Entropy Power Inequalities: Results and Speculation

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Contributions involve joint work with Varun Jog

Outline

1. Setting the stage
2. The Brunn Minkowski inequality
3. Proofs of the EPI
4. Mrs. Gerber's Lemma
5. Young's inequality
6. Versions of the EPI for discrete random variables
7. EPI for Groups or Order 2^n
8. The Fourier transform on a finite abelian group
9. Young's inequality on a finite abelian group
10. Brunn Minkowski inequality on a finite abelian group
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- The *entropy power* of $\mathbf{X}$ is

$$N(\mathbf{X}) := \frac{1}{2\pi e} e^{\frac{2}{n} h(\mathbf{X})} \ .$$

Shannon's entropy power inequality

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Theorem (Entropy Power Inequality)

If $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent* $\mathbb{R}^n$*-valued random variables, then*

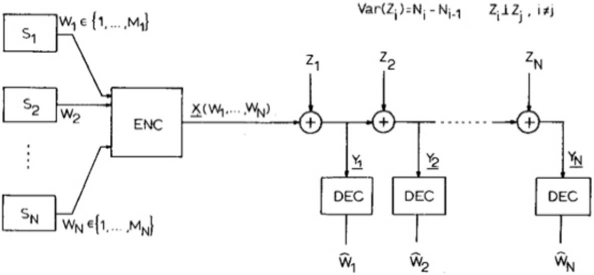
$$e^{\frac{2}{n}h(\mathbf{X}+\mathbf{Y})} \geq e^{\frac{2}{n}h(\mathbf{X})} + e^{\frac{2}{n}h(\mathbf{Y})},$$

with equality if and only if $\mathbf{X}$ *and* $\mathbf{Y}$ *are Gaussian with proportional covariance matrices.*

Notable applications of the EPI-1

A Simple Converse for Broadcast Channels with Additive White Gaussian Noise

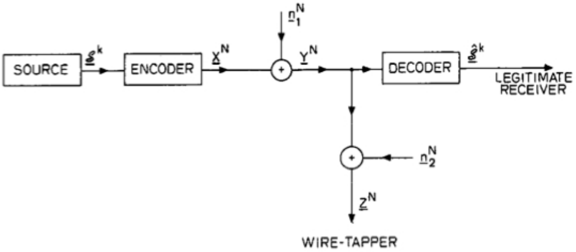
PATRICK P. BERGMANS



Notable applications of the EPI-2

The Gaussian Wire-Tap Channel

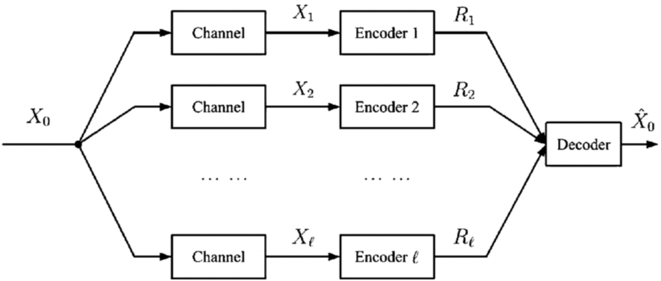
S. K. LEUNG-YAN-CHEONG, MEMBER, IEEE, AND MARTIN E. HELLMAN, MEMBER, IEEE



Notable applications of the EPI-3

The Rate-Distortion Function for the Quadratic Gaussian CEO Problem

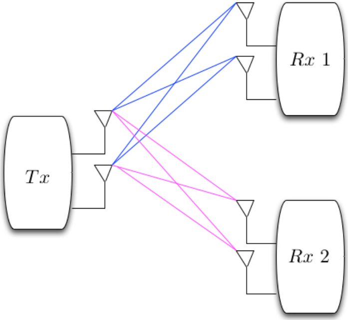
Yasutada Oohama



Notable applications of the EPI-4

The Capacity Region of the Gaussian Multiple-Input Multiple-Output Broadcast Channel

