A POISSON STRUCTURE ON COMPACT SYMMETRIC SPACES

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Abstract. We present some basic results on a natural Poisson structure on any compact symmetric space. The symplectic leaves of this structure are related to the orbits of the corresponding real semisimple group on the complex flag manifold.

1. Introduction and the Poisson structure π_0 on U/K_0.

Let $\mathfrak{g}_0$ be a real semi-simple Lie algebra, and let $\mathfrak{g}$ be its complexification. Fix a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$, and let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}$ given by $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$. Let G be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, and let G_0, K_0, and U be the connected subgroups of G with Lie algebras $\mathfrak{g}_0, \mathfrak{k}_0$, and $\mathfrak{u}$ respectively. Then $K_0 = G_0 \cap U$, and U/K_0 is the compact dual of the non-compact Riemannian symmetric space G_0/K_0. In this paper, we will define a Poisson structure π_0 on U/K_0 and study some of its properties.

The definition of π_0 depends on a choice of an *Iwasawa–Borel* subalgebra of $\mathfrak{g}$ relative to $\mathfrak{g}_0$. Recall [5] that a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is said to be Iwasawa relative to $\mathfrak{g}_0$ if $\mathfrak{b} \supset \mathfrak{a}_0 + \mathfrak{n}_0$ for some Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ of $\mathfrak{g}_0$. Let Y be the variety of all Borel subalgebras of $\mathfrak{g}$. Then G acts transitively on Y by conjugations, and $\mathfrak{b} \in Y$ is Iwasawa relative to $\mathfrak{g}_0$ if and only if it lies in the unique closed orbit of G_0 on Y [5]. Denote by τ and θ the complex conjugations on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}$ respectively. Throughout this paper, we will fix an Iwasawa–Borel subalgebra $\mathfrak{b}$ relative to $\mathfrak{g}_0$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ of $\mathfrak{g}$ that is stable under both τ and θ. Let Δ^+ be the set of roots for $\mathfrak{h}$ determined by $\mathfrak{b}$, and let $\mathfrak{n}$ be the complex span of root vectors for roots in Δ^+, so that $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Let $\mathfrak{a} = \{x \in \mathfrak{h} : \theta(x) = -x\}$. Let $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0$ and $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ is an Iwasawa decomposition of $\mathfrak{g}_0$.

We can define a Poisson structure π_0 on U/K_0 as follows: let $\ll\, , \gg$ be the Killing form of $\mathfrak{g}$. For each $\alpha \in \Delta^+$, choose a root vector E_α such that $\ll E_\alpha, \theta(E_\alpha) \gg = -1$. Let $E_{-\alpha} = -\theta(E_\alpha)$, and let $X_\alpha = E_\alpha - E_{-\alpha}$ and $Y_\alpha = i(E_\alpha + E_{-\alpha})$. Then $X_\alpha, Y_\alpha \in \mathfrak{u}$ for each $\alpha \in \Delta^+$. Set

$$\Lambda = \frac{1}{4} \sum_{\alpha \in \Delta^+} X_\alpha \wedge Y_\alpha \in \mathfrak{u} \wedge \mathfrak{u},$$

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and define the bi-vector field π_U on U by

$$\pi_U = \Lambda^r - \Lambda^l,$$

where Λ^r and Λ^l are respectively the right and left invariant bi-vector fields on U with value Λ at the identity element. Then π_U is a Poisson bivector field, and (U, π_U) is the Poisson-Lie group defined by the Manin triple $(\mathfrak{g}, \mathfrak{u}, \mathfrak{a} + \mathfrak{n})$ [12].

The group G acts on U from the right via $u^g = u_1$, if $ug = bu_1$ for some $b \in AN$, where $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$. Therefore every subgroup of G, for example AN or G_0, also acts on U. The symplectic leaves of π_U are precisely the orbits of the right AN-action. These leaves are parameterized by the torus $T = \exp(i\mathfrak{a})$ and the Weyl group W of $(U, \mathfrak{h})$. The Poisson structure π_U is both left and right T-invariant, and it descends to the so-called Bruhat Poisson structure on $T\backslash U$, whose symplectic leaves are precisely the Bruhat cells of $T\backslash U \cong B\backslash G$ as the orbits of the Borel group $B = TAN$. We refer to [12] and [15] for details.

Proposition 1.1. *There exists a Poisson structure π_0 on U/K_0 such that the natural projection $p : (U, \pi_U) \to (U/K_0, \pi_0)$ is a Poisson map. The symplectic leaves of the Poisson structure π_0 are precisely the projections of the G_0-orbits on U via the map p.*

Proof. To show that the Poisson structure π_U descends to the quotient U/K_0, it is enough to show that the annihilator space $\mathfrak{k}_0^\perp$ of $\mathfrak{k}_0$ inside $\mathfrak{u}^*$, which is identified with $\mathfrak{a}+\mathfrak{n}$, is a Lie subalgebra of $\mathfrak{a}+\mathfrak{n}$. The bilinear form which is used in this identification is the imaginary part of the Killing form $\ll\,,\gg$ of $\mathfrak{g}$. We observe that being a real form of $\mathfrak{g}$, $\mathfrak{g}_0$ is isotropic with respect to $\mathrm{Im} \ll\,,\gg$, which implies that $\mathfrak{k}_0^\perp \subset \mathfrak{a}_0 + \mathfrak{n}_0$. It then follows for dimension reason that $\mathfrak{k}_0^\perp = \mathfrak{a}_0 + \mathfrak{n}_0$, which is a Lie subalgebra of $\mathfrak{a}+\mathfrak{n}$.

