ON A CLASS OF DOUBLE COSETS IN REDUCTIVE ALGEBRAIC GROUPS

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Abstract. We study a class of double coset spaces $R_{\mathcal{A}}\backslash G_1 \times G_2/R_{\mathcal{C}}$, where G_1 and G_2 are connected reductive algebraic groups, and $R_{\mathcal{A}}$ and $R_{\mathcal{C}}$ are certain spherical subgroups of $G_1 \times G_2$ obtained by "identifying" Levi factors of parabolic subgroups in G_1 and G_2. Such double cosets naturally appear in the symplectic leaf decompositions of Poisson homogeneous spaces of complex reductive groups with the Belavin–Drinfeld Poisson structures. They also appear in orbit decompositions of the De Concini–Procesi compactifications of semi-simple groups of adjoint type. We find explicit parametrizations of the double coset spaces and describe the double cosets as homogeneous spaces of $R_{\mathcal{A}} \times R_{\mathcal{C}}$. We further show that all such double cosets give rise to set-theoretical solutions to the quantum Yang–Baxter equation on unipotent algebraic groups.

1. The setup

1.1. **The setup.** Let G_1 and G_2 be two connected reductive algebraic groups over an algebraically closed base field k. For $i = 1, 2$, we will fix a maximal torus H_i in G_i and a choice Δ_i^+ of positive roots in the set Δ_i of all roots for G_i with respect to H_i. For each $\alpha \in \Delta_i$, we will use U_i^α to denote the one-parameter unipotent subgroup of G_i determined by α. Let Γ_i be the set of simple roots in Δ_i^+. For a subset A_i of Γ_i, let P_{A_i} be the standard parabolic subgroup of G_i containing the Borel subgroup $B_i = H_i U_i$, where $U_i = \prod_{\alpha \in \Delta_i^+} U_i^\alpha$. Let M_{A_i} and U_{A_i} be respectively the Levi factor of P_{A_i} containing H_i and the unipotent radical of P_{A_i}. Then $P_{A_i} = M_{A_i} U_{A_i}$ and M_{A_i} and U_{A_i} intersect trivially. Denote by M'_{A_i} the derived subgroups of the Levi factors M_{A_i}. We will also use $[A_i]$ to denote the set of roots in Δ_i that are in the linear span of A_i.

Definition 1.1. Identify Γ_1 and Γ_2 with the Dynkin diagrams of G_1 and G_2 respectively. By a *partial isometry* from Γ_1 to Γ_2 we mean an isometry from a subdiagram of Γ_1 to a subdiagram of Γ_2. We will denote by $P(\Gamma_1, \Gamma_2)$ the set of all partial isometries from Γ_1 to Γ_2. If $a \in P(\Gamma_1, \Gamma_2)$, and if A_1 and A_2 are the domain and the range of a respectively, we define a *generalized a-graph* to be an abstract (not necessarily algebraic) subgroup K of $M_{A_1} \times M_{A_2}$ such that $K \cap (U_1^\alpha \times U_2^{a(\alpha)}) \subset U_1^\alpha \times U_2^{a(\alpha)}$ is the graph of a group isomorphism from U_1^α to $U_2^{a(\alpha)}$ for every $\alpha \in [A_1]$.

By an *admissible pair* for $G_1 \times G_2$ we mean a pair (a, K), where $a \in P(\Gamma_1, \Gamma_2)$ and K is a generalized a-graph. If $\mathcal{A} = (a, K)$ is an admissible pair, we define the subgroup $R_{\mathcal{A}}$ of $P_{A_1} \times P_{A_2}$ by

$$R_{\mathcal{A}} = K(U_{A_1} \times U_{A_2}). \tag{1.1}$$

1.2. **An Example.** Assume that G_1 and G_2 are connected and simply-connected. Fix a partial isometry $a \in P(\Gamma_1, \Gamma_2)$ with domain A_1 and range A_2. Then the derived subgroups M'_{A_i} of the Levi factors M_{A_i} are connected and simply-connected [22, Corollary 5.4], and a can be lifted to a group isomorphism $\theta_a \colon M'_{A_1} \to M'_{A_2}$. Let

$$\mathrm{Graph}(\theta_a) = \{(m_1, \theta_a(m_1)) \mid m_1 \in M'_{A_1}\} \subset M'_{A_1} \times M'_{A_2}$$

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be the graph of θ_a, and let Z_{A_i} be the center of M_{A_i} for $i = 1, 2$. Then, any abstract subgroup X of $Z_{A_1} \times Z_{A_2}$ gives rise to the group $K = X\mathrm{Graph}(\theta_a)$ which can be easily seen to be a generalized a-graph for $G_1 \times G_2$, and for the corresponding admissible pair $\mathcal{A} = (a, K)$, we have

$$R_{\mathcal{A}} = X\mathrm{Graph}(\theta_a)\,(U_{A_1} \times U_{A_2}).$$

As a direct consequence of Lemma 3.2 in §3, one sees that, in the case when G_1 and G_2 are connected and simply-connected, all admissible pairs $\mathcal{A}$ and groups $R_{\mathcal{A}}$ are of the above type.

1.3. **Main problem.** Fix two admissible pairs $\mathcal{A} = (a, K)$ and $\mathcal{C} = (c, L)$. In this paper, we obtain the solutions to the following problems:

1.) *Find an explicit parametrization of all $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$.*
2.) *Describe explicitly each $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset as a homogeneous space of $R_{\mathcal{A}} \times R_{\mathcal{C}}$.*
3.) *Find a closed formula for the dimension of each $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset.*

Remark 1.2. For an admissible pair $\mathcal{A} = (a, K)$, we will set

$$R_{\mathcal{A}}^- = K(U_{A_1} \times U_{A_2}^-), \tag{1.2}$$

where $U_{A_2}^-$ is the subgroup of G_2 generated by the one-parameter subgroups U_2^{α} for $\alpha \in -[A_2]$. It is easy to see that solutions to the problems in §1.3 would lead easily to solutions to the corresponding problems for the $(R_{\mathcal{A}}^-, R_{\mathcal{C}}^-)$ double cosets in $G_1 \times G_2$.

1.4. **Motivation.** Let G be a complex semi-simple Lie group of adjoint type. The wonderful compactifications of G, constructed by De Concini and Procesi in [4], are smooth $(G \times G)$-varieties with finitely many $G \times G$ orbits. It is shown in [21] that every $G \times G$ orbit in these compactifications of G is isomorphic to $(G \times G)/R_{\mathcal{C}}^-$ for an admissible pair $\mathcal{C} = (c, L)$ for $G \times G$, where c is the restriction of an automorphism of the Dynkin diagram of G to some subdiagram. Let $\mathfrak{g}$ be the Lie algebra of G. It was observed in [12] that each wonderful compactification of G naturally lies in the variety $\mathcal{L}_{\mathfrak{g}}$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$. Here, a subalgebra $\mathfrak{l}$ of $\mathfrak{g} \oplus \mathfrak{g}$ is said to be Lagrangian if $\dim \mathfrak{l} = \dim \mathfrak{g}$ and if $\mathfrak{l}$ is isotropic with respect to the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$ given by

$$\langle x_1 + y_1, x_2 + y_2 \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \qquad x_1, y_1, x_2, y_2 \in \mathfrak{g},$$

where $\ll\, , \,\gg$ denotes the Killing form of $\mathfrak{g}$. The group $G \times G$ acts on $\mathcal{L}_{\mathfrak{g}}$ through the adjoint action. It is shown in [12] that all $G \times G$ orbits in $\mathcal{L}_{\mathfrak{g}}$ are again of the type $(G \times G)/R_{\mathcal{C}}^-$ for some admissible pairs $\mathcal{C} = (c, L)$ for $G \times G$, but this time the partial isometries c can be arbitrary. Thus, solutions to the double coset problems in §1.3 can be used to classify $R_{\mathcal{A}}^-$ orbits in $\mathcal{L}_{\mathfrak{g}}$ for any admissible pair $\mathcal{A}$.

