MINIMAL RATIONAL CURVES ON MODULI SPACES OF STABLE BUNDLES

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Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ be a line bundle on C of degree d. Let $M := \mathcal{U}_C(r, \mathcal{L})$ be the moduli space of stable vector bundles on C of rank r and with the fixed determinant $\mathcal{L}$. Assume that $(r, d) = 1$, then M is a smooth projective Fano variety with Picard number 1. For any projective curve on M, we can define its degree with respect to the ample anti-canonical line bundle $-K_M$. The first result of this paper determines all rational curves of *minimal degree* passing through a generic point of M, which answers a question of Jun-Muk Hwang (see Question 1 in [Hw]).

Theorem 1. *Any rational curve in M passing through the generic point has degree at least $2r$. If $g \geq 3$, then it has degree $2r$ if and only if it is a Hecke curve.*

On the other hand, a general problem (see Problem 1.13 of [Ko]) about low degree rational curves on Fano varieties is: Does there exist a rational curve ℓ, on any smooth Fano variety X with Picard number 1, such that $-K_X \cdot \ell$ equals to the index of X ? According to [Ko], we call such curve a *line* on X. The existence of *lines* is already implicit in Section 2 of [Ra] and part of Lemma 3.1 was made there (thanks to J.-M. Hwang and S. Ramanan for pointing this out). By using the proof of Theorem 1, we determine all *lines* on the moduli space M. There are unique $0 < r_1 < r$, d_1 such that $r_1 d - r d_1 = 1$. Let $r_2 = r - r_1$, $d_2 = d - d_1$ and $\mathcal{U}_C(r_1, d_1)$ (resp. $\mathcal{U}_C(r_2, d_2)$) be the moduli space of stable vector bundles with rank r_1 (resp. r_2) and degree d_1 (resp. d_2). Let $\mathcal{R} \subset \mathcal{U}_C(r_1, d_1) \times \mathcal{U}_C(r_2, d_2)$ be the closed subvariety consisting of (V_1, V_2) satisfying $det(V_1) \otimes det(V_2) = \mathcal{L}$. We construct a projective bundle $q : P \to \mathcal{R}$. The lines in its fibers $q^{-1}(\bullet) \cong \mathbb{P}^{r_1 r_2 (g-1)}$ are simply called *lines* on P.

Theorem 2. *There exist a morphism $\Phi : P \to M$ such that for any line $\mathbb{P}^1 \subset P$ its image $\Phi(\mathbb{P}^1)$ is a line on M and $\Phi|_{\mathbb{P}^1} : \mathbb{P}^1 \to \Phi(\mathbb{P}^1)$ is its normalization. Conversely, for any line $\ell \subset M$ on M, there is a line $\mathbb{P}^1 \subset P$ on P such that $\Phi(\mathbb{P}^1) = \ell$.*

When $g \geq 4$, the variety of Hecke curves passing through a generic point $[W] \in M$ is isomorphic to a (double) projective bundle $\mathbb{P}(\Omega_W)$ over C. Thus Theorem 1 can be used to give a simple proof of non-abelian Torelli theorem (Corollary 1.3) and the description of automorphisms of $\mathcal{U}_C(r, \mathcal{L})$ (Corollary 1.4).

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The proofs of our theorems are elementary. If E is a vector bundle on $X = C\times\mathbb{P}^1$ that induces the morphism of $\mathbb{P}^1$ to M. Then a simple computation shows that its degree equals to the second Chern class of $\mathcal{E}nd(E)$. If the restriction of E to the generic fiber of ruled surface $f : X \to C$ is semistable, then one sees easily that $c_2(\mathcal{E}nd(E))$ is at least $2r$, and it is $2r$ if and only if $c_2(E) = 1$ (after tensoring E by suitable line bundle pulling back from $\mathbb{P}^1$). This will force E to be an extension

$$0 \to f^*V \xrightarrow{i} E \xrightarrow{\phi} \mathcal{O}_{\{p\}\times\mathbb{P}^1}(-1) \to 0,$$

where V is a bundle on C. That is, after performing elementary transformation on E *once* along *one* fiber, E becomes a pullback of a vector bundle on C. For any $x \in \mathbb{P}^1$, restricting above sequence to $C \times \{x\}$ and denote $E|_{C\times\{x\}}$ by E_x, we have

$$0 \to V \xrightarrow{i_x} E_x \xrightarrow{\phi_x} \mathcal{O}_{\{p\}\times\mathbb{P}^1}(-1)_x \to 0.$$

Let $\iota_x : V_p \to E_x|_p = E_{(p,x)}$ be the homomorphism between the fibers at p induced by the sheaf map i_x. Then the *right* Hecke modifications $\{(\widetilde{W}^{ker(\iota_x)})^\vee;\ x \in \mathbb{P}^1\}$ of V along $\{ker(\iota_x) \subset V_p;\ x \in \mathbb{P}^1\}$ are exactly $\{E_x;\ x \in \mathbb{P}^1\}$. Thus the given curve is a Hecke curve by definition. If the restriction of E to the generic fiber is not semistable, then using relative Harder-Narasimhan filtration we are able to prove that $c_2(\mathcal{E}nd(E)) > 2r$ when $g \geq 3$. In the case $g = 2$, we can only prove that $c_2(\mathcal{E}nd(E)) \geq 2r$.