Hanan Weingarten, *Student Member, IEEE*, Yossef Steinberg, *Member, IEEE*, and Shlomo Shamai (Shitz), *Fellow, IEEE*



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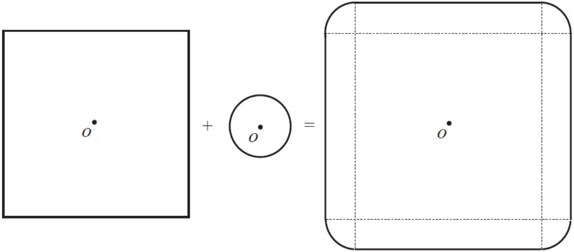

Minkowski sum of a square and a circle

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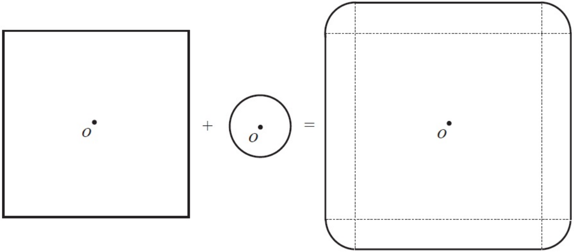

Minkowski sum of a square and a circle

- In general the Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is

$$A + B := \{a + b \ : \ a \in A \text{ and } b \in B\} \ .$$

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- An equivalent form is that for convex bodies $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$ we have

$$\mathrm{Vol}((1-\lambda)A + \lambda B)^{\frac{1}{n}} \geq (1-\lambda)\mathrm{Vol}(A)^{\frac{1}{n}} + \lambda \mathrm{Vol}(B)^{\frac{1}{n}} \ .$$

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- Another equivalent form is that for convex bodies $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$ we have
$$\text{Vol}((1-\lambda)A + \lambda B)^{\frac{1}{n}} \geq \min(\text{Vol}(A), \text{Vol}(B)) .$$

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- The distribution of a random variable corresponds to a (measurable) subset of $\mathbb{R}^n$.
- Addition of independent random variables corresponds to the Minkowski sum.
- Balls play the role of spherically symmetric Gaussians.
- For instance, an equivalent form of the EPI is that if $\mathbf{X}$ and $\mathbf{Y}$ are independent $\mathbb{R}^n$-valued random variables, and $0 < \lambda < 1$, then

$$h(\sqrt{\lambda}\mathbf{X} + \sqrt{1-\lambda}\mathbf{Y}) \geq \lambda h(\mathbf{X}) + (1-\lambda) h(\mathbf{Y}) \ .$$

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- Stam gave the first rigorous proof of the EPI in 1959, using de Bruijn's identity which relates Fisher information and differential entropy.
- Blachman's 1965 paper gives a simplified and succinct version of Stam's proof.
- Subsequently, several proofs have appeared, some of which will be mentioned here for our story.

Blachman's (1965) proof strategy for the EPI following Stam (1959)

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De Bruijn's Identity

$$\frac{d}{dt}h(X_t) = \frac{1}{2}J(X_t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x_t}p_t(x_t)\right)^2 \frac{dx_t}{p_t(x_t)}$$

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- The proof follows from differentiating

$$h(X_t) = -\int_{-\infty}^{\infty} p_t(x_t)\log p_t(x_t)dx_t,$$

with respect to t, and using the heat equation

$$\frac{\partial}{\partial t}p_t(x_t) = \frac{1}{2}\frac{\partial^2}{\partial t^2}p_t(x_t).$$

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$\frac{1}{J(Z)} \geq \frac{1}{J(X)} + \frac{1}{J(Y)}$

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- The proof goes by showing that

$$\mathbb{E}\left(a\frac{p'_X(x)}{p_X(x)} + b\frac{p'_Y(y)}{p_Y(y)}|Z = z\right) = (a+b)\frac{p'_Z(z)}{p_Z(z)},$$

for all a, b. Then squaring both sides, using Jensen's inequality and optimizing over the choice of a, b establishes the inequality.