Our motivation for studying $R_{\mathcal{A}}^-$ orbits in $\mathcal{L}_{\mathfrak{g}}$ comes from the theory of Poisson Lie groups. In [1], Belavin and Drinfeld classified all quasi-triangular Poisson structures on G. Denote such a Poisson structure on G by π_{BD}. The Poisson Lie groups (G, π_{BD}) have received a lot of attention since the appearance of [1]. They were explicitly quantized by Etingof–Schedler–Schiffmann in [10] and their symplectic leaves were studied by the second author in [23]. One important problem regarding the Poisson Lie groups (G, π_{BD}) is the study of their Poisson homogeneous spaces [7]. The variety $\mathcal{L}_{\mathfrak{g}}$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ serves as a "master" variety of Poisson homogeneous spaces of (G, π_{BD}). Indeed, identify G with the diagonal subgroup G_{Δ} of $G \times G$. It is shown in [11] that $\mathcal{L}_{\mathfrak{g}}$ carries a natural Poisson structure Π_{BD}, such that every G_{Δ} orbit in $\mathcal{L}_{\mathfrak{g}}$, when equipped with Π_{BD}, is a Poisson homogeneous space of (G, π_{BD}). To identify these homogeneous spaces of G, one needs to classify G_{Δ} orbits in $\mathcal{L}_{\mathfrak{g}}$, and to study their symplectic leaf decompositions with respect to Π_{BD}, one needs to classify $N(G^*_{\mathrm{BD}})$ orbits in $L_{\mathfrak{g}}$, where G^*_{BD} is the dual Poisson Lie group of (G, π_{BD}) inside $G \times G$, and $N(G^*_{\mathrm{BD}})$ is the normalizer subgroup of G^*_{BD} in $G \times G$. Both G^*_{BD}

and $N(G^*_{\mathrm{BD}})$ are of the type $R^-_{\mathcal{A}}$, where the partial isometry a in $\mathcal{A} = (a, K)$ is the Belavin-Drinfeld triple defined in [1]. In the forthcoming paper [18], we will show that intersections of G_Δ and $N(G^*_{\mathrm{BD}})$ orbits in $\mathcal{L}_{\mathfrak{g}}$ are all regular Poisson subvarieties of $(\mathcal{L}_{\mathfrak{g}}, \Pi_{\mathrm{BD}})$. Solutions to the problems in §1.3 will be used in [18] to classify such orbits and to compute the ranks of the Poisson structures Π_{BD}. Based on the results of this paper, in [18] we will also treat the more general class of Poisson structures on $\mathcal{L}_{\mathfrak{g}}$, obtained from arbitrary lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$. The latter were classified by P. Delorme in [5].

For the so-called standard Poisson structure on G, we have $N(G^*_{\mathrm{BD}}) = B \times B^-$, where (B, B^-) is a pair of opposite Borel subgroups of G. In [21], T. Springer classified the $B \times B^-$ orbits in a wonderful compactification $\overline{G}$ of G and studied the intersection cohomology of the $(B \times B^-)$-orbit closures. The Poisson structures Π_{BD} on $\overline{G} \subset \mathcal{L}_{\mathfrak{g}}$ point to the finer decomposition of $\overline{G}$ into the intersections of G_Δ and $N(G^*_{\mathrm{BD}})$ orbits. We hope that these finer decompositions of $\overline{G}$ will be useful in the study of $\overline{G}$, and especially in understanding the closures of conjugacy classes of G inside $\overline{G}$. We also point out that intersections of G_Δ and $B \times B^-$ orbits in the closed $G \times G$ orbits in $\mathcal{L}_{\mathfrak{g}}$ are closely related to the double Bruhat cells in G and intersections of dual Schubert cells on the flag variety of G (see [2, 12, 13]). It would be very interesting to understand intersections of arbitrary G_Δ and $N(G^*_{\mathrm{BD}})$ orbits in $\mathcal{L}_{\mathfrak{g}}$ in the framework of the theory of cluster algebras of Fomin and Zelevinsky [14], and to find toric charts on the symplectic leaves of $(\mathcal{L}_{\mathfrak{g}}, \Pi_{\mathrm{BD}})$ using the methods of Kogan and Zelevinsky [17].

Our approach to the double coset space $R_{\mathcal{A}} \backslash G_1 \times G_2 / R_{\mathcal{C}}$ relies on an inductive argument that relates an $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset in $G_1 \times G_2$ to cosets of similar type in the product $M_1 \times M_2$ of Levi subgroups $M_1 \subset G_1$ and $M_2 \subset G_2$. Such an argument first appeared in the work of the second author [23] on the symplectic leaf decomposition of the Poisson Lie group (G, π_{BD}). It dealt with the case $G_1 = G_2 = G$ and $\mathcal{A} = \mathcal{C}$. Recently the first author and S. Evens [12] extended the methods of [23] to the case $G_1 = G_2 = G$ and $R_{\mathcal{A}} = G_\Delta$. The advantage of the approach in this paper to the one in [23] is that in this paper we find an explicit relation between the double cosets in $G_1 \times G_2$ and the double cosets in $M_1 \times M_2$, while [23] deals with various complicated Zariski open subsets of the double cosets in $G_1 \times G_2$. In particular, the iteration of the inductive procedure in [23] is only helpful for studying Zariski open subsets of the symplectic leaves of π_{BD}, while we expect that the results of this paper will be useful in understanding globally all the symplectic leaves of $(\mathcal{L}_{\mathfrak{g}}, \Pi_{BD})$.

The roots of the methods used in this paper can be traced back to the inductive procedure of Duflo [8] and Moeglin–Rentschler [20] for describing the primitive spectrum of the universal enveloping algebra of a Lie algebra which is in general neither solvable nor semi-simple. The reason for this relation is that the Poisson structures Π_{BD} on $\mathcal{L}_{\mathfrak{g}}$ vanish at many points and the corresponding linearizations give rise to Lie algebras that are rarely semi-simple or solvable. Due to the Kirillov–Kostant orbit method, one expects a close relation between the primitive spectrum of the quantized algebras of coordinate rings of various affine subvarieties of $\mathcal{L}_{\mathfrak{g}}$ (which are in fact deformations of the universal enveloping algebras of the linearizations) and the symplectic leaves of such affine Poisson subvarieties of $\mathcal{L}_{\mathfrak{g}}$. From this point of view, our method can be considered as a nonlinear quasiclassical analog of [8, 20].

