Mean square estimate for twisted automorphic L-functions on weight aspect

Yuk-Kam Lau
Email: yklau@maths.hku.hk
Department of Mathematics, The University of Hong Kong
Pokfulam Road, Hong Kong

Abstract We study the mean square estimate for the twisted automorphic L-functions averaged over Hecke eigencuspforms at large weight. The upper bound obtained is quite sharp, and a direct application yields an unconditional version for a result of Kohnen and Sengupta.

1. Introduction. Let k be an even positive integer. The space of all holomorphic cusp forms of weight k with respect to the full modular group has a basis $\mathcal{B}_k$ of normalized Hecke eigencuspforms. More explicitly, for $f \in \mathcal{B}_k$, $T_n f = \lambda_f(n) n^{(k-1)/2} f$ where T_n $(n = 1, 2, \cdots)$ are the Hecke operators, and f has the Fourier series

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz)$$

where $e(\alpha) = e^{2\pi i \alpha}$. The eigenvalues $\lambda_f(n)$ are all real, and furthermore, $\lambda_f(1) = 1$ and

$$|\lambda_f(n)| \le d(n) \tag{1.1}$$

(Deligne's bound) where $d(n) = \sum_{d|n} 1$ is the divisor function.

Let χ be a primitive Dirichlet character of conductor $D \ge 1$. We associate each f with a twisted L-function

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \chi(n) \lambda_f(n) n^{-s} \qquad (\Re e\, s > 1). \tag{1.2}$$

Analogously to the classical L-functions, the twisted automorphic L-function factors into an Euler product. Moreover, let us define

$$\Lambda(f \otimes \chi, s) = (\frac{D}{2\pi})^s \Gamma(s + (k-1)/2) L(f \otimes \chi, s).$$

The completed function $\Lambda(f \otimes \chi, s)$ can be holomorphically continued to the whole $\mathbb{C}$. It is bounded on any vertical strip, and satisfies the functional equation

$$\Lambda(f \otimes \chi, s) = \epsilon_k(\chi) \Lambda(f \otimes \overline{\chi}, 1 - s) \tag{1.3}$$

where $\epsilon_k(\chi) = i^k\tau(\chi)^2/D$ and $\tau(\chi)$ is the Gaussian sum (see [3, Theorem 7.6]). In addition to this similarity, it is *conjectured* that the Grand Riemann Hypothesis holds, and

$$L(f\otimes\chi, 1/2+it) \ll_\epsilon (k(1+|t|))^\epsilon \tag{1.4}$$

for any $\epsilon > 0$. These (twisted) automorphic L-functions play an important role in GL_2 theory and are interesting; for example, their values at the central point reveal arithmetical information.

When χ is quadratic, Kohnen and Sengupta [6] proved that

$$\sum_{f\in\mathcal{B}_k} L(f\otimes\chi, 1/2) \ll_{\epsilon,D} k^{1+\epsilon} \qquad (k\to\infty,\ i^kD>0), \tag{1.5}$$

and deduced that under the assumption $L(f\otimes\chi,1/2)\ll_{\delta,D} k^\delta$ where $\delta\geq 0$,

$$\#\{f\in\mathcal{B}_k : L(f\otimes\chi,1/2)\neq 0\} \gg_{\delta,D} k^{1-\delta}/\log k. \tag{1.6}$$

Our main objective is to establish a mean square estimate for $L(f\otimes\chi,1/2+it)$ averaging over the basis $\mathcal{B}_k$ on the weight aspect. We obtain an estimate (see Theorem 2 below) which supports the validity of (1.4) on the weight aspect. An application is to yield an unconditional lower bound for (1.6).

Now let us fix our notation: denote by $D(k)$ a positive increasing function such that $\log D(k) = o(\log k)$, and define for $f\in\mathcal{B}_k$,

$$w_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|^2}$$

where $\|f\|^2 = \int_F y^{k-2}|f(z)|^2\,dxdy$. ($F$ is the fundamental domain for the full modular group.) The first theorem is a weighted form for (1.5) in the critical strip, and the next theorem on mean square estimate is our main result.

Theorem 1 *Let any $A\geq 0$ and any arbitrarily small $\epsilon>0$ be given. Let k be any sufficiently large even integer. Suppose $1/2\leq \Re e\, s\leq 1$ and χ is a primitive character of conductor D with $1\leq D\leq D(k)$. Then,*

$$\begin{aligned}\sum_{f\in\mathcal{B}_k} w_f L(f\otimes\chi, s) &= 1+\epsilon_k(\chi)\Big(\frac{D}{2\pi}\Big)^{1-2s}\frac{\Gamma(1-s+(k-1)/2)}{\Gamma(s+(k-1)/2)}\\ &\qquad + O(|s|^\epsilon k^{-A} + |s|^{2\Re e\, s+A+\epsilon}k^{-2(\Re e\, s+A)})\end{aligned}$$

where the implied constant in the O-term depends on ϵ and A but is uniform for $1\leq D\leq D(k)$. (Recall $\mathcal{B}_k$ is the basis containing all normalized Hecke eigencuspforms.)

Remark. Due to (1.3), there is no loss of generality to assume $1/2 \le \Re e\, s \le 1$. Taking $s = 1/2 + it$, the second term in Theorem 1 is $\ll 1$. Hence, for any $A \ge 0$,

$$\sum_{f \in \mathcal{B}_k} w_f L(f \otimes \chi, 1/2 + it) \ll_{\epsilon, A} |t|^{\epsilon} + 1 \qquad (|t| \le k^{1+A/(1+A)}),$$

i.e. the left-side satisfies the Lindelof hypothesis for $|t| \le k^{2-\epsilon}$ by setting $A = (1-\epsilon)/\epsilon$. Here and in the sequel, we write $\ll_*$ to specify the dependence of the implied constant on $*$.

Theorem 2 *Under the same assumptions as in Theorem 1, we have for any $\epsilon > 0$,*

$$\sum_{f \in \mathcal{B}_k} w_f |L(f \otimes \chi, 1/2 + it)|^2 \ll_{\epsilon} k^{\epsilon} (|t| + 1)^{2+\epsilon} \qquad (t \in \mathbb{R})$$

where the implied constant depends only on ϵ but is uniform in t.

