LOGARITHMIC HEAT PROJECTIVE OPERATORS

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Abstract. The sheaf of differential operators on log-schemes is defined and studied. Then logarithmic differential operators on compactified Jacobian of singular curves are studied. In particular, logarithmic heat equation for theta functions is produced geometrically.

Introduction

Let $\mathcal{C} \to S$ be a proper flat family of stable curves, smooth over $S_0 = S \smallsetminus \Delta$. One can associate a flat family $f : \mathcal{SU}_{\mathcal{C}}(r,d) \to S_0$ of moduli spaces $\mathcal{SU}_{\mathcal{C}_s}(r,d)$ of semistable vector bundles of rank r and degree d with fixed determinant over the curves $\mathcal{C}_s$, and also a line bundle Θ on $\mathcal{SU}_{\mathcal{C}}(r.d)$ such that its restriction to each fibre is the line bundle on $\mathcal{SU}_{\mathcal{C}_s}(r,d)$ defined by the theta divisors. It is now well known that for any positive integer k the direct image $\mathcal{E}_0 := f_*\Theta^k$ is a vector bundle on S_0 and have a flat projective connection, which is in fact given by the heat operator on Θ^k. Our motivation is to understand geometrically the behaviours of the operator when the curves degenerate to singular curves (See also [Hi], p. 350). More precisely, we can formulate the question as following

Problem. *For the family $\mathcal{U}_{S_0} \to S_0$ of moduli spaces of semistable vector bundles and the relative theta line bundle Θ_{S_0}, find the 'correct' degeneration $(\mathcal{U}_S, \Theta_S)$ of moduli spaces and theta line bundles (in other words, the 'correct' algeo-geometric analogy of spaces of conformal blocks on singular curves) such that the direct image of Θ_S^k is a vector bundle on S with a flat logarithmic projective connection.*

When the curves degenerate to singular curves, the moduli spaces usally degenerate to some singular varieties. The existence of a (projective) heat operator requires in general some geometric properties for the variety. In this sense, it has independent interests to figure out some global geometric properties that the degeneration of moduli spaces may have. Then it becomes clear for the degeneration of moduli spaces that we should work at least in log-geometry.

We first consider the problem in a general situation (forgeting moduli spaces). Let $f : X \to S$ be a flat family of (reduced) normal crossing varieties of dimension d, and X, S smooth. Assume that $\Delta \subset S$ is a normal crossing divisor and $Y := f^{-1}(\Delta) \subset X$ is also a normal crossing divisor such that $f : X \smallsetminus Y \to S \smallsetminus \Delta$ is smooth and $(X, logY) \xrightarrow{f} (S, log\Delta)$ is log smooth. Let $\mathcal{L}$ be a line bundle on X. Then we defined the logarithmic analogies (see Definition 2.2 and Definition 2.3) of (projective) heat operators of [GJ] and figured out the sufficient conditions (see Theorem 2.1) of existence of a projective logarithmic heat operator on $\mathcal{L}$ over S, which, similar to [GJ], gave a logarithmic projective connection on $f_*\mathcal{L}$. In this step, we need a correct logarithmic analogy of *sheaf of differential operators* and have to work in logarithmic algebraic geometry. We defined the *sheaf of logarithmic differential operators on a log scheme*, which works well for the log schemes we concern in this paper (see Proposition 2.1). These materials may be well known to experts but I am not able to find a reference satisfying our requirements. Then we checked the conditions in Theorem 2.1 for the rank one case, namely, we proved that a family of moduli spaces of torsion free sheaves of rank one over nodal curves satisfies the conditions in Theorem 2.1, thus there exists a projective logarithmic heat operator on the relative theta line bundle Θ^k (see Theorem 3.1).

We developed the necessary technique tools, especially the sheaf of differential operators in Section 1. Then, in Section 2, we figured out the sufficient conditions of existence of a projective logarithmic heat operator (thus the conditions of existence of the required projective logarithmic connection) in the general situation, and we also gave some descriptions of the conditions. Finally, in Section 3, we verified the conditions figured out in §2 for a family of generalized Jacobians (moduli spaces of torsion free sheaves of rank one), and thus showed the existence of projective logarithmic heat operator in this case.

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§1 Logarithmic schemes and logarithmic operators

In this preliminary section, we recall the so called logarithmic structures (or log structures for simplicity) on schemes (see [KK]), and define the sheaves of differential operators on logarithmic schemes. All monoids M are commutative monoids with unit element and $M^{gp} = \{ab^{-1}\}$ is the associated group.

By a pre-log structure M on a scheme X, we mean a sheaf of monoids M on the étale site X_{et} endowed with a homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication on $\mathcal{O}_X$. A morphism

$$f : X^\dagger := (X, M) \to Y^\dagger := (Y, N)$$

of schemes with pre-log structures is defined to be a pair (f, h) of a morphism of schemes $f : X \to Y$ and a homomorphism $h : f^{-1}(N) \to M$ such that

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{h} & M \\ \Big\downarrow & & \Big\downarrow \\ f^{-1}(\mathcal{O}_Y) & \longrightarrow & \mathcal{O}_X \end{array}$$

is commutative. A pre-log structure (M, α) is called a logarithmic structure if

$$\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^* \quad \text{via} \quad \alpha$$

where $\mathcal{O}_X^*$ denotes the group of invertible elements of $\mathcal{O}_X$. A morphism of schemes with log structures is defined as a morphism of schemes with pre-log structures. These schemes are called log schemes.

For a pre-log structure (M, α) on X, one can define its associated log structure (M^a, α^a) by

$$M^a := (M \oplus \mathcal{O}_X^*)/P, \quad \alpha^a(x, u) = u \cdot \alpha(x)$$

where $P = \{(x, \alpha(x)^{-1}) \,|x \in \alpha^{-1}(\mathcal{O}_X^*)\}$. Let $f : X \to Y$ be a morphism of schemes. For a log structure M on X, we can define a log structure on Y called the direct image of M, to be the fibre product of sheaves

$$\begin{array}{ccc} & & f_*M \\ & & \downarrow \\ \mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_X \end{array} .$$

For a log structure N on Y, we define a log structure f^*N on X called the inverse image of N to be the log structure associated to the pre-log structure

$$f^{-1}(N) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X.$$

Definition 1.1. *Let $\alpha : M \to \mathcal{O}_X$ and $\beta : N \to \mathcal{O}_Y$ be pre-log structures and $f : (X, M) \to (Y, N)$ be a morphism of log schemes. Then the $\mathcal{O}_X$-module $\Omega^1_{X/Y}(log(M/N))$ called logarithmic differential sheaf is defined to be the quotient of*

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{gp})$$

($\Omega^1_{X/Y}$ is the usual relative differential module) divided by the $\mathcal{O}_X$-submodule generated locally by local sections of the following forms

(1) *$(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in M$.*
(2) *$(0, 1 \otimes a)$ with $a \in Image(f^{-1}(N) \xrightarrow{h} M)$.*

It is easily seen that if M^a and N^a denote the associated log structures respectively, we have

$$\Omega^1_{X/Y}(log(M/N)) = \Omega^1_{X/Y}(log(M^a/N)) = \Omega^1_{X/Y}(log(M^a/N^a)).$$

We collect some easy facts in the following proposition which may be useful in the paper.

Proposition 1.1. *Let $f : X \to Y$ be a morphism of schemes, and N^a the log structure associated to a pre-log structure N on Y. Then*

(1) *$f^*(N^a)$ coincides with the log structure associated to the pre-log structure $f^{-1}(N) \to \mathcal{O}_X$.*
(2) *If $f : (X, M^a) \to (Y, N^a)$ is a morphism of log schemes such that*

$$f^{-1}(N) \to M$$

is surjective, we have $\Omega^1_{X/Y}(log(M/N)) = \Omega^1_{X/Y}$ (the usual relative differential sheaf).
(3) *If we have a cartesian diagram of log schemes*

$$\begin{array}{ccc} (X', M') & \xrightarrow{f} & (X, M) \\ \downarrow & & \downarrow \\ (Y', N') & \longrightarrow & (Y, N), \end{array}$$

we have an isomorphism $f^*\Omega^1_{X/Y}(log(M/N)) \cong \Omega^1_{X'/Y'}(log(M'/N'))$.

Now we are going to introduce the sheaf of differential operators on general log schemes although we need it only for some special log structures, we hope this general treatment to be useful in the future. Fix a morphism $X^\dagger = (X, M) \to Y^\dagger = (Y, N)$ of log schemes and denote the sheaf of $\mathcal{O}_Y$-derivations of $\mathcal{O}_X$ by $T_{X/Y}$. We write $\Omega^1_{X/Y}(log)$ simply for $\Omega^1_{X/Y}(log(M/N))$, and

$$\bar{d} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \to \Omega^1_{X/Y}(log)$$

denotes the canonical logarithmic derivation.

Definition 1.2. *A derivation $\delta \in T_{X/Y}$ is called a logarithmic derivation if there exists a $\theta \in Hom_{\mathcal{O}_X}(\Omega^1_{X/Y}(log), \mathcal{O}_X)$ such that*

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\delta"] \arrow[d, "\bar{d}"'] & \mathcal{O}_X \\ \Omega^1_{X/Y}(log) \arrow[r, equal] & \Omega^1_{X/Y}(log) \arrow[u, "\theta"'] \end{tikzcd}$$

is commutative.

Remark 1.1. The sheaf of logarithmic derivations, denoted by $T_{X/Y}(log)$, is a subsheaf of $T_{X/Y}$. By definition, we have a surjection

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}(log), \mathcal{O}_X) \to T_{X/Y}(log)$$
$$u \mapsto u \circ \bar{d},$$

which is not injective in general since $\Omega^1_{X/Y}(log)$ is not generated by $\{\bar{d}f\}_{f \in \mathcal{O}_X}$. However, we will see in Lemma 1.2 that for all the logarithmic structures we concern in this paper the above surjection is actually an isomorphism.

Let $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{O}_X)$ be sheaf of $\mathcal{O}_Y$-linear maps and $\mathcal{O}_X \subset \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{O}_X)$ be the subsheaf of maps multiplying by elements of $\mathcal{O}_X$. It is clear that $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{O}_X)$ is a sheaf of noncommutative rings, thus it has two $\mathcal{O}_X$-module structures (left and right multipications).

Definition 1.3. *The sheaf $\mathcal{D}_{X^\dagger/Y^\dagger} \subset \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{O}_X)$ of subrings generated by $\mathcal{O}_X$ and $T_{X/Y}(log)$ is called the sheaf of logarithmic differential operators on log scheme $X^\dagger = (X, M)$ over $Y^\dagger = (Y, N)$. We will simply call it the sheaf of log differential operators. For any integer $k > 0$, we define inductively the sheaf of k-th log differential operators:*

$$\mathcal{D}^0_{X^\dagger/Y^\dagger} := \mathcal{O}_X,$$

$$\mathcal{D}^k_{X^\dagger/Y^\dagger} := \{D \in \mathcal{D}_{X^\dagger/Y^\dagger} \mid [D, f] \in \mathcal{D}^{k-1}_{X^\dagger/Y^\dagger}, \textit{for any } f \in \mathcal{O}_X\}.$$

If $\Omega^1_{X/Y}(log)$ is locally free, we can describe $\mathcal{D}_{X^\dagger/Y^\dagger}$ locally. Let

$$dlog : \mathcal{O}_X \otimes_{\mathbb{Z}} M^{gp} \to \Omega^1_{X/Y}(log)$$

be the surjection and choose locally $t_1, ..., t_r \in M$ such that $\{dlog(t_i)\}_{1 \le i \le r}$ is an $\mathcal{O}_X$-base of $\Omega^1_{X/Y}(log)$. Let $f_i = \alpha(t_i) \in \mathcal{O}_X$ $(i = 1, ..., r)$, where $\alpha : M \to \mathcal{O}_X$

is the log structure. Then there exists locally a system of generators $\{\partial_i\}_{1\leq i\leq r}$ of $T_{X/Y}(log)$ such that

$$\partial_i(f_j) = \begin{cases} f_i & j = i, \\ 0 & j \neq i. \end{cases}$$

All $[\partial_i, \partial_j]$ vanish on the subring $\mathcal{O}' \subset \mathcal{O}_X$ generated by $f_1, ..., f_r$ and $\mathcal{O}_Y$, which means that $[\partial_i, \partial_j]$ are derivations of $\mathcal{O}_X$ over $\mathcal{O}'$, thus $[\partial_i, \partial_j]$ vanish on $\mathcal{O}_X$ since $\Omega^1_{\mathcal{O}_X/\mathcal{O}'}$ is a torsion sheaf. Therefore any local section $D \in \mathcal{D}_{X^\dagger/Y^\dagger}$ can be expressed as a finite sum

$$D = \sum \lambda_{\beta_1,...,\beta_r} \partial_1^{\beta_1} \cdots \partial_r^{\beta_r}. \tag{1.1}$$