1.5. **Organization of paper.** The statements of the main results in this paper are given in §2. The two sections, §3 and §4, serve as preparations for the proofs of the main results which are given in §5 and §6. In §7, we describe some solutions to the set-theoretical Yang–Baxter equation that arise in our solution to 2) of the main problems in §1.3. In §8, we classify $(R_{\mathcal{A}}, P)$ double cosets in $G_1 \times G_2$ for any admissible pair $\mathcal{A}$ and any standard parabolic subgroup P of $G_1 \times G_2$, and we show in particular that $R_{\mathcal{A}}$ is a spherical subgroup of $G_1 \times G_2$.

1.6. **Notation.** Throughout this paper, the letter e always denotes the identity element of any group. We will denote by Ad the conjugation action of a group G on itself, given by $\mathrm{Ad}_g(h) = ghg^{-1}$ for $g, h \in G$. If M_1 and M_2 are two sets and if $K \subset M_1 \times M_2$, we can think of K as a correspondence between M_1 and M_2. If M_3 is another set and if $L \subset M_2 \times M_3$, then one defines the composition of L and K to be

$$(1.3)\quad L \circ K = \{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ such that } (m_1, m_2) \in K,\ (m_2, m_3) \in L\}.$$

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2. Statements of the main results

The main parameters that come into our classification of $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$ are certain elements in the Weyl groups of G_1 and G_2 and certain "twisted conjugacy classes" in Levi subgroups of G_1.

2.1. **Minimal length representatives.** For $i = 1, 2$, let W_i be the Weyl group of Γ_i. For a subset D_i of Γ_i, we will use W_{D_i} to denote the subgroup of W_i generated by D_i. Let $W_i^{D_i} \subset W_i$ and ${}^{D_i}W_i \subset W_i$ be respectively the set of minimal length representatives of the cosets from W_i/W_{D_i} and $W_{D_i} \backslash W_i$. It is well-known that

$$w_i \in W_i^{D_i} \qquad \text{if and only if} \qquad w_i(D_i) \subset \Delta_i^+$$

and

$$w_i \in {}^{D_i}W_i \qquad \text{if and only if} \qquad w_i^{-1}(D_i) \subset \Delta_i^+.$$

2.2. **Twisted conjugacy classes.**

Definition 2.1. Let D_1 be a subset of Γ_1, and let d be a partial isometry from Γ_1 to Γ_1 with D_1 as both its domain and its range. If $J \subset M_{D_1} \times M_{D_1}$ is a generalized d-graph, we let J act on M_{D_1} from the left by

$$J \times M_{D_1} \to M_{D_1} \colon ((l, n), m) \mapsto lmn^{-1}, \qquad (l, n) \in J,\ m \in M_{D_1}.$$

By a *d-twisted conjugacy class* in M_{D_1} we mean a J orbit in M_{D_1} for any generalized d-graph J. If J is the graph of a group automorphism $j \colon M_{D_1} \to M_{D_1}$, the action of J on M_{D_1} becomes the usual *j-twisted conjugation action* of M_{D_1} on itself:

$$M_{D_1} \times M_{D_1} \to M_{D_1} \colon (l, m) \mapsto lmj(l)^{-1}, \qquad l, m \in M_{D_1}$$

whose orbits will be referred to as *j-twisted conjugacy classes*.

2.3. **Classification of double cosets.** Let $\mathcal{A} = (a, K)$ and $\mathcal{C} = (c, L)$ be two admissible pairs, and let A_1 and A_2 be the domain and the range of a and C_1 and C_2 the domain and the range of c respectively. We now state our classification of $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$. In doing so, we will need to refer to several lemmas that will be proved in §3-§6.

For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, consider the sets

$$A_1(v_1, v_2) = \{\alpha \in A_1 \mid (v_1 c^{-1} v_2^{-1} a)^n \alpha \text{ is defined and is in } A_1 \text{ for } n = 0, 1, \ldots\} \tag{2.1}$$

$$C_2(v_1, v_2) = \{\beta \in C_2 \mid (v_2^{-1} a v_1 c^{-1})^n \beta \text{ is defined and is in } C_2 \text{ for } n = 0, 1, \ldots\} \tag{2.2}$$

Then $A_1(v_1, v_2)$ is the largest subset of A_1 that is invariant under $v_1 c^{-1} v_2^{-1} a$, and $C_2(v_1, v_2)$ is the largest subset of C_2 that is invariant under $v_2^{-1} a v_1 c^{-1}$. Note that

$$v_2^{-1} a \text{ and } c v_1^{-1} \colon A_1(v_1, v_2) \to C_2(v_1, v_2) \tag{2.3}$$

are both partial isometries from Γ_1 to Γ_2. Fix a representative $\dot{v}_i$ in the normalizer of H_i in G_i, and define

$$K(v_1, v_2) = \left(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}\right) \cap \mathrm{Ad}_{(e,\dot{v}_2)}^{-1} K \tag{2.4}$$

$$L(v_1, v_2) = \left(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}\right) \cap \mathrm{Ad}_{(\dot{v}_1,e)} L. \tag{2.5}$$

It is easy to see that $K(v_1, v_2)$ is a generalized $(v_2^{-1}a)$-graph, while $L(v_1, v_2)$ is a generalized (cv_1^{-1})-graph for the partial isometries in (2.3). Define

$$J(v_1, v_2) = \{(m, m') \in M_{A_1(v_1,v_2)} \times M_{A_1(v_1,v_2)} \mid \exists n \in M_{C_2(v_1,v_2)} \text{ such that } (m, n) \in K(v_1, v_2) \text{ and } (m', n) \in L(v_1, v_2)\}. \tag{2.6}$$

Then $J(v_1, v_2)$ is a generalized $(v_1 c^{-1} v_2^{-1} a)$-graph for the partial isometry

$$v_1 c^{-1} v_2^{-1} a \colon A_1(v_1, v_2) \to A_1(v_1, v_2).$$

This follows from Lemma 3.5 in §3 and the facts that if σ is the map

$$\sigma \colon M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)} \to M_{C_2(v_1,v_2)} \times M_{A_1(v_1,v_2)} \colon (m, n) \mapsto (n, m),$$

then $\sigma(L(v_1, v_2))$ is a generalized $(v_1 c^{-1})$-graph, and $J(v_1, v_2)$ is the composition

$$J(v_1, v_2) = \sigma(L(v_1, v_2)) \circ K(v_1, v_2) \tag{2.7}$$

(of group correspondences, see §1.6). Let $J(v_1, v_2)$ act on $M_{A_1(v_1,v_2)}$ from the left by

$$(l_1, n_1) \cdot m_1 := l_1 m_1 n_1^{-1}, \qquad (l_1, n_1) \in J(v_1, v_2),\ m_1 \in M_{A_1(v_1,v_2)}. \tag{2.8}$$

By Definition 2.1, $J(v_1, v_2)$ orbits in $M_{A_1(v_1,v_2)}$ are $(v_1 c^{-1} v_2^{-1} a)$-twisted conjugacy classes in $M_{A_1(v_1,v_2)}$. Let $Z_{C_2(v_1,v_2)}$ be the center of $M_{C_2(v_1,v_2)}$, let $\eta_2 \colon G_1 \times G_2 \to G_2$ be the projection, and let

$$Z(v_1, v_2) = Z_{C_2(v_1,v_2)} / \left(Z_{C_2(v_1,v_2)} \cap v_2^{-1}(\eta_2(K))\right) \left(Z_{C_2(v_1,v_2)} \cap \eta_2(L)\right).$$

For each $s \in Z(v_1, v_2)$, fix a representative $\dot{s}$ of s in $Z_{C_2(v_1,v_2)}$. For $(g_1, g_2) \in G_1 \times G_2$, we will use $[(g_1, g_2)]$ to denote the double coset $R_{\mathcal{A}}(g_1, g_2) R_{\mathcal{C}}$ in $G_1 \times G_2$. We can now state our main theorem on the classification of $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$.