For the statement concerning the symplectic leaves of π_0, we observe that (X, π_0) is a (U, π_U)-Poisson homogeneous space whose Drinfeld Lagrangian subalgebra at the base point $eK_0 \in U/K_0$ is $\mathfrak{g}_0$, and then apply [11, Theorem 7.2].

Q.E.D.

Remark 1.2. For the case when the Satake diagram of $\mathfrak{g}_0$ has no black dots, the Poisson structure π_0 was considered by Fernandes in [4].

In this paper, we will study some properties of the symplectic leaves of π_0. Recall that Y is the variety of all Borel subalgebras of $\mathfrak{g}$. We will show that the set of symplectic leaves of π_0 is essentially parameterized by the set of G_0-orbits in Y, which have been studied extensively because of their importance in the representation theory of G_0. More precisely, let $q : U \to Y$ be surjective map $u \mapsto \mathrm{Ad}_u^{-1}\mathfrak{b} \in Y$. Then the map $\mathcal{O} \mapsto p(q^{-1}(\mathcal{O}))$ gives a bijective correspondence between the set of G_0-orbits in Y and the set of T-orbits of symplectic leaves in U/K_0. In particular, there are finitely many families of symplectic leaves. In each family leaves are translates of one another by elements in T. Moreover, π_0 has open symplectic leaves if and only if $\mathfrak{g}_0$ has a compact Cartan subalgebra, in which case, the number of open symplectic leaves is the same as the number of open G_0-orbits in Y, and each open symplectic leaf is diffeomorphic to G_0/K_0. When X is Hermitian symmetric, the Poisson structure π_0 is shown to be the sum of the Bruhat Poisson structure [12], [15] and a multiple of any non-degenerate invariant Poisson structure.

We also show that the U-invariant Poisson cohomology $H^{\bullet}_{\pi_0,U}(U/K_0)$ is isomorphic to the De Rham cohomology of U/K_0. The full Poisson cohomology and some further properties of π_0 will be studied in a future paper.

Throughout the paper, if Z is a set and if σ is an involution on Z, we will use Z^σ to denote the fixed point set of σ in Z.

2. Symplectic leaves of π_0 and G_0-orbits in Y.

By Proposition 1.1, symplectic leaves of π_0 are precisely the projections to U/K_0 of G_0-orbits in U. Here, recall that G_0 acts on U as a subgroup of G, and G acts on U from the right by

$$u^g = u_1, \quad \text{if} \quad ug = bu_1 \ \text{ for } \ b \in AN, \tag{2.1}$$

where $u \in U$ and $g \in G$. It is easy to see that the above right action of G on U descends to an action of G on $T\backslash U$. On the other hand, the map $U \to Y : u \mapsto \mathrm{Ad}_u^{-1}\mathfrak{b}$ gives a G-equivariant identification of Y with $T\backslash U$. This identification will be used throughout the paper. The G_0-orbits on Y have been studied extensively (see, for example, [13] and [17]). In particular, there are finitely many G_0-orbits in Y. We will now formulate a precise connection between symplectic leaves of π_0 and G_0-orbits in Y.

Let $X = U/K_0$. For $x \in X$, let L_x be the symplectic leaf of π_0 through x. Since T acts by Poisson diffeomorphisms, for each $t \in T$, the set $tL_x = \{tx_1 : x_1 \in L_x\}$ is again a symplectic leaf of π_0. Let

$$\mathcal{S}_x = \bigcup_{t \in T} tL_x \subset X.$$

For $y \in Y$, let $\mathcal{O}_y$ be the G_0-orbit in Y through y. Let $p : U \to X = U/K_0$ and $q : U \to Y = T\backslash U$ be the natural projections.

Proposition 2.1. *Let $x \in X$ and $y \in Y$ be such that $p^{-1}(x) \cap q^{-1}(y) \neq \emptyset$. Then*

$$p(q^{-1}(\mathcal{O}_y)) = \mathcal{S}_x, \quad \text{and} \quad q(p^{-1}(\mathcal{S}_x)) = \mathcal{O}_y.$$

Proof. Let $u \in p^{-1}(x) \cap q^{-1}(y)$, and let u^{G_0} be the G_0-orbit in U through u. It is easy to show that

$$q^{-1}(\mathcal{O}_y) = p^{-1}(\mathcal{S}_x) = \bigcup_{t \in T} t(u^{G_0}).$$

Thus,

$$p(q^{-1}(\mathcal{O}_y)) = \bigcup_{t \in T} tp(u^{G_0}) = \mathcal{S}_x,$$

and

$$q(p^{-1}(\mathcal{S}_x)) = q(u^{G_0}) = \mathcal{O}_y.$$

Q.E.D.