We introduce a notation $[D, a_1 \star \cdots \star a_n]$ for any local section $D \in \mathcal{D}_{X^\dagger/Y^\dagger}$ and $a_1, ..., a_n \in \mathcal{O}_X$. The $[D, a_1 \star \cdots \star a_n] \in \mathcal{D}_{X^\dagger/Y^\dagger}$ is defined inductively by

$$[D, a_1 \star \cdots \star a_n] := [[D, a_1 \star \cdots \star a_{n-1}], a_n].$$

If $a_1 = \cdots = a_n$, we write $[D, a_1 \star \cdots \star a_n] = [D, a_1^{\star n}]$, thus the notation $[D, a_1^{\star i_1} \star a_2^{\star i_2} \star \cdots \star a_k^{\star i_k}]$ is clear. Notice that $[\partial_i, f_j] = 0$ for $i \neq j$, one checks easily that

$$[\partial_1^{\beta_1} \cdots \partial_r^{\beta_r}, f_i^{\star\beta_i}] = \partial_1^{\beta_1} \cdots \partial_{i-1}^{\beta_{i-1}} \cdot [\partial_i^{\beta_i}, f_i^{\star\beta_i}] \cdot \partial_{i+1}^{\beta_{i+1}} \cdots \partial_r^{\beta_r}.$$

Thus $[\partial_1^{\beta_1} \cdots \partial_r^{\beta_r}, f_1^{\star\beta_1} \star \cdots \star f_r^{\star\beta_r}] = [\partial_1^{\beta_1}, f_1^{\star\beta_1}] \cdots [\partial_r^{\beta_r}, f_r^{\star\beta_r}]$. One computes that $[\partial_i^{\beta_i}, f_i^{\star\beta_i}] = (\beta_i!) f_i^{\beta_i}$ and $[\partial_i^{k_i}, f_i^{\star\beta_i}] = 0$ if $k_i < \beta_i$. For any $D \in \mathcal{D}^k_{X^\dagger/Y^\dagger}$, let $\lambda_{\alpha_1,...,\alpha_r} \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r}$ be the first term of D in (1.1) (namely $\alpha_1 \geq \beta_1, ..., \alpha_r \geq \beta_r$ and at least one inequality is strict). Then

$$\begin{aligned} [D, f_1^{\star\alpha_1} \star \cdots \star f_r^{\star\alpha_r}] &= \lambda_{\alpha_1,...,\alpha_r} [\partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r}, f_1^{\star\alpha_1} \star \cdots \star f_r^{\star\alpha_r}] \\ &= (\alpha_1!) \cdots (\alpha_r!) \lambda_{\alpha_1,...,\alpha_r} f_1^{\alpha_1} \cdots f_r^{\alpha_r}. \end{aligned}$$

If $\alpha_1 + \cdots + \alpha_r > k$, then $[D, f_1^{\star\alpha_1} \star \cdots \star f_r^{\star\alpha_r}] = 0$ by definition of $\mathcal{D}^k_{X^\dagger/Y^\dagger}$, thus

$$\lambda_{\alpha_1,...,\alpha_r} \cdot f_1^{\alpha_1} \cdots f_r^{\alpha_r} = 0. \tag{1.2}$$

Proposition 1.2. *Let $Gr_i(\mathcal{D}_{X^\dagger/Y^\dagger}) = \mathcal{D}^i_{X^\dagger/Y^\dagger}/\mathcal{D}^{i-1}_{X^\dagger/Y^\dagger}$ and $T(Gr_1(\mathcal{D}_{X^\dagger/Y^\dagger}))$ be the tensor algebra. Then*

(1) *The symbol map $\sigma : Gr_1(\mathcal{D}_{X^\dagger/Y^\dagger}) \to T_{X/Y}(log)$, defined by $\sigma(D)(f) = [D, f](1)$, is an isomorphism.*
(2) *$\mathcal{D}_{X^\dagger/Y^\dagger} = \bigcup_{i=0}^\infty \mathcal{D}^i_{X^\dagger/Y^\dagger}$, $\mathcal{D}^i_{X^\dagger/Y^\dagger} \cdot \mathcal{D}^j_{X^\dagger/Y^\dagger} \subset \mathcal{D}^{i+j}_{X^\dagger/Y^\dagger}$ and $f^{-1}(\mathcal{O}_Y)$ is in the center of $\mathcal{D}_{X^\dagger/Y^\dagger}$.*
(3) *The left and right $\mathcal{O}_X$-module structures on $Gr_i(\mathcal{D}_{X^\dagger/Y^\dagger})$ coincide*
(4) *The natural map*

$$T(Gr_1(\mathcal{D}_{X^\dagger/Y^\dagger})) \to Gr(\mathcal{D}_{X^\dagger/Y^\dagger}) := \bigoplus_{i=0}^\infty Gr_i(\mathcal{D}_{X^\dagger/Y^\dagger})$$

is surjective.

Proof. One checks that $\sigma(D)$ is a derivation of $\mathcal{O}_X$. To see it factorizing through $\bar{d}:\mathcal{O}_X\to\Omega^1_{X/Y}(log)$, it is enough to show that for any $m\in M$ there exists a unique $\theta(dlog(m))\in\mathcal{O}_X$ such that $\sigma(D)(\alpha(m))=\alpha(m)\theta(dlog(m))$, which can be checked by using the fact that D is a map composed by logarithmic derivations.

The proof of others is easy, we will omit it but just remark that we are not able to claim the natural map in (4) induces surjections

$$T^i(Gr_1(\mathcal{D}_{X^\dagger/Y^\dagger}))\to Gr_i(\mathcal{D}_{X^\dagger/Y^\dagger}).$$

A reduced scheme Y is called a normal crossing variety of dimension d if the completion of local ring $\mathcal{O}_{Y,y}$ at each point y is isomorphic to $\mathbb{C}\{x_0,...,x_d\}/(x_0\cdots x_r)$ for some $r=r(y)$ such that $0\le r\le d$. Now we discuss the log structures on normal crossing varieties and on smooth varieties induced by normal crossing divisors. These are the only log structures we will concern in this paper.

Lemma 1.1. *Let $f:X\to S$ be a flat family of (reduced) normal crossing varieties of dimension d, and X, S smooth. Assume that $\Delta\subset S$ is a normal crossing divisor and $Y:=f^{-1}(\Delta)\subset X$ such that $f:X\smallsetminus Y\to S\smallsetminus\Delta$ is smooth. Then, for any $x\in X$, we can choose isomorphisms*

$$\hat{\mathcal{O}}_{X,x}\cong\mathbb{C}\{x_1,...,x_{d+1},...,x_{d+m}\},\quad \hat{\mathcal{O}}_{S,f(x)}\cong\mathbb{C}\{\pi_1,...,\pi_m\}$$

such that

$$f^\sharp:\mathbb{C}\{\pi_1,...,\pi_m\}\to\mathbb{C}\{x_1,...,x_{d+1},...,x_{d+m}\}$$

is a $\mathbb{C}$-algebra homomorphism with

$$f^\sharp(\pi_2)=x_{d+2},\quad ...\quad ,f^\sharp(\pi_m)=x_{d+m}$$

and the local equation of Δ at $f(x)$ is

$$\pi_{i_1}\cdot\pi_{i_2}\cdots\pi_{i_s}=0.$$

Proof. Let $\hat{\mathcal{O}}_{S,f(x)}\cong\mathbb{C}\{\pi_1,...,\pi_m\}$ and $\hat{\mathcal{O}}_{X,x}\cong\mathbb{C}\{z_1,...,z_{d+m}\}$. Then, by definition,

$$\varphi:\frac{\mathbb{C}\{y_1,...,y_{d+1}\}}{(y_1\cdots y_{d+1})}\cong\frac{\mathbb{C}\{z_1,...,z_{d+m}\}}{(f^\sharp(\pi_1),...,f^\sharp(\pi_m))}.$$

We can assume that $r>1$ (otherwise, f is smooth at x, the lemma is clear). Let

$$\varphi(\bar{y}_i)=\overline{\varphi_i(z_1,...,z_{d+m})}\in\frac{\mathbb{C}\{z_1,...,z_{d+m}\}}{(f^\sharp(\pi_1),...,f^\sharp(\pi_m))},$$

we can write $\varphi_i(z_1,...,z_{d+m})=\sum_{j=1}^{d+m}a_{ij}z_j+\varphi_i^{\ge2}$, where $\varphi_i^{\ge2}$ denotes the part of $\varphi_i(z_1,...,z_{d+m})$ with order ≥2. Then

$$\frac{\partial(\varphi_1,...,\varphi_{d+1})}{\partial(z_1,...,z_{d+m})}\mid_{(0,...,0)}=\begin{pmatrix}a_{11}&\cdots&a_{1\,d+m}\\ \vdots&\ddots&\vdots\\ a_{d+1\,1}&\cdots&a_{d+1\,d+m}\end{pmatrix}$$

has rank $d+1$. Otherwise, there is $0 \neq (k_1, ..., k_{d+1}) \in \mathbb{C}^{d+1}$ such that

$$\sum_{i=1}^{d+1} k_i \varphi_i(z_1, ..., z_{d+m}) = \sum_{i=1}^{d+1} k_i \varphi_i^{\geq 2} \in (z_1, ..., z_{d+m})^2,$$

which implies that

$$\sum_{i=1}^{d+1} k_i \bar{y}_i = \varphi^{-1}(\overline{\sum_{i=1}^{d+1} k_i \varphi_i(z_1, ..., z_{d+m})}) \in (\bar{y}_1, ..., \bar{y}_{d+1})^2.$$

Thus there is $g(y_1, ..., y_{d+1}) \in (y_1, ..., y_{d+1})^2$ such that

$$\sum_{i=1}^{d+1} k_i y_i \; - \; g(y_1, ..., y_{d+1}) \in (y_1 \cdots y_r),$$

which is impossible since $r > 1$. Replaceing some $z_{j_1}, ..., z_{j_{d+1}}$ by $\varphi_1(z_1, ..., z_{d+m})$, ..., $\varphi_{d+1}(z_1, ..., z_{d+m})$, we can assume that

$$\varphi : \frac{\mathbb{C}\{y_1, ..., y_{d+1}\}}{(y_1 \cdots y_{d+1})} \cong \frac{\mathbb{C}\{z_1, ..., z_{d+m}\}}{(f^\sharp(\pi_1), ..., f^\sharp(\pi_m))}$$

such that $\varphi(\bar{y}_i) = \bar{z}_i$ $(i = 1, ..., d+1)$. Let

$$\varphi^{-1}(\bar{z}_{d+j}) = \overline{g_j(y_1, ..., y_{d+1})} \in \frac{\mathbb{C}\{y_1, ..., y_{d+1}\}}{(y_1 \cdots y_r)},$$

then $z_{d+j} - g_j(z_1, ..., z_{d+1}) \in (f^\sharp(\pi_1), ..., f^\sharp(\pi_m))$. If we write that (for $j = 2, ..., m$)

$$z_{d+j} - g_j(z_1, ..., z_{d+1}) = \sum_{i=1}^{m} a_{ji} f^\sharp(\pi_i) \; + \text{higher order terms},$$

then

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} a_{21} & \dots & a_{2\,m} \\ \vdots & \ddots & \vdots \\ a_{m\,1} & \dots & a_{m\,m} \end{pmatrix} \cdot \frac{\partial(f^\sharp(\pi_1), ..., f^\sharp(\pi_m))}{\partial(z_{d+2}, ..., z_{d+m})} \mid_{(0,...,0)} .$$

Thus

$$\frac{\partial(f^\sharp(\pi_1), ..., f^\sharp(\pi_m))}{\partial(z_{d+2}, ..., z_{d+m})} \mid_{(0,...,0)}$$

has rank $m-1$, and we can choose the isomorphism

$$\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}\{x_1, ..., x_{d+1}, ..., x_{d+m}\}$$

such that $f^\sharp(\pi_2) = x_{d+2}$, ..., $f^\sharp(\pi_m) = x_{d+m}$ by changing the order of π_1, ..., π_m.

We are not able to prove in Lemma 1.1 that $Y = f^{-1}(\Delta)$ is a normal crossing divisor in X even if each fibre is a normal crossing variety. Assume that Y is a normal crossing divisor, we have canonical log structures on X and S

$$logY = \{g \in \mathcal{O}_X \mid g \text{ is invertible outside } Y\} \subset \mathcal{O}_X$$

$$log\Delta = \{g \in \mathcal{O}_S \mid g \text{ is invertible outside } \Delta\} \subset \mathcal{O}_S.$$

These are fine log structures, and if one writes locally that

$$Y = \bigcup_{i=1}^{r} \{x_i = 0\} \quad \Delta = \bigcup_{i=1}^{s} \{\pi_i = 0\},$$

then $logY$ and $log\Delta$ are associated to the pre-log. structures ([KK]):

$$\begin{array}{cc} \mathbb{N}^r \to \mathcal{O}_X & \mathbb{N}^s \to \mathcal{O}_S \\ (n_i)_{1\le i\le r} \mapsto \prod x_i^{n_i}, & (n_i)_{1\le i\le s} \mapsto \prod \pi_i^{n_i}. \end{array}$$

For the local descriptions and properties of $\Omega^1_S(log\Delta)$ and $\Omega^1_X(logY)$, we refer to [EV1] and [EV2]. We will use $\Omega^1_{X/S}(logY)$ to denote $\Omega^1_{X/S}(logY/\Delta)$.