The case $t = 0$ yields (1.5) immediately by the Cauchy-Schwarz inequality and $\sum_{f \in \mathcal{B}_k} w_f \ll 1$, see (2.6) below. Another consequence is about the non-vanishing of the central values.

Corollary 3 *Under the assumptions in Theorem 1, it holds that for any $\epsilon > 0$*

$$\#\{f \in \mathcal{B}_k : L(f \otimes \chi, 1/2) \ne 0\} \gg_{\epsilon} |1 + \epsilon_k(\chi)|^2 k^{1-\epsilon}.$$

Proof. Taking $s = 1/2$ in Theorem 1 and using the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} |1 + \epsilon_k(\chi)|^2 - O(k^{-1}) &\le \left| \sum_{f \in \mathcal{B}_k} w_f L(f \otimes \chi, 1/2) \right|^2 \\ &\le \Big(\sum_{\substack{f \in \mathcal{B}_k \\ L(f \otimes \chi, 1/2) \ne 0}} w_f \Big) \Big(\sum_{f \in \mathcal{B}_k} w_f |L(f \otimes \chi, 1/2)|^2 \Big). \end{aligned}$$

Together with the bound $w_f \ll \log k / k$ (see [6]), it follows that for any $\epsilon > 0$,

$$\sum_{\substack{f \in \mathcal{B}_k \\ L(f \otimes \chi, 1/2) \ne 0}} 1 \gg_{\epsilon} |1 + \epsilon_k(\chi)|^2 k^{1-\epsilon}.$$

This completes the proof.

Remark. Note that $\epsilon_k(\chi) = i^k D/|D|$ for real χ. This gives (1.6) an unconditional lower bound with essentially the same quality.

Finally we outline the proof of the main result Theorem 2. We represent the value of $L(f\otimes\chi, 1/2+it)$ by a fast convergent series. Averaging over $\mathcal{B}_k$ enables us to apply the Petersson trace formula. It turns up a sum that involves the Kloosterman sum and the Bessel function. The next process is to delve the possible cancellations, like [1] and [5] but these articles focus on the level aspect, not on the weight. In [4] there is a tool to treat the case of averaging over weights but not for individual. Hence we need different auxiliary tools for our situation, as follows: Using the periodicity of the Kloosterman sum, we pass to an exponential sum over an arithmetic progression. Then the resulting exponential integral will be handled in Lemma 2.1 by a 'saddle-point' theorem [2, Theorem 2.2].

2. Some Preparations. Our key tool is the Petersson trace formula [3, Theorem 3.6]:

$$\sum_{f\in\mathcal{B}_k} w_f\lambda_f(m)\lambda_f(n) = \delta_{m,n} + 2\pi i^{-k}\sum_{c\geq 1} c^{-1}S(m,n,c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{2.1}$$

where $\delta_{m,n}$ is the Kronecker delta, $S(m,n,c)$ is the Kloosterman sum and J_{k-1} is the Bessel function of order $k-1$. Note that

$$|S(m,n,c)| \leq (m,n,c)^{1/2}c^{1/2}d(c). \tag{2.2}$$

($d(\cdot)$ is the divisor function.) We derive some estimates for the Bessel function with large k, based on the integral representations:

(i) $J_{k-1}(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{-i(k-1)\theta+ix\sin\theta}\,d\theta$ for $x>0$ (see [8, §17.23]),

(ii) $J_{k-1}(x) = \dfrac{1}{\sqrt{\pi}\Gamma(k-1/2)}\left(\dfrac{x}{2}\right)^{k-1}\displaystyle\int_{-1}^{1}(1-t^2)^{k-3/2}\cos(xt)\,dt$ from [8, §17.3 Corollary].

Writing $f(\theta) = (k-1)\theta + x\sin\theta$, then f' is monotonic and $f'(\theta)\geq k-1$ for $x>0$ and $\theta\in[0,\pi/2]$. This yields

$$\int_0^{\pi/2} e^{\pm if(\theta)}\,d\theta \ll \sup_{\theta\in[0,\pi/2]}|f'(\theta)|^{-1} \ll k^{-1} \tag{2.3}$$

by the first derivative test ([7, Lemma 4.3]). In addition, it follows from the same argument that, for $x\ll k^{1+\delta}$ where $0<\delta<1$,

$$\int_{\pi/2-k^{-4\delta}}^{\pi/2} e^{ix\sin\theta-i(k-1)\theta}\,d\theta \ll k^{-1}.$$

(Note that $x\cos\theta-(k-1)\le -k+O(k^{1-3\delta})\le -k/2$ this time.) Dividing the θ-integral in (i) into suitable ranges, we see that for any $0<\delta<1$ and $0<x\ll k^{1+\delta}$,

$$J_{k-1}(x) \;=\; \Re e\,\frac{1}{\pi}\int_0^{\pi/2-k^{-4\delta}} e^{-i(k-1)\theta+ix\sin\theta}\,d\theta+O(k^{-1}). \tag{2.4}$$

Stirling's formula [2, (A.33)] gives $\Gamma(k-1/2)\gg 3^{-k}k^{k-1}$. Apparently (ii) yields

$$J_{k-1}(x)\ll\frac{1}{\Gamma(k-1/2)}\big(\frac{x}{2}\big)^{k-1}\ll\big(\frac{2x}{k}\big)^{k-1}. \tag{2.5}$$

As $\lambda_f(1)=1$, taking $m=n=1$ in (2.1), it follows immediately from (2.2) and (2.5)

$$\sum_{f\in\mathcal{B}_k} w_f\ll 1+\big(\frac{2}{k}\big)^{k-1}\sum_{c\ge1}c^{1/2-k}d(c)\ll 1. \tag{2.6}$$

The next lemma is another main tool, used in the proof of Theorem 2.

Lemma 2.1 *Let $p,\ q\in\mathbb{C}$ with $\Re e\,p,\Re e\,q\ll 1$ and $0<\theta<1$ be any fixed number. Suppose $1\le c^2\le Q\le K^\theta$. Then, for any $\epsilon>0$,*

$$\int_0^1\Bigg|\sum_{\substack{K<m,n\le 2K\\ m\equiv a,\,n\equiv b\,(Q)}} m^{-1/2-p}n^{-q}e\big(\frac{2\lambda}{c}\sqrt{mn}\big)\Bigg|\,d\lambda \;\ll\; (|p|+1)(|q|+1)\frac{Q}{c}K^{1/2+\epsilon-\Re e(p+q)}$$

where the implied constant depends on ϵ and θ only.