In Section 1, we recall the definitions of Hecke curves and show the two applications of Theorem 1. We prove Theorem 1 and Theorem 2 respectively in Section 2 and Section 3.

§1 Hecke curves

For a vector bundle V on a smooth curve C and a subspace $K \subset V_p$, where V_p is the fiber of V at a point $p \in C$, there two canonical constructions called Hecke modifications defined as follows:
(I) We call $V^L := Ker(V \to V_p \to V_p/K)$ the $left$ Hecke modification of V along $K \subset V_p$ at $p \in C$, which is the vector bundle satisfying

$$0 \to V^L \xrightarrow{\phi} V \to (V_p/K) \otimes \mathcal{O}_p \to 0$$

with $\phi_p(V^L_p) = K$.
(II) Let $(V^\vee)^L$ be the $left$ Hecke modification of $V^\vee$ along $(V_p/K)^\vee \subset V_p^\vee$. Note that $(V_p/K)^\vee = K^\perp$, the subspace annihilated by K. We call its dual, denoted by V^R, the *right* Hecke modification of V along K at $p \in C$, which satisfies

$$0 \to V \xrightarrow{\phi} V^R \to (V_p^R/(K^\perp)^\vee) \otimes \mathcal{O}_p \to 0$$

with $ker(\phi_p) = K$.

In what follows, we adopt notations of [Hw] and [HR]. For any $[W] \in M$, let $\mathbb{P}(W)$ be the projective bundle consisting of lines through the origin on each fiber. For $p \in C$ and $\zeta \in \mathbb{P}(W_p^\vee)$, define a vector bundle W^ζ, which is the $left$ Hecke modification of W along $\zeta^\perp \subset W_p$, by

$$0 \to W^\zeta \to W \to (W_p/\zeta^\perp) \otimes \mathcal{O}_p \to 0 \tag{1.1}$$

where $\zeta^{\perp}$ denotes the hyperplane in W_p annihilated by ζ. Let $\iota : W^{\zeta}_p \to W_p$ be the homomorphism between the fibers at p induced by the sheaf injection $W^{\zeta} \to W$. The kernel $ker(\iota)$ of ι is a 1-dimensional subspace of W^{ζ}_p and W is in fact the *right* Hecke modification of W^{ζ} along $ker(\iota) \subset W^{\zeta}_p$. Let $\mathcal{H}$ be a line in $\mathbb{P}(W^{\zeta}_p)$ containing the point $[ker(\iota)]$. For each point $[l] \in \mathcal{H}$ corresponding to a 1-dimensional subspace $l \subset W^{\zeta}_p$, define a vector bundle $\widetilde{W}^l$ by

$$0 \to \widetilde{W}^l \to (W^{\zeta})^{\vee} \to ((W^{\zeta})^{\vee}_p / l^{\perp})) \otimes \mathcal{O}_p \to 0 \tag{1.2}$$

where $l^{\perp} \subset (W^{\zeta})^{\vee}_p$ is the hyperplane annihilating l. The bundle $(\widetilde{W}^l)^{\vee}$ is the *right* Hecke modification of W^{ζ} along $l \subset W^{\zeta}_p$ and, for $l = ker(\iota)$,

$$\widetilde{W}^{ker(\iota)} \cong W^{\vee}. \tag{1.3}$$

Thus, if it happens that $(\widetilde{W}^l)^{\vee}$ is a stable bundle for each $[l] \in \mathcal{H}$, then

$$\{(\widetilde{W}^l)^{\vee} \, ; \, l \in \mathcal{H}\}$$

will define a rational curve passing through $[W] \in M$. Such a rational curve in M is called a Hecke curve. By [NR], it can be shown that a Hecke curve is smooth and has degree $2r$ with respect to $-K_M$. We will see in the following that, for generic $[W] \in M$, $(\widetilde{W}^l)^{\vee}$ is always a stable bundle for each $[l] \in \mathcal{H}$.

Given two nonnegative integers k, ℓ, a vector bundle W of rank r and degree d on C is (k, ℓ)-stable, if, for each proper subbundle W' of W, we have

$$\frac{deg(W') + k}{rk(W')} < \frac{deg(W) + k - \ell}{r}.$$

The usual stability is equivalent to $(0, 0)$-stability. The dual bundle of a (k, ℓ)-stable bundle is (ℓ, k)-stable. The proofs of following lemmas are easy and elementary.