Proposition 1.3. *Let $f : X \to S$ be a flat family of normal crossing varieties of dimension d, X and S be smooth. Assume that $Y := f^{-1}(\Delta)$ is a normal crossing divisor such that*

$$f : X \setminus Y \to S \setminus \Delta$$

is smooth. Then we have the associated exact sequence

$$0 \to f^*\Omega^1_S(log\Delta) \xrightarrow{j} \Omega^1_X(logY) \to \Omega^1_{X/S}(logY) \to 0,$$

and the following are equivalent

1. *$(X, logY) \xrightarrow{f} (S, log\Delta)$ is log smooth.*
2. *The image of j is locally a direct summand.*
3. *For any singular point $x \in X$ of f, we can choose coordinates*

$$\hat{\mathcal{O}}_{S,f(x)} \cong \mathbb{C}\{\pi_1, ..., \pi_m\} \hookrightarrow \mathbb{C}\{x_1, ..., x_{d+1}, \pi_2, ..., \pi_m\} \cong \hat{\mathcal{O}}_{X,x}$$

   such that $\pi_1 = x_1 \cdots x_r$ for some $1 \le r \le d+1$ and

$$\pi_1 \cdot \pi_2 \cdots \pi_s = 0$$

   is the local equation of Δ at $f(x)$.
4. *$\Omega^1_{X/S}(logY)$ is locally free.*

Proof. The map j has to be injective since it is injective at the generic point of S and $\Omega^1_{X/S}(log\Delta)$ is locally free.

The (1) $\Leftrightarrow$ (2) follows the Proposition (3.12) of [KK], and (2) $\Leftrightarrow$ (4) is obvious. We prove that (3) $\Leftrightarrow$ (4). Firstly, (3) $\Rightarrow$ (4) is clear since $\Omega^1_{X/S}(logY)$ is locally isomorphic to

$$\frac{\hat{\mathcal{O}}_{X,x}\{dx_1, ..., dx_{d+1}, e_1, ..., e_r\}}{(dx_1 - x_1e_1, ..., dx_r - x_re_r, e_1 + \cdots + e_r)},$$

which is a free module generated by $\{\frac{1}{x_1}dx_1, ..., \frac{1}{x_{r-1}}dx_{r-1}, dx_{r+1}, ..., dx_{d+1}\}$. To prove (4) $\Rightarrow$ (3), we only need to show that local equation of Δ is divisible by π_1 (assume that we choose the coordinates as in Lemma 1.1). If it is not so, we can assume the local equation of Δ to be

$$\pi_2 \cdots \pi_s = 0.$$

Then, as in the Lemma 1.1, $f^\sharp(\pi_2 \cdots \pi_s) = x_{d+2} \cdots x_{d+s}$ is the local equation of Y at x, and $\Omega^1_{X/S}(logY)$ is locally isomorphisc to

$$\frac{\hat{\mathcal{O}}_{X,x}\{dx_1, ..., dx_{d+1}\}}{\sum_{i=1}^{d+1} \frac{\partial f^\sharp(\pi_1)}{\partial x_i} dx_i}.$$

It is not locally free except one of $\frac{\partial f^\sharp(\pi_1)}{\partial x_i}$ is invertible, which means that f is smooth at x. But (3) is clear at this case.

Let X_s ($s \in \Delta$) be a fibre of $f : X \to S$, then X_s is a normal crossing variety with log structure $M := (logY)|_{X_s} \xrightarrow{\alpha} \mathcal{O}_{X_s}$ such that $X_s^\dagger := (X_s, M)$ is a log smooth variety. We will describe M locally and to show how it gives a log structure in the sense of [KN].

For $x \in X_s$ a singular point of X_s, there is a neighbourhood $U_\lambda \subset X$ of x and holomorphic functions $x_i^\lambda \in \mathcal{O}_X(U_\lambda)$ such that

$$\begin{aligned} &U_\lambda \to \mathbb{C}^{d+m} \\ &p \mapsto (x_1^\lambda(p), ..., x_{d+m}^\lambda) \end{aligned}$$

is an open embedding and $X_s \cap U_\lambda \subset U_\lambda$ is defined by

$$x_1^\lambda \cdots x_r^\lambda = 0, \quad x_{d+2}^\lambda = \cdots = x_{d+m}^\lambda = 0,$$

$Y \cap U_\lambda$ is defined by $x_1^\lambda \cdots x_r^\lambda \cdot x_{d+i_1}^\lambda \cdots x_{d+i_s}^\lambda = 0$, for some $1 < r \leq d+1$ and $2 \leq i_1 < \cdots < i_s \leq m$. The log structure $logY|_{U_\lambda}$ on U_λ is associated to the pre-log structure

$$\begin{aligned} \mathbb{N}^{r+s} &\to \mathcal{O}_{U_\lambda} \\ (n_1, ..., n_r, n_{d+i_1}, ..., n_{d+i_s}) &\mapsto (x_1^\lambda)^{n_1} \cdots (x_r^\lambda)^{n_r} \cdot (x_{d+i_1}^\lambda)^{n_{d+i_1}} \cdots (x_{d+i_s}^\lambda)^{n_{d+i_s}}. \end{aligned}$$

Let $z_i^\lambda = x_i^\lambda|_{X_s \cap U_\lambda} \in \mathcal{O}_{X_s}(X_s \cap U_\lambda)$, then $M|_{X_s \cap U_\lambda}$ is associated the pre-log structure

$$\begin{aligned} \mathbb{N}^r \oplus \mathbb{N}^s = \mathbb{N}^{r+s} &\xrightarrow{\upsilon} \mathcal{O}_{U_\lambda \cap X_s} \\ (n_1, ..., n_r, n_{d+i_1}, ..., n_{d+i_s}) &\mapsto (z_1^\lambda)^{n_1} \cdots (z_r^\lambda)^{n_r} \cdot (z_{d+i_1}^\lambda)^{n_{d+i_1}} \cdots (z_{d+i_s}^\lambda)^{n_{d+i_s}}. \end{aligned}$$

Clearly $ker(\upsilon) = \mathbb{N}^s$ and $\upsilon^*(\mathcal{O}^*_{U_\lambda \cap X_s}) = \{(0, ..., 0) \in \mathbb{N}^{r+s}\}$, so $M|_{X_s \cap U_\lambda} = \mathbb{N}^{r+s} \oplus \mathcal{O}^*_{U_\lambda \cap X_s}$ and $\alpha_\lambda : M|_{X_s \cap U_\lambda} \to \mathcal{O}_{U_\lambda \cap X_s}$ is defined by $\alpha_\lambda(\vec{n}, g) = \upsilon(\vec{n}) \cdot g$. Thus we get a partial open cpvering $\{V_\lambda := U_\lambda \cap X_s\}$ of X_s containing the singular locus of X_s and systems of holomorphic functions

$$z_i^{(\lambda)} := \alpha_\lambda(e_i, 1) \in \mathcal{O}_{X_s}(V_\lambda),$$

where $i = 1, ..., r_\lambda$ and $e_i = (0, ..., 1, ..., 0) \in \mathbb{N}^{r_\lambda + s}$. These functions satisfy

(1) There is an isomorphism φ_λ from V_λ to an open neighborhood of $(0, ..., 0)$ of the variety

$$\{(x_1, ..., x_{d+1}) \in \mathbb{C}^{d+1} \,|\, x_1 \cdots x_{r_\lambda} = 0\}$$

such that $\varphi_\lambda^*(x_j) = z_j^{(\lambda)}$ for $1 \le j \le r_\lambda$.

(2) If $V_\lambda \cap V_\mu \neq$ ($r_\lambda = r_\mu$ at this case), then there exist invertible holomorphic functions $u_j^{(\lambda\mu)}$ ($1 \le j \le r_\lambda$) on $V_\lambda \cap V_\mu$ and a permutation $\sigma \in S_{r_\lambda}$ such that

$$z_{\sigma(j)}^{(\lambda)} = u_j^{(\lambda\mu)} z_j^{(\mu)} \text{ and } u_1^{(\lambda\mu)} \cdots u_{r_\lambda}^{(\lambda\mu)} = 1 \text{ on } V_\lambda \cap V_\mu.$$

Thus we have a *log atlas* in the sense of [KN]. To check (2), noting that

$$\begin{aligned} Y &= \bigcup_{i=1}^{r_\lambda} \{x_i^\lambda = 0\} \cup \bigcup_{k=1}^{s} \{x_{d+i_k}^\lambda = 0\} \\ &= \bigcup_{i=1}^{r_\lambda} \{x_i^\mu = 0\} \cup \bigcup_{k=1}^{s} \{x_{d+i_k}^\mu = 0\} \end{aligned}$$

on $U_\lambda \cap U_\mu$, we have $\sigma \in S_{r_\lambda + s}$ and invertible holomorphic functions $u_j^{(\lambda\mu)} \in \mathcal{O}_X(U_\lambda \cap U_\mu)$ such that

$$x_{\sigma(j)}^\lambda = u_j^{(\lambda\mu)} x_j^\mu.$$

Since $x_{d+i_k}^\lambda = x_{d+i_k}^\mu = 0$ on $X_s \cap U_\lambda \cap U_\mu$, we have $\sigma(j) \in \{1, ..., r_\lambda\}$ if $j \in \{1, ..., r_\lambda\}$. On the other hand, for fixed $(\pi_1, ..., \pi_m) = m_s$, we have $\pi_1 = x_1^\lambda \cdots x_{r_\lambda}^\lambda = x_1^\mu \cdots x_{r_\lambda}^\mu$, thus $u_1^{(\lambda\mu)} \cdots u_{r_\lambda}^{(\lambda\mu)} = 1$. The restrictions of $\{u_j^{(\lambda\mu)}\}$ to $X_s \cap U_\lambda \cap U_\mu$ give the required functions in (2).

Lemma 1.2. *For the smooth variety X with log structure $logY$ given by a normal crossing divisor Y, and for the normal crossing variety X_s with log structure in the sense of [KN] (in particular, for $logY|_{X_s}$), the surjections*

$$\begin{aligned} \Omega^1_{X/S}(logY)^* &\to T_{X/S}(logY) \qquad & \Omega^1_{X_s}(log)^* &\to T_{X_s}(log) \\ u &\mapsto u \circ \bar{d} & u &\mapsto u \circ \bar{d} \end{aligned}$$

are isomorphisms, where $\mathcal{E}^*$ denotes the dual of $\mathcal{E}$.

Proof. It is enough to check the injectivity of the above morphisms, which is a local problem. Thus we may assume that

$$\mathcal{O}_{X_s} = \frac{\mathbb{C}[[x_1, ..., x_r, ..., x_n]]}{(x_1 \cdots x_r)}, \quad \Omega^1_{X_s}(log) = \frac{\mathcal{O}_{X_s}\{\frac{\bar{d}x_1}{x_1}, ..., \frac{\bar{d}x_r}{x_r}, \bar{d}x_{r+1}, ..., \bar{d}x_n\}}{(\frac{\bar{d}x_1}{x_1} + \cdots + \frac{\bar{d}x_r}{x_r})}.$$

For any $u : \Omega^1_{X_s}(log) \to \mathcal{O}_{X_s}$ such that $u(\bar{d}f) = 0$ for all $f \in \mathcal{O}_{X_s}$, we need to show that for $i = 1, ..., r$

$$\bar{a}_i := u(\frac{1}{x_i}\bar{d}x_i) = 0.$$

Firstly, $a_i = b_i(x_1 \cdots \hat{x}_i \cdots x_r)$ for some $b_i \in \mathbb{C}[[x_1, ..., x_n]]$ since $\bar{a}_i \bar{x}_i = u(\bar{d}x_i) = 0$. Secondly, $a_1 + \cdots + a_r$ is divisible by $x_1 \cdots x_r$ since

$$\bar{a}_1 + \cdots + \bar{a}_r = u(\frac{1}{x_1}\bar{d}x_1 + \cdots + \frac{1}{x_r}\bar{d}x_r) = 0.$$

These facts imply that b_i is dvisible by x_i $(i = 1, ..., r)$, which means all $\bar{a}_i$ are zero.

When X is a smooth variety, local ring (analytic) at any point of X is integral, the injectivity is obvious from the above proof.

Thus for a logarithmic derivation θ, we will write $< \theta, \cdot >$ denoting the unique element of $\Omega^1_{X_s}(log)^*$ such that $< \theta, \bar{d}a >= \theta(a)$.