Proof. We only need to consider the case for large K. Stieltjes integration with by parts gives

$$\sum_{\substack{A<h\le B\\ h\equiv\gamma\,(\Delta)}} h^{-(u+v)}e(\sqrt{h}\phi)=\int_A^B y^{-v}\,d\sum_{\substack{A<h\le y\\ h\equiv\gamma\,(\Delta)}} h^{-u}e(\sqrt{h}\phi)$$

$$=\;B^{-v}\sum_{\substack{A<h\le B\\ h\equiv\gamma\,(\Delta)}} h^{-u}e(\sqrt{h}\phi)+v\int_A^B y^{-v-1}\sum_{\substack{A<h\le y\\ h\equiv\gamma\,(\Delta)}} h^{-u}e(\sqrt{h}\phi)\,dy.$$

Applying it with $u=1/2$, $v=p$ for the sum over m, and $u=0$, $v=q$ for the sum over

n, the integrand is expressed as

$$\begin{aligned}
&\sum_{\substack{K<m,n\le 2K\\ m\equiv a,\, n\equiv b\,(Q)}} m^{-1/2-p}n^{-q}e(\frac{2\lambda}{c}\sqrt{mn})\\
=\ & (2K)^{-(p+q)}\sum_{\substack{K<m\le 2K\\ m\equiv a\,(Q)}} m^{-1/2}\sum_{\substack{K<n\le 2K\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn})\\
&+p(2K)^{-q}\int_K^{2K} x^{-1-p}\sum_{\substack{K<m\le x\\ m\equiv a\,(Q)}} m^{-1/2}\sum_{\substack{K<n\le 2K\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn})\,dx\\
&+q(2K)^{-p}\int_K^{2K} y^{-1-q}\sum_{\substack{K<m\le 2K\\ m\equiv a\,(Q)}} m^{-1/2}\sum_{\substack{K<n\le y\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn})\,dy\\
&+pq\int_K^{2K}\int_K^{2K} x^{-1-p}y^{-1-q}\sum_{\substack{K<m\le x\\ m\equiv a\,(Q)}} m^{-1/2}\sum_{\substack{K<n\le y\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn})\,dx\,dy
\end{aligned}$$

Integrating with respect to λ, we obtain for some $M, N \in [K, 2K]$,

$$\begin{aligned}
&\int_0^1\left|\sum_{\substack{K<m,n\le 2K\\ m\equiv a,\, n\equiv b\,(Q)}} m^{-1/2-p}n^{-q}e(\frac{2\lambda}{c}\sqrt{mn})\right|d\lambda\\
\ll\ & (|p|+1)(|q|+1)K^{-\Re e(p+q)}\int_0^1\left|\sum_{\substack{K<m\le M\\ m\equiv a\,(Q)}} m^{-1/2}\sum_{\substack{K<n\le N\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn})\right|d\lambda. \quad (2.7)
\end{aligned}$$

We proceed to transform the sum over n in (2.7) by an extension of [7, Lemma 4.7], namely, for any $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{N}$,

$$\begin{aligned}
\sum_{\substack{a<n<b\\ n\equiv\alpha\,(Q)}} e(g(n)) \ =\ & \sum_{Qg'(b)-1/2<\nu<Qg'(a)+1/2}\int_a^b e(g(x)-\frac{\nu x}{Q})\,dx\cdot\frac{1}{Q}e(\frac{\alpha\nu}{Q})\\
&+O(\log(Q(g'(a)-g'(b))+2))
\end{aligned}$$

where g is twice continuously differentiable with decreasing g'. This yields

$$\begin{aligned}
\sum_{\substack{K<n\le N\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn}) \ =\ & Q^{-1}\sum_{\mathcal{U}(m)<\nu<\mathcal{V}(m)} e(\frac{\nu b}{Q})\int_K^N e(\frac{2\lambda}{c}\sqrt{mt}-\frac{\nu t}{Q})\,dt\\
&+O(\log K) \quad (2.8)
\end{aligned}$$

(as $c^2 \le Q \le K^\theta$, $K < m \le 2K$) where

$$\mathcal{U}(m) = \frac{\lambda Q}{c}\sqrt{\frac{m}{N}} - \frac{1}{2} \qquad \text{and} \qquad \mathcal{V}(m) = \frac{\lambda Q}{c}\sqrt{\frac{m}{K}} + \frac{1}{2} \ll \frac{Q}{c}. \quad (2.9)$$

Let us remind the values of different parameters:

$$K \le m \le M \le 2K,\ K \le n \le N \le 2K,\ 1 \le c^2 \le Q \le K^\theta\ (0 < \theta < 1),\ 0 < \lambda < 1.$$

The term for $\nu = 0$ (if exists) is handled, as follows:

$$\int_K^N e(\frac{2\lambda}{c}\sqrt{mt})\,dt \ll \min(\frac{c}{|\lambda|}\sqrt{\frac{N}{m}}, N) \ll \min(K, c|\lambda|^{-1})$$

by the first derivative test (see (2.3)) and the trivial estimate. Thus, (2.8) becomes

$$\begin{aligned}\sum_{\substack{K<n\le N\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn}) &= Q^{-1}\sum_{\substack{\nu\ge 1\\ \mathcal{U}(m)<\nu\le\mathcal{V}(m)}} e(\frac{\nu b}{Q})\int_K^N e(\frac{2\lambda}{c}\sqrt{mt}-\frac{\nu t}{Q})\,dt \\ &\quad + O(K^\epsilon \min(K, c|\lambda|^{-1})). \qquad (2.10)\end{aligned}$$