Lemma 1.1 ([NR]). *If $g \geq 4$, a generic point $[W] \in M$ corresponds to a $(1, 1)$-stable bundle W.*

Lemma 1.2 ([NR]). *Let $0 \to V \to W \to \mathcal{O}_p \to 0$ be an exact sequence, where $\mathcal{O}_p$ is the 1-dimensional skyscraper sheaf at $p \in C$. If W is (k, ℓ)-stable, then V is $(k, \ell - 1)$-stable.*

If we choose a generic point $[W] \in M$ such that W is a $(1, 1)$-stable bundle, then W^{ζ} is a $(1, 0)$-stable bundle by Lemma 1.2 and $(W^{\zeta})^{\vee}$ is a $(0, 1)$-stable bundle. Thus $\{(\widetilde{W}^l)^{\vee} \, ; \, l \in \mathcal{H}\}$ is a family of stable bundles, which defines a Hecke curve passing through $[W] \in M$. Let $\mathbb{P}(W^{\vee}) \to C$ be the projection and Ω_W be its relative cotangent bundle. The projective bundle $\mathbb{P}(\Omega_W)$ over $\mathbb{P}(W^{\vee})$ is a smooth projective variety of dimension $2r - 2$. Then the variety of all Hecke curves through $[W] \in M$ is naturally isomorphic to $\mathbb{P}(\Omega_W) \xrightarrow{p} C$. Thus Theorem 1 can be used to prove the following known results (see [NRa], [KP] and [HR]).

Corollary 1.3. *Let C and C' be two smooth projective curves of genus $g \geq 4$. If $\mathcal{U}_C(r, \mathcal{L}) \cong \mathcal{U}_{C'}(r, \mathcal{L}')$, then $C \cong C'$.*

Proof. Let $[W'] \in \mathcal{U}_{C'}(r, \mathcal{L}')$ be the image of $[W]$. Then $\mathcal{U}_C(r, \mathcal{L}) \cong \mathcal{U}_{C'}(r, \mathcal{L}')$ induces an isomorphism between the varieties of rational curves of degree $2r$ passing through $[W]$, $[W']$ respectively. By Theorem 1, it induces an isomorphism $\mathbb{P}(\Omega_W) \cong \mathbb{P}(\Omega_{W'})$ between the varieties of Hecke curves passing through $[W]$, $[W']$ respectively. Thus it induces an isomorphism $C \cong C'$.

Corollary 1.4. *Let C be a smooth projective curves of genus $g \geq 4$. If $r > 2$, then the group of automorphisms of $\mathcal{U}_C(r, \mathcal{L})$ is generated by automorphisms of the following two types:*

(1) *$W \mapsto \gamma^* W$ where γ is an automorphism of C.*
(2) *$W \mapsto W \otimes \tau$ where τ is an r-torsion of the Jacobian J^0_C.*

When $r = 2$, additional generators of the type (3) are needed: (3) $W \mapsto W^\vee \otimes L$ where L is a line bundle of degree d with $L^{\otimes 2} = \mathcal{L}^{\otimes 2}$.

Proof. Let σ be an automorphism of $M = \mathcal{U}_C(r, \mathcal{L})$ and $[W] \in M$ a generic point. Then σ induces an isomorphism $G : \mathbb{P}(\Omega_W) \cong \mathbb{P}(\Omega_{\sigma(W)})$. Thus there is an automorphism $\gamma : C \cong C$ (being independent of generic $[W]$ since $Aut(C)$ is finite) such that

$$\begin{array}{ccc} \mathbb{P}(\Omega_W) & \xrightarrow{G} & \mathbb{P}(\Omega_{\sigma(W)}) \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p'}\big\downarrow \\ C & \xrightarrow{\gamma} & C \end{array}$$

and G induces either $\mathbb{P}(W^\vee) \cong \mathbb{P}(\sigma(W)^\vee)$ or $\mathbb{P}(W^\vee) \cong \mathbb{P}(\sigma(W))$ (see Lemma 5.4 of [HR]). Thus either $W \cong \gamma^*\sigma(W) \otimes \tau$ or $W \cong \gamma^*\sigma(W)^\vee \otimes L$ for lines bundles τ, L. Since W and $\sigma(W)$ have the fixed determinant $\mathcal{L}$, τ, L must satisfy the requirements in the corollary. The proof is finished.

