P_0(x^{n-1}) \end{array} \right]$$

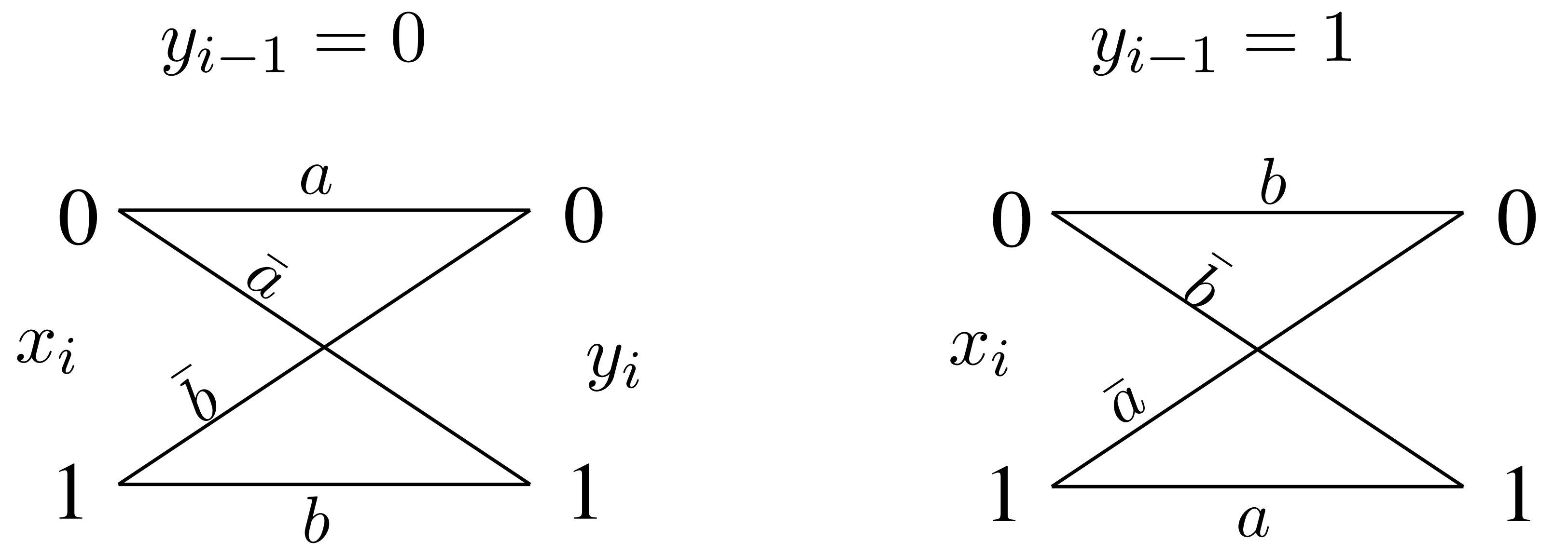
Feedback does not increase capacity of POST(α)

The feedback and the non-feedback capacity of POST(α) channel is the same as of the memoryless Z channel with parameter α, which is $C = -\log_2 c$ where

$$c = (1 + \bar{\alpha}\alpha^{\frac{\alpha}{\bar{\alpha}}})^{-1}$$



POST(a, b) channel



the input distribution

$$P_0(x^n) = \frac{1}{(a+b-1)(\gamma+1)} \left[\begin{array}{c} b\gamma P_0(x^{n-1}) - \bar{b} P_1(x^{n-1}) \\ -\bar{a}\gamma P_0(x^{n-1}) + a P_1(x^{n-1}) \end{array} \right]$$

$$P_1(x^n) = \frac{1}{(a+b-1)(\gamma+1)} \left[\begin{array}{c} a P_0(x^{n-1}) - \bar{a}\gamma P_1(x^{n-1}) \\ -\bar{b} P_0(x^{n-1}) + b\gamma P_1(x^{n-1}) \end{array} \right]$$

$$\gamma = 2^{\frac{H(b)-H(a)}{a+b-1}}.$$

In order to prove that $P(x^n)$ is valid we needed:

- $\gamma \geq \frac{\bar{b}}{b}$
- $\gamma \leq \frac{a}{\bar{a}}$
- $\gamma \geq \frac{a}{b}$ for $a \geq \bar{b}$
- $\gamma^2 \leq \frac{a^2}{b\bar{a}}$ for $a \geq \bar{b}$
- $\frac{\gamma(\bar{a}+b)}{2\bar{b}} \geq 1$ for $a \geq \bar{b}$ and $a\bar{a} \leq b\bar{b}$
- $\gamma^2(\bar{a}+b)^2 - 4a\bar{b} \geq 0$
- $\gamma(\bar{a}+b) - \sqrt{\gamma^2(\bar{a}+b)^2 - 4a\bar{b}} \leq 2\bar{b}$, for $a \geq \bar{b}$ and $a\bar{a} \leq b\bar{b}$