To make use of the sum over ν, we need to evaluate the exponential integral in a quite precise form. To this end, we apply [2, Theorem 2.2] with $\Phi(x) = 1$, $F(x) = \lambda K/c$ and $\mu(x) = x/2$ to the complex conjugate of the integral in (2.10). It is not difficult to see that $x_0 = m(\lambda Q/(c\nu))^2$ in [2, Theorem 2.2], and on the right-side of [2, (2.16)], the first O-term is

$$\ll \int_K^N \exp(-C_0\nu x/Q)\,dx \ll Q\nu^{-1}\exp(-C_0\nu K^{1-\theta}) \ll (\frac{Q}{\nu})^2\frac{1}{\sqrt{cK}}$$

($\nu \ll Q$ by (2.9)) for some absolute constant C_0 and the second O-term is

$$\ll \frac{m}{K}(\frac{Q}{\nu})^2\sqrt{\frac{\lambda}{cK}} \ll (\frac{Q}{\nu})^2\frac{1}{\sqrt{cK}}.$$

Taking complex conjugate again, it follows from [2, Theorem2.2] that

$$\begin{aligned}\int_K^N e(\frac{2\lambda}{c}\sqrt{mt}-\frac{\nu t}{Q})\,dt &= \sqrt{2m}\frac{\lambda}{c}\left(\frac{Q}{\nu}\right)^{3/2} e(\frac{Q\lambda^2}{\nu c^2}m - \frac{1}{8}) \\ &\quad + O(\frac{Q^2}{\nu^2}\frac{1}{\sqrt{cK}} + E_{\nu,m}(\lambda, K) + E_{\nu,m}(\lambda, N)) \qquad (2.11)\end{aligned}$$

where, after some simplification, we can take

$$E_{\nu,m}(\lambda, h) = c\left(\left|\lambda - \frac{c\nu}{Q}\sqrt{\frac{h}{m}}\right| + \sqrt{\frac{\lambda}{K}}\right)^{-1}. \qquad (2.12)$$

(Remark. The classical formula [7, Lemma 4.6] is not sufficient, its error term involving the third derivative is too big.)

Inserting (2.11) into (2.10), we have

$$\int_0^1 \Big| \sum_{\substack{K<m\le M\\ m\equiv a\,(Q)}} m^{-1/2} \sum_{\substack{K<n\le N\\ n\equiv b\,(Q)}} e(\frac{2\lambda}{c}\sqrt{mn})\Big|\, d\lambda \ll \frac{\sqrt{Q}}{c} I_1 + K^\epsilon I_2 + Q^{-1} I_3 \tag{2.13}$$

where

$$I_1 = \int_0^1 \lambda \Big| \sum_{\substack{K<m\le M\\ m\equiv a\,(Q)}} \sum_{\substack{\nu\ge 1\\ \mathcal{U}(m)<\nu\le\mathcal{V}(m)}} \nu^{-3/2} e(\frac{Q\lambda^2}{\nu c^2} m + \frac{\nu b}{Q})\Big|\, d\lambda \tag{2.14}$$

$$I_2 = \sum_{K<m\le 2K} m^{-1/2} \int_0^1 \Big(\min(K, \frac{c}{|\lambda|}) + \frac{Q}{\sqrt{cK}}\Big)\, d\lambda$$

$$I_3 = \sum_{K<m\le 2K} m^{-1/2} \int_0^1 \sum_{\substack{\nu\ge 1\\ \mathcal{U}(m)<\nu<\mathcal{V}(m)}} (E_{\nu,m}(\lambda, K) + E_{\nu,m}(\lambda, N))\, d\lambda \tag{2.15}$$

Apparently $I_2 \ll cK^{1/2+\epsilon} + Q/\sqrt{c}$. Observing that for any real number H,

$$\begin{aligned}\int_0^1 \frac{d\lambda}{|\lambda - H| + \sqrt{\lambda/K}} &\ll K^{1/2} \int_{\substack{0<\lambda<1\\ |\lambda-H|\le K^{-1}}} \frac{d\lambda}{\sqrt{\lambda}} + \int_{\substack{0<\lambda<1\\ |\lambda-H|\ge K^{-1}}} \frac{d\lambda}{|\lambda-H|}\\ &\ll \log K,\end{aligned}$$

we infer that $I_3 \ll QK^{1/2+\epsilon}$ by (2.9), (2.12) and (2.15), hence

$$K^\epsilon I_2 + Q^{-1} I_3 \ll cK^{1/2+\epsilon} + Q/\sqrt{c} \ll K^{1/2+\epsilon} Q/c. \tag{2.16}$$

Finally we estimate I_1. Interchanging the summations, the double sum in (2.14) equals

$$\sum_{\substack{\nu\ge 1\\ \mathcal{U}(K)<\nu\le\mathcal{V}(M)}} \nu^{-3/2} e(\frac{\nu b}{Q}) \sum_{m\in\mathcal{I}} e(\frac{Q\lambda^2}{\nu c^2} m) \tag{2.17}$$

where m runs over the arithmetic progression $\mathcal{I} = \{m \equiv a \pmod Q) : X < m \le Y\}$, with

$$K \le X = \max\{K, (\frac{c(\nu - 1/2)}{Q\lambda})^2 K\}\ ,\ Y = \min\{M, (\frac{c(\nu + 1/2)}{Q\lambda})^2 N\} \le 2K.$$

It is not hard to show that for any $\alpha \in \mathbb{Z}$ and $\phi \in \mathbb{R}$,

$$\sum_{\substack{n\le x\\ n\equiv\alpha\,(Q)}} e(n\phi) \ll \min(1 + |x-\alpha|/Q, |\sin(\pi Q\phi)|^{-1}).$$

Applying it to (2.17), it follows that the double sum (in (2.14)) is

$$\ll \sum_{\substack{\nu\ge 1\\ \mathcal{U}(K)<\nu\le\mathcal{V}(M)}} \nu^{-3/2} \min(K, |\sin(\pi \frac{Q^2\lambda^2}{\nu c^2})|^{-1}).$$

We conclude that

$$
\begin{aligned}
I_1 &\ll \int_0^1 \sum_{\substack{\nu\geq 1\\ \mathcal{U}(K)<\nu\leq\mathcal{V}(M)}} \nu^{-3/2}\min(K,|\sin(\frac{\pi Q^2}{\nu c^2}\lambda^2)|^{-1}))\lambda\,d\lambda \\
&\ll \log K. \qquad (2.18)
\end{aligned}
$$

The last line follows from $Q^2/(\nu c^2)\ll 1$ and the estimate

$$
\int_0^1 \lambda\min(K,|\sin(\pi\alpha\lambda^2)|^{-1})\,d\lambda \ll \frac{\alpha+1}{\alpha}\int_0^{\pi/2}\min(K,|\sin u|^{-1})\,du \ll \log K
$$

for any $\alpha\gg 1$. This completes the proof in view of (2.7), (2.13), (2.16) and (2.18).