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- Let $f(t)$ and $g(t)$ be increasing functions tending to infinity with t, and consider the function

$$s(t) = \frac{e^{2h(X_f)} + e^{2h(Y_g)}}{e^{2h(Z_{f+g})}}.$$

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- Intuitively, $\lim_{t\to\infty} s(t) = 1$, since both initial distributions become increasingly Gaussian as time progresses along the flow.
- Choosing $f'(t) = exp(2h(X_f))$ and $g'(t) = exp(2h(Y_g))$, and using Stam's inequality gives $s'(t) \geq 0$. This gives $s(0) \leq 1$, proving the EPI.

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- Reparametrization of the de Bruijn identity gives the Guo-Shamai-Verdú I-MMSE relation:

$$\frac{d}{d\gamma} I(X; \sqrt{\gamma} X + N) = \frac{1}{2} \mathrm{mmse}(X, \gamma),$$

and thus

$$h(X) = \frac{1}{2} \log 2\pi e - \frac{1}{2} \int_0^\infty \frac{1}{1+\gamma} - \mathrm{mmse}(X, \gamma) d\gamma.$$

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- The EPI in its equivalent form,

$$h(X_1 \cos\alpha + X_2 \sin\alpha) \geq \cos^2\alpha\, h(X_1) + \sin^2\alpha\, h(X_2),$$

  can be proved using these ideas.

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$$+ \sin^2\alpha \;\text{mmse}(X_2, \gamma)\Big) d\gamma$$

- Consider $Z_1 = \sqrt{\gamma}X_1 + N_1$, $Z_2 = \sqrt{\gamma}X_2 + N_2$ and $Z = Z_1 \cos\alpha + Z_2 \sin\alpha$. Then

$$\text{mmse}(X_1 \cos\alpha + X_2 \sin\alpha | Z) \geq \text{mmse}(X | Z_1, Z_2)$$

immediately gives the term inside the integral is ≥ 0.

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- The restricted Minkowski sum of sets A and B, based on $\Theta \subseteq A \times B$ is defined as

$$A +_{\Theta} B := \{x + y : (x, y) \in \Theta\} .$$

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$$A +_\Theta B := \{x + y : (x, y) \in \Theta\} \ .$$

- Szarek and Voiculescu's restricted B-M inequality says: For every ϵ, there exists δ such that if Θ is large enough, viz $V_{2n}(\Theta) \geq (1-\delta)^n V_n(A) V_n(B)$, then

$$V_n(A +_\Theta B)^{\frac{2}{n}} \geq (1-\epsilon)(V_n(A)^{\frac{2}{n}} + V_n(B)^{\frac{2}{n}}).$$

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- To prove the EPI, Szarek and Voiculescu replace A and B by typical sets for n i.i.d. copies of X and Y respectively, and define Θ as all pairs (x^n, y^n) (each marginal typical) such that $x^n + y^n$ is typical for $X + Y$.

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- For this choice of restriction, they determine suitable sequence of typicality defining constants (ϵ_n, δ_n) going to 0, thus proving the EPI.

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- Note that
$$(x, y) \mapsto \frac{n}{2}\log(e^{\frac{2}{n}x} + e^{\frac{2}{n}y})$$
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- Note that
$$(x, y) \mapsto \frac{n}{2}\log(e^{\frac{2}{n}x} + e^{\frac{2}{n}y})$$
is convex in (x, y).
- In particular, for fixed x,
$$y \mapsto \frac{n}{2}\log(e^{\frac{2}{n}x} + e^{\frac{2}{n}y})$$
is convex in y and for fixed y,
$$x \mapsto \frac{n}{2}\log(e^{\frac{2}{n}x} + e^{\frac{2}{n}y})$$
is convex in x.

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- This leads to *Young's inequality*. If $p, q, r > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then

$$\|f * g\|_r \le \frac{C_p C_q}{C_r} \|f\|_p \|g\|_q \ , \quad \text{if } p, q, r \ge 1 \ ,$$

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$$\|f * g\|_r \geq \frac{C_p C_q}{C_r} \|f\|_p \|g\|_q \ , \quad \text{if } 1 \geq p, q, r \ .$$

- The second half of this is called the *reverse* Young's inequality.