Corollary 2.2. *Let $\mathcal{O}_Y$ be the collection of G_0-orbits in Y, and let $\mathcal{S}_X$ be the collection of all the subsets $\mathcal{S}_x, x \in X$. Then the map*

$$\mathcal{O}_Y \longrightarrow \mathcal{S}_X : \quad \mathcal{O} \longmapsto p(q^{-1}(\mathcal{O}))$$

is a bijection with the inverse given by $\mathcal{S} \mapsto q(p^{-1}(\mathcal{S}))$.

We now recall some facts about G_0-orbits in Y from [14] which we will use to compute the dimensions of symplectic leaves of π_0. Since [14] is based on the choice of a Borel subalgebra in an open G_0-orbit in Y, we will restate the relevant results from [14] in Proposition 2.3 to fit our set-up.

Let $\mathfrak{t} = i\mathfrak{a}$ be the Lie algebra of T, and let $N_U(\mathfrak{t})$ be the normalizer subgroup of $\mathfrak{t}$ in U. Set

$$\mathcal{V} = \{u \in U : \, u\tau(u)^{-1} \in N_U(\mathfrak{t})\}.$$

Then $u \in \mathcal{V}$ if and only if $\mathrm{Ad}_u^{-1}\mathfrak{h}$ is τ-stable. Clearly $\mathcal{V}$ is invariant under the left translations by elements in T and the right translations by elements in K_0. Set

$$V = T\backslash\mathcal{V}/K_0.$$

Then we have a well-defined map

$$V \longrightarrow \mathcal{O}_Y : \; v \longmapsto \mathcal{O}(v),$$

where for $v = TuK_0 \in V$, $\mathcal{O}(v)$ is the G_0-orbit in Y through the point $\mathrm{Ad}_u^{-1}\mathfrak{b} \in Y$. Let $W = N_U(\mathfrak{t})/T$ be the Weyl group. Then we also have the well-defined map

$$\psi : \; V \longrightarrow W : \; v = TuK_0 \longmapsto u\tau(u)^{-1}T \in W.$$

For $w \in W$, let $l(w)$ be the length of w.

Proposition 2.3. *1) The map $v \mapsto \mathcal{O}(v)$ is a bijection between the set V and the set $\mathcal{O}_Y$ of all G_0-orbits in Y;*

2) For $v \in V$, the co-dimension of $\mathcal{O}(v)$ in Y is equal to $l(\psi(v)w_b w_0)$, where w_0 is the longest element of W, and w_b is the longest element of the subgroup of W generated by the black dots of the Satake diagram of $\mathfrak{g}_0$.

Remarks 2.4. 1) Since τ leaves $\mathfrak{a}$ invariant, it acts on the set of roots for $\mathfrak{h}$ by $(\tau\alpha)(x) = \alpha(\tau(x))$ for $x \in \mathfrak{a}$. We know from [1] that the black dots in the Satake diagram of $\mathfrak{g}_0$ correspond precisely to the simple roots α in Δ^+ such that $\tau(\alpha) = -\alpha$. Moreover, if $\alpha \in \Delta^+$ and if $\tau(\alpha) \neq -\alpha$, then $\tau(\alpha) \in \Delta^+$;

2) We now point out how Proposition 2.3 follows from results in [14]. Let $u_0 \in U$ be such that $\mathfrak{b}' := \mathrm{Ad}_{u_0}\mathfrak{b}$ lies in an open G_0-orbit in Y and $\mathfrak{h}' := \mathrm{Ad}_{u_0}\mathfrak{h}$ is τ-stable. The

pair $(\mathfrak{g}_0, \mathfrak{b}')$ is called a *standard pair* in the terminology of [14, No.1.2]. Let $\mathfrak{t}' = \mathrm{Ad}_{u_0}\mathfrak{t}$, $T' = u_0 T u_0^{-1}$, and $N_U(\mathfrak{t}') = u_0 N_U(\mathfrak{t}) u_0^{-1}$. Let