Proposition 1.4. *Let $W \subset X_s$ be the singular locus of X_s and $\mathcal{I}_W \subset \mathcal{O}_{X_s}$ the ideal sheaf of W. Then every local section $D \in \mathcal{D}^k_{X_s^\dagger}$ can be expressed into*

$$D = \sum_{\beta_1+\cdots+\beta_d \le k} \lambda_{\beta_1,...,\beta_d} \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$$

for a local basis ∂_1, ..., ∂_d of $T_{X_s}(log)$ and $D(\mathcal{I}_W) \subset I_W$. In particular, we have the canonical exact sequence

$$0 \to \mathcal{D}^{k-1}_{X_s^\dagger} \to \mathcal{D}^k_{X_s^\dagger} \xrightarrow{\sigma_k} S^k T_{X_s}(log) \to 0, \tag{1.3}$$

where $S^k T_{X_s}(log)$ is the subsheaf of symmetric tensors of $T^k(T_{X_s}(log))$ and σ_k will be defined in the proof.

Proof. Locally, $\hat{\mathcal{O}}_{X_s,x} = \mathbb{C}\{x_1, ..., x_{d+1}\}/(x_1 \cdots x_r)$, and $T_{X_s}(log)$ is locally generated by

$$x_1\frac{\partial}{\partial x_1}, ..., x_r\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, ..., \frac{\partial}{\partial x_{d+1}}$$

with a relation

$$x_1\frac{\partial}{\partial x_1} + \cdots + x_r\frac{\partial}{\partial x_r} = 0.$$

If we take the local basis ∂_1, ..., ∂_d of $T_{X_s}(log)$ to be

$$\partial_1 = x_1\frac{\partial}{\partial x_1}, ..., \partial_{r-1} = x_{r-1}\frac{\partial}{\partial x_{r-1}}, \partial_r = \frac{\partial}{\partial x_{r+1}}, ..., \partial_d = \frac{\partial}{\partial x_{d+1}},$$

one can check that $\partial_i^{\alpha_i} = x_i g_i(x_i, \frac{\partial}{\partial x_i})$ when $\alpha_i > 0$ and $1 \le i < r$, where $g_i(x, y)$ is a polynoimal. Thus the first term of (1.1) becomes into

$$\lambda_{\alpha_1,...,\alpha_d} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} = \lambda_{\alpha_1,...,\alpha_d} x_{i_1} \cdots x_{i_t} \cdot g_{i_1} \cdots g_{i_t} \cdot \partial_r^{\alpha_r} \cdots \partial_d^{\alpha_d},$$

and the equality (1.2) becomes into $\lambda_{\alpha_1,...,\alpha_d} x_{i_1}^{\alpha_{i_1}} \cdots x_{i_t}^{\alpha_{i_t}} = 0$, where $\alpha_{i_1}, ..., \alpha_{i_t}$ are nonzero integers of $\alpha_1, ..., \alpha_{r-1}$. This implies that $\lambda_{\alpha_1,...,\alpha_d}$ is divisible by all x_i with $i \in \{1, ..., r\} \setminus \{i_1, ..., i_t\}$, namely $\lambda_{\alpha_1,...,\alpha_d} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} = 0$, which means that any $D \in \mathcal{D}^k_{X_s^\dagger}$ can be expressed into

$$D = \sum_{\beta_1+\cdots+\beta_d \le k} \lambda_{\beta_1,...,\beta_d} \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}.$$

On the other hand, $\mathcal{I}_W$ is locally generated by $\frac{x_1\cdots x_r}{x_i}$ $(i=1,...,r)$, and

$$\partial_j(\frac{x_1\cdots x_r}{x_i}) = \begin{cases} 0, & \text{if } j=i \text{ or } j\geq r \\ \frac{x_1\cdots x_r}{x_i}. & \text{if } i\neq j<r \end{cases}$$

Thus $D(\mathcal{I}_W)\subset\mathcal{I}_W$. Now we see that the natural map in Proposition 1.2 (4) induces isomorphisms (it is surjective by the above proof, and injective since it is generically injective and $Gr_1(\mathcal{D}_{X^\dagger/Y^\dagger})$ locally free)

$$S^k(Gr_1(\mathcal{D}_{X^\dagger/Y^\dagger}))\to Gr_k(\mathcal{D}_{X^\dagger/Y^\dagger}).$$

For any local section $D\in\mathcal{D}^k_{X_s^\dagger}$, let $D_{<k}$ denote the lower order part and write

$$D=D_{<k}+\sum_{\beta_1+\cdots+\beta_d=k}\lambda_{\beta_1,...,\beta_d}\partial_1^{\beta_1}\cdots\partial_d^{\beta_d},$$

$$D(\omega_1,...,\omega_k)=\sum_{\beta_1+\cdots+\beta_d=k}\lambda_{\beta_1,...,\beta_d}\left(\prod_1^{\beta_1}<\sigma(\partial_1),\omega_i>\cdots\prod_{\beta_{d-1}+1}^{\beta_d}<\sigma(\partial_d),\omega_i>\right)$$

where $\omega_1,...,\omega_k$ are elements of $\Omega^1_{X_s}(log)$. The symbol $\sigma_k(D)$ of D as a symmetric function on $\otimes^k\Omega^1_{X_s}(log)$ is defined to be

$$\sigma_k(D)(\omega_1,...,\omega_k)=\sum_{\tau\in S_k}D(\omega_{\tau(1)},...,\omega_{\tau(k)}).$$

This gives the exact sequence (1.3) and coincides the definition of [GJ] and [We] in smooth case (see Remark 2.2.4 of [GJ]).

§2 Logarithmic heat operators and logarithmic connections

In this section, we generalize the definitions and arguments about heat operators and connections in [GJ] and [We] to the logarithmic case. Our task here is to figure out the conditions for the existence of a projective logarithmic heat operator.

Let $f:X\to S$ be a flat family of normal crossing varieties of dimension d satisfying the assumpations of Proposition 1.3. Since X is smooth, the (1.2) will imply that $\lambda_{\alpha_1,...,\alpha_r}=0$. Thus, for any local section $D\in\mathcal{D}^k_{X^\dagger/S^\dagger}$, we have

$$D=\sum_{\beta_1+\cdots+\beta_r\leq k}\lambda_{\beta_1,...,\beta_r}\partial_1^{\beta_1}\cdots\partial_r^{\beta_r}.$$

Namely, in this case, we have $(T_{X/S}(logY)$ isomorphic to the dual of $\Omega^1_{X/S}(logY))$

$$Sym^i_{\mathcal{O}_X}(T_{X/S}(logY))\cong Gr_i(\mathcal{D}_{X^\dagger/S^\dagger}),$$

which means that we have the canonical exact sequences (Proposition 1.4)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^{k-1}_{X^\dagger} & \longrightarrow & \mathcal{D}^k_{X^\dagger} & \longrightarrow & S^kT_X(logY) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{D}^{k-1}_{X^\dagger/S^\dagger} & \longrightarrow & \mathcal{D}^k_{X^\dagger/S^\dagger} & \longrightarrow & S^kT_{X/S}(logY) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.1}$$

For convenience of applications, we summarize the discussions in section 1 for the special logarithmic varieties we consider in this paper. Let $f : Z \to T$ denote the logarithmic schemes: (1) $Z = X$ and $T = S$ with log structures $logY$ and $log\Delta$, (2) $Z = X_s$ and $T = \{s\}$ with log structures $logY|_{X_s}$ and $log\Delta|_{\{s\}}$. Let Λ and Λ_i denote the corresponding sheaf of logarithmic differential operators on Z. Then, according to [Si], the following proposition (which is the definition of [Si]) assures that Λ is *a sheaf of split almost polynomial rings of differential operators on* Z/T. Thus Λ and Λ_i are endowed with all the nice properties such as compatible with base changes and Λ generated by Λ_1 as a ring (see [Si]).

Proposition 2.1. *The sheaf Λ of rings of logarithmic differential operators on Z over T is a sheaf of $\mathcal{O}_Z$-algebra Λ over T with a filtration*

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_i \subset \cdots ,$$

which satisfies

(1) $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$ *and* $\Lambda_i \cdot \Lambda_j \subset \Lambda_{i+j}$.
(2) *The image of the morphism* $\mathcal{O}_Z \to \Lambda$ *is equal to* Λ_0.
(3) *The image of* $f^{-1}(\mathcal{O}_T)$ *in* $\mathcal{O}_Z$ *is contained in the center of* Λ.
(4) *The left and right* $\mathcal{O}_Z$*-module structures on* $Gr_i(\Lambda) := \Lambda_i/\Lambda_{i-1}$ *are equal.*
(5) *The sheaves of* $\mathcal{O}_Z$*-modules* $Gr_i(\Lambda)$ *are coherent.*
(6) *The sheaf of graded* $\mathcal{O}_Z$*-algebra* $Gr(\Lambda) := \bigoplus_{i=0}^{\infty} Gr_i(\Lambda)$ *is generated by* $Gr_1(\Lambda)$ *in the sense that the morphism of sheaves*

$$Gr_1(\Lambda) \otimes_{\mathcal{O}_Z} \cdots \otimes_{\mathcal{O}_Z} Gr_1(\Lambda) \to Gr_i(\Lambda)$$

is surjective.
(7) $\Lambda_0 = \mathcal{O}_Z$, $Gr_1(\Lambda)$ *is locally free and* $Gr(\Lambda)$ *is the symmetric algebra on* $Gr_1(\Lambda)$.
(8) *There is a morphism* $\xi : Gr_1(\Lambda) \to \Lambda_1$ *of left* $\mathcal{O}_Z$*-modules splitting the projection* $\Lambda_1 \to Gr_1(\Lambda)$.

Definition 2.1. *For any coherent* $\mathcal{O}_X$*-modules* E_1 *and* E_2*, we define*

$$\mathcal{D}^k_{X^\dagger/S^\dagger}(E_1, E_2) := E_2 \otimes_{\mathcal{O}_X} \mathcal{D}^k_{X^\dagger/S^\dagger} \otimes_{\mathcal{O}_X} E_1^*$$

$$\mathcal{D}^k_{X^\dagger}(E_1, E_2) := E_2 \otimes_{\mathcal{O}_X} \mathcal{D}^k_{X^\dagger} \otimes_{\mathcal{O}_X} E_1^*,$$

where $E_1^* = \mathcal{H}om_{\mathcal{O}_X}(E_1, \mathcal{O}_X)$*, and the notation* $\mathcal{D}^k_{X^\dagger/S^\dagger} \otimes_{\mathcal{O}_X} E_1^*$ *(resp.* $\mathcal{D}^k_{X^\dagger} \otimes_{\mathcal{O}_X} E_1^*$ *means that we use the right* $\mathcal{O}_X$*-module structure of* $\mathcal{D}^k_{X^\dagger/S^\dagger}$ *(resp.* $\mathcal{D}^k_{X^\dagger}$*). If* $E_1 = E_2 = E$*, we simply write* $\mathcal{D}^k_{X^\dagger/S^\dagger}(E) = \mathcal{D}^k_{X^\dagger/S^\dagger}(E_1, E_2)$ *and* $\mathcal{D}^k_{X^\dagger}(E) = \mathcal{D}^k_{X^\dagger}(E_1, E_2)$.