Remark: In view of (2.16), it is possible to get a better upper estimate in Lemma 2.1, but we take this for simplicity.

Lemma 2.2 *Let B be a fixed positive number. Let $H\geq 2B+4$ be any large number. Suppose α, $\beta\in\mathbb{C}$ satisfy $\Re e\,\alpha,\Re e\,\beta\in[-B,B]$. Then,*

$$
\frac{\Gamma(H+\beta+\alpha)}{\Gamma(H+\beta)} \ll_{\epsilon,B} (|\beta|+1)^{\epsilon}(H+|\beta|)^{\Re e\,\alpha}(|\mathrm{Im}\,\alpha|+1)^{1/2}e^{\pi|\mathrm{Im}\,\alpha|/2}
$$

where $0<\epsilon<1$ is arbitrary and the implied constant depends on ϵ and B.

Proof. Firstly we note that for any σ, $t\in\mathbb{R}$ with $|\sigma|\leq B$,

$$
\frac{\Gamma(H+\sigma+it)}{\Gamma(H+it)} \ll_B (H+|t|)^{\sigma}, \qquad (2.19)
$$

with the implied constant uniform in t. This is done by applying Stirling's formula (see (A.33) in the appendix of [2]) to $\Re e\big(\log\Gamma(H+\sigma+it)-\log\Gamma(H+it)\big)$.

It is known that for $\Re e\,x$ and $\Re e\,y>0$, $\Gamma(x)\Gamma(y)/\Gamma(x+y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, see [2, (A.31)]. Let $a=\Re e\,\alpha+\epsilon$. Then,

$$
\begin{aligned}
\Big|\frac{\Gamma(H+\beta+\alpha)\Gamma(a-\alpha)}{\Gamma(H+\beta+a)}\Big| &= \Big|\int_0^1 t^{a-\alpha-1}(1-t)^{H+\beta+\alpha-1}\,dt\Big| \\
\leq \int_0^1 t^{\epsilon-1}(1-t)^{H+\Re e\,(\beta+\alpha)-1}\,dt &= \frac{\Gamma(H+\Re e\,(\beta+\alpha))}{\Gamma(H+\Re e\,\beta+a)}\Gamma(\epsilon).
\end{aligned}
$$

Multiplying both sides by $|\Gamma(H+\beta+a)|/|\Gamma(H+\beta)\Gamma(a-\alpha)|$ gives

$$
\frac{\Gamma(H+\beta+\alpha)}{\Gamma(H+\beta)} \ll_{\epsilon} \frac{\Gamma(H+\Re e\,(\beta+\alpha))}{\Gamma(H+\Re e\,\beta+a)}\frac{\Gamma((H+\Re e\,\beta)+a+i\mathrm{Im}\,\beta)}{\Gamma((H+\Re e\,\beta)+i\mathrm{Im}\,\beta)}|\Gamma(\epsilon-i\,\mathrm{Im}\,\alpha)|^{-1}.
$$

By (2.19), the first two fractions on the right are respectively $\ll_\epsilon H^{\Re e\,\alpha-a}$, and $(H+|\beta|)^a$, which is $\ll (H(|\beta|+1))^{a-\Re e\,\alpha}(H+|\beta|)^{\Re e\,\alpha}$. Our result follows after evaluating the last Gamma factor with [2, (A.34)] – for $|\sigma|\ll 1$ and $|t|\gg 1$,

$$
|\Gamma(\sigma+it)| \asymp |t|^{\sigma-1/2}e^{-\pi|t|/2}. \qquad (2.20)
$$

Remark: From Lemma 2.2 and (1.3), it follows that $L(f\otimes\chi, -\epsilon+it) \ll (k+|t|)^{1+\epsilon}$. Then Phragmén-Lindelöf theorem yields the convexity bound

$$L(f\otimes\chi, 1/2+it) \ll_\epsilon (k+|t|)^{1/2+\epsilon}. \tag{2.21}$$

3. Proof of Theorems. We assume $1/2 \le \Re e\, s \le 1$. Given any number $A \ge 0$, we set $A_0 = (2+\delta+A)/\delta$ $(> 3+A)$ where $0<\delta<1$ is a parameter to be chosen at our disposal. Then we consider the integral

$$\frac{1}{2\pi i}\int_{\mathcal{R}} \Lambda(f\otimes\chi, 1/2+w)G(w)\,dw \quad \text{where} \quad G(w) = \frac{1}{w}\frac{\Gamma(A_0-w)\Gamma(A_0+w)}{\Gamma(A_0)^2}.$$

Here the contour $\mathcal{R}$ is the positively oriented rectangle with vertices at $\pm 2 \pm iT$.

Inside the strip $-A_0 < \Re e\, w < A_0$, $G(w)$ has only one simple pole at $w=0$ and by (2.20), we have the rapid decay of $G(w)$ along $\mathrm{Im}\, w$,

$$G(w) \ll |w|^{-1}(|\mathrm{Im}\, w|+1)^{2A_0-1}e^{-\pi|\mathrm{Im}\, w|}, \tag{3.1}$$

which assures the damping down of the Gamma factors from $\Lambda(f\otimes\chi,\cdot)$, see (3.4).