$$\mathcal{V}' = \{u' \in U : \ u'\tau(u')^{-1} \in N_U(\mathfrak{t}')\},$$

and let $V' = T'\backslash\mathcal{V}'/K_0$. For $v' = T'u'K_0$, let $\mathcal{O}(v')$ be the G_0-orbit in Y through the point $\mathrm{Ad}_{u'}^{-1}\mathfrak{b}' \in Y$. Then [14, Theorem 6.1.4(3)] says that the map $V' \to \mathcal{O}_Y : v' \to \mathcal{O}(v')$ is a bijection between the set V' and the set $\mathcal{O}_Y$ of G_0-orbits in Y, and [14, Theorem 6.4.2] says that the co-dimension of $\mathcal{O}(v')$ in Y is the length of the element $\phi(v')$ in the Weyl group $W' = N_U(\mathfrak{t}')/T'$ defined by $u'\tau(u')^{-1} \in N_U(\mathfrak{t}')$. Since $\mathfrak{b} = \mathrm{Ad}_{u_0}^{-1}\mathfrak{b}'$ lies in the unique closed G_0-orbit in Y, it follows from [14, No. 1.6] that $u_0\tau(u_0)^{-1} \in N_U(\mathfrak{t}')$ defines the element in W' that corresponds to $w_b w_0 \in W$ under the natural identification of W and W'. It is also easy to see that $\mathcal{V}' = u_0\mathcal{V}$, and if $v' = T'u'K_0 \in V'$ and $v = T(u_0^{-1}u')K_0 \in V$ for $u' \in \mathcal{V}'$, then $\mathcal{O}(v') = \mathcal{O}(v)$, and $\phi(v') \in W'$ corresponds to $\psi(v)w_b w_0 \in W$ under the natural identification of W and W'. It is now clear that Proposition 2.3 holds. Statement 2) of Proposition 2.3 can also be seen directly from Lemma 3.2 below;

3) Starting from a complete collection of representatives of equivalence classes of strongly orthogonal real roots for the Cartan subalgebra $\mathfrak{h}^\tau$ of $\mathfrak{g}_0$, it is possible, by using Cayley transforms, to explicitly construct a set of representatives of V in $\mathcal{V}$. This is done in [13, Theorem 3].

4) The three involutions τ, w_0 and w_b on $\Delta = \Delta^+ \cup (-\Delta^+)$ commute with each other. Indeed, since τ commutes with the reflection defined by every black dot on the Satake diagram, τ commutes with w_b. We know from Remark (2.4) that $\tau w_b(\Delta^+) = \Delta^+$, so τw_b defines an automorphism of the Dynkin diagram of $\mathfrak{g}$. It is well-known that $-w_0$ is in the center of the group of all automorphisms of the Dynkin diagram of $\mathfrak{g}$ (this can be checked, for example, case by case). Thus w_0 commutes with τw_b. To see that w_0 commutes with w_b, note by directly checking case by case that $-w_0$ maps a simple black root on the Satake diagram of $\mathfrak{g}_0$ to another such simple black root. Thus $w_0 w_b w_0$ is still in the subgroup W_b of W generated by the set of all black simple roots. It follows that $w_0 w_b$ and $w_b w_0 = w_0(w_0 w_b w_0)$ are in the same right W_b coset in W. Since $l(w_0 w_b) = l(w_b w_0) = l(w_0) - l(w_b)$, we know that $w_0 w_b = w_b w_0$ by the uniqueness of minimal length representatives of right W_b cosets in W. Thus w_0 commutes with both τ and w_b. These remarks will be used in the proof of Lemma 3.2.

3. Symplectic leaves of π_0.

Recall that $p : U \to U/K_0$ and $q : U \to Y = T\backslash U$ are the natural projections. For each $v \in V = T\backslash\mathcal{V}/K_0$, set

$$\mathcal{S}(v) = p(q^{-1}(\mathcal{O}(v))) \subset U/K_0.$$

By Corollary 2.2, we have a disjoint union

$$U/K_0 = \bigcup_{v \in V} \mathcal{S}(v).$$

Moreover, each $\mathcal{S}(v)$ is a union of symplectic leaves of π_0, all of which are translates of each other by elements in T. Thus it is enough to understand one single leaf in $\mathcal{S}(v)$. Recall that G acts on U from the right by $(u, g) \mapsto u^g$ as described in (2.1).

Lemma 3.1. *For every $u \in U$, the map*

$$(G_0 \cap u^{-1}(AN)u)\backslash G_0/K_0 \longrightarrow U/K_0 : \ (G_0 \cap u^{-1}(AN)u)g_0K_0 \longmapsto u^{g_0}K_0, \qquad g_0 \in G_0,$$

gives a diffeomorphism between the double coset space $(G_0 \cap u^{-1}(AN)u)\backslash G_0/K_0$ and the symplectic leaf of π_0 through the point $uK_0 \in U/K_0$.

Proof. Lemma 3.1 follows immediately from a theorem of Karolinsky [6] or Corollary 7.3 of [11]. To see it directly, consider the G_0-action on U as a subgroup of G. By (2.1), the induced action of K_0 on U is by left translations. It is easy to see that the stabilizer subgroup of G_0 at u is $G_0 \cap u^{-1}(AN)u$. Let u^{G_0} be the G_0-orbit in U through u. Then

$$u^{G_0} \cong (G_0 \cap u^{-1}(AN)u)\backslash G_0.$$

Since the action of K_0 on u^{G_0} by left translations is free, we see that the double coset space $(G_0 \cap u^{-1}(AN)u)\backslash G_0/K_0$ is smooth. Lemma 3.1 now follows from Proposition 1.1.

Q.E.D.