Let $\mathcal{L}$ be a line bundle on X, and define the subsheaf $\mathcal{W}_{X/S}(\mathcal{L})$ of $\mathcal{D}^2_{X^\dagger}(\mathcal{L})$ to be

$$\mathcal{W}_{X/S}(\mathcal{L}) := \mathcal{D}^1_{X^\dagger}(\mathcal{L}) + \mathcal{D}^2_{X^\dagger/S^\dagger}(\mathcal{L}).$$

Tensor firstly the sequence (2.1) on the right by $\mathcal{L}^*$ as a right $\mathcal{O}_X$-module then on the left by $\mathcal{L}$ as a left $\mathcal{O}_X$-module, we get the commutative diagram for $k=2$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}^1_{X^\dagger}(\mathcal{L}) \arrow[r] & \mathcal{D}^2_{X^\dagger}(\mathcal{L}) \arrow[r, "\sigma"] & S^2T_X(logY) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{D}^1_{X^\dagger}(\mathcal{L}) \arrow[r] \arrow[u, equal] & \mathcal{W}_{X/S}(\mathcal{L}) \arrow[r, "\sigma"] \arrow[u] & S^2T_{X/S}(logY) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[r] \arrow[u] & \mathcal{D}^2_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[r, "\sigma"] \arrow[u] & S^2T_{X/S}(logY) \arrow[r] \arrow[u, equal] & 0,
\end{tikzcd}$$

and the commutative diagram for $k=1$

$$\begin{tikzcd}
 & & & f^*T_S(log\Delta) & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{D}^1_{X^\dagger}(\mathcal{L}) \arrow[r, "\sigma_1"] & T_X(logY) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[u, equal] & \mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[r, "\sigma_1"] \arrow[u] & T_{X/S}(logY) \arrow[r] \arrow[u] & 0,
\end{tikzcd} \tag{2.2}$$

where the third vertical is the canonical exact sequence

$$0 \to T_{X/S}(logY) \to T_X(logY) \to f^*T_S(log\Delta) \to 0. \tag{2.3}$$

Let $\varepsilon : \mathcal{D}^1_{X^\dagger}(\mathcal{L}) \xrightarrow{\sigma_1} T_X(logY) \to f^*T_S(log\Delta)$ be the composition of canonical maps, one can see easily from the diagram (2.2) that

$$ker(\varepsilon) = \mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}).$$

Thus we have a surjection $\bar{\varepsilon} : \mathcal{W}_{X/S}(\mathcal{L}) \to f^*T_S(log\Delta)$ such that the following diagram is commutative

$$\begin{tikzcd}
f^*T_S(log\Delta) \arrow[r, equal] & f^*T_S(log\Delta) & & \\
\mathcal{D}^1_{X^\dagger}(\mathcal{L}) \arrow[r, "i"] \arrow[u, "\varepsilon"] & \mathcal{W}_{X/S}(\mathcal{L}) \arrow[r, "\sigma"] \arrow[u, "\bar{\varepsilon}"] & S^2T_{X/S}(logY) \arrow[r] & 0 \\
\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[r] \arrow[u] & \mathcal{D}^2_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[r, "\sigma"] \arrow[u] & S^2T_{X/S}(logY) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \tag{2.4}$$

Let $\mathcal{W}_{X/S}(\mathcal{L}) \xrightarrow{\sigma\oplus\bar{\varepsilon}} S^2T_{X/S}(logY) \oplus f^*T_S(log\Delta)$ be the surjection defined by $\sigma \oplus \bar{\varepsilon}(D) := \sigma(D) \oplus \bar{\varepsilon}(D)$ for any local section $D \in \mathcal{W}_{X/S}(\mathcal{L})$. Then we have the exact sequence

$$0 \to \mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \to \mathcal{W}_{X/S}(\mathcal{L}) \xrightarrow{\sigma\oplus\bar{\varepsilon}} S^2T_{X/S}(logY) \oplus f^*T_S(log\Delta) \to 0. \tag{2.5}$$

Definition 2.2. *A logarithmic heat operator on $\mathcal{L}$ over S is an $\mathcal{O}_S$-module homomorphism*

$$H : T_S(log\Delta) \to f_*\mathcal{W}_{X/S}(\mathcal{L}) \subset \mathcal{D}^2_{X^\dagger}(\mathcal{L})$$

such that

$$T_S(log\Delta) \xrightarrow{H} f_*\mathcal{W}_{X/S}(\mathcal{L}) \xrightarrow{f_*\bar{\varepsilon}} T_S(log\Delta) \tag{2.6}$$

is the identity map. A logarithmic heat operator H on $\mathcal{L}$ is called flat if

$$H([\theta_1, \theta_2]) = [H(\theta_1), H(\theta_2)]$$

for any local sections $\theta_1, \theta_2 \in T_S(log\Delta)(U)$.

Any $\mathcal{O}_S$-linear map $\widetilde{H} : T_S(log\Delta) \to f_*\mathcal{W}_{X/S}(\mathcal{L})/\mathcal{O}_S$ has local lifting. Namely, there exists an open covering $\bigcup U = S$ such that for each open set U there is an $\mathcal{O}_U$-linear map $H_U : T_S(log\Delta) \to f_*\mathcal{W}_{X/S}(\mathcal{L})|_U$ which reduces to $\widetilde{H}|_U$.

Definition 2.3. *A projective logarithmic heat operator $\widetilde{H}$ on $\mathcal{L}$ over S is an $\mathcal{O}_S$-linear map*

$$\widetilde{H} : T_S(log\Delta) \to \frac{f_*\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_S}$$

such that any local lifting H_U is a logarithmic heat operator on $\mathcal{L}|_{f^{-1}(U)}$ over U. $\widetilde{H}$ is called projectively flat if any of the local lifts H_U satisfies

$$H_U([\theta_1, \theta_2]) = h_{\theta_1,\theta_2} + [H_U(\theta_1), H_U(\theta_2)]$$

for some function $h_{\theta_1,\theta_2} \in \mathcal{O}_S(V)$, where $V \subset U$ is any open set of U and $\theta_1, \theta_2 \in T_S(log\Delta)(V)$.

In the following, we will figure out the conditions under which a projective logarithmic heat operator on $\mathcal{L}$ over S do exist. As the same as in [GJ], one can see that a (projective) logarithmic heat operator of $\mathcal{L}$ over S gives a (projective) logarithmic connection on $f_*\mathcal{L}$. Firstly, it is clear that the map

$$f_*\sigma : f_*\mathcal{W}_{X/S}(\mathcal{L}) \to f_*S^2T_{X/S}(logY)$$

factors through $f_*\mathcal{W}_{X/S}(\mathcal{L})/\mathcal{O}_S$, thus we have the map

$$\rho_{\widetilde{H}} : T_S(log\Delta) \xrightarrow{\widetilde{H}} f_*\mathcal{W}_{X/S}(\mathcal{L})/\mathcal{O}_S \xrightarrow{f_*\sigma} f_*S^2T_{X/S}(logY),$$

which is called the *symbol* of $\widetilde{H}$. By taking the direct image of

$$0 \to \mathcal{O}_X \to \mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \xrightarrow{\sigma_1} T_{X/S}(logY) \to 0, \tag{2.7}$$

we have the connecting map $f_*T_{X/S}(logY) \xrightarrow{\cup[\mathcal{L}]} R^1f_*\mathcal{O}_X$ and the map

$$R^1f_*\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \xrightarrow{R^1f_*\sigma_1} R^1f_*T_{X/S}(logY)$$

induced by the symbol map $\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \xrightarrow{\sigma_1} T_{X/S}(logY)$. Similarly, from

$$0 \to \mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \to \mathcal{D}^2_{X^\dagger/S^\dagger}(\mathcal{L}) \xrightarrow{\sigma} S^2T_{X/S}(logY) \to 0, \tag{2.8}$$

we have the connecting map $f_*S^2T_{X/S}(logY) \xrightarrow{c} R^1f_*\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})$ and thus

$$\mu_{\mathcal{L}} : f_*S^2T_{X/S}(logY) \xrightarrow{c} R^1f_*\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \xrightarrow{R^1f_*\sigma_1} R^1f_*T_{X/S}(logY).$$

¿From the canonical exact sequence (2.3), we get the connecting map

$$\kappa_{X/S} : T_S(log\Delta) \to R^1f_*T_{X/S}(logY),$$

which is the Kodaira-Spencer map of the family X/S.

Theorem 2.1. *Let $f : X \to S$, Δ and $Y = f^{-1}(\Delta)$ satisfy the assumptions of Proprosition 1.3 and $f_*\mathcal{O}_X = \mathcal{O}_S$, $\mathcal{L}$ a line bundle on X. If there exists a symbol*

$$\rho : T_S(log\Delta) \to f_*S^2T_{X/S}(logY)$$

such that the following two conditions hold

1. $\mu_{\mathcal{L}} \cdot \rho + \kappa_{X/S} = 0$,
2. $f_*T_{X/S}(logY) \xrightarrow{\cup[\mathcal{L}]} R^1f_*\mathcal{O}_X$ *is an isomorphism.*

Then there exists a unique projective logarithmic heat operator

$$\widetilde{H} : T_S(log\Delta) \to f_*\mathcal{W}_{X/S}(\mathcal{L})/\mathcal{O}_S$$

such that $\rho_{\widetilde{H}} = \rho$. In particular, there exists a projective logarithmic connection on $f_*\mathcal{L}$.

Proof. It is enough to prove that for any $\theta \in T_S(log\Delta)(U)$ there exists a unique lifting of $\rho(\theta) \oplus \theta$ to $f_*\mathcal{W}_{X/S}(\mathcal{L})(U)$ up to a section of $\mathcal{O}_S(U)$. Thus we consider the commutative diagram

$$\begin{tikzcd}
T_{X/S}(logY) \arrow[r] & \frac{\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_X} \arrow[r, "\sigma\oplus\bar{\varepsilon}"] & S^2T_{X/S}(logY) \oplus f^*T_S(log\Delta) \\
\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[u, "\sigma_1"] \arrow[r] & \mathcal{W}_{X/S}(\mathcal{L}) \arrow[u] \arrow[r, "\sigma\oplus\bar{\varepsilon}"] & S^2T_{X/S}(logY) \oplus f^*T_S(log\Delta) \arrow[u, equal] \\
\mathcal{O}_X \arrow[u] \arrow[r, equal] & \mathcal{O}_X, \arrow[u] &
\end{tikzcd}$$

which gives the induced commutative diagram

$$\begin{tikzcd}
R^1f_*\mathcal{O}_X \arrow[r, equal] & R^1f_*\mathcal{O}_X & & \\
f_*T_{X/S}(logY) \arrow[u, "\cup[\mathcal{L}]"] \arrow[r] & f_*\frac{\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_X} \arrow[u, "\nu"] \arrow[r] & f_*S^2T_{X/S}(logY) \oplus T_S(log\Delta) \arrow[r, "o"] & {} \\
f_*\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L}) \arrow[u] \arrow[r] & f_*\mathcal{W}_{X/S}(\mathcal{L}) \arrow[u] \arrow[r] & f_*S^2T_{X/S}(logY) \oplus T_S(log\Delta) \arrow[u, equal] \arrow[r] & {} \\
f_*\mathcal{O}_X \arrow[u] \arrow[r, equal] & f_*\mathcal{O}_X, \arrow[u] & &
\end{tikzcd}$$

where $f_*S^2T_{X/S}(logY) \oplus T_S(log\Delta) \xrightarrow{o} R^1f_*T_{X/S}(logY)$ is the connecting map. We claim that

$$o(\rho(\theta) \oplus \theta) = \mu_{\mathcal{L}} \cdot \rho(\theta) + \kappa_{X/S}(\theta).$$

If it is true, by the condition (1), we will have a lifting $\widetilde{H}_U(\theta) \in f_*\frac{\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_X}(U)$. By the surjectivity in condition (2), there exists a section $s \in f_*T_{X/S}(logY)(U)$ such that $s \cup [\mathcal{L}] = \nu(\widetilde{H}_U(\theta))$. Thus there exists a $H_U(\theta) \in f_*\mathcal{W}_{X/S}(\mathcal{L})(U)$ such that

$\widetilde{H_U(\theta)} = \widetilde{H}_U(\theta) - s$, which is also a lifting of $\rho(\theta) \oplus \theta$. The injectivity in condition (2) implies that such $H_U(\theta)$ is unique up to a section of $f_*\mathcal{O}_X(U) = \mathcal{O}_S(U)$. This gives a unique projective logarithmic heat operator

$$\widetilde{H} : T_S(log\Delta) \to \frac{f_*\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_S}.$$

Now we show the claim by considering the following commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \arrow[r] & \frac{\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_X} \arrow[r, "\sigma\oplus\bar{\varepsilon}"] & S^2T_{X/S}(logY) \oplus f^*T_S(log\Delta) \\
0 \arrow[r] & \frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \arrow[r] \arrow[u, equal] & \frac{\mathcal{D}^2_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \arrow[r, "\sigma"] \arrow[u] & S^2T_{X/S}(logY) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \arrow[r] & \frac{\mathcal{W}_{X/S}(\mathcal{L})}{\mathcal{O}_X} \arrow[r, "\sigma\oplus\bar{\varepsilon}"] & S^2T_{X/S}(logY) \oplus f^*T_S(log\Delta) \\
0 \arrow[r] & \frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \arrow[r] \arrow[u, equal] \arrow[d, "\sigma_1"'] & \frac{\mathcal{D}^1_{X^\dagger}(\mathcal{L})}{\mathcal{O}_X} \arrow[r, "\varepsilon"] \arrow[u, "i"] \arrow[d, "\sigma_1"'] & f^*T_S(log\Delta) \arrow[u] \arrow[d, equal] \\
0 \arrow[r] & T_{X/S}(logY) \arrow[r] & T_X(logY) \arrow[r] & f^*T_S(log\Delta),
\end{tikzcd}$$

from which we have commutative diagrams for the connecting maps

$$\begin{tikzcd}
f_*S^2T_{X/S}(logY) \oplus T_S(log\Delta) \arrow[r, "o"] & R^1f_*\frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \cong R^1f_*T_{X/S}(logY) \\
f_*S^2T_{X/S}(logY) \arrow[u] \arrow[r, "\mu_{\mathcal{L}}"] & R^1f_*\frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \cong R^1f_*T_{X/S}(logY) \arrow[u, equal]
\end{tikzcd}$$

and

$$\begin{tikzcd}
f_*S^2T_{X/S}(logY) \oplus T_S(log\Delta) \arrow[r, "o"] & R^1f_*\frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \cong R^1f_*T_{X/S}(logY) \\
T_S(log\Delta) \arrow[u] \arrow[r, "\kappa_{X/S}"] & R^1f_*\frac{\mathcal{D}^1_{X^\dagger/S^\dagger}(\mathcal{L})}{\mathcal{O}_X} \cong R^1f_*T_{X/S}(logY). \arrow[u, equal]
\end{tikzcd}$$

Thus the claim $o(\rho(\theta) \oplus \theta) = \mu_{\mathcal{L}} \cdot \rho(\theta) + \kappa_{X/S}(\theta)$ is indeed true.