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- $N_p(\mathbf{X}) := \frac{1}{2\pi} p^{-p'/p} \|f\|_p^{-2p'/n}$ is the *Renyi entropy power of* $\mathbf{X} \sim f$.

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- $N_p(\mathbf{X}) := \frac{1}{2\pi} p^{-p'/p}\|f\|_p^{-2p'/n}$ is the *Renyi entropy power of* $\mathbf{X} \sim f$.
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- Optimize over λ and let $r \to 1$ to get the EPI.

Brunn Minkowski inequality from the reverse Young's inequality

Brunn Minkowski inequality from the reverse Young's inequality

- The limit as $r \to 0$ in Young's inequality leads to the *Prékopa-Leindler inequality*: If $0 < \lambda < 1$ and f, g, h are nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx \right)^{\lambda} .$$

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- Setting $f := 1_A$, $g := 1_B$, and $h := 1_{(1-\lambda)A+\lambda B}$, this gives

$$\mathrm{Vol}((1-\lambda)A+\lambda B) \geq \mathrm{Vol}(A)^{1-\lambda}\mathrm{Vol}(B)^{\lambda} \geq \min(\mathrm{Vol}(A), \mathrm{Vol}(B)) .$$

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Mrs. Gerber's lemma

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- Let h be the binary entropy function, and h^{-1} be its inverse. Wyner and Ziv (1973) showed that for a binary X and arbitrary U, if $Z \sim Bern(p)$ is independent of (X, U) then

$$h(X \oplus Z|U) \geq h(h^{-1}(H(X|U) \star p).$$

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$$h(X \oplus Z|U) \geq h(h^{-1}(H(X|U) \star p).$$

- The result follows immediately from the following lemma, using Jensen's inequality -

Lemma

The function

$$x \mapsto h(h^{-1}(x) \star p)$$

is convex on $0 \leq x \leq \log 2$.

Wyner and Ziv's proof of Mrs. Gerber's lemma

Wyner and Ziv's proof of Mrs. Gerber's lemma

Introduce the parametrization $\alpha = \frac{1-h^{-1}(x)}{2}$, let $p = \frac{1-a}{2}$, and differentiate wrt α, to get that

$$f''(x) \geq 0 \iff a(1-\alpha^2)\log\frac{1+\alpha}{1-\alpha} \leq (1-a\alpha)^2\log\frac{1+a\alpha}{1-a\alpha},$$

which can be verified using the series expansion of $\log\frac{1+x}{1-x}$.

Shamai and Wyner's EPI

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- Shamai and Wyner show that -

EPI for binary sequences

For *binary independent processes* with entropies $H(X)$ and $H(Y)$,

$$\sigma(Z) \geq \sigma(X) \star \sigma(Y),$$

where $Z = X \oplus Y$ and $\sigma(X) = h^{-1}(H(X)), \sigma(Y) = h^{-1}(H(Y)$ and $\sigma(Z) = h^{-1}(H(Z))$.

Shamai and Wyner's proof strategy for the binary EPI

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\begin{aligned}
H(Z_0|Z_{-n}^{-1}) \geq H(Z_0|Z_{-n}^{-1}, X_{-n}^{-1}, Y_{-n}^{-1}) &= H(Z_0|X_{-n}^{-1}, Y_{-n}^{-1}), \\
&= \sum P(X_{-n}^{-1} = x)P(Y_{-n}^{-1} = y)H(Z_0|X_{-n}^{-1} = x, Y_{-n}^{-1} = y) \\
&= \sum P(X_{-n}^{-1} = x)P(Y_{-n}^{-1} = y)h(\alpha(x) \star \beta(y))
\end{aligned}
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Where $\alpha(x) = P(X_0 = 1|X_{-n}^{-1} = x)$, $\beta(x) = P(Y_0 = 1|Y_{-n}^{-1} = y)$.