where

$$\gamma = 2^{\frac{H(b)-H(a)}{a+b-1}}.$$

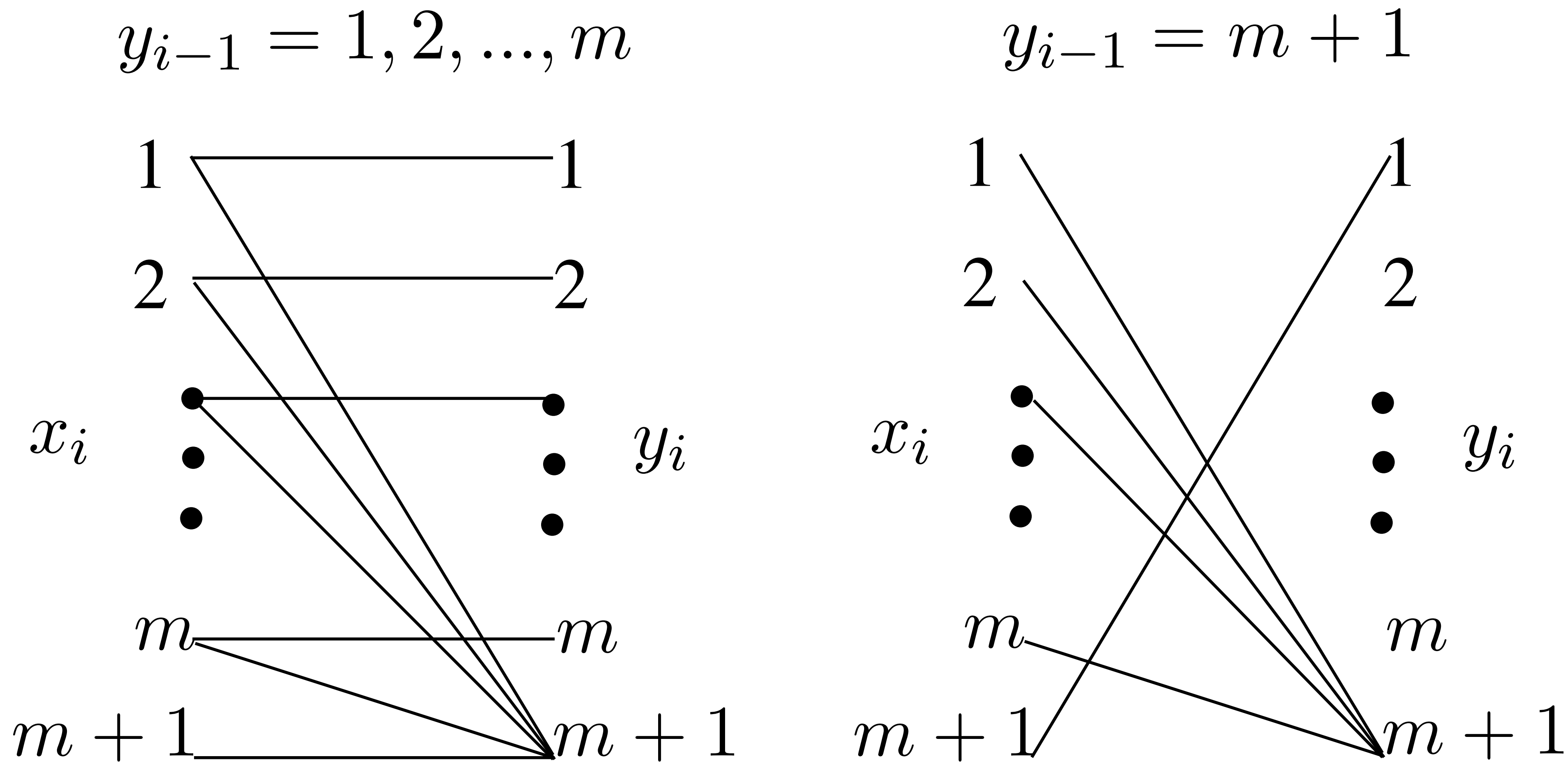
Feedback does not increase capacity of a POST(a, b) channel