Remark 2.1. From the proof, we see that the local lifting exists when the map in condition (2) is surjective, and the injectivity was only used to assure the uniqueness of the local lifting. Thus in some cases the map in condition (2) is only surjective but one has a natural way to choose the lifting uniquely, we still have the heat operator. For example, the map in [GJ] is zero but one can choose uniquely the $\mathcal{G}$-invariant lifting.

We are now going to describe the maps $\cup[\mathcal{L}]$ and $\mu_{\mathcal{L}}$. For any $s \in \Delta$, $\{s\} = \mathrm{Spec}k(s)$ has the induced log structure, we denote this logarithmic point by $s^\dagger$, then the logarithmic fibre $X_s^\dagger$ over $s^\dagger$ is $(X_s, logY|_{X_s})$. Thus

$$\Omega^1_{X_s}(log) = \Omega^1_{X/S}(logY)|_{X_s}, \quad \mathcal{D}^k_{X_s^\dagger} = \mathcal{D}^k_{X^\dagger/S^\dagger}|_{X_s}$$

and, if the dimensions of $H^0(T_{X_s}(log))$, $H^0(S^2T_{X_s}(log))$, $H^1(\mathcal{O}_{X_s})$, $H^1(T_{X_s}(log))$ are constant (for s), then fibrewisely the maps $\cup[\mathcal{L}]$ and $\mu_{\mathcal{L}}$ are the following maps

$$\cup[\mathcal{L}_s] : H^0(T_{X_s}(log)) \to H^1(\mathcal{O}_{X_s})$$

$$\mu_{\mathcal{L}_s} : H^0(S^2T_{X_s}(log)) \to H^1(T_{X_s}(log))$$

where $\mathcal{L}_s = \mathcal{L}|_{X_s}$ and $\cup[\mathcal{L}_s]$ is the connecting map of

$$0 \to \mathcal{O}_{X_s} \to \mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s) \xrightarrow{\sigma_1} T_{X_s}(log) \to 0 \tag{2.9}$$

and $\mu_{\mathcal{L}_s}$ is the connecting map $H^0(S^2T_{X_s}(log)) \to H^1(\mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s))$ of

$$0 \to \mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s) \to \mathcal{D}^2_{X_s^\dagger}(\mathcal{L}_s) \xrightarrow{\sigma_2} S^2T_{X_s}(log) \to 0, \tag{2.10}$$

composing with the natural map $H^1(\mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s)) \xrightarrow{H^1(\sigma_1)} H^1(T_{X_s}(log))$.

Let $[\mathcal{L}_s] \in H^1(\Omega^1_{X_s}(log))$ denote the extension class of (2.9), then the map $\cup[\mathcal{L}_s]$ means the cup product. In general, for any class $cl \in H^1(\Omega^1_{X_s}(log))$, one has the natural cup product map

$$H^0(\otimes^k T_{X_s}(log)) \xrightarrow{\cup cl} H^1(\otimes^{k-1} T_{X_s}(log))$$

and for any $\omega \in H^0(S^kT_{X_s}(log))$ the symbol $\omega \cup cl$ means that we consider ω as a symmetric tensor. For any line bundle L on X_s, we define the Chern class $c_1(L) \in H^1(\Omega^1_{X_s}(log))$ of L to be the image of usual Chern class of L under the natural map $H^1(\Omega^1_{X_s}) \to H^1(\Omega^1_{X_s}(log))$. More precisely, let $\bar{d} : \mathcal{O}_{X_s} \xrightarrow{d} \Omega^1_{X_s} \to \Omega^1_{X_s}(log)$ and $dl : \mathcal{O}^*_{X_s} \to \Omega^1_{X_s}(log)$ be defined as $dl(u) = \frac{1}{u}\bar{d}u$. Then dl is a morphism of abelian sheaves and induces a morphism

$$H^1(\mathcal{O}^*_{X_s}) \xrightarrow{c_1} H^1(\Omega^1_{X_s}(log))$$

of abelian groups. The Chern class $c_1(L)$ of L is defined to be the image of this morphism. With these notation, we have

Proposition 2.2. *The extension class $[\mathcal{L}_s] \in H^1(\Omega^1_{X_s}(log))$ of*

$$0 \to \mathcal{O}_{X_s} \to \mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s) \xrightarrow{\sigma_1} T_{X_s}(log) \to 0 \tag{2.11}$$

is equal to the Chern class $c_1(\mathcal{L}_s)$ and for any $\omega \in H^0(S^2T_{X_s}(log))$, we have

$$\mu_{\mathcal{L}_s}(\omega) = -\omega \cup c_1(\mathcal{L}_s) + \mu_{\mathcal{O}_{X_s}}(\omega).$$

Proof. We check firstly the following descriptions about the symbol maps.

(1) If D is a local section of $\mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s)$, its image $\sigma_1(D) \in T_{X_s}(log)$ is determined by the requirement, for all $a \in \mathcal{O}_{X_s}$ and all $s \in \mathcal{L}_s$

$$< \sigma_1(D), \bar{d}a > \cdot s = \sigma_1(D)(a) \cdot s = D(a \cdot s) - a \cdot D(s). \tag{2.12}$$

(2) If D is a local section of $\mathcal{D}^2_{X_s^\dagger}(\mathcal{L}_s)$, its image $\sigma_2(D) \in S^2T_{X_s}(log)$ is characterized by the formula, for all a, $b \in \mathcal{O}_{X_s}$ and all $s \in \mathcal{L}_s$

$$\sigma_2(D)(a,b) \cdot s = D(ab \cdot s) - a \cdot D(b \cdot s) - b \cdot D(a \cdot s) + ab \cdot D(s). \tag{2.13}$$

(1) is clear by the definition $\sigma_1(D)(a) \cdot s = [D,a](1) \cdot s = D(a \cdot s) - a \cdot D(s)$ in Proposition 1.2 (1). To check (2), one can write $D = D_{<2} + \sum \lambda_{ij}\partial_i\partial_j$, where $D_{<2}$ denotes the part with order smaller than 2. Then, by definition in Proposition 1.4,

$$< \sigma_2(D), \bar{d}a \otimes \bar{d}b > \cdot s := \sigma_2(D)(a,b) = (\sum \lambda_{ij}[\partial_i, a][\partial_j, b] + \sum \lambda_{ij}[\partial_i, b][\partial_j, a])(1) \cdot s.$$

Thus (2) is clear from the following computations

$$\begin{aligned} \sum \lambda_{ij}[\partial_i, a][\partial_j, b] &= \sum \lambda_{ij}\partial_i a \partial_j b - \sum \lambda_{ij} a \partial_i \partial_j b - \sum \lambda_{ij}\partial_i a b \partial_j + \sum \lambda_{ij} a \partial_i b \partial_j \\ &= Dab - aDb - bDa + abD + b[D_{<2}, a] - [D_{<2}, a]b \\ &\quad - [\sum \lambda_{ij}\partial_i[\partial_j, a], b] - [\sum \lambda_{ij}[\partial_i, b]\partial_j, a] + \sum \lambda_{ij}[\partial_i, b][\partial_j, a] \\ &= Dab - aDb - bDa + abD - \sum \lambda_{ij}[\partial_i, b][\partial_j, a]. \end{aligned}$$

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open cover of X_s trivializing $\mathcal{L}_s$ and $s_i : \mathcal{O}_{U_i} \cong \mathcal{L}_s|_{U_i}$, $s_j = u_{ij} \cdot s_i$ on $U_{ij} = U_i \cap U_j$. Then $c_1(\mathcal{L}_s) \in H^1(\Omega^1_{X_s}(log))$ is given by the 1-cocycle

$$\{\frac{\bar{d}u_{ij}}{u_{ij}}\} \in C^1(\mathcal{U}, \Omega^1_{X_s}(log)).$$

The sequence (2.11) is locally splitting, and there exist morphisms of $\mathcal{O}_{U_i}$-modoules

$$\rho_i : T_{X_s}(log)(U_i) \to \mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s)(U_i)$$

such that $\sigma_1 \circ \rho_i(\theta) = \theta$ for any $\theta \in T_{X_s}(log)(U_i)$. Let $\rho_i(\theta)(s_i) = \omega_i(\theta) \cdot s_i$, then $\omega_i \in T_{X_s}(log)(U_i)^* = \Omega^1_{X_s}(log)(U_i)$ since ρ_i is a morphism of $\mathcal{O}_{U_i}$-modules. For any $\theta \in T_{X_s}(log)(U_{ij})$

$$\rho_i(\theta)(s_i) - \rho_j(\theta)(s_i) =< \theta, \omega_i - \omega_j + \frac{\bar{d}u_{ij}}{u_{ij}} > \cdot s_i.$$

Thus the extension class of (2.11) is $c_1(\mathcal{L}_s)$.

Given $\omega \in H^0(S^2T_{X_s}(log))$, let $D_i \in H^0(U_i, \mathcal{D}^2_{X_s^\dagger}(\mathcal{O}_{X_s}))$ be the lifting of $\omega_i = \omega|_{U_i}$, thus

$$\{D_{ij}\}_{i<j} = \{D_j - D_i\}_{i<j} \in C^1(\mathcal{U}, \mathcal{D}^1_{X_s^\dagger}(\mathcal{O}_{X_s})).$$

Then $\mu_{\mathcal{O}_{X_s}}(\omega) \in H^1(T_{X_s}(log))$ is given by the 1-cocycle $\{v_{ij}\} \in C^1(\mathcal{U}, T_{X_s}(log))$, where $v_{ij} \in H^0(U_{ij}, T_{X_s}(log))$ is the image $\sigma_1(D_{ij})$ of $D_{ij} = D_j - D_i$, namely, for all $a \in \mathcal{O}_{X_s}(U_{ij})$,

$$< v_{ij}, \bar{d}a >= v_{ij}(a) = D_{ij}(a) - aD_{ij}(1).$$

To compute $\mu_{\mathcal{L}_s}(\omega)$, we see that by definition $\widetilde{D}_i = s_i \otimes D_i \otimes s_i^* \in H^0(U_i, \mathcal{D}^2_{X_s^\dagger}(\mathcal{L}_s)$ is a lifting of ω_i, and thus $\widetilde{D}_i(a \cdot s_i) = D_i(a) \cdot s_i$ and

$$\{\widetilde{D}_{ij}\}_{i<j} = \{\widetilde{D}_j - \widetilde{D}_i\}_{i<j} \in C^1(\mathcal{U}, \mathcal{D}^1_{X_s^\dagger}(\mathcal{L}_s)).$$

Then $\mu_{\mathcal{L}_s}(\omega) \in H^1(T_{X_s}(log))$ is given by the 1-cocycle $\{\widetilde{v}_{ij}\} \in C^1(\mathcal{U}, T_{X_s}(log))$, where $\widetilde{v}_{ij} \in H^0(U_{ij}, T_{X_s}(log))$ is the image $\sigma_1(\widetilde{D}_{ij})$ of $\widetilde{D}_{ij}$, namely, for all $a \in \mathcal{O}_{X_s}(U_{ij})$, by using (2.13), we have

$$\begin{aligned} < \widetilde{v}_{ij}, \bar{d}a > \cdot s_j &= \widetilde{v}_{ij}(a) = \widetilde{D}_{ij}(a \cdot s_j) - a \cdot \widetilde{D}_{ij}(s_j) \\ &= D_j(a) \cdot s_j - D_i(au_{ij}) \cdot s_i - aD_j(1) \cdot s_j + aD_i(U_{ij}) \cdot s_i \\ &= (v_{ij}(a) - < \omega, \bar{d}a \otimes \frac{\bar{d}u_{ij}}{u_{ij}} >) \cdot s_j. \end{aligned}$$

Hence $\widetilde{v}_{ij} = v_{ij} - \omega \cup \frac{\bar{d}u_{ij}}{u_{ij}}$ and $\mu_{\mathcal{L}_s}(\omega) = \mu_{\mathcal{O}_{X_s}}(\omega) - \omega \cup c_1(\mathcal{L}_s)$.

§3 Logarithmic operators on generalized Jacobians

In this section, we will verify the conditions in Theorem 2.1 for a family of generalized Jacobians of stable curves, and thus show the existence of logarithmic heat operator. Let $(\mathcal{C}, \mathcal{C}_\Delta) \to (S, \Delta)$ be a flat family of stable curves satisfying the assumpations in Proposition 1.3, namely, S, $\mathcal{C}$ are regular schemes and Δ a (reduced) normal crossing divisor. It is well known there exists a projective S-scheme $f : J(\mathcal{C}) \to S$ such that for any $s \in S$ the fibre $J(\mathcal{C})_s$ is the generalized Jacobian $J(\mathcal{C}_s)$ of $\mathcal{C}_s$.