§2 GENERIC MINIMAL RATIONAL CURVES ON THE MODULI SPACE

For any rational curve $\mathbb{P}^1 \subset M$ through a general point of M, let E be the vector bundle on $X := C \times \mathbb{P}^1$, which induces the embedding $\mathbb{P}^1 \subset M$. Let $\pi : X = C \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection and $\mathbb{E} \subset \mathcal{E}nd(E)$ be the subbundle of trace free. Then, since $\pi_*(\mathbb{E}) = 0$, we have $T_M|_{\mathbb{P}^1} = R^1\pi_*\mathbb{E}$ and, by using Leray spectral sequence and Riemann-Roch theorem,

$$-\chi(\mathbb{E}) = \chi(R^1\pi_*\mathbb{E}) = -K_M \cdot \mathbb{P}^1 + (r^2 - 1)(g - 1).$$

By using $\chi(\mathbb{E}) = deg(ch(\mathbb{E}) \cdot td(T_X))_2$, noting $c_1(\mathbb{E}) = c_1(\mathcal{E}nd(E)) = 0$, we get

$$-K_M \cdot \mathbb{P}^1 = c_2(\mathbb{E}) = 2rc_2(E) - (r-1)c_1(E)^2 := \Delta(E). \tag{2.1}$$

Let $f : X = C \times \mathbb{P}^1 \to C$ be the projection. Then, for any torsion free sheaf E on the ruled surface X, its restriction to a generic fiber $f^{-1}(\xi) = X_\xi$ has the form

$$E|_{X_\xi} = \bigoplus_{i=1}^{n} \mathcal{O}_{X_\xi}(\alpha_i)^{\oplus r_i}, \quad \alpha_1 > \cdots > \alpha_n.$$

The $\alpha = (\alpha_1^{\oplus r_1}, ..., \alpha_n^{\oplus r_n})$ is called the generic splitting type of E. In our case, tensoring E by $\pi^*\mathcal{O}(-\alpha_n)$, we can (and we will) assume that $\alpha_n = 0$. Any such E admits a relative Hardar-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

of which the quotient sheaves $F_i = E_i/E_{i-1}$ are torsion free with generic splitting type $(\alpha_i^{\oplus r_i})$ respectively. Then it is easy to see that

$$\begin{aligned} 2c_2(E) &= 2\sum_{i=1}^n c_2(F_i) + 2\sum_{i=1}^n c_1(E_{i-1})c_1(F_i) \\ &= 2\sum_{i=1}^n c_2(F_i) + c_1(E)^2 - \sum_{i=1}^n c_1(F_i)^2. \end{aligned}$$

Thus

$$\Delta(E) = 2r\sum_{i=1}^n c_2(F_i) + c_1(E)^2 - r\sum_{i=1}^n c_1(F_i)^2.$$

Let $F_i' = F_i \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-\alpha_i)$ $(i = 1, ..., n)$, thus they have generic splitting type $(0^{\oplus r_i})$ respectively. Let $c_1(F_i) = f^*\mathcal{O}_C(d_i) + \pi^*\mathcal{O}_{\mathbb{P}^1}(r_i\alpha_i)$, where $\mathcal{O}_C(d_i)$, $\mathcal{O}_{\mathbb{P}^1}(r_i\alpha_i)$ are divisors of degree d_i, $r_i\alpha_i$ on C, $\mathbb{P}^1$ respectively. Here we remark that for any torsion free sheaf F_i on X we have $c_1(F_i)|_{f^{-1}(\bullet)} = c_1(F_i|_{f^{-1}(\bullet)})$ for general points on C (resp. $c_1(F_i)|_{\pi^{-1}(\bullet)} = c_1(F_i|_{\pi^{-1}(\bullet)})$ for general points on $\mathbb{P}^1$). Therefore d_i are the degrees of F_i on the general fiber of π respectively. Without confusion, we denote the degree of F_i (resp. E_i) on the generic fiber of π by $deg(F_i)$ (resp. $deg(E_i)$). Consequently, $\mu(E_i)$, $\mu(E)$ denote the slope of restrictions of E_i, E to the generic fiber of π respectively. Note that

$$c_2(F_i') = c_2(F_i) - (r_i - 1)c_1(F_i)\pi^*\mathcal{O}_{\mathbb{P}^1}(\alpha_i) = c_2(F_i) - (r_i - 1)d_i\alpha_i,$$

$c_1(F_i)^2 = 2r_id_i\alpha_i$ and $c_1(E)^2 = 2d\sum_{i=1}^n r_i\alpha_i$, we have

$$\Delta(E) = 2r\left(\sum_{i=1}^n c_2(F_i') + \mu(E)\sum_{i=1}^n r_i\alpha_i - \sum_{i=1}^n d_i\alpha_i\right).$$

Let $rk(E_i)$ denote the rank of E_i, note that $r_i = rk(E_i) - rk(E_{i-1})$ and $d_i = deg(E_i) - deg(E_{i-1})$, we have

$$\Delta(E) = 2r\left(\sum_{i=1}^n c_2(F_i') + \sum_{i=1}^{n-1}(\mu(E) - \mu(E_i))(\alpha_i - \alpha_{i+1})rk(E_i)\right). \tag{2.2}$$

Lemma 2.1. *Any torsion free sheaf $\mathcal{E}$ of rank r on a ruled surface $f : X \to C$, with generic splitting type $(0^{\oplus r})$, must have $c_2(\mathcal{E}) \geq 0$ and $c_2(\mathcal{E}) = 0$ if and only if $\mathcal{E} = f^*V$ where V is a locally free sheaf on C.*

Proof. The argument is in fact contained in the proof of Lemma 1.4 of [GL]. If $\mathcal{E}$ has rank $r = 1$, then $c_2(\mathcal{E}) = \ell(\mathcal{E}^{\vee\vee}/\mathcal{E}) \geq 0$ and $c_2(\mathcal{E}) = 0$ if and only if $\mathcal{E} = \mathcal{E}^{\vee\vee}$ is a pullback of line bundle on C.