Assume now that $u \in \mathcal{V}$. To better understand the group $G_0 \cap u^{-1}(AN)u$, we introduce the involution τ_u on $\mathfrak{g}$:

$$\tau_u = \mathrm{Ad}_u \tau \mathrm{Ad}_u^{-1} = \mathrm{Ad}_{u\tau(u^{-1})}\tau : \ \mathfrak{g} \longrightarrow \mathfrak{g}.$$

The fixed point set of τ_u in $\mathfrak{g}$ is the real form $\mathrm{Ad}_u \mathfrak{g}_0$ of $\mathfrak{g}$. We will use the same letter for the lifting of τ_u to G. Since τ_u leaves $\mathfrak{a}$ invariant, it acts on the set of roots for $\mathfrak{h}$ by $(\tau_u \alpha)(x) = \alpha(\tau_u(x))$ for $x \in \mathfrak{a}$. Recall that associated to $v = TuK_0 \in V$ we have the Weyl group element $\psi(v)w_{\mathfrak{b}}w_0$. Let

$$N_v = N \cap (\dot{w}N^- \dot{w}^{-1}),$$

where $\dot{w} \in U$ is any representative of $\psi(v)w_{\mathfrak{b}}w_0 \in W$.

Lemma 3.2. *For any $u \in \mathcal{V}$ and $v = TuK_0 \in V$,*

1) $\Delta^+ \cap \tau_u(\Delta^+) = \Delta^+ \cap (\psi(v)w_{\mathfrak{b}}w_0)(-\Delta^+)$;

2) N_v is τ_u-invariant and $G_0 \cap u^{-1}Nu = u^{-1}(N_v)^{\tau_u}u = (u^{-1}N_v u)^\tau$ is connected;

3) the map

$$M : \ (G_0 \cap u^{-1}Tu) \times (G_0 \cap u^{-1}Au) \times (G_0 \cap u^{-1}Nu) \longrightarrow G_0 \cap u^{-1}(TAN)u \tag{3.1}$$

given by $M(g_1, g_2, g_3) = g_1g_2g_3$ is a diffeomorphism.

Proof. 1) Recall that $\psi(v) \in W$ is the element defined by $u\tau(u)^{-1} \in N_U(\mathfrak{t})$. Then $\tau_u(\alpha) = \psi(v)\tau(\alpha)$ for every $\alpha \in \Delta$. Thus $\tau_u(\alpha) \in \Delta^+$ if and only if $\psi(v)\tau(\alpha) \in \Delta^+$, which is in turn equivalent to $w_0\tau w_{\mathfrak{b}}\psi(v)\tau(\alpha) \in -\Delta^+$ because $w_0\tau w_{\mathfrak{b}}(\Delta^+) = -\Delta^+$. Since the three involutions w_0, τ and w_b commute with each other by Remark 2.4, we have $w_0\tau w_b\psi(v)\tau = (\psi(v)w_bw_0)^{-1}$. This proves 1).

2) We know from 1) that $\Delta^+ \cap (\psi(v)w_bw_0)(-\Delta^+)$ is τ_u-invariant. Thus N_v is τ_u-invariant. Clearly $u^{-1}(N_v)^{\tau_u}u \subset G_0 \cap u^{-1}Nu$. Let $N_v' = N \cap \dot{w}N\dot{w}^{-1}$. Then $N = N_vN_v'$ is a direct product, and we know from 1) that $\tau_u(N_v') \subset N^-$. Suppose now that $n \in N$ is such that $u^{-1}nu \in G_0 \cap u^{-1}Nu$. Write $n = mm'$ with $m \in N_v$ and $m' \in N_v'$. Then from $\tau_u(n) = n$ we get $\tau_u(m') = \tau_u(m^{-1})n \in N^- \cap N = \{e\}$. Thus $m' = e$, and $n = m \in (N_v)^{\tau_u}$. Since the exponential map for the group $u^{-1}(AN)u$ is a diffeomorphism, $(u^{-1}(AN)u)^\tau$ is the connected subgroup of $u^{-1}(AN)u$ with Lie algebra $(\mathrm{Ad}_u^{-1}(\mathfrak{a} + \mathfrak{n}))^\tau$. This shows 2).

We now prove 3). Since $\mathrm{Ad}_u^{-1}\mathfrak{h}$ is τ-invariant, the Lie algebra $\mathfrak{g}_0 \cap \mathrm{Ad}_u^{-1}\mathfrak{b}$ of $G_0 \cap u^{-1}(TAN)u$ is the direct sum of the Lie algebras of the three subgroups on the left hand side of (3.1). Thus the map M is a local diffeomorphism. It is also easy to see that M is one-to-one. Thus it remains to show that M is onto. Suppose that $h \in TA$ and $n \in N$ are such that $u^{-1}(hn)u \in G_0$. Then $\tau_u(hn) = hn$. Write $n = mm'$ with $m \in N_v$ and $m' \in N_v'$. Then from $\tau_u(hn) = hn$ we get $\tau_u(m') = \tau_u(m^{-1})\tau_u(h^{-1})hn \in N^- \cap HN = \{e\}$. Thus $m' = e$, and $\tau_u(h) = h$ and $n = m \in (N_v)^{\tau_u}$. If $h = ta$ with $t \in T$ and $a \in A$, it is also easy to see that $\tau(h) = h$ implies that $\tau_u(t) = t$ and $\tau_u(a) = a$.