Theorem 2.2. *Let $\mathcal{A} = (a, K)$ and $\mathcal{C} = (c, L)$ be any two admissible pairs for $G_1 \times G_2$. Then every $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset of $G_1 \times G_2$ is of the form*

$$[(m_1 \dot{v}_1, \dot{v}_2 \dot{s})] \textit{ for some } v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2,\ m_1 \in M_{A_1(v_1,v_2)}, s \in Z(v_1, v_2).$$

Two such double cosets $[(m_1 \dot{v}_1, \dot{v}_2 \dot{s})]$ and $[(m_1' \dot{v}_1', \dot{v}_2' \dot{s}')]$ coincide if and only if $v_i' = v_i$ for $i = 1, 2$, $s = s'$, and m_1 and m_1' are in the same $J(v_1, v_2)$ orbit in $M_{A_1(v_1,v_2)}$.

Example 2.3. If $\eta_2(K) = M_{A_2}$ or $\eta_2(L) = M_{C_2}$, then $Z(v_1, v_2) = \{e\}$ for all choices of (v_1, v_2). In particular, every $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset of $G_1 \times G_2$ is of the form

$$[(m_1\dot{v}_1, \dot{v}_2)] \text{ for some } v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2,\ m_1 \in M_{A_1(v_1,v_2)}.$$

Two cosets of this type $[(m_1\dot{v}_1, \dot{v}_2)]$ and $[(m_1'\dot{v}_1', \dot{v}_2')]$ coincide if and only if $v_i' = v_i$ for $i = 1, 2$, and m_1 and m_1' are in the same $J(v_1, v_2)$ orbit in $M_{A_1(v_1,v_2)}$.

Example 2.4. If K and L are respectively the graphs of group isomorphisms

$$\theta_a \colon M_{A_1} \to M_{A_2} \quad \text{ and } \quad \theta_c \colon M_{C_1} \to M_{C_2},$$

the group $J(v_1, v_2)$ is the graph of the group automorphism

$$j(v_1, v_2) := \mathrm{Ad}_{\dot{v}_1}\theta_c^{-1}\mathrm{Ad}_{\dot{v}_2}^{-1}\theta_a \colon M_{A_1(v_1,v_2)} \to M_{A_1(v_1,v_2)}.$$

Then every $(R_{\mathcal{A}}, R_{\mathcal{C}})$ orbit in $G_1 \times G_2$ is of the form $[(m_1\dot{v}_1, \dot{v}_2)]$ for a unique pair $(v_1, v_2) \in W_1^{C_1} \times {}^{A_2}W_2$ and some $m_1 \in M_{A_1(v_1,v_2)}$. Two such cosets $[(m_1\dot{v}_1, \dot{v}_2)]$ and $[(m_1'\dot{v}_1, \dot{v}_2)]$ coincide if and only if m_1 and m_1' are in the same $j(v_1, v_2)$-twisted conjugacy class (see §2.2).

2.4. **Structure of double cosets.** Let $\mathcal{A} = (a, K)$ and $\mathcal{C} = (c, L)$ be two admissible pairs for $G_1 \times G_2$. For $q = (g_1, g_2) \in G_1 \times G_2$, set

$$\mathrm{Stab}(q) = \mathrm{Ad}_{(e,g_2)}^{-1}(R_{\mathcal{A}}) \cap \mathrm{Ad}_{(g_1,e)}(R_{\mathcal{C}}).$$

Then the double coset $R_{\mathcal{A}}qR_{\mathcal{C}}$ in $G_1 \times G_2$, considered as an $R_{\mathcal{A}} \times R_{\mathcal{C}}$-space under the action $(r_{\mathcal{A}}, r_{\mathcal{C}}) \cdot q' = r_{\mathcal{A}}q'r_{\mathcal{C}}^{-1}$, is given by

$$R_{\mathcal{A}}qR_{\mathcal{C}} = (R_{\mathcal{A}} \times R_{\mathcal{C}})/\{(\mathrm{Ad}_{(e,g_2)}(r), \mathrm{Ad}_{(g_1,e)}^{-1}(r)) \mid r \in \mathrm{Stab}(q)\}.$$

For $q = (m_1\dot{v}_1, \dot{v}_2 s_2)$ as in Theorem 2.2, where $m_1 \in M_{A_1(v_1,v_2)}, v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2$, and $s_2 \in Z_{C_2(v_1,v_2)}$, we will now give an explicit description of $\mathrm{Stab}(q)$. Let $\pi_{A_1} \colon P_{A_1} \to M_{A_1}$ and $\pi_{C_2} \colon P_{C_2} \to M_{C_2}$ be, respectively, the projections with respect to the decompositions $P_{A_1} = M_{A_1}N_{A_1}$ and $P_{C_2} = M_{C_2}N_{C_2}$. By 2) of Lemma 3.4 in §3 (by taking $D_1 = \emptyset$ therein), the group K gives rise to a group isomorphism $\theta_a \colon U_1 \cap M_{A_1} \to U_2 \cap M_{A_2}$ such that $\theta_a(U_1^{\alpha}) = U_2^{a(\alpha)}$ for every $\alpha \in [A_1]$. Similarly, we have the group isomorphism $\theta_c \colon U_1 \cap M_{C_1} \to U_2 \cap M_{C_2}$ induced from the group L. The group $\mathrm{Stab}(q)$ will lie in the product group

$$\left(P_{A_1(v_1,v_2)} \cap v_1(P_{C_1})\right) \times \left(P_{C_2(v_1,v_2)} \cap v_2^{-1}(P_{A_2})\right),$$

and we will use the maps $\pi_{A_1}, \pi_{C_2}, \theta_a$, and θ_c to describe $\mathrm{Stab}(q)$.