If $\mathcal{E}$ has rank $r > 1$, one can choose a rank 1 subsheaf $\mathcal{O}(D) \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{O}(D)$ is torsion free and $c_1(\mathcal{O}(D)) = D$ consists of fibers. Since $\mathcal{E}/\mathcal{O}(D)$ has generic splitting type $(0^{\oplus(r-1)})$, by induction hypothesis on rank, we can assume that $c_2(\mathcal{E}/\mathcal{O}(D)) \geq 0$ and it is zero if and only if $\mathcal{E}/\mathcal{O}(D)$ is the pullback of a local free sheaf V_1 on C. Hence

$$c_2(\mathcal{E}) = c_2(\mathcal{O}(D)) + c_2(\mathcal{E}/\mathcal{O}(D)) + D\cdot(c_1(\mathcal{E}) - D) = c_2(\mathcal{E}/\mathcal{O}(D)) + c_2(\mathcal{O}(D)) \geq 0,$$

and $c_2(\mathcal{E}) = 0$ if and only if $c_2(\mathcal{O}(D)) = c_2(\mathcal{E}/\mathcal{O}(D)) = 0$. Then $\mathcal{O}(D) = \mathcal{O}_X(D)$, $\mathcal{E}/\mathcal{O}(D) = f^*V_1$, which imply that $\mathcal{E}$ has constant splitting type $(0^{\oplus r})$ on each fiber. Thus $\mathcal{E} = f^*V$ for some locally free sheaf V on C.

Proposition 2.2. *If the rational curve passes through the generic point, then*

$$\Delta(E) \geq 2r.$$

When $g \geq 3$, then the equality holds if and only if E has generic splitting type $(0^{\oplus r})$ and $c_2(E) = 1$.

Proof. If $\Delta(E) < 2r$, then, by the equality (2.2), we have $n \geq 2$ and

$$\sum_{i=1}^{n} c_2(F_i') = 0, \quad \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i))(\alpha_i - \alpha_{i+1}) rk(E_i) < 1. \tag{2.3}$$

By Lemma 2.1, there are vector bundles V_i on C such that $F_i = f^*V_i \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(\alpha_i)$, where V_i has degree d_i and rank r_i. Thus the rational curve $\mathbb{P}^1$ parametrizes a family of stable bundles that are obtained by iterating extensions of V_i and V_{i+1}. Such bundles in M form a locally closed subset $\mathcal{R}^{(d_i)}_{(r_i)}$ of codimension at least

$$(g-1)\sum_{i=1}^{n-1}(r_{i+1}+\cdots+r_n)r_i + n - 1 + \sum_{i=1}^{n-1}\left(\frac{d_1+\cdots+d_i}{r_1+\cdots+r_i} - \frac{d}{r}\right)(r_1+\cdots+r_i)(r_{i+1}+r_i).$$

By (2.3), $deg(E_i) = d_1 + \cdots + d_i$, $rk(E_i) = r_1 + \cdots + r_i$, we have

$$\sum_{i=1}^{n-1}\left(\frac{d_1+\cdots+d_i}{r_1+\cdots+r_i} - \frac{d}{r}\right)(r_1+\cdots+r_i)(r_{i+1}+r_i) > -r_{i_0+1} + r_{i_0} \tag{2.4}$$

where $r_{i_0+1} + r_{i_0} = \max\{r_{i+1} + r_i \,|\, i = 1, ..., n\}$. Thus, by using the fact that $xy \geq x + y - 1$ for any positive integers x and y, we have

$$Codim(\mathcal{R}^{(d_i)}_{(r_i)}) > (g-2)\sum_{i=1}^{n-1}(r_{i+1}+\cdots+r_n)r_i + \sum_{i\neq i_0}^{n-1}(r_{i+1}+\cdots+r_n)r_i + n - 2 \geq 0.$$

For all possible $\{r_i\}_i$, $\{d_i\}_i$ satisfying (2.4), we get a countable locally closed subsets $\mathcal{R}^{(d_i)}_{(r_i)}$ of positive codimensions. What we proved above is that if $\Delta(E) < 2r$, then the rational curve falls in these given locally closed subsets of positive codimension. Thus if the rational curve passes through the generic point, then $\Delta(E) \geq 2r$.