Q.E.D.

In particular, we see that $G_0 \cap u^{-1}(AN)u$ is a contractible subgroup of G_0. Since Lemma 3.1 states that the symplectic leaf of π_0 through the point uK_0 is diffeomorphic to $(G_0 \cap u^{-1}(AN)u)\backslash G_0/K_0$, we see that this leaf is the base space of a smooth fibration with contractible total space and fiber. Thus we have:

Proposition 3.3. *Each symplectic leaf of the Poisson structure π_0 is contractible.*

Remark 3.4. Since $\dim(Y) = \dim((G_0 \cap u^{-1}(TA)u)\backslash G_0)$, it is also clear from 3) of Lemma 3.2 that the codimension of $\mathcal{O}(v)$ in Y is $l(\psi(v)w_bw_0)$. See Proposition 2.3.

It is a basic fact [17] that associated to each G_0-orbit in Y there is a unique G_0-conjugacy class of τ-stable Cartan subalgebras of $\mathfrak{g}$. For $u \in \mathcal{V}$ and $v = TuK_0 \in V$, the G_0-conjugacy class of τ-stable Cartan subalgebras of $\mathfrak{g}$ associated to $\mathcal{O}(v)$ is that defined by $\mathrm{Ad}_u^{-1}\mathfrak{h}$. The intersection $(\mathrm{Ad}_u^{-1}\mathfrak{h}) \cap \mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Regard both τ and $\psi(v)$ as maps on $\mathfrak{h}$ so that $\psi(v)\tau = \tau_u|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$. Then we have

$$(\mathrm{Ad}_u^{-1}\mathfrak{h}) \cap \mathfrak{g}_0 = (\mathrm{Ad}_u^{-1}\mathfrak{h})^\tau = \mathrm{Ad}_u^{-1}(\mathfrak{h}^{\psi(v)\tau}).$$

Since $\psi(v)\tau$ commutes with θ, it leaves both $\mathfrak{t} = \mathfrak{h}^\theta$ and $\mathfrak{a} = \mathfrak{h}^{-\theta}$ invariant, and we have

$$(\mathrm{Ad}_u^{-1}\mathfrak{h}) \cap \mathfrak{g}_0 = \mathrm{Ad}_u^{-1}(\mathfrak{t}^{\psi(v)\tau} + \mathfrak{a}^{\psi(v)\tau}).$$

The subspaces $\mathrm{Ad}_u^{-1}(\mathfrak{t}^{\psi(v)\tau})$ and $\mathrm{Ad}_u^{-1}(\mathfrak{a}^{\psi(v)\tau})$ are respectively the toral and vector parts of the Cartan subalgebra $(\mathrm{Ad}_u^{-1}\mathfrak{h}) \cap \mathfrak{g}_0$ of $\mathfrak{g}_0$. Set

$$t(v) = \dim(\mathfrak{t}^{\psi(v)\tau}) = \dim(\mathrm{Ad}_u^{-1}(\mathfrak{t}^{\psi(v)\tau})) = \dim(G_0 \cap u^{-1}Tu) \tag{3.2}$$

$$a(v) = \dim(\mathfrak{t}^{\psi(v)\tau}) = \dim(\mathrm{Ad}_u^{-1}(\mathfrak{a}^{\psi(v)\tau})) = \dim(G_0 \cap u^{-1}Au). \tag{3.3}$$

Theorem 3.5. *For every $v \in V$,*

1) every symplectic leaf L in $\mathcal{S}(v)$ has dimension

$$\dim L = \dim(\mathcal{O}(v)) - \dim(K_0) + t(v),$$

so the co-dimension of L in U/K_0 is $a(v) + l(\psi(v)w_\flat w_0)$;

2) the family of symplectic leaves in $\mathcal{S}(v)$ is parameterized by the quotient torus $T/T^{\psi(v)\tau}$.

Proof. Let u be a representative of v in $\mathcal{V} \subset U$. Let $x = uK_0 \in U/K_0$, and let L_x be the symplectic leaf of π_0 through x. We only need to compute the dimension of L_x. Let u^{G_0} be the G_0-orbit in U through u. We know from Lemma 3.1 that $u^{G_0} \cong (G_0 \cap u^{-1}(AN)u)\backslash G_0$, and that u^{G_0} fibers over L_x with fiber K_0. Thus $\dim L_x = \dim u^{G_0} - \dim K_0$. On the other hand, since

$$\mathcal{O}(v) \cong (G_0 \cap u^{-1}(TAN)u)\backslash G_0,$$

we know that u^{G_0} fibers over $\mathcal{O}(v)$ with fiber $(G_0 \cap u^{-1}(TAN)u)/(G_0 \cap u^{-1}(AN)u)$, which is diffeomorphic to $G_0 \cap u^{-1}Tu$ by Lemma 3.2. Thus $\dim u^{G_0} = \dim \mathcal{O}(v) + t(v)$, and we have

$$\dim L_x = \dim(\mathcal{O}(v)) - \dim(K_0) + t(v).$$

The formula for the co-dimension of L_x in U/K now follows from the facts that the co-dimension of $\mathcal{O}(v)$ in Y is $l(\psi(v)w_\flat w_0)$ and that $t(v) + \alpha(v) = \dim T$.