Note that $v_1 \in {}^{A_1(v_1,v_2)}W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2^{C_2(v_1,v_2)}$ because

$$v_1^{-1}(A_1(v_1, v_2)) = c^{-1}(C_2(v_1, v_2)) \subset C_1 \quad \text{and} \quad v_2(C_2(v_1, v_2)) = a(A_1(v_1, v_2)) \subset A_2. \tag{2.9}$$

By Lemma 4.2 in §4, we have

$$P_{A_1(v_1,v_2)} \cap v_1(P_{C_1}) = M_{A_1(v_1,v_2)}\left(U_{A_1(v_1,v_2)} \cap v_1(U_1)\right),$$
$$P_{C_2(v_1,v_2)} \cap v_2^{-1}(P_{A_2}) = M_{C_2(v_1,v_2)}\left(U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)\right).$$

Consider the group homomorphisms

$$\phi_1 = \mathrm{Ad}_{m_1\dot{v}_1}\theta_c^{-1}\pi_{C_2} \colon\ U_2 \to \mathrm{Ad}_{m_1}(U_1), \tag{2.10}$$

$$\phi_2 = \mathrm{Ad}_{\dot{v}_2 s_2}^{-1}\theta_a\pi_{A_1} \colon\ U_1 \to U_2. \tag{2.11}$$

In §6 we will show that

$$\sigma_1 = \phi_1\phi_2 \in \mathrm{End}(U_{A_1(v_1,v_2)} \cap v_1(U_1)), \tag{2.12}$$

$$\sigma_2 = \phi_2\phi_1 \in \mathrm{End}(U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)) \tag{2.13}$$

are well-defined and that $\sigma_1^{k+1} \equiv e$ and $\sigma_2^{k+1} \equiv e$ for some integer $k \geq 1$, where e stands for the trivial endomorphism. Consider the subgroups

$$U_1 \cap v_1(U_{C_1}) = U_{A_1(v_1,v_2)} \cap v_1(U_{C_1}) \subset U_{A_1(v_1,v_2)} \cap v_1(U_1), \tag{2.14}$$

$$U_2 \cap v_2^{-1}(U_{A_2}) = U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_{A_2}) \subset U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2), \tag{2.15}$$

and define

$$\psi_1\colon\ (U_1 \cap v_1(U_{C_1})) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right) \to U_{A_1(v_1,v_2)} \cap v_1(U_1),$$
$$\psi_2\colon\ (U_1 \cap v_1(U_{C_1})) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right) \to U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2),$$

respectively by

$$\psi_1(n_1,n_2) = \phi_1(n_2)\sigma_1(n_1\phi_1(n_2))\sigma_1^2(n_1\phi_1(n_2)) \cdots \sigma_1^k(n_1\phi_1(n_2)), \tag{2.16}$$

$$\psi_2(n_1,n_2) = \phi_2(n_1)\sigma_2(n_2\phi_2(n_1))\sigma_2^2(n_2\phi_2(n_1)) \cdots \sigma_2^k(n_2\phi_2(n_1)). \tag{2.17}$$

Theorem 2.5. *Let the notation be as in Theorem 2.2. For $q = (m_1\dot{v}_1, \dot{v}_2 s_2)$, where $m_1 \in M_{A_1(v_1,v_2)}, v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2$, and $s_2 \in Z_{C_2(v_1,v_2)}$, we have the semi-direct product decomposition*

$$\mathrm{Stab}(q) = \mathrm{Stab}_M(q)\mathrm{Stab}_U(q),$$

where

$$\begin{aligned}
\mathrm{Stab}_M(q) &= K(v_1,v_2) \cap \mathrm{Ad}_{(m_1,e)} L(v_1,v_2)\\
&= \left(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}\right) \cap \mathrm{Ad}_{(e,\dot{v}_2)}^{-1} K \cap \mathrm{Ad}_{(m_1\dot{v}_1,e)} L,\\
\mathrm{Stab}_U(q) &= \{(n_1\psi_1(n_1,n_2),\ n_2\psi_2(n_1,n_2)) \mid (n_1,n_2) \in (U_1 \cap v_1(U_{C_1})) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right)\}\\
&\subset \left(U_{A_1(v_1,v_2)} \cap v_1(U_1)\right) \times \left(U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)\right).
\end{aligned}$$

Remark 2.6. For $m_1 \in M_{A_1(v_1,v_2)}$, let $J_{m_1}(v_1,v_2)$ be the stabilizer subgroup of $J(v_1,v_2)$ at m_1 for the action of $J(v_1,v_2)$ on $M_{A_1(v_1,v_2)}$ given by (2.8), cf. Definition 2.1. We will show in Lemma 6.2 in §6 that $\mathrm{Stab}_M(q)$ is a central extension of $J_{m_1}(v_1,v_2)$ by

$$(\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K) \subset \{e\} \times Z_{C_2(v_1,v_2)},$$

where, for $i = 1,2$, $\eta_i\colon G_1 \times G_2 \to G_i$ is the projection.

2.5. A dimension formula.

Theorem 2.7. *In the notation of Theorem 2.5, if K and L are algebraic subgroups of $G_1 \times G_2$, or if the base field is $\mathbb{C}$ and K and L are Lie subgroups of $G_1 \times G_2$, then the double coset $[(m_1\dot{v}_1, \dot{v}_2\dot{s})]$ is smooth and has dimension equal to*

$$\begin{aligned}
&l(v_1) + l(v_2) + \dim P_{A_1} - \dim M_{A_1(v_1,v_2)} + \dim P_{C_2} - \dim Z_{C_2(v_1,v_2)}\\
&\quad + \dim \eta_2\left(KL \cap (v_1^{-1} Z_{A_1(v_1,v_2)} \times v_2 Z_{C_2(v_1,v_2)})\right) + \dim(J(v_1,v_2)\cdot m_1),
\end{aligned}$$

where $l(\cdot)$ denotes the length function on the Weyl groups W_1 and W_2, and $J(v_1,v_2)\cdot m_1$ is the $J(v_1,v_2)$ orbit through m_1 for the $J(v_1,v_2)$ action on $M_{A_1(v_1,v_2)}$ given in (2.8).

Let us note that in the two cases in Theorem 2.7, the double coset $[(m_1\dot{v}_1, \dot{v}_2\dot{s})]$ is a locally closed algebraic subset of $G_1 \times G_2$ (as an orbit of an algebraic group) or a submanifold of $G_1 \times G_2$. It will be shown in §6 that in these two cases $J(v_1,v_2)$ is respectively an algebraic group or a Lie group. Note that $\eta_2\left(KL \cap (v_1^{-1} Z_{A_1(v_1,v_2)} \times v_2 Z_{C_2(v_1,v_2)})\right)$ is also an algebraic group or a Lie group because it is a quotient of $KL \cap (v_1^{-1} Z_{A_1(v_1,v_2)} \times v_2 Z_{C_2(v_1,v_2)})$.

3. Properties of the groups $R_{\mathcal{A}}$

Let $a \in P(\Gamma_1, \Gamma_2)$ be a partial isometry from Γ_1 to Γ_2, and let A_1 and A_2 be respectively the domain and the range of a. In this section, we first give a characterization of those subgroups K of $M_{A_1} \times M_{A_2}$ that are generalized a-graphs. We will then prove some properties of the groups $R_{\mathcal{A}}$ associated to an admissible pairs $\mathcal{A} = (a, K)$. These properties are crucial in the proofs of the main results in this paper.

Recall that for $i = 1, 2$, $\eta_i \colon G_1 \times G_2 \to G_i$ denote the projections to the i'th factor. For a subset D_i of Γ_i, M'_{D_i} denotes the derived subgroup of M_{D_i}. We will denote by Z_{D_i} the center of M_{D_i}.