If E has generic splitting type $(0^{\oplus r})$ and $c_2(E) = 1$, then it is obvious that $\Delta(E) = 2r$. Conversely, if $\Delta(E) = 2r$ and the rational curve passes through the generic point, then it is easy to see that $n = 1$ under the assumption $g \geq 3$. Otherwise the rational curve will fall in a $\mathcal{R}^{(d_i)}_{(r_i)}$ of positive codimension. The proof is finished.

From now on, we assume that E has generic splitting type $(0^{\oplus r})$. If E has a jumping line $X_p = f^{-1}(p)$ $(p \in C)$, i.e.,

$$E|_{X_p} = \bigoplus_{i=1}^{n} \mathcal{O}_{X_p}(\beta_i)^{\oplus r_i}, \quad \beta_1 > \cdots > \beta_n$$

with the type $(\beta_1^{\oplus r_1}, \cdots, \beta_n^{\oplus r_n})$ different from $(0^{\oplus r})$. Then we can perform the elementary transformation on E along X_p by taking F to be the kernel of the (unique surjective) homomorphism $\phi : E \to E|_{X_p} \to \mathcal{O}_{X_p}(\beta_n)^{\oplus r_n}$. Clearly,

$$0 \to F \to E \xrightarrow{\phi} \mathcal{O}_{X_p}(\beta_n)^{\oplus r_n} \to 0. \tag{2.5}$$

An easy calculation yields

Lemma 2.3. *$c_1(F) = c_1(E) - r_n X_p$ and $c_2(F) = c_2(E) + r_n\beta_n$.*

Proof. By the exact sequence (2.5), the computation is straightforward.

Lemma 2.4. *If $c_2(E) = 1$ and E has generic splitting type $(0^{\oplus r})$, then E has exactly one jumping line X_p and the elementary transformation F along X_p is isomorphic to f^*V for a vector bundle V over C.*

Proof. The E has at least one jumping line. Otherwise, E will be a pullback of a vector bundle over C, which is impossible. At any jumping line X_p, with splitting type $(\beta_1^{\oplus r_1}, \cdots, \beta_n^{\oplus r_n})$, we must have $\beta_n < 0$. Hence, by Lemma 2.3 and Lemma 2.1, E has a unique jumping line X_p with $\beta_n = -1$ and $r_n = 1$. Then F has no jumping line, thus $F = f^*V$ for a vector bundle V over C.

Therefore by Proposition 2.2 and Lemma 2.4, if $\Delta(E) = 2r$ and $g \geq 3$, we have

$$0 \to f^*V \to E \xrightarrow{\phi} \mathcal{O}_{X_p}(-1) \to 0. \tag{2.6}$$

Proposition 2.5. *If $g \geq 3$ and $\Delta(E) = 2r$, then the rational curve is a Hecke curve.*

Proof. For any $x \in \mathbb{P}^1$, let E_x denote $E|_{C\times\{x\}}$. Restrict the sequence (2.6) to $\pi^{-1}(x) = C \times \{x\}$, we get

$$0 \to V \to E_x \xrightarrow{\phi_x} \mathcal{O}_{X_p}(-1)_x \to 0. \tag{2.7}$$

Let $\iota_x : V_p \to E_x|_p = E_{(p,x)}$ be the homomorphism between the fibers at p induced by the sheaf injection $V \to E_x$ in sequence (2.7). Then the kernel $ker(\iota_x)$ is a 1-dimensional subspace of V_p. When x moves on $\mathbb{P}^1$, these $[ker(\iota_x)] \in \mathbb{P}(V_p)$ form a line $\mathcal{H} \subset \mathbb{P}(V_p)$. Note that here V corresponds to W^ζ in (1.1). It is easy to check that, as the same as (1.3), for any $x \in \mathbb{P}^1$

$$\widetilde{W}^{ker(\iota_x)} \cong E_x^\vee.$$

Thus $\{(\widetilde{W}^{ker(\iota_x)})^\vee;\ [ker(\iota_x)] \in \mathcal{H}\}$ defines the given rational curve. That is, the given rational curve is a Hecke curve.

Theorem 2.6. *Any rational curve of M passing through the generic point of M has at least degree $2r$ with respect to $-K_M$. If $g \geq 3$, then it has degree $2r$ if and only if it is a Hecke curve.*

Proof. By (2.1), the degree $-K_M \cdot \mathbb{P}^1$ equals to $\Delta(E)$. Then, by Proposition 2.2, it has degree at least $2r$. If it has degree $2r$, then by Proposition 2.5 it must be a Hecke curve. It was known that any Hecke curve has degree $2r$. We are done