Now we apply the residue theorem to express $L(f\otimes\chi, s)$ as a convergent series. From [3, Theorem 7.6], $\Lambda(f\otimes\chi,\cdot)$ is bounded and $\mathcal{R}$ is inside the strip $-A_0 < \Re e\, w < A_0$. We take $T\to\infty$, and apply the functional equation (1.3) to get

$$\begin{aligned} &(D/(2\pi))^s\Gamma(s+(k-1)/2)L(f\otimes\chi, s) \\ = \; &\frac{1}{2\pi i}\int_{(2)} \Lambda(f\otimes\chi, s+w)G(w)\,dw + \epsilon_k(\chi)\frac{1}{2\pi i}\int_{(2)}\Lambda(f\otimes\overline{\chi}, 1-s+w)G(w)\,dw. \end{aligned}$$

With (1.2), this leads to

$$L(f\otimes\chi, s) \;=\; \sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi(n)}{n^s}V_s(n) + \epsilon_k(\chi)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\overline{\chi}(n)}{n^{1-s}}V_{1-s}(n) \tag{3.2}$$

where

$$V_z(y) = \frac{1}{2\pi i}\int_{(2)}\left(\frac{D}{2\pi}\right)^{w+z-s}\frac{\Gamma(z+w+(k-1)/2)}{\Gamma(s+(k-1)/2)}y^{-w}G(w)\,dw. \tag{3.3}$$

Observing $|\Gamma(z+w+(k-1)/2)| = |\Gamma((k-1)/2+s+(\overline{w}+\Re e\,(z-s)))|$ for the case $z=1-s$, Lemma 2.2 implies that for $z=s$ or $1-s$, and $|\Re e\, w| < A_0$,

$$\frac{\Gamma(z+w+(k-1)/2)}{\Gamma(s+(k-1)/2)} \ll_{\epsilon, A_0} |s|^\epsilon (k+|s|)^{\Re e(w+z-s)}(|\mathrm{Im}\, w|+1)^{1/2}e^{\pi|\mathrm{Im}\, w|/2} \tag{3.4}$$

for all large enough $k(>2A_0+4)$. In view of (3.1) and (3.4), we obtain by moving the line of integration to $\Re e\, w=\sigma\in(0,A_0)$ that as $D^\sigma\le D(k)^\sigma\ll_{\epsilon,A_0}k^\epsilon$,

$$V_z(y)\ll_{\sigma,\epsilon,A_0}(k|s|)^\epsilon y^{-\sigma}(k+|s|)^{\sigma+\Re e(z-s)}. \tag{3.5}$$

Moreover, shifting the integral path to $\Re e\, w=-A-1>-A_0$, we obtain by (3.4),

$$V_z(y) \;=\; \Big(\frac{D}{2\pi}\Big)^{z-s}\frac{\Gamma(z+(k-1)/2)}{\Gamma(s+(k-1)/2)}+O(y^{A+1}k^{\Re e(z-s)-A-1}). \tag{3.6}$$

The first term comes from the simple pole of $G(w)$.

We truncate the series in (3.2) and estimate the contribution of the tails. By (3.5) with $\Re e\, s<\sigma<A_0$ and (1.1), the tails of the sums in (3.2) are

$$\begin{aligned}
&\ll (k|s|)^\epsilon(k+|s|)^\sigma\sum_{n\ge N}d(n)n^{-(\Re e\, s+\sigma)}\\
&\quad+(k|s|)^\epsilon(k+|s|)^{1-2\Re e\, s+\sigma}\sum_{n\ge N}d(n)n^{-(1-\Re e\, s+\sigma)}\\
&\ll (k|s|)^\epsilon(\log N)\big((k+|s|)^\sigma N^{1-\Re e\, s-\sigma}+(k+|s|)^{1-2\Re e\, s+\sigma}N^{\Re e\, s-\sigma}\big)\\
&\ll (k|s|)^\epsilon(k+|s|)^\sigma N^{\Re e\, s-\sigma}\log N.
\end{aligned}\tag{3.7}$$

The last estimate is a bit loose but convenient for later calculation. Note also that for $\Re e\, s\ge 1/2$, $\Re e\, s+\sigma\ge 1-\Re e\, s+\sigma>1$ which is needed for the convergence of (3.7). With the choice $N=k^{1+\delta}$ (for the same δ as above), it follows from (3.2) that

$$\begin{aligned}
L(f\otimes\chi,s) \;=\;& \sum_{n\le k^{1+\delta}}\frac{\lambda_f(n)\chi(n)}{n^s}V_s(n)\\
&+\epsilon_k(\chi)\sum_{n\le k^{1+\delta}}\frac{\lambda_f(n)\overline{\chi}(n)}{n^{1-s}}V_{1-s}(n)+E(|s|,k)
\end{aligned}\tag{3.8}$$

where $E(|s|,k)\ll(k|s|)^\epsilon(k+|s|)^\sigma k^{(1+\delta)(\Re e\, s-\sigma)}$ and $\Re e\, s\le\sigma<A_0$.

To prove Theorem 1, we choose $\delta=(A+1)/(A+1+\epsilon)$, so δ is quite close to 1. (Small ϵ is assumed.) Choosing $\sigma=(A+\epsilon+(1+\delta)\Re e\, s)/\delta$, then $\Re e\, s<\sigma<2\Re e\, s+A+3\epsilon<A_0$ and $\sigma+(1+\delta)(\Re e\, s-\sigma)=-A-\epsilon$. Hence

$$E(|s|,k)\ll(k|s|)^\epsilon(1+(|s|/k)^\sigma)k^{\sigma+(1+\delta)(\Re e\, s-\sigma)}\ll|s|^{4\epsilon}(1+(|s|/k)^{2\Re e\, s+A})k^{-A}. \tag{3.9}$$

Summing over $f\in\mathcal{B}_k$, we obtain by (2.1)

$$\begin{aligned}
\sum_{f\in\mathcal{B}_k}w_fL(f\otimes\chi,s) \;=\;& V_s(1)+\epsilon_k(\chi)V_{1-s}(1)\\
&+2\pi i^{-k}\sum_{n\le k^{1+\delta}}\Big(\frac{\chi(n)}{n^s}V_s(n)+\epsilon_k(\chi)\frac{\overline{\chi}(n)}{n^{1-s}}V_{1-s}(n)\Big)\\
&\times\sum_{c\ge1}c^{-1}S(1,n,c)J_{k-1}(\frac{4\pi\sqrt{n}}{c})+E(|s|,k).
\end{aligned}$$