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Where $\alpha(x) = P(X_0 = 1|X_{-n}^{-1} = x)$, $\beta(x) = P(Y_0 = 1|Y_{-n}^{-1} = y)$.

- Using convexity in MGL twice, summing over x and then y, we get

$$H(Z_0|Z_{-n}^{-1}) \geq h(h^{-1}(H(X_0|X_{-n}^{-1}) \star h^{-1}(H(Y_0|Y_{-n}^{-1})).$$

Taking $n \to \infty$ proves the result.

Harremoës's EPI for the Binomial family

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- Let $X_n \sim Binomial(n, \frac{1}{2})$, then for all $m, n \geq 1$,

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- The proof follows by showing that $Y_n = \frac{e^{2H(X_n)}}{n}$ is an increasing sequence, and thus is super additive, i.e $Y_{m+n} \geq Y_m + Y_n$.

n	1	2	3	4	5	6
$\frac{e^{2H(X_n)}}{n}$	4	4	4.105	4.173	4.212	4.233

Figure : Values of Y_n

Our approach

Our approach

Suppose the random variables take values on a finite abelian group G. We define the function $f_G : [0, \log |G|] \times [0, \log |G|] \to [0, \log |G|]$ by

$$f_G(x, y) = \min_{H(X)=x, H(Y)=y} H(X + Y)$$

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We try to exploit the group structure to answer these questions.

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- Perhaps the analogs of the Gaussian distributions are the distributions of the form

$$p(0) = 1 - \epsilon \ , \quad p(g) = \frac{1}{|G| - 1}\epsilon \text{ for } g \neq 0 \ , \quad 0 \leq \epsilon \leq \frac{|G| - 1}{|G|} \ ?$$

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- But these are not extremal for the EPI. For instance, for $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, if ϵ is chosen so that $h(\epsilon) + \epsilon \log 3 = \log 2$ (note that $\epsilon < \frac{3}{4}$), then we have

$$h(1 - 2\epsilon + \frac{4}{3}\epsilon^2) + 2\epsilon(1 - \frac{2}{3}\epsilon) \log 3 > \log 2 \ .$$

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- However, for the uniform distribution on a subgroup, its convolution with itself has entropy $\log 2$.

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- There is a version of Stam's inequality for finite abelian groups (Gibilisco and Isola, 2008). Given a set of generators $\Gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ for the group G, define the "Fisher information" of a random variable X by

$$J(X) := \sum_{\gamma \in \Gamma} \sum_{g \in G} \left(\frac{p_X(g) - p_X(\gamma^{-1} g)}{p_X(g)} \right)^2 p_X(g) \ .$$

Then one has $\frac{1}{J(X_1+X_2)} \geq \frac{1}{J(X_1)} + \frac{1}{J(X_2)}$ for independent G-valued random variables X_1 and X_2.

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  Then one has $\frac{1}{J(X_1+X_2)} \geq \frac{1}{J(X_1)} + \frac{1}{J(X_2)}$ for independent G-valued random variables X_1 and X_2.
- However this notion of "Fisher information" is merely mimicking formal properties of the Fisher information from the continuous case. It seems to have little to do with estimation, which, for finite groups ought to refer to likelihood ratio type quantities.

Mimicking Szarek and Voiculescu's proof?

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- The question remains, what is the correct analog of the restricted Minkowski sum?
- The key lemma driving the proof of Szarek and Voiculescu appears to be Euclidean in character.

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- We also describe the minimum entropy achieving distributions on G, these distributions are analogs of Gaussians in this sense.
- The $f_G(x, y)$ function obtained has the property that it is convex in x for a fixed y. This is yet another version of Mrs. Gerber's Lemma.