Let $t \in T$. Then $tL_x = L_x$ if and only if there exists $g_0 \in G_0$ such that $tuK_0 = u^{g_0}K_0 \in U/K_0$. By replacing g_0 by a product of g_0 with some $k_0 \in K_0$, we see that $tL_x = L_x$ if and only if there exists $g_0 \in G_0$ such that $tu = u^{g_0}$, which is equivalent to $bt \in uG_0u^{-1}$ for some $b \in AN$. By Lemma 3.2, this is equivalent to $t \in T \cap uG_0u^{-1} = T^{\psi(v)\tau}$.

Q.E.D.

By [16, Proposition 1.3.1.3], for every $v \in V$, we can always choose $u \in \mathcal{V}$ representing v such that $\mathfrak{g}_0 \cap \mathrm{Ad}_u^{-1}\mathfrak{a} = (\mathrm{Ad}_u^{-1}\mathfrak{a})^\tau \subset \mathfrak{a}^\tau$. When $\mathcal{O}(v)$ is open in Y, $\mathfrak{g}_0 \cap \mathrm{Ad}_u^{-1}\mathfrak{h}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$ [17], which is unique up to G_0-conjugation. Let $\mathfrak{h}_1$ be any maximally compact Cartan subalgebra of $\mathfrak{g}_0$ whose vector part $\mathfrak{a}_1$ lies in $\mathfrak{a}_0 = \mathfrak{a}^\tau$, and let $\mathfrak{a}_0'$ be any complement of $\mathfrak{a}_1$ in $\mathfrak{a}_0$. Let $A_0' = \exp \mathfrak{a}_0' \subset A_0$. We have the following corollary of Lemma 3.1 and Theorem 3.5.

Corollary 3.6. *A symplectic leaf of π_0 has the largest dimension among all symplectic leaves if and only if it lies in $\mathcal{S}(v)$ corresponding to an open G_0-orbit $\mathcal{O}(v)$. Such a leaf is diffeomorphic to $A_0'N_0$.*

Corollary 3.7. *The Poisson structure π_0 has open symplectic leaves if and only if $\mathfrak{g}_0$ has a compact Cartan subalgebra. In this case the number of open symplectic leaves of π_0 is the same as the number of open G_0-orbits in Y, and each open symplectic leaf is diffeomorphic to G_0/K_0.*

For the rest of this section we assume that $X = U/K_0$ is an irreducible Hermitian symmetric space. In this case, there is a parabolic subgroup P of G containing $B = TAN$ such that $u_0 K_0 u_0^{-1} = U \cap P$ for some $u_0 \in U$. It is proved in [12] that the Poisson structure π_U on U projects to a Poisson structure on $U/(U \cap P)$, which can be regarded as a Poisson structure on U/K_0, denoted by π_∞, via the U-equivariant identification

$$X = U/K_0 \longrightarrow U/(U \cap P) : \ uK_0 \longmapsto uu_0^{-1}(U \cap P).$$

Since (X, π_∞) is also (U, π_U)-homogeneous, the difference $\pi_0 - \pi_\infty$ is a U-invariant bivector field on X. On the other hand, X carried a U-invariant symplectic structure which is unique up to scalar multiples. Let $\omega_{\rm inv}$ be such a symplectic structure, and let $\pi_{\rm inv}$ be the corresponding Poisson bi-vector field. Then since every U-invariant bi-vector field on X is a scalar multiple of $\pi_{\rm inv}$, we have

Lemma 3.8. *There exists $b \in \mathbb{R}$ such that $\pi_0 = \pi_\infty + b \cdot \pi_{\rm inv}$.*

The family of Poisson structures $\pi_\infty + b \cdot \pi_{\rm inv}$, $b \in \mathbb{R}$, has been studied in [7]. We also remark that when X is Hermitian symmetric, it is shown in [14] that there is a way of parameterizing the G_0-orbits in Y, and thus symplectic leaves of π_0 in X, using only the Weyl group W. We refer the interested reader to [14, Section 5].