§3 LINES ON THE MODULI SPACES

Since $(r, d) = 1$, it is easy to see that there are unique d_1 and $0 < r_1 < r$ such that $r_1 d - r d_1 = 1$. Let $r_2 = r - r_1$ and $d_2 = d - d_1$. Then

$$r_1 d - r d_1 = 1, \quad r_1 d_2 - d_1 r_2 = 1. \tag{3.1}$$

Let $\mathcal{U}_C(r_1, d_1)$ (resp. $\mathcal{U}_C(r_2, d_2)$) be the moduli space of stable vector bundles with rank r_1 (resp. r_2) and degree d_1 (resp. d_2). Then, by (3.1), they are smooth projective varieties and there are universal vector bundles $\mathcal{V}_1$, $\mathcal{V}_2$ on $C\times\mathcal{U}_C(r_1,d_1)$ and $C\times\mathcal{U}_C(r_2,d_2)$ respectively. Consider the morphism

$$\mathcal{U}_C(r_1,d_1)\times\mathcal{U}_C(r_2,d_2)\xrightarrow{det(\bullet)\times det(\bullet)}J_C^{d_1}\times J_C^{d_2}\xrightarrow{(\bullet)\otimes(\bullet)}J_C^d$$

and let $\mathcal{R}$ be its fiber at $[\mathcal{L}]\in J_C^d$. We still use $\mathcal{V}_1$, $\mathcal{V}_2$ to denote the pullback on $C\times\mathcal{R}$ by the projection $C\times\mathcal{R}\to C\times\mathcal{U}_C(r_i,d_i)$ $(i=1,2)$ respectively. Let $p:C\times\mathcal{R}\to\mathcal{R}$ and $\mathcal{G}=R^1p_*(\mathcal{V}_2^\vee\otimes\mathcal{V}_1)$. Then, by (3.1), $\mathcal{G}$ is a vector bundle of rank $r_1r_2(g-1)+1$. Let $q:P=\mathbb{P}(\mathcal{G}^\vee)\to\mathcal{R}$ be the projective bundle parametrzing 1-dimensional quotients of $\mathcal{G}^\vee$. Let

$$f:C\times P\to C,\quad \pi:C\times P\to P$$

be the projections. Then there exists a universal extension

$$0\to(id\times q)^*\mathcal{V}_1\otimes\pi^*\mathcal{O}_P(1)\to\mathcal{E}\to(id\times q)^*\mathcal{V}_2\to 0$$

on $C\times P$ such that for any $x=([V_1],[V_2],[e])\in P$, where $[V_i]\in\mathcal{U}_C(r_i,d_i)$ with $det(V_1)\otimes det(V_2)=\mathcal{L}$ and $[e]\subset H^1(C,V_2^\vee\otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}|_{C\times\{x\}}$ is the isomorphic class of vector bundles E given by extensions

$$0\to V_1\to E\to V_2\to 0 \tag{3.2}$$

that defined by vectors on the line $[e]\subset H^1(C,V_2^\vee\otimes V_1)$.

Lemma 3.1. *Let V_i be vector bundles of rank r_i and degree d_i $(i=1,2)$, where r_i, d_i satisfy (3.1). Let $0\to V_1\to V\to V_2\to 0$ be a non-trivial extension. Then V is stable if and only if V_1 and V_2 are stable bundles.*

Proof. Assume that V_1, V_2 are stable bundles, we prove that V is a stable bundle. Let $V'\subset V$ be a proper subbundle of rank r' and $V_2'\subset V_2$ be its image with rank r_2'. Then we have $0\to V_1'\to V'\to V_2'\to 0$, where $V_1'\subset V_1$ has rank r_1'. If $V_1'=0$, then $V'\cong V_2'$ is a proper subsheaf of V_2 since the extension is non-trivial. Thus $r_2r_2'(\mu(V_2)-\mu(V_2'))\geq 1$ by the stability of V_2 (note that the left side is an integer). On the other hand, by (3.1), we have

$$\mu(V_1)=\mu(V)-\frac{1}{r_1r},\quad \mu(V_2)=\mu(V)+\frac{1}{r_2r}. \tag{3.3}$$

Therefore, $\mu(V')=\mu(V_2')\leq\mu(V)+1/r_2r-1/r_2r_2'<\mu(V)$. If $V_2'=0$, it is clear that $\mu(V')<\mu(V)$. Then we assume that $V_i'\neq 0$ $(i=1,2)$. If $V_2'=V_2$, then $V_1'\neq V_1$ and $\mu(V_1')\leq\mu(V_1)-1/r_1'r_1$. Thus, combining with (3.3), we have

$$\mu(V')=\mu(V_1')\frac{r_1'}{r'}+\mu(V_2')\frac{r_2'}{r'}\leq\mu(V)-\frac{r_1'}{r_1rr'}+\frac{1}{rr'}-\frac{1}{r_1r'}<\mu(V).$$

If $V_1'=V_1$, one can check that $\mu(V')<\mu(V)$ similarly. Thus we assume that $V_i'\neq V_i$ $(i=1,2)$. Then $\mu(V_i')\leq\mu(V_i)-1/r_i'r_i$ $(i=1,2)$. By (3.3), we have

$$\mu(V')=\mu(V_1')\frac{r_1'}{r'}+\mu(V_2')\frac{r_2'}{r'}\leq\mu(V)-\frac{1}{rr'}<\mu(V).$$