(Note that $\sum_f w_f \ll 1$ by (2.6).) From (2.5), we see that for $n \le k^{1+\delta}$,

$$J_{k-1}(4\pi\sqrt{n}/c) \ll 2^k k^{-(1-\delta)k/2} c^{-1}.$$

Also, $V_s(n), V_{1-s}(n) \ll (k|s|)^{2\epsilon} n^{-\epsilon}$ by (3.5) with $\sigma = \epsilon$, the double sum above is hence

$$\ll_{\epsilon,A} |s|^{2\epsilon} 2^k k^{-(1-\delta)k/2} \sum_{n \le k^{1+\delta}} n^{-1-\epsilon} \sum_{c\ge 1} d(c) c^{-3/2} \ll |s|^{2\epsilon} k^{-A},$$

with (2.2), for sufficiently large $k \ge k_0(\epsilon, A)$. By (3.6) and (3.9), we obtain

$$\begin{aligned} \sum_{f\in\mathcal{B}_k} w_f L(f\otimes\chi, s) &= V_s(1) + \epsilon_k(\chi) V_{1-s}(1) + E(|s|, k) \\ &= 1 + \epsilon_k(\chi)\Big(\frac{D}{2\pi}\Big)^{1-2s} \frac{\Gamma(1-s+(k-1)/2)}{\Gamma(s+(k-1)/2)} + O(|s|^\epsilon (1 + (|s|/k)^{2\Re e\, s + A}) k^{-A}) \end{aligned}$$

after replacing ϵ by $\epsilon/4$. This completes the proof of Theorem 1.

Next we prove Theorem 2 and let $s = 1/2 + it$. It suffices to consider the case $|t| \le k^{1/2}$, for otherwise, $L(f\otimes\chi, 1/2+it) \ll |t|^{1+\epsilon}$ by (2.21) and Theorem 2 follows immediately with (2.6).

We shall take $\delta = \epsilon$ (close to 0) and fix $A = 1$, so $A_0 = 1 + 3/\delta$ which is very large. For $s = 1/2 + it$ ($\ll k^{1/2}$), the last term of (3.8) is $E(|s|, k) \ll_\delta (k|s|)^\epsilon$ by selecting $\sigma = \sigma' := (1+\delta)/(2\delta) \in (0, A_0)$, i.e. $\sigma' + (1+\delta)(1/2 - \sigma') = 0$. For simplicity, we write $W_\sigma(m,n) = V_s(m)\overline{V_s(n)} = V_s(m) V_{\overline{s}}(n)$, that is,

$$\begin{aligned} W_\sigma(m,n) &= \frac{1}{(2\pi i)^2} \int_{(\sigma)} \int_{(\sigma)} \frac{\Gamma(k/2+it+w_1)\Gamma(k/2-it+w_2)}{|\Gamma(k/2+it)|^2} \Big(\frac{D}{2\pi}\Big)^{w_1+w_2} \\ &\qquad \times G(w_1) G(w_2) m^{-(w_1+it)} n^{-(w_2-it)}\, dw_1 dw_2 \end{aligned} \tag{3.10}$$

and denote

$$|S_K|^2 = \sum_{K<m,n\le 2K} \frac{\chi(m)\overline{\chi}(n)}{\sqrt{mn}} W_\sigma(m,n) \sum_{f\in\mathcal{B}_k} w_f \lambda_f(m)\lambda_f(n). \tag{3.11}$$

Apparently, we can shift the line of integration in (3.10) to conclude $W_{\sigma'} = W_\sigma$ for any $0 < \sigma < A_0$. By $D \ll k^\epsilon$ and again $|s| \le k^{1/2}$,

$$W_\sigma(m,n) \ll_{\sigma,\epsilon} (mn)^{-\sigma} k^{2\sigma+2\epsilon}. \tag{3.12}$$

We divide the sums in (3.8) of the form $\sum_{n\le k^{1+\delta}}$ into $\sum_{K=2^r<k^{1+\delta}} \sum_{K<n\le 2K}$ where K runs over all powers of two in $[2^{-1}, k^{1+\delta})$. Cauchy-Schwarz inequality implies $|\sum_{n\le k^{1+\delta}}|^2 \le \log k \sum_{K=2^r<k^{1+\delta}} |\sum_{K<n\le 2K}|^2$. Hence, we obtain by (3.8),

$$\sum_{f\in\mathcal{B}_k} w_f |L(f\otimes\chi, 1/2+it)|^2 \ll \sum_{K=2^r<k^{1+\delta}} |S_K|^2 \log k + k^{3\epsilon}. \tag{3.13}$$

In view of (3.11), it follows by (2.1) that

$$\begin{aligned}|S_K|^2 &= \sum_{K<n\le 2K} \frac{|\chi(n)|^2}{n} W_\sigma(n,n) \\ &\quad + 2\pi i^{-k} \sum_{K<m,n\le 2K} \chi(m)\overline{\chi}(n) \frac{W_\sigma(m,n)}{\sqrt{mn}} \sum_{c\ge 1} \frac{S(m,n,c)}{c} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}). \quad (3.14)\end{aligned}$$

Taking $\sigma = \epsilon$ in (3.12), the first sum in (3.14) is $\ll k^\epsilon$. When $K \le k^{1-2\delta}$, we take $\sigma = 1$ in (3.12) and obtain by (2.5),

$$\begin{aligned}\sum_{K<m,n\le 2K} \sum_{c\ge 1} &\ll k^{2+2\epsilon}(8\pi k^{-\delta})^{k-1} \sum_{m,n\le k^{1-2\delta}} (mn)^{-3/2} \sum_{c\ge 1} d(c) c^{1-k} \\ &\ll k^{-1} \quad (3.15)\end{aligned}$$

for all sufficiently large $k \ge k_0(\delta)$. It is enough for our purpose.