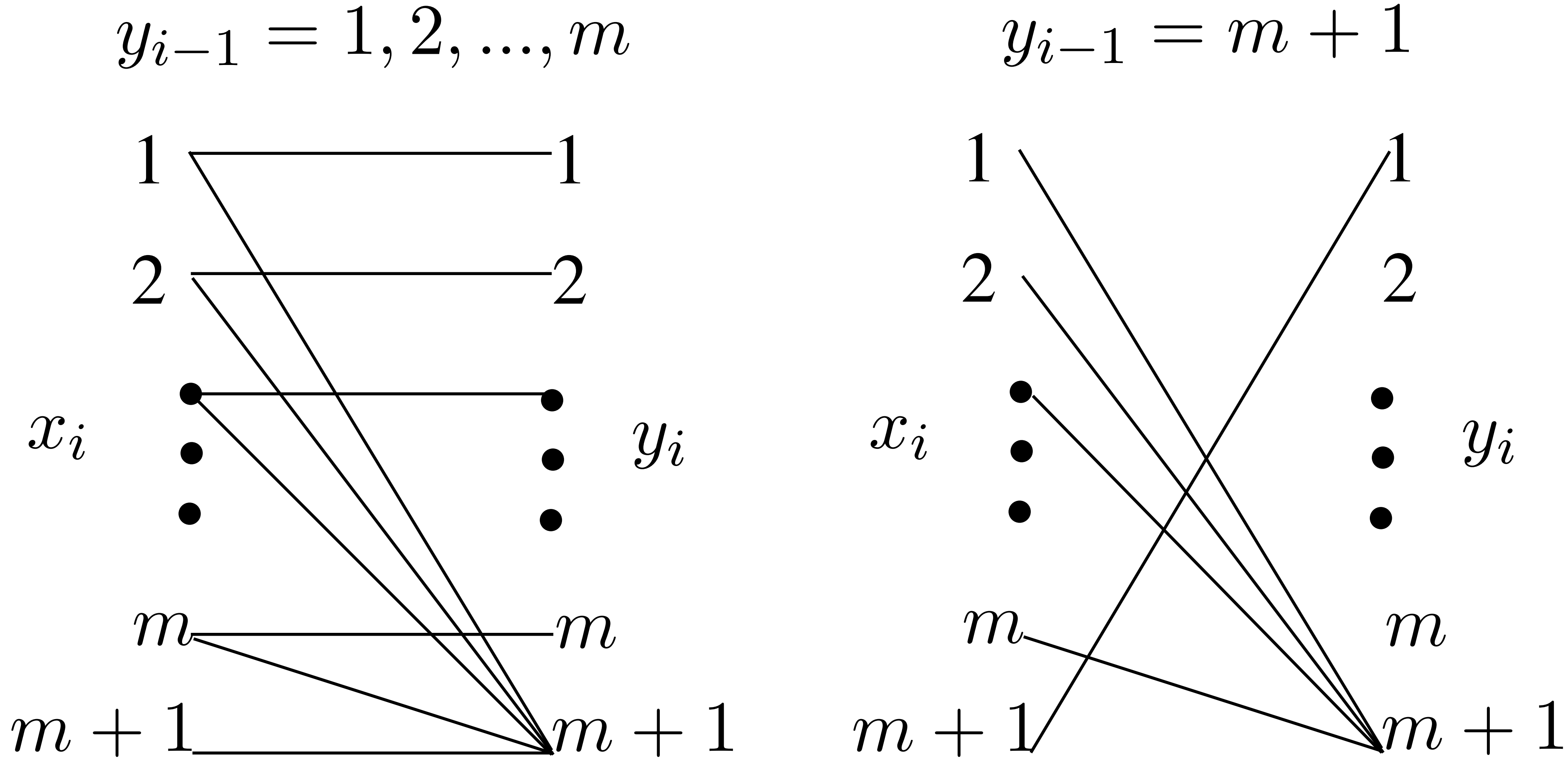
The feedback and the non-feedback capacity of POST(a, b) channel is the same as of a binary DMC channel with parameters (a, b), which is given by

$$C = \log\left[2^{\frac{\bar{a}H_b(b) - bH_b(a)}{a+b-1}} + 2^{\frac{\bar{b}H_b(a) - aH_b(b)}{a+b-1}}\right].$$

can feedback help for a general POST channel?

large alphabet





m	upper bound on capacity $\frac{1}{6}\max_{s_0}\max_{P(x^6)} I(X^6;Y^6\|s_0)$	lower bound on C_{fb} $R = \frac{\log_2 m}{3}$
2^9	2.5376	3.0000

is the sufficient condition necessary?

- we didn’t know
- numerics in binary case were inconclusive

condition is necessary and sufficient

Theorem:

Let $C_{FB} = \max_{p(x_1|y_0)} I(X_1; Y_1|Y_0)$ and let $p^*(y_0)$ be induced by $p^*(x_1|y_0)$.
$C_{FB} = C_{NFB}$ <u>if and only if</u> $\forall n$:
Can induce $p^*_{fb}(y^n)$ without feedback and without knowledge of $Y_0 \sim p^*(y_0)$.

implication

$$\exists \text{ binary POST channels with } C_{FB} > C_{NFB}$$

conclusions

- $C_{FB} = C_{NFB} \Leftrightarrow$ can induce $p^*_{fb}(y^n)$ without feedback $\forall n$
- $C_{FB} = C_{NFB}$ for binary $\mathrm{POST}(a, b)$ channels
- $\exists$ POST channels with $C_{FB} > C_{NFB}$

questions

- crisper necessary and sufficient condition for $C_{FB} = C_{NFB}$
- capacity achieving schemes