Statement of our results

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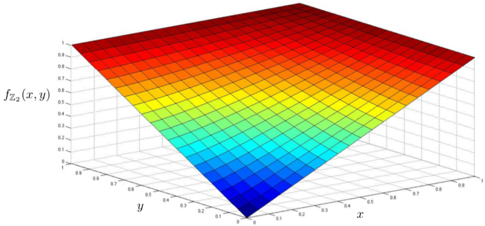
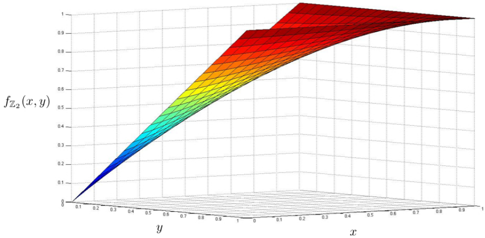
Theorem

f_G depends only on the size of G, and is denoted by f_{2^n}, where

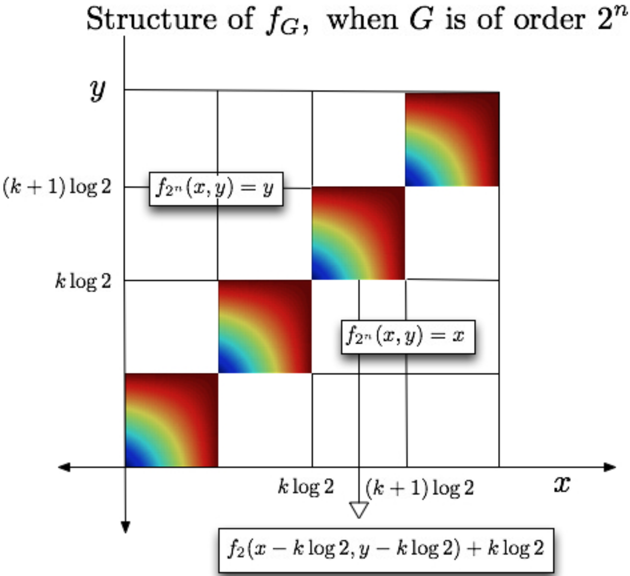
$$f_{2^n}(x,y) = \begin{cases} f_2(x - k\log 2, y - k\log 2) + k\log 2, \\ \qquad \text{if } k\log 2 \le x, y \le (k+1)\log 2 \ , \\ \max(x,y) \quad \text{otherwise.} \end{cases}$$

Visualizing our results





Visualizing our results (contd.)



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- If one is less optimistic, one might make this conjecture only for finite abelian groups.
- Simulations for $\mathbb{Z}_3$ and $\mathbb{Z}_5$ seem to support this conjecture.
- We saw already that f_{2^n} satisfies the conjecture.

EPI for abelian groups of order 2^n: proof strategy

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- First observe that for $\mathbb{Z}_2$ the function $f_{\mathbb{Z}_2}(x, y)$ is explicitly given by
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- The proof proceeds by first proving the EPI for $\mathbb{Z}_4$, using convexity properties of $f_{\mathbb{Z}_2}$ and concavity of entropy.
- We then use induction, where we assume that the result holds for all groups of size 2^n and prove it groups of size 2^{n+1}.
- The induction part of the proof has almost exactly the same structure as the proof for $\mathbb{Z}_4$. Thus proving the result for $\mathbb{Z}_4$ is the key step in our proof.

Proof for $\mathbb{Z}_4$

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- For $\mathbb{Z}_4$, splitting the support of distributions on $\mathbb{Z}_4$ into two parts, $\{0, 2\}$ (even part) and $\{1, 3\}$ (odd part) and using precisely two ingredients: concavity of entropy, and convexity in MGL we arrive at the lower bound

Lower bound

$$f_4(x, y) \geq \min_{u,v} f_2(u, v) + f_2(x - u, y - v)$$

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Lower bound

$$f_4(x,y) \geq \min_{u,v} f_2(u,v) + f_2(x-u,y-v)$$

- To evaluate the minimum, we prove some additional properties of $f_{\mathbb{Z}_2}$, specifically regarding its behavior along lines through the origin. The key Lemma we use is

Lemma

$\frac{\partial f_{\mathbb{Z}_2}}{\partial x}$ (and by symmetry, $\frac{\partial f_{\mathbb{Z}_2}}{\partial y}$) strictly decreases along lines through the origin