It remains to consider the range $k^{1-2\delta} < K < k^{1+\delta}$. Similarly to (3.15), one can see (with $\sigma = 1$) that the contribution from large c is negligible,

$$\begin{aligned}\sum_{K<m,n\le 2K} \sum_{c\ge k^{3\delta}} &\ll k^{2+2\epsilon} \sum_{m,n\le k^{1+\delta}} (mn)^{-3/2} \sum_{c\ge k^{3\delta}} \frac{d(c)}{\sqrt{c}} \Big(\frac{8\pi k^\delta}{c}\Big)^{k-1} \\ &\ll (8\pi)^k k^{1+\delta} k^{-\delta k} \ll k^{-1}. \quad (3.16)\end{aligned}$$

The remnant of the double sum in (3.14), which we need to consider, is

$$\sum_{K<m,n\le 2K} \sum_{c<k^{3\delta}} \ll |\mathcal{J}| + k^{-1} \sum_{K<m,n\le 2K} \sum_{c<k^{3\delta}} \frac{|W_\epsilon(m,n)|}{\sqrt{mn}} \frac{|S(m,n,c)|}{c} \quad (3.17)$$

by (2.4). Writing $f_{m,n}(\theta) = 4\pi c^{-1}\sqrt{mn}\sin\theta - (k-1)\theta$, $\mathcal{J}$ is given by

$$\begin{aligned}\mathcal{J} &= \sum_{K<m,n\le 2K} \chi(m)\overline{\chi}(n) \frac{W_\epsilon(m,n)}{\sqrt{mn}} \\ &\quad \times \sum_{c<k^{3\delta}} \frac{S(m,n,c)}{c} \int_0^{\pi/2-k^{-4\delta}} \exp(i f_{m,n}(\theta))\, d\theta, \quad (3.18)\end{aligned}$$

the treatment of which is the main difficulty in the proof. The other summand can be easily handled: by (3.12) with $\sigma = \epsilon$, the second term on the right side of (3.17) is

$$\begin{aligned}&\ll k^{2\epsilon-1} \sum_{K<m,n\le 2K} (mn)^{-1/2} \sum_{c<k^{3\delta}} d(c) c^{-1/2} \\ &\ll K k^{3\delta+2\epsilon-1} \log k \ll k^{4\delta+3\epsilon}. \quad (3.19)\end{aligned}$$

To deal with $\mathcal{J}$, the periodicity of $S(\cdot,\cdot,c)$ and $\chi(\cdot)$ enables us to express (3.18) as

$$\begin{aligned}\mathcal{J} &= \sum_{\substack{c<k^{3\delta}\\ \alpha_1,\alpha_2\,(c)}} c^{-1}S(\alpha_1,\alpha_2,c) \sum_{\beta_1,\beta_2\,(D)} \chi(\beta_1)\overline{\chi}(\beta_2)\\ &\quad\times \sum_{\substack{(*)\\ K<m,n\le 2K}} \frac{W_\epsilon(m,n)}{\sqrt{mn}} \int_0^{\pi/2-k^{-4\delta}} e(\frac{2\sqrt{mn}}{c}\sin\theta)e^{-i(k-1)\theta}\,d\theta \qquad (3.20)\end{aligned}$$

where the condition $(*)$ denotes $m \equiv \alpha_1$ (c), $m \equiv \beta_1$ (D) and $n \equiv \alpha_2$ (c), $n \equiv \beta_1$ (D). The congruence system $m \equiv \alpha_1$ (c) and $m \equiv \beta_1$ (D) is solvable if and only if $(c,D)|(\alpha_1-\beta_1)$. Assume $(c,D)|(\alpha_1-\beta_1)$. The solution is given by $m \equiv \gamma_1$ (cd) for some γ_1 where $d = D/(c,D)$. Similarly, the system $n \equiv \alpha_2$ (c) and $n \equiv \beta_2$ (D) is given by $n \equiv \gamma_2$ (cd) for some γ_2. Hence, with (3.10) we can write the sum over m, n in (3.20) into

$$\begin{aligned}&\frac{1}{(2\pi i)^2}\int_{(\epsilon)}\int_{(\epsilon)} \frac{\Gamma(k/2+it+w_1)\Gamma(k/2-it+w_2)}{|\Gamma(k/2+it)|^2}\\ &\quad\times\left(\frac{D}{2\pi}\right)^{w_1+w_2} H(w_1,w_2)G(w_1)G(w_2)\,dw_1dw_2 \qquad (3.21)\end{aligned}$$

where $H(w_1,w_2)$ denotes the integral

$$\int_0^{\pi/2-k^{-4\delta}} \sum_{\substack{K<m,n\le 2K\\ m\equiv\gamma_1,n\equiv\gamma_2\,(cd)}} m^{-1/2-it-w_1}n^{-1/2+it-w_2}e(\frac{2\sqrt{mn}}{c}\sin\theta)e^{-i(k-1)\theta}\,d\theta$$

with $d = D/(c,D)$. Substituting $\lambda = \sin\theta$, we have $d\theta = (\cos\theta)^{-1}\,d\lambda = O(k^{4\delta})\,d\lambda$ as $0<\theta<\pi/2-k^{-4\delta}$, and hence

$$\begin{aligned}&H(w_1,w_2)\\ &\ll k^{4\delta}\int_0^1\Big|\sum_{\substack{K<m,n\le 2K\\ m\equiv\gamma_1,n\equiv\gamma_2\,(cd)}} m^{-1/2-it-w_1}n^{-1/2+it-w_2}e(\frac{2\lambda}{c}\sqrt{mn})\Big|\,d\lambda\\ &\ll (|w_1|+|t|)(|w_2|+|t|)Dk^{4\delta}K^{\epsilon-\Re e(w_1+w_2)}\end{aligned}$$

by Lemma 2.1 and $d \le D$. This gives by Lemma 2.2 and our choice of $G(w)$ that the integral in (3.21) is $\ll Dk^{4\delta+2\epsilon}(|t|+1)^{2+4\epsilon}$. Substituting into (3.20), we obtain

$$\mathcal{J} \ll D^3k^{4\delta+2\epsilon}(|t|+1)^{2+4\epsilon}\sum_{c<k^{3\delta}} c^{-1}\sum_{\alpha_1,\alpha_2\,(c)}|S(\alpha_1,\alpha_2,c)| \ll k^{12\delta+2\epsilon}(|t|+1)^{2+4\epsilon}$$

by (2.2) and $\log D = o(\log k)$. Together with (3.19), we see that the right-side of (3.17) has an upper bound in the above fashion. In view of (3.14) and (3.15), $|S_K|^2 \ll$

$k^{12\delta+2\epsilon}(|t|+1)^{2+4\epsilon}$ for all $K \leq k^{1+\delta}$. Our result follows from (3.13) by the choice $\delta = \epsilon$ and then $\epsilon/15$ in place of ϵ.

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