Definition 3.1. By an a-quintuple we mean a quintuple $(K_1, X_1, K_2, X_2, \theta)$, where, for $i = 1, 2$, K_i is an abstract subgroup of M_{A_i} containing M'_{A_i}, X_i is an abstract subgroup of $K_i \cap Z_{A_i}$, and $\theta \colon K_1/X_1 \to K_2/X_2$ is a group isomorphism that maps U_1^α to $U_2^{a(\alpha)}$ for each $\alpha \in [A_1]$, where U_1^α is identified with its image in K_1/X_1 and similarly for $U_2^{a(\alpha)}$.

Lemma 3.2. *Let $K \subset M_{A_1} \times M_{A_2}$ be a generalized a-graph. Define*

$$K_1 = \eta_1(K) \subset M_{A_1}, \quad X_1 = \eta_1(\ker(\eta_2|_K)) = \{x_1 \in M_{A_1} \mid (x_1, e) \in K\} \subset K_1$$
$$K_2 = \eta_2(K) \subset M_{A_2}, \quad X_2 = \eta_2(\ker(\eta_1|_K)) = \{x_2 \in M_{A_2} \mid (e, x_2) \in K\} \subset K_2,$$

and let

$$\theta \colon K_1/X_1 \to K_2/X_2 \;\; \textit{where} \;\; \theta(k_1X_1) = k_2X_2 \quad \textit{if} \;\; (k_1, k_2) \in K. \tag{3.1}$$

Then θ is well-defined, and $\Theta(K) := (K_1, X_1, K_2, X_2, \theta)$ is an a-quintuple. Moreover, K can be expressed in terms of $\Theta(K)$ as

$$K = \{(k_1, k_2) \in K_1 \times K_2 \mid \theta(k_1X_1) = k_2X_2\}. \tag{3.2}$$

The assignment $K \mapsto \Theta(K)$ is a one to one correspondence between the set of generalized a-graphs and the set of a-quintuples.

Proof. Let $i = 1, 2$. Since K_i contains all one-parameter unipotent subgroups U_i^α of M_{A_i} and the latter generate M'_{A_i} as an abstract group, we have that $K_i \supset M'_{A_i}$. Moreover X_i is an abstract normal subgroup of K_i which does not intersect any one-parameter unipotent subgroup U_i^α of M_{A_i} because of the main condition for an a-graph, cf. Definition 1.1.

First we show that $X_i \subset Z_{A_i}$. We will make use of the following fact which can be found e.g. in [15, Corollary 29.5].

$(*)$ *Any simple algebraic group of adjoint type (over an algebraically closed field) is simple as an abstract group.*

Assume that X_i is not a subgroup of Z_{A_i}. Because of $K_i \supset M'_{A_i}$, we have that $M_{A_i} = K_iZ_{A_i}$. Since X_i is normal in K_i, $X_iZ_{A_i}/Z_{A_i}$ is a nontrivial abstract normal subgroup of M_{A_i}/Z_{A_i}. The group M_{A_i}/Z_{A_i} is semi-simple and has trivial center, so it is a direct product of simple algebraic groups of adjoint types. We can now apply $(*)$ to get that $X_iZ_{A_i}/Z_{A_i}$ contains a simple factor of M_{A_i}/Z_{A_i}. Therefore $(X_iZ_{A_i})'$ contains M'_{D_i} for some nontrivial subset D_i of A_i. (Here $(.)'$ refers to the derived subgroup of an abstract group.) As a consequence $X_i \supset X'_i = (X_iZ_{A_i})' \supset M'_{D_i}$ which contradicts with the fact that X_i does not intersect any one-parameter unipotent subgroup of M_{A_i}. This shows that $X_i \subset Z_{A_i}$.

It is easy to see that θ is a well-defined group isomorphism and that (3.2) holds. The definition of K implies that θ induces a group isomorphism from U_1^α to $U_2^{a(\alpha)}$ for each $\alpha \in [A_1]$. Thus $\Theta(K)$ is an a-quintuple. It is straightforward to check that the map $K \mapsto \Theta(K)$ is bijective. □

Notation 3.3. For $i = 1, 2$ and for a subset $D_i \subset A_i$, let

$$P_{D_i}^{A_i} = P_{D_i} \cap M_{A_i}, \qquad U_{D_i}^{A_i} = U_{D_i} \cap M_{A_i}.$$

For subsets $S_i \subset M_{A_i}$, we will set

$$K(S_1) = \{m_2 \in M_{A_2} \mid \exists\, m_1 \in S_1 \text{ such that } (m_1, m_2) \in K\} \tag{3.3}$$

$$K(S_2) = \{m_1 \in M_{A_1} \mid \exists\, m_2 \in S_2 \text{ such that } (m_1, m_2) \in K\}. \tag{3.4}$$

Lemma 3.4. *For any subset D_1 of A_1,*

1) $K(M_{D_1}) \subset M_{a(D_1)}$, $K(M_{a(D_1)}) \subset M_{D_1}$, and the intersection $(M_{D_1} \times M_{a(D_1)}) \cap K$ is generalized $a|_{D_1}$-graph;

2) the intersection $(U_{D_1}^{A_1} \times U_{a(D_1)}^{A_2}) \cap K$ is the graph of a group isomorphism $\theta\colon U_{D_1}^{A_1} \to U_{a(D_1)}^{A_2}$;

3) we have the decompositions

$$(P_{D_1} \times M_{A_2}) \cap K = (M_{A_1} \times P_{a(D_1)}) \cap K = (P_{D_1} \times P_{a(D_1)}) \cap K \tag{3.5}$$

$$= \big((M_{D_1} \times M_{a(D_1)}) \cap K\big)\left((U_{D_1}^{A_1} \times U_{a(D_1)}^{A_2}) \cap K\right) \tag{3.6}$$

$$(P_{D_1} \times P_{A_2}) \cap R_{\mathcal{A}} = (P_{A_1} \times P_{a(D_1)}) \cap R_{\mathcal{A}} = (P_{D_1} \times P_{a(D_1)}) \cap R_{\mathcal{A}} \tag{3.7}$$

$$= \big((M_{D_1} \times M_{a(D_1)}) \cap R_{\mathcal{A}}\big)\big((U_{D_1} \times U_{a(D_1)}) \cap R_{\mathcal{A}}\big) \tag{3.8}$$

$$= \big((P_{D_1} \times P_{a(D_1)}) \cap K\big)\,(U_{A_1} \times U_{A_2})\,. \tag{3.9}$$

Proof. We first prove that $K(H_1) \subset H_2$ (corresponding to the special case when $D_1 = \emptyset$). Assume that $(h_1, m_2) \in (H_1 \times M_{A_2}) \cap K$. For $\beta \in [A_2]$, let $\alpha = a^{-1}(\beta)$, and let $\theta_\alpha\colon U_1^\alpha \to U_2^\beta$ be the group isomorphism whose graph is $(U_1^\alpha \times U_2^\beta) \cap K$. For every $(u_1, u_2) \in (U_1^\alpha \times U_2^\beta) \cap K$, we have

$$(h_1 u_1 h_1^{-1},\ m_2 u_2 m_2^{-1}) \in K, \quad \text{and} \quad (h_1 u_1 h_1^{-1}, \theta_\alpha(h_1 u_1 h_1^{-1})) \in K.$$

Thus $m_2 u_2 m_2^{-1} \theta_\alpha(h_1 u_1 h_1^{-1})^{-1} \in X_2 \subset Z_{A_2}$, so $m_2 u_2 m_2^{-1} \in H_2 U_2^\beta$. Thus m_2 normalizes both Borel subgroups of M_{A_2} defined by $[A_2] \cap \Delta_2^+$ and by $[A_2] \cap (-\Delta_2^+)$. Thus $m_2 \in H_2$. This shows that $K(H_1) \subset H_2$. Similarly, one shows that $K(H_2) \subset H_1$.