Lemma 3.1. *If $(\mathcal{C}, \mathcal{C}_\Delta) \to (S, \Delta)$ satisfies the assumpations of Proposition 1.3, then so do $f : J(\mathcal{C}) \to S$.*

Proof. By deformation theory of torsion free sheaves with rank one, for any point $y \in J(\mathcal{C})$ corresponds to a torsion free sheaf $\mathcal{F}$ on $\mathcal{C}_{f(y)}$ such that $\mathcal{F}$ is not locally free at a double point $x \in \mathcal{C}_{f(y)}$, there are integers l_1, l_2 such that

$$\hat{\mathcal{O}}_{J(\mathcal{C}),y}[[u_1, ..., u_{l_1}]] \cong \hat{\mathcal{O}}_{\mathcal{C},x}[[v_1, ..., v_{l_2}]].$$

Thus $f : J(\mathcal{C}) \to S$ satisfies the assumpations in Proposition 1.3 if $\mathcal{C}/S$ satisfies them. In particular, $J(\mathcal{C})$ is regular and all fibres $J(\mathcal{C}_s)$ are normal crossing varieties.

For simplicity, we assume that all fibres $\mathcal{C}_s = C$ $(s \in \Delta)$ are irreducible and smooth except one node x_0 and the family has a rational section. We recall briefly some facts about the so called generalized Jacobian $J^d(C)$ (we write $J(C)$ for $J^0(C)$) of a projective (singular) curve C of (arithmetic) genus g).

The $J^d(C)$ is defined to be the moduli space of rank one torsion free sheaves with degree d. There is a natural ample line bundle

$$\Theta_{J^d(C)} = detH^*(\mathcal{F})^{-1} \otimes (det\mathcal{F}_y)^{d+1-g}$$

called theta line bundle on $J^d(C)$, where $\mathcal{F}$ is a universal family on $C \times J^d(C)$, $detH^*(\mathcal{F})$ is the determinant of cohomology and $\mathcal{F}_y$ denotes the restriction of $\mathcal{F}$ to $\{y\} \times J^d(C)$ for a fixed smooth point $y \in C$. This construction can be generalized to relative case, namely, for a family of curves $\mathcal{C}/T$, one can construct a family of generalized Jacobians $J^d(\mathcal{C})/T$ and a line bundle Θ on $J^d(\mathcal{C})$ such that each fibre $J^d(\mathcal{C})_t$ is the generalized Jacobian $J^d(\mathcal{C}_t)$ and the restriction of Θ to $J^d(\mathcal{C})_t$ is the theta line bundle $\Theta_{J^d(\mathcal{C}_t)}$.

Let $\pi : \widetilde{C} \to C$ be the normalization and $\pi^{-1}(x_0) = \{x_1, x_2\}$, let $P = \mathbb{P}(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2})$ and $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{O}(1) \to 0$ be the universal quotient on P, where $\mathcal{E}$ is a univeral line bundle over $\widetilde{C} \times J^d(\widetilde{C})$. We consider the diagram

$$\begin{array}{ccc} P = \mathbb{P}(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}) & \xrightarrow{\phi} & J^d(C) \\ {\scriptstyle\rho}\big\downarrow & & \\ J^d(\widetilde{C}) & & \end{array}$$

where ρ is the natural projection, and ϕ is defined as follows: for any $(L, q) := (L, L_{x_1} \oplus L_{x_2} \xrightarrow{q} \mathbb{C}) \in P$, $\phi(L, q)$ is the kernel of $\pi_*L \xrightarrow{q} {}_{x_0}\mathbb{C} \to 0$.

Lemma 3.2. *Let $W \subset J^d(C)$ be the reduced subscheme of non-locally free sheaves and D_1, D_2 be the sections of $P \xrightarrow{\rho} J^d(\widetilde{C})$ given by projections $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{E}_{x_1}$ and $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2} \to \mathcal{E}_{x_2}$. Then*

(1) *$P \xrightarrow{\phi} J^d(C)$ is the normalization of $J^d(C)$, and W is the non-normal locus of $J^d(C)$.*
(2) *$\phi^{-1}(W) = D_1 + D_2$ and $\phi|_{D_i} : D_i \to W$ $(i = 1, 2)$ are isomorphisms.*
(3) *For any integer $k > 0$, $\Theta_P := \phi^*(\Theta^k_{J^d(C)}) = \mathcal{O}(1)^k \otimes \rho^*\mathcal{E}_y^{-k} \otimes \rho^*\Theta^k_{J^d(\widetilde{C})}$ and*

$$\rho_*\mathcal{O}(1)^k = \bigoplus_{j=0}^{k} \mathcal{E}^j_{x_1} \otimes \mathcal{E}^{k-j}_{x_2}.$$

Proof. This is the special case (rank one) of [NR] and [Su].

Lemma 3.3. *Fix a line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$ and two points $p_1 = (1, 0)$, $p_2 = (0, 1)$ of $X = \mathbb{P}^1$, which give a logarithmic structure on X. For any $D \in H^0(\mathcal{D}^1_{X^\dagger}(\mathcal{L}))$, if there exist $c \in \mathbb{C}^*$ and a nonzero section $s \in H^0(\mathcal{L})$ satisfying $s(p_2) = c \cdot s(p_1)$, such that*

$$D(s)(p_2) = c \cdot D(s)(p_1).$$

Then the symbol of D is trivial.

Proof. $\mathbb{P}^1$ is covered by $V_1 = \mathbb{P}^1 \setminus \{p_2\} = Spec\mathbb{C}[\frac{x_2}{x_1}]$ and $V_2 = \mathbb{P}^1 \setminus \{p_1\} = Spec\mathbb{C}[\frac{x_1}{x_2}]$, and there is a global vector field $\partial \in H^0(T_X(-p_1 - p_2))$ such that

$$\partial_1 := \partial|_{V_1} = u\frac{\partial}{\partial u}, \quad \partial_2 := \partial|_{V_2} = -v\frac{\partial}{\partial v}$$

where $u = \frac{x_2}{x_1}$, $v = \frac{x_1}{x_2} = \frac{1}{u}$. The space $H^0(T_X(-p_1-p_2))$ is generated by ∂.

We see that $\mathcal{L}|_{V_1} = \mathbb{C}[u]\cdot x_1$ and $\mathcal{L}|_{V_2} = \mathbb{C}[v]\cdot x_2$, thus any section $s \in H^0(\mathcal{L})$ has the form

$$s_1 := s|_{V_1} = (a_0 + a_1 u)\cdot x_1, \quad s_2 := s|_{V_2} = (a_1 + a_0 v)\cdot x_2,$$

where $(a_0, a_1) = (s(p_1), s(p_2)) \in \mathbb{C}^2$. Therefore, for any $D \in H^0(\mathcal{D}^1_{X^\dagger}(\mathcal{L}))$, there exists $(b_0, b_1) \in \mathbb{C}^2$ such that

$$D(s)|_{V_1} = (b_0 + b_1 u)\cdot x_1, \quad D(s)|_{V_2} = (b_1 + b_0 v)\cdot x_2.$$

If the symbol of D is $k\cdot\partial$, by the definition of symbol, we have

$$\begin{aligned} D(s)|_{V_1} &= D(s_1) = (a_0 + a_1 u)\cdot D(x_1) + k a_1 u \cdot x_1 \\ D(s)|_{V_2} &= D(s_2) = (a_1 + a_0 v)\cdot D(x_2) - k a_0 v \cdot x_2. \end{aligned}$$

Thus D is determined by any given number $D(1) \in \mathbb{C}$ such that

$$D(x_1) = D(1)\cdot x_1, \quad D(x_2) = (D(1) + k)\cdot x_2,$$

and one checks that for any given number $D(1) \in \mathbb{C}$ the above definition gives indeed a global differential operator of $\mathcal{L}$ with symbol $k\cdot\partial$. It is easy to see that for any $s \in H^0(\mathcal{L})$ and $c \in \mathbb{C}^*$

$$D(s)(p_2) - c\cdot D(s)(p_1) = (s(p_2) - c\cdot s(p_1))D(1) + k\cdot s(p_2).$$

Thus, if there exist a nonzero s and c such that $s(p_2) = c\cdot s(p_1)$, we have

$$D(s)(p_2) - c\cdot D(s)(p_1) = k s(p_2),$$

which is nonzero except $k = 0$ since $s(p_2) \neq 0$ (otherwise $s(p_1) = 0$ and s will have at least two zero points).

The fact that $X = J^d(C)$ is a degenerating fibre of flat family means that X is more special than usual normal crossing varieties. For example, its cohomology has low bound ($h^1(\mathcal{O}_X) \geq g$) and there is a logarithmic structure on it. Moreover, we have

Proposition 3.1. *Let $X = J^d(C)$ be the moduli space of torsion free sheaves on C with rank one and degree d, and $\mathcal{L}$ be the theta line bundle on X. Then for any logarithmic structure on X in the sense of [KN] and any integer $k > 0$*

$$H^0(\mathcal{D}^1_{X^\dagger}(\mathcal{L}^k)) \cong \mathbb{C}, \quad H^0(T_X(log)) \cong H^1(\mathcal{O}_X) \cong \mathbb{C}^g.$$

Proof. It is enough to prove Proposition 3.1 for $k = 1$ since $c_1(\mathcal{L}^k) = kc_1(\mathcal{L})$ and thus we have isomorphism $\mathcal{D}^1_{X^\dagger}(\mathcal{L}) \cong \mathcal{D}^1_{X^\dagger}(\mathcal{L}^k)$ by Proposition 2.2.

Local computation shows that $\phi^*\Omega^1_X(log) \cong \Omega^1_P(log(D_1 + D_2))$ and thus the natural map $T_{P^\dagger} := T_P(log(D_1 + D_2)) \to \phi^* T_X(log)$ is an isomorphism, where $P^\dagger = (P, log(D_1 + D_2))$, and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_P & \longrightarrow & \mathcal{D}^1_{P^\dagger}(\phi^*\mathcal{L}) & \longrightarrow & T_{P^\dagger} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \phi^*\mathcal{O}_X & \longrightarrow & \phi^*\mathcal{D}^1_{X^\dagger}(\mathcal{L}) & \longrightarrow & \phi^* T_X(log) & \longrightarrow & 0 \end{array}$$

implies that $\mathcal{D}^1_{P^\dagger}(\Theta_P) \to \phi^*\mathcal{D}^1_{X^\dagger}(\mathcal{L})$ is an isomorphism. Hence we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi_*\mathcal{O}_P & \longrightarrow & \phi_*\mathcal{D}^1_{P^\dagger}(\Theta_P) & \longrightarrow & \phi_*T_{P^\dagger} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{D}^1_{X^\dagger}(\mathcal{L}) & \longrightarrow & T_X(log) & \longrightarrow & 0
\end{array}$$