Example 3.9. Consider the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$. We have $U = \mathrm{SU}(2)$, and K_0 is the subgroup of U isomorphic to S^1 given by:

$$K_0 = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R} \right\}.$$

The space $X = U/K_0$ can be naturally identified with the Riemann sphere S^2 via the map

$$M = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto z = \frac{-\mathrm{Im}(a) + i \cdot \mathrm{Im}(b)}{\mathrm{Re}(a) + i \cdot \mathrm{Re}(b)},$$

where $M \in \mathrm{SU}(2)$ with $|a|^2 + |b|^2 = 1$ and z is a holomorphic coordinate on $X \setminus \{\mathrm{pt}\}$. Then the Poisson structure π_0 is given by

$$\pi_0 = i(1 - |z|^4) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

Therefore there are two open symplectic leaves for π_0, which can be thought of as the Northern and the Southern hemispheres. Every point in the Equator, corresponding to $|z| = 1$, is a symplectic leaf as well. It is interesting to notice that the image of a symplectic leaf in U given by:

$$\frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} z & 1 \\ -1 & \bar{z} \end{pmatrix}, \quad z \in \mathbb{C}$$

is the union of the Northern and the Southern hemispheres and a point in the Equator. All three are Poisson submanifolds of S^2.

Remark 3.10. Let $\mathcal{L}$ be the variety of Lagrangian subalgebras of $\mathfrak{g}$ with respect to the pairing $\mathrm{Im} \ll \, , \gg$, as defined in [3]. Then G acts on $\mathcal{L}$ by conjugating the subalgebras. The variety $\mathcal{L}$ carries a Poisson structure Π defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{u} + (\mathfrak{a} + \mathfrak{n})$ such that every U-orbit (as well as every AN-orbit) is a Poisson subvariety of $(\mathcal{L}, \Pi)$. Consider the point $\mathfrak{g}_0$ of $\mathcal{L}$ and let X' be the U-orbit in $\mathcal{L}$ through $\mathfrak{g}_0$. Then we have a natural map

$$\mathcal{J} : \ U/K_0 \longrightarrow X'.$$

The normalizer subgroup of $\mathfrak{g}_0$ in U is not necessarily connected but always has K_0 as its connected component. Thus $\mathcal{J}$ is a finite covering map. It follows from [3] that the map $\mathcal{J}$ is Poisson with respect to the Poisson structure Π on X'.

4. Invariant Poisson cohomology of $(U/K_0, \pi_0)$.

Let $\chi^\bullet(X)$ stand for the graded vector space of the multi-vector fields on X. Recall that the Poisson coboundary operator, introduced by Lichnerowicz [10], is given by:

$$d_{\pi_0} : \ \chi^i(X) \to \chi^{i+1}(X), \quad d_{\pi_0}(V) = [\pi_0, V],$$

where $[\cdot, \cdot]$ is the Schouten bracket of the multi-vector fields [8]. The Poisson cohomology of (X, π_0) is defined to be the cohomology of the cochain complex $(\chi^\bullet(X), d_{\pi_0})$ and is denoted by $H^\bullet_{\pi_0}(X)$. By [11], the space $(\chi^\bullet(X))^U$ of U-invariant multi-vector fields on X is closed under d_{π_0}. The cohomology of the cochain sub-complex $((\chi^\bullet(X))^U, d_{\pi_0})$ is called the U-invariant Poisson cohomology of (X, π_0) and we denote it by $H^\bullet_{\pi_0, U}(X)$. We have the following result from [11, Theorem 7.5], adapted to our situation $X = U/K_0$, which relates the Poisson cohomology of a Poisson homogeneous space with certain relative Lie algebra cohomology. Recall that G_0, as a subgroup of G, acts on U by (2.1), and thus $C^\infty(U)$ can be viewed as a $\mathfrak{g}_0$-module. We also treat $\mathbb{R}$ as the trivial $\mathfrak{g}_0$-module:

Proposition 4.1. [11]

$$H^\bullet_{\pi_0}(X) \simeq H^\bullet(\mathfrak{g}_0, \mathfrak{k}_0, C^\infty(U)), \ \textit{and}\ H^\bullet_{\pi_0, U}(X) \simeq H^\bullet(\mathfrak{g}_0, \mathfrak{k}_0, \mathbb{R}),$$

We will compute the cohomology space $H^\bullet_{\pi_0}(X)$ in a future paper. The Poisson homology of π_0 for $X = \mathbb{CP}^n$ was computed in [9]. For the U-invariant Poisson cohomology, we have

Theorem 4.2. *The U-invariant Poisson cohomology of $(U/K_0, \pi_0)$ is isomorphic to the De Rham cohomology $H^\bullet(X)$, or, equivalently, to the space of G_0-invariant differential forms on the non-compact dual symmetric space G_0/K_0.*

Proof. By [2, Corollary II.3.2], $H^q(\mathfrak{g}_0, \mathfrak{k}_0, \mathbb{R})$ is isomorphic to $(\wedge^q \mathfrak{q}_0^*)^{\mathfrak{k}_0}$, where $\mathfrak{q}_0$ is the radial part in the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{q}_0$. This space is isomorphic to the space of G_0-invariant differential q-forms on the space G_0/K_0. Since $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{q}_0$, and U is compact, we obtain

$$H^q(\mathfrak{g}_0, \mathfrak{k}_0, \mathbb{R}) \simeq H^q(\mathfrak{u}, \mathfrak{k}_0, \mathbb{R}) \simeq H^q(U/K_0).$$

Q.E.D.

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