Let now D_1 be any subset of A_1. Assume that $(m_1, m_2) \in (M_{D_1} \times M_{A_2}) \cap K$. Write $m_1 = h_1 m_1'$, where $h_1 \in H_1$ and $m_1' \in M_{D_1}'$. Since M_{D_1}' is generated by the U_1^α's for $\alpha \in [D_1]$, there exists $m_2' \in M_{a(D_1)}'$ such that $(m_1', m_2') \in K$. Thus $(h_1, m_2(m_2')^{-1}) \in K$. It follows from $K(H_1) \subset H_2$ that $m_2(m_2')^{-1} \in H_2$. Thus $m_2 \in H_2 M_{a(D_1)}' = M_{a(D_1)}$. This shows that $K(M_{D_1}) \subset M_{a(D_1)}$. One proves similarly that $K(M_{a(D_1)}) \subset M_{D_1}$. It is clear from the definition that $(M_{D_1} \times M_{a(D_1)}) \cap K$ is a generalized $a|_{D_1}$-graph. This proves 1).

Let $\theta\colon K_1/X_1 \to K_2/X_2$ be given as in (3.1). Since $X_1 \cap U_{D_1}^{A_1} = \{e\}$ and $X_2 \cap U_{a(D_1)}^{A_2} = \{e\}$, we can embed $U_{D_1}^{A_1}$ into K_1/X_1 and $U_{a(D_1)}^{A_2}$ into K_2/X_2. Then θ induces a group isomorphism from $U_{D_1}^{A_1}$ to $U_{a(D_1)}^{A_2}$ whose graph is the intersection $(U_{D_1}^{A_1} \times U_{a(D_1)}^{A_2}) \cap K$. This proves 2).

To prove (3.5) and (3.6), assume that $(p_1, m_2) \in (P_{D_1} \times M_{A_2}) \cap K$. Write $p_1 = m_1 u_1$, where $m_1 \in M_{D_1}$ and $u_1 \in U_{D_1}$. Note that $u_1 \in U_{D_1}^{A_1}$ because $p_1, m_1 \in M_{A_1}$. By 2), there exists $u_2 \in U_{a(D_1)}^{A_2}$ such that $(u_1, u_2) \in K$. Thus $(m_1, m_2 u_2^{-1}) \in K$. By 1), $m_2 u_2^{-1} \in M_{a(D_1)}$. Thus $m_2 \in M_{a(D_1)} U_{a(D_1)}^{A_2} \in P_{a(D_1)}$, and

$$(p_1, m_2) = (m_1, m_2 u_2^{-1})(u_1, u_2) \in \big((M_{D_1} \times M_{a(D_1)}) \cap K\big)\left((U_{D_1}^{A_1} \times U_{a(D_1)}^{A_2}) \cap K\right).$$

This shows that

$$(P_{D_1} \times M_{A_2}) \cap K = (P_{D_1} \times P_{a(D_1)}) \cap K = \big((M_{D_1} \times M_{a(D_1)}) \cap K\big)\left((U_{D_1}^{A_1} \times U_{a(D_1)}^{A_2}) \cap K\right).$$

Similarly, one proves that $(M_{A_1} \times P_{a(D_1)}) \cap K$ is also equal to any one of the above three groups. This proves (3.5) and (3.6).

The identities in (3.7) follow directly from those in (3.5). To prove (3.8) and (3.9), assume now that $(p_1, p_2) \in (P_{D_1} \times P_{a(D_1)}) \cap R_{\mathcal{A}}$. Write $p_1 = m_1 u_1$ and $p_2 = m_2 u_2$, where

$m_1 \in M_{D_1}, u_1 \in U_{D_1}, m_2 \in M_{a(D_1)}$ and $u_2 \in U_{a(D_1)}$. Further write

$$u_1 = u_{D_1}^{A_1} u_{A_1}, \quad u_2 = u_{a(D_1)}^{A_2} u_{A_2},$$

where

$$u_{D_1}^{A_1} \in U_{D_1}^{A_1}, \quad u_{A_1} \in U_{A_1}, \quad u_{a(D_1)}^{A_2} \in u_{a(D_1)}^{A_2}, \quad u_{A_2} \in U_{A_2}.$$

Since $(u_{A_1}, u_{A_2}) \in R_{\mathcal{A}}$, we have

$$\Big(m_1, u_{D_1}^{A_1},\, m_2 u_{a(D_1)}^{A_2}\Big) \in \Big(P_{D_1}^{A_1} \times P_{a(D_1)}^{A_2}\Big) \cap R_{\mathcal{A}} = \Big(P_{D_1}^{A_1} \times P_{a(D_1)}^{A_2}\Big) \cap K.$$

By (3.6), we see that $(m_1, m_2) \in R_{\mathcal{A}}$. This shows (3.8). Note that $(M_{D_1} \times M_{a(D_1)}) \cap R_{\mathcal{A}} = (M_{D_1} \times M_{a(D_1)}) \cap K$. It is also easy to show that

$$(U_{D_1} \times U_{a(D_1)}) \cap R_{\mathcal{A}} = \Big((U_{D_1}^{A_1} \times U_{a(D_1)}^{A_2}) \cap K\Big) (U_{A_1} \times U_{A_2})\,.$$

Now (3.9) follows from (3.6). □

Finally we use Lemma 3.2 to treat compositions of generalized graphs.

Lemma 3.5. *Let G_1, G_2 and G_3 be connected reductive algebraic groups (with fixed choices of maximal tori and Borel subgroups). Assume that (a, K) and (c, L) are two admissible pairs for $G_1 \times G_2$ and $G_2 \times G_3$, respectively, such that the domain of c coincides with the range of a. Then the composition $(ca, L \circ K)$ is an admissible pair for $G_1 \times G_3$.*

Proof. Lemma 3.2 implies that for any root α of G_1 in the span of the domain of a, there exist group isomorphisms $\theta\colon U_1^\alpha \to U_2^{a(\alpha)}$ and $\varphi\colon U_2^{a(\alpha)} \to U_3^{ca(\alpha)}$ such that

$$K \cap (U_1^\alpha \times G_2) = (\{e\} \times X_2)\, (\mathrm{Id} \times \theta)(U_1^\alpha) \quad \text{and}$$
$$L \cap (G_2 \times U_3^{ca(\alpha)}) = (Y_2 \times \{e\})\, (\mathrm{Id} \times \varphi)(U_3^{a(\alpha)})$$

for some subgroups X_2 and Y_2 of the fixed maximal torus of G_2. Then

$$(L \circ K) \cap (U_1^\alpha \times U_3^{ca(\alpha)}) = (\mathrm{Id} \times \varphi\theta)(U_1^\alpha).$$

Therefore $L \circ K$ is a generalized ca-graph and $(ca, L \circ K)$ is an admissible pair for $G_1 \times G_3$. Let us note that it is not hard to determine the ca-quintuple corresponding to the composition $L \circ K$, cf. Lemma 3.2, but it will not be needed and will omit it. □

4. Some facts on Weyl groups and intersections of parabolic subgroups

4.1. **The Bruhat Lemma.** Fix a connected reductive algebraic group G over k with a maximal torus H and a set Δ^+ of positive roots for (G, H). Let Γ be the set of simple roots in Δ^+. We will denote by W the Weyl group of (G, H) and by $l(\cdot)$ the standard length function on W. For $w \in W$, $\dot{w}$ will denote a representative of w in the normalizer of H in G.