Any operator $D \in H^0(\mathcal{D}^1_{X^\dagger}(\mathcal{L}))$ with nonzero symbol will give an operator $D \in H^0(\mathcal{D}^1_{P^\dagger}(\Theta_P))$ with nonzero symbol $\sigma(D) \in H^0(T_{P^\dagger})$. It is easy to see that

$$\mathcal{D}^1_{P^\dagger}(\Theta_P) \hookrightarrow \mathcal{D}^1_P(\Theta_P) \hookrightarrow \mathcal{D}^1_P(\Theta_P) \otimes \mathcal{O}(D_1) = \mathcal{O}(D_1) \otimes \mathcal{D}^1_P(\rho^*\widetilde{\mathcal{L}}),$$

where $\mathcal{O}(D_1) = \mathcal{O}(1) \otimes \rho^*\mathcal{E}^{-1}_{x_1}$ and $\widetilde{\mathcal{L}} = \Theta_{J^d(\widetilde{C})} \otimes \mathcal{E}^{-1}_y \otimes \mathcal{E}_{x_1}$, which is algebraically equivalent to $\Theta_{J^d(\widetilde{C})}$. On the other hand, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_P & \longrightarrow & \mathcal{D}^1_P(\rho^*\widetilde{\mathcal{L}}) & \longrightarrow & T_P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \rho^*\mathcal{O}_{J^d(\widetilde{C})} & \longrightarrow & \rho^*\mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}}) & \longrightarrow & \rho^*T_{J^d(\widetilde{C})} & \longrightarrow & 0
\end{array}$$

and Consider $0 \to T_{P/J^d(\widetilde{C})}(log) \to T_{P^\dagger} \to \rho^*T_{J^d(\widetilde{C})}$, if the image of $\sigma(D)$ in $H^0(\rho^*T_{J^d(\widetilde{C})})$ is nonzero, then the connecting map of

$$0 \to \mathcal{O}(D_1) \otimes \rho^*\mathcal{O}_{J^d(\widetilde{C})} \to \mathcal{O}(D_1) \otimes \rho^*\mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}}) \to \mathcal{O}(D_1) \otimes \rho^*T_{J^d(\widetilde{C})} \to 0$$

is not injective, which is impossible since the space

$$\begin{aligned}
&H^0(\mathcal{O}(D_1) \otimes \rho^*\mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}})) \\
&= H^0(\mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}})) \oplus H^0(\mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}} \otimes \mathcal{E}_{x_2} \otimes \mathcal{E}^{-1}_{x_1}))
\end{aligned}$$

and the space

$$H^0(\mathcal{O}(D_1) \otimes \rho^*\mathcal{O}_{J^d(\widetilde{C})}) = H^0(\mathcal{O}_{J^d(\widetilde{C})}) \oplus H^0(\mathcal{O}_{J^d(\widetilde{C})} \otimes \mathcal{E}^{-1}_{x_1} \otimes \mathcal{E}_{x_2})$$

have the same dimension. In fact, when $\mathcal{E}_{x_2} \otimes \mathcal{E}^{-1}_{x_1} = \mathcal{O}_{J^d(\widetilde{C})}$, they are two dimensional spaces (see [We]), and if $\mathcal{E}_{x_2} \otimes \mathcal{E}^{-1}_{x_1} \neq \mathcal{O}_{J^d(\widetilde{C})}$, one has

$$H^0(\mathcal{E}_{x_2} \otimes \mathcal{E}^{-1}_{x_1}) = H^0(\mathcal{E}^{-1}_{x_1} \otimes \mathcal{E}_{x_2}) = 0$$

since $\mathcal{E}_{x_2} \otimes \mathcal{E}^{-1}_{x_1}$ is algebraically equivalent to zero, which implies that

$$H^0(\mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}}, \widetilde{\mathcal{L}} \otimes \mathcal{E}_{x_2} \otimes \mathcal{E}^{-1}_{x_1})) = 0.$$

To see it, we tensor the exact sequence

$$0 \to \mathcal{O}_{J^d(\widetilde{C})} \to \mathcal{D}^1_{J^d(\widetilde{C})}(\widetilde{\mathcal{L}}) \to T_{J^d(\widetilde{C})} \to 0$$

by $\mathcal{E}_{x_2} \otimes \mathcal{E}_{x_1}^{-1}$ (as left modules) and use the fact that

$$\mathcal{E}_{x_2} \otimes \mathcal{E}_{x_1}^{-1} \otimes T_{J^d(\widetilde{C})} = (\mathcal{E}_{x_2} \otimes \mathcal{E}_{x_1}^{-1})^{\oplus g(\widetilde{C})}.$$

Thus if $D \in H^0(\mathcal{D}^1_{X^\dagger}(\mathcal{L}))$ has nonzero symbol, then D gives an operator $D \in H^0(\mathcal{D}^1_{P^\dagger/J^d(\widetilde{C})}(\Theta_P))$, which induces an operator $D \in H^0(\mathcal{D}^1_{F^\dagger}(\widetilde{L}))$ with nonzero symbol for general fibre $F = \mathbb{P}^1$ of $\rho : P \to J^d(\widetilde{C})$ and the $\widetilde{L} = \mathcal{O}_{\mathbb{P}^1}(1)$. By using Lemma 3.3, we will show that it is impossible. In fact, for any fibre F, let $p_1 = F \cap D_1$, $p_2 = F \cap D_2$. Since $H^0(X, \mathcal{L})$ has dimension 1 and any nonzero section $s \in H^0(\mathcal{L})$ does not vanish on W, we can find a fibre F such that $s(p_1) \neq 0$, $s(p_2) \neq 0$. Then we find a $c := s(p_2)/s(p_1)$ and a nonzero $s|_F \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ satisfying that $s(p_2) = c \cdot s(p_1)$. Since D has to induce a morphism $\mathcal{L} \to \mathcal{L}$ of abelian group sheaves, $D(s) \in H^0(\mathcal{L}) = \mathbb{C} \cdot s$ has to satisfy $D(s)(p_2) = c \cdot D(s)(p_1)$, which means that D has zero symbol by Lemma 3.3.

To see that $H^0(T_X(log)) = H^1(\mathcal{O}_X) = \mathbb{C}^g$, we remark that both spaces have at least dimension g, then we only need to check that $dim\, H^1(\mathcal{O}_X) \leq g$. This is easy to see by using

$$0 \to \mathcal{O}_X \to \phi_*\mathcal{O}_P \to \mathcal{O}_W \to 0$$

and $H^1(\phi_*\mathcal{O}_P) = H^1(\mathcal{O}_P) = H^1(\rho_*\mathcal{O}_P) = H^1(\mathcal{O}_{J^d(\widetilde{C})}) = \mathbb{C}^{g-1}$.

Lemma 3.4. *For any logarithmic structure on $X = J^d(C)$ in the sense of [KN], we have $H^0(S^2T_X(log)) = S^2H^0(T_X(log))$.*

Proof. It is enough to show that

$$h^0(S^2T_X(log)) := dim\, H^0(S^2T_X(log)) \leq dim\, S^2H^0(T_X(log)) = \frac{g(g+1)}{2}.$$

To prove it, let $\mathcal{F} := \phi^*T_X(log) = T_P(log(D_1 + D_2))$, $\mathcal{F}' := T_{P/J^d(\widetilde{C})}(-D_1 - D_2)$, $\mathcal{F}" := \rho^*T_{J^d(\widetilde{C})}$ and use the exact squence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}" \to 0,$$

one has $h^0(S^2T_X(log)) \leq h^0(S^2\mathcal{F})$ and the following two exact sequences

$$0 \to \mathcal{G} \to S^2\mathcal{F} \to S^2(\mathcal{F}") \to 0,$$

$$0 \to S^2(\mathcal{F}') \to \mathcal{G} \to \mathcal{F}' \otimes \mathcal{F}" \to 0.$$

Thus, by using $h^0(S^2(\mathcal{F}")) = h^0(S^2T_{J^d(\widetilde{C})}) = \frac{g(g-1)}{2}$, we have

$$h^0(S^2T_X(log)) \leq \frac{g(g-1)}{2} + h^0(S^2\mathcal{F}') + h^0(T_{J^d(\widetilde{C})} \otimes \rho_*\mathcal{F}').$$

To compute $\mathcal{F}'$, noting that $\mathcal{O}_P(D_i) = \mathcal{O}(1) \otimes \rho^*\mathcal{E}_{x_i}^{-1}$ and using the exact sequence

$$0 \to \mathcal{O}_P \to \mathcal{O}(1) \otimes \rho^*(\mathcal{E}_{x_1}^{-1} \oplus \mathcal{E}_{x_2}^{-1}) \to T_{P/J^d(\widetilde{C})} \to,$$

we get $\mathcal{F}' = T_{P/J^d(\widetilde{C})}(-D_1 - D_2) = \mathcal{O}_P$ and hence

$$h^0(S^2T_X(log)) \leq \frac{g(g-1)}{2} + h^0(T_{J^d(\widetilde{C})}) + 1 = \frac{g(g+1)}{2}.$$

Proposition 3.2. *Let $\mathcal{C}/S$ be a flat family of proper curves satisfing the assumpations of Proposition 1.3 and such that $\mathcal{C}_s$ ($s \in \Delta \subset S$) are irreducible curves of one node. Let $f : J(\mathcal{C}) \to S$ be the associated family of moduli spaces of torsion free sheaves of rank* 1 *and degree* 0*, and $\mathcal{L}$ be the relative theta line bundle on $J(\mathcal{C})/S$. Then for any integer $k > 0$ and $s \in \Delta$*

(1) $H^0(T_{J(\mathcal{C}_s)}(log)) \xrightarrow{\cup c_1(\mathcal{L}_s^k)} H^1(\mathcal{O}_{J(\mathcal{C}_s)})$ *is an isomorphism.*
(2) $\mu_{\mathcal{O}_{J(\mathcal{C}_s)}} = 0$ *and* $H^0(\mathcal{D}^2_{J(\mathcal{C}_s)^\dagger}(\mathcal{L}_s^k)) \cong \mathbb{C}$.

Proof. From the discussions in Section 1, the log structure on $J(\mathcal{C}_s)$ induced by $logf^{-1}(\Delta)$ is a logarithmic structure in the sense of [KN], thus we can use our Proposition 3.1 and Lemma 3.4, the (1) is a corollary of Proposition 3.1.

The claim $\mu_{\mathcal{O}_{J(\mathcal{C}_s)}} = 0$ is equivalent to that

$$h^0(\mathcal{D}^2_{J(\mathcal{C}_s)^\dagger}(\mathcal{O}_{J(\mathcal{C}_s)})) = h^0(\mathcal{D}^1_{J(\mathcal{C}_s)^\dagger}(\mathcal{O}_{J(\mathcal{C}_s)})) + h^0(S^2T_{J(\mathcal{C}_s)}(log)),$$

which is true for $s \in S \smallsetminus \Delta$ (see [We]). Therefore, by using the semicontinuity and Lemma 3.4, it is true for all $s \in S$ if we remark that $h^0(\mathcal{D}^1_{J(\mathcal{C}_s)^\dagger}(\mathcal{O}_{J(\mathcal{C}_s)}))$ is constant for all $s \in S$ since the canonical exact sequence

$$0 \to \mathcal{O}_{J(\mathcal{C}_s)} \to \mathcal{D}^1_{J(\mathcal{C}_s)^\dagger}(\mathcal{O}_{J(\mathcal{C}_s)}) \to T_{J(\mathcal{C}_s)}(log) \to 0$$

is splitting by Proposition 2.2 and $c_1(\mathcal{O}_{J(\mathcal{C}_s)}) = 0$. By using again Proposition 2.2 and the above (1), we know that $\mu_{\mathcal{L}_s^k} = -\cup c_1(\mathcal{L}_s^k)$ is injective. Hence

$$H^0(\mathcal{D}^2_{J(\mathcal{C}_s)^\dagger}(\mathcal{L}_s^k)) = H^0(\mathcal{D}^1_{J(\mathcal{C}_s)^\dagger}(\mathcal{L}_s^k)) \cong \mathbb{C}.$$

Theorem 3.1. *Let $f : J(\mathcal{C}) \to S$ be the family of generalized Jacobians in Proposition 3.2, $Y = f^{-1}(\Delta)$ and $\mathcal{L}$ be the relative theta line bundles. Then there exists a symbol*

$$\rho : T_S(log\Delta) \to f_*S^2T_{J(\mathcal{C})/S}(logY)$$

such that the following two conditions hold

(1) $\mu_{\mathcal{L}^k} \cdot \rho + \kappa_{J(\mathcal{C})/S} = 0$,
(2) $f_*T_{J(\mathcal{C})/S}(logY) \xrightarrow{\cup c_1(\mathcal{L}^k)} R^1f_*\mathcal{O}_{J(\mathcal{C})}$ *is an isomorphism.*

In particular, there exists a unique projective logarithmic heat operator

$$\widetilde{H} : T_S(log\Delta) \to f_*\mathcal{W}_{J(\mathcal{C})/S}(\mathcal{L}^k)/\mathcal{O}_S$$

such that $\rho_{\widetilde{H}} = \rho$, and thus there exists a projective logarithmic connection on $f_*\mathcal{L}^k$.

Proof. It is clear that we only need to check (1) since (2) has been shown in Proposition 3.2, namely, we need to find a solution of $\mu_{\mathcal{L}^k} \cdot \rho + \kappa_{J(\mathcal{C})/S} = 0$. By (2), we have the isomorphism

$$f_*T_{J(\mathcal{C})/S}(logY) \otimes f_*T_{J(\mathcal{C})/S}(logY) \xrightarrow{\cup c_1(\mathcal{L}^k)} R^1f_*T_{J(\mathcal{C})/S}(logY).$$

Let $\rho = (\cup c_1(\mathcal{L}^k))^{-1} \circ \kappa_{J(\mathcal{C})/S} : T_S(log\Delta) \to f_*T_{J(\mathcal{C})/S}(logY) \otimes f_*T_{J(\mathcal{C})/S}(logY)$, which, over the open set $S \smallsetminus \Delta$, is a map into $f_*S^2T_{J(\mathcal{C})/S}(logY)$ (see §2.3.8 of [GJ] or [We]), thus it is a map

$$\rho = (\cup c_1(\mathcal{L}^k))^{-1} \circ \kappa_{J(\mathcal{C})/S} : T_S(log\Delta) \to f_*S^2T_{J(\mathcal{C})/S}(logY).$$

By Proposition 2.2, $\mu_{\mathcal{L}^k)} = -\cup c_1(\mathcal{L}^k)$ and ρ is a solution of $\mu_{\mathcal{L}^k} \cdot \rho + \kappa_{J(\mathcal{C})/S} = 0$.

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