Proving the lemma

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- We use the parametrization $h^{-1}(x) = p$ and $h^{-1}(y) = q$. Our strategy involves brute force differentiation, and a series of steps where we conclude that proving the lemma is equivalent to proving

$$
\begin{aligned}
F(p) = p^2(\log(p))^2 - (1-p)^2(\log(1-p))^2 + \\
(1-2p)\left(\log(p)\log(1-p) + p\log(p) + (1-p)\log(1-p)\right) \\
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- We show that proving $F(p) \leq 0$ is the same as proving $\frac{d^5}{dp^5}F(p) \leq 0$, we differentiate F five times, use a polynomial approximation of log, and finally use Sturm's theorem to prove that the resulting polynomial is non-positive.

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Norm of the Fourier transform on a finite abelian group

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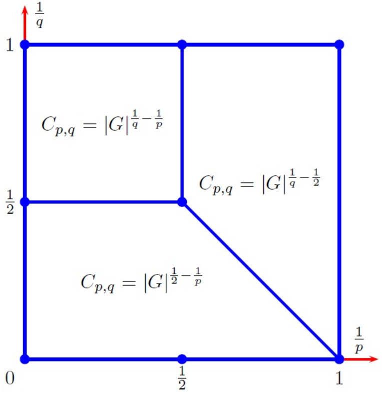
- Resolved by Gilbert and Rzeszotnik (2010).

Visualizing the norm of the FT on a finite abelian group



Norm of the Fourier Transform on a finite abelian group

Outline

1. Setting the stage
2. The Brunn Minkowski inequality
3. Proofs of the EPI
4. Mrs. Gerber's Lemma
5. Young's inequality
6. Versions of the EPI for discrete random variables
7. EPI for Groups or Order 2^n
8. The Fourier transform on a finite abelian group
9. Young's inequality on a finite abelian group
10. Brunn Minkowski inequality on a finite abelian group
11. Speculation

Young's inequality on a finite abelian group

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- There is a forward Young's inequality, which reads: For every $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ we have

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \ .$$

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$$\mu_G(r,s) := \min\{|A+B| \ : \ A, B \subset G, |A| = r, |B| = s\} \ .$$

Brunn Minkowski inequality on a finite abelian group

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$$\mu_G(r,s) := \min\{|A+B| \ : \ A,B \subset G, |A|=r, |B|=s\} \ .$$

- Then we have $\mu_G(r,s) = \min_{d \text{ divides } |G|} \left(\lceil \frac{r}{d} \rceil + \lceil \frac{s}{d} \rceil - 1 \right) d$.

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- In particular for G a finite abelian group of size $n = 2^k$, we have

$$\mu_G(r,s) = r \circ s$$

where $r \circ s$ is the *Hopf-Stiefel function* defined as the smallest positive integer m for which the polynomial $(x+y)^m$ falls in the ideal generated by x^r and y^s in the polynomial ring $\mathbb{F}_2[x,y]$.

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- Involved history of partial results. See Eliahu and Kervaire (2007) for the history.

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- Is the EPI we proved for finite abelian groups of size 2^k derivable as a limit from the Young's inequality for such groups?
- Is the Brunn Minkowski inequality for finite abelian groups of size 2^k (expressed via the Hopf-Stiefel function) derivable as a limit from the appropriate reverse Young's inequality for such groups?

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- Is the Brunn Minkowski inequality for finite abelian groups of size 2^k (expressed via the Hopf-Stiefel function) derivable as a limit from the appropriate reverse Young's inequality for such groups?
- Is the general Brunn Minkowski inequality for finite abelian groups derivable as a limit from the appropriate reverse Young's inequality for such groups?

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- Is the Brunn Minkowski inequality for finite abelian groups of size 2^k (expressed via the Hopf-Stiefel function) derivable as a limit from the appropriate reverse Young's inequality for such groups?
- Is the general Brunn Minkowski inequality for finite abelian groups derivable as a limit from the appropriate reverse Young's inequality for such groups?
- Will a limit of the Young's inequality for a general finite Abelian group give rise to an EPI for each such group?

The End