Assume that V is stable, we show that V_1, V_2 must be stable. For any proper subbundle $V_1' \subset V_1$ of rank r_1', by using stability of V and (3.3), we have

$$\mu(V_1') \leq \mu(V) - \frac{1}{r_1' r} = \mu(V_1) + \frac{1}{r_1 r} - \frac{1}{r_1' r} < \mu(V_1).$$

Thus V_1 is stable. For any proper subbundle $V_2' \subset V_2$ of rank r_2', let $V' \subset V$ be defined such that $0 \to V_1 \to V' \to V_2' \to 0$ being exact. Then, by using (3.3) and the stability of V: $\mu(V') \leq \mu(V) - 1/r'r$ where $r' = rk(V')$, we have

$$\mu(V_2') = \frac{r'}{r_2'}\mu(V') - \frac{r_1}{r_2'}\mu(V) + \frac{1}{r_2' r} \leq \mu(V) < \mu(V_2).$$

Thus V_2 is stable. We are done.

By the Lemma 3.1, the vector bundle $\mathcal{E}$ given by the universal extension on $C \times P$ defines a morphism

$$\Phi : P \to \mathcal{U}_C(r, \mathcal{L}) = M. \tag{3.4}$$

Definition 3.2. *A smooth rational curve $\mathbb{P}^1 \subset P$ is called a line on P if*

$$\mathcal{O}_P(1)|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1).$$

Thus it is contained in a fiber of $q : P \to \mathcal{R}$.

Lemma 3.3. *For any line $\mathbb{P}^1 \subset P$, its image $\Phi(\mathbb{P}^1)$ is a line on M and*

$$\Phi|_{\mathbb{P}^1} : \mathbb{P}^1 \to \Phi(\mathbb{P}^1)$$

is the normalization of $\Phi(\mathbb{P}^1)$.

Proof. Let $q(\mathbb{P}^1) = (V_1, V_2) \in \mathcal{R}$ and $E = \mathcal{E}|_{C\times\mathbb{P}^1}$. Then the morphism

$$\Phi|_{\mathbb{P}^1} : \mathbb{P}^1 \to M$$

is defined by E, which fits in the exact sequence

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0, \tag{3.5}$$

where $f : C \times \mathbb{P}^1 \to C$ and $\pi : C \times \mathbb{P}^1 \to \mathbb{P}^1$ are the projections. Thus

$$c_1(E)^2 = 2r_1 d, \quad c_2(E) = r_1 d - d_1.$$

Then, by (2.1), the degree of $\Phi^*(-K_M)|_{\mathbb{P}^1}$ equals to

$$\Delta(E) = c_2(\mathcal{E}nd(E)) = 2rc_2(E) - (r-1)c_1(E)^2 = 2(r_1 d - r d_1) = 2,$$

which in particular implies that $\Phi(\mathbb{P}^1)$ is a curve on M of degree 2 and $\Phi|_{\mathbb{P}^1}$ is its normalization morphism. We finished the proof.

Theorem 3.4. *There exist lines on the moduli space M. For any line $\ell \subset M$, there is a line $\mathbb{P}^1 \subset P$ such that $\Phi(\mathbb{P}^1) = \ell$.*

Proof. The existence is just Lemma 3.3. For any line ℓ on M, let $\phi : \mathbb{P}^1 \to \ell \subset M$ be its normalization. Let E be the vector bundle on $C \times \mathbb{P}^1$ that defines the morphism ϕ. Then we have $\Delta(E) = \phi^*(-K_M) = -K_M \cdot \ell = 2$. Using the equality (2.2),

$$r\sum_{i=1}^{n} c_2(F'_i) + r\sum_{i=1}^{n-1}(\mu(E) - \mu(E_i))(\alpha_i - \alpha_{i+1})rk(E_i) = 1. \tag{3.6}$$

Then we must have that $n = 2$, $\alpha_1 = 1$ (E is choosed so that $\alpha_n = 0$), and $c_2(F'_1) = c_2(F'_2) = 0$. By Lemma 2.1, there are vector bundles V_1, V_2 on C such that $F_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) = F'_1 = f^*V_1$, $F_2 = F'_2 = f^*V_2$. Thus E satisfies

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0$$

where, by (3.6), V_1, V_2 must have rank r_1, r_2 and degree d_1, d_2 satisfying (3.1). Then, by Lemma 3.1, V_1 and V_2 must be stable bundles satisfying

$$det(V_1) \otimes det(V_2) = \mathcal{L}$$

since $E|_{C\times\{x\}}$ are stable ($x \in \mathbb{P}^1$) with determinant $\mathcal{L}$. Thus $\phi : \mathbb{P}^1 \to M$ factors through $\Phi : P \to M$. We are done.

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