Given $A \subset \Gamma$, we will denote by W_A the subgroup of the Weyl group W generated by elements in A. For $A \subset \Gamma$, let P_A be the corresponding parabolic subgroup of G containing the Borel subgroup of G determined by Δ^+, and let M_A and U_A be respectively its Levi factor containing H and its unipotent radical. For $A, C \subset \Gamma$, the standard Bruhat decomposition for (P_A, P_C) double cosets of G states that

$$G = \coprod_{[w] \in W_A \backslash W / W_C} P_A \dot{w} P_C. \tag{4.1}$$

We note that if $\widetilde{P}_A$ is a subgroup of P_A such that $P_A = \widetilde{P}_A H$, then we also have

$$G = \coprod_{[w] \in W_A \backslash W / W_C} \widetilde{P}_A \dot{w} P_C, \tag{4.2}$$

which is a fact that will be used later in the paper.

4.2. **Minimal length representatives of double cosets.** Let $A, C \subset \Gamma$. Each (W_A, W_C) double coset of W contains a unique element of minimal length, see e.g. [3, Proposition 2.7.3]. The set of minimal length representatives for (W_A, W_C) double cosets will be denoted by ${}^A W^C$. The latter set consists of exactly those elements $w \in W$ with the properties

$$w^{-1}(A) \subset \Delta^+ \text{ and } w(C) \subset \Delta^+, \tag{4.3}$$

see e.g. [3, §2.7]. When A is empty, we set ${}^A W^C = W^C$. If D and E are two subsets of $A \subset \Gamma$, the set of minimal length representatives in W_A for the double cosets from $W_D \backslash W_A / W_E$ will be denoted by ${}^D W_A^E$. Elements in ${}^D W_A^E$ will be thought of as elements of W via the inclusion of W_A in W.

Proposition 4.1. *Fix three subsets A, D, and C of Γ with $D \subset A$. Then every element $v \in {}^D W^C$ is represented in a unique way as a product*

$$v = uw \ \text{ for some } \ w \in {}^A W^C, \ u \in {}^D W_A^{A \cap w(C)}, \tag{4.4}$$

and all such products belong to ${}^D W^C$. Moreover, $l(v) = l(u) + l(w)$ for u, v, and w in (4.4).

Note that the defining set for u in (4.4) depends on the first element w. A proof of this result can be found in [3, Proposition 2.7.5], see also [23, Lemma 4.3].

By taking inverses in (4.4), one sees that for three subsets A, E, and C of Γ with $E \subset C$, every element $v \in {}^A W^E$ is represented in a unique way as a product

$$v = wu \text{ for some } w \in {}^A W^C, \ u \in {}^{w^{-1}(A) \cap C} W_C^E, \tag{4.5}$$

and all such products belong to ${}^A W^D$. We also note that if $w \in {}^A W^C$, the minimality conditions on w imply that

$$[A]^+ \cap w([C]^+) = [A \cap w(C)]^+ \quad \text{ and } \quad A \cap w([C]^+) = A \cap w(C), \tag{4.6}$$

where for $D \subset \Gamma$, we have $[D]^+ = \Delta^+ \cap \mathbb{Z}[D]$.

4.3. **Intersections of parabolic subgroups.** For $A \subset \Gamma$ and $w \in W$, we set

$$w(P_A) = \mathrm{Ad}_{\dot{w}}(P_A), \ w(M_A) = \mathrm{Ad}_{\dot{w}}(M_A), \ w(U_A) = \mathrm{Ad}_{\dot{w}}(U_A) \tag{4.7}$$

all of which are independent of the choice of the representative $\dot{w}$. We will use π_A to denote the projection $P_A \to M_A$ with respect to the decomposition $P_A = M_A U_A$. For a subset D of A, we will set $P_D^A = P_D \cap M_A$ and $U_D^A = U_D \cap M_A$.

Lemma 4.2. *Assume that $A, C \subset \Gamma$ and $w \in {}^A W^C$. Then*

1) we have the direct product decompositions

$$\begin{aligned}
P_A \cap w(P_C) &= (M_A \cap w(M_C))\,(M_A \cap w(U_C))\,(U_A \cap w(M_C))\,(U_A \cap w(U_C)) && (4.8)\\
&= M_{A \cap w(C)}\, U^A_{A \cap w(C)} \left(w(U^C_{w^{-1}(A) \cap C}) \right) (U_A \cap w(U_C)) && (4.9)\\
&= M_{A \cap w(C)} \left(U_{A \cap w(C)} \cap w(U) \right) && (4.10)\\
&= M_{A \cap w(C)} \left(U_{A \cap w(C)} \cap w(U_{w^{-1}(A) \cap C}) \right) && (4.11)
\end{aligned}$$

where $M_A \cap w(M_C) = M_{A \cap w(C)}$ normalizes all terms. In particular, $P_A \cap w(P_C) \subset P_{A \cap w(C)}$ and $\pi_A(P_A \cap w(P_C)) = P^A_{A \cap w(C)}$;

2) the following dimension formulas hold for the groups in 1):

$$\dim U^A_{A \cap w(C)} + \dim(U_A \cap w(U_C)) = \dim U_C - l(w),$$
$$\dim w(U^C_{w^{-1}(A) \cap C}) + \dim(U_A \cap w(U_C)) = \dim U_A - l(w).$$

Proof. The first part of Lemma 4.2 is Theorem 2.8.7 from [3]. It only uses (4.3) and (4.6).

For Part 2), we only prove the second formula since the proof of the first one is similar. To this end, note that $w(U^C_{w^{-1}(A)\cap C})(U_A \cap w(U_C)) \subset U_A$ and that it is the product of the one parameter unipotent subgroups of G, corresponding to roots

$$\alpha \in \Delta^+ - [A]^+ \text{ such that } w^{-1}(\alpha) \in \Delta^+.$$

Since $w \in {}^AW$, one has $w^{-1}(\alpha) \in \Delta^+$ for any $\alpha \in [A]^+$. Thus the codimension of $w(U^C_{w^{-1}(A)\cap C})(U_A \cap w(U_C))$ in U_A is equal to the cardinality of the set $\Delta^+ \cap w(-\Delta^+)$ which is equal to the length of w, and hence the second formula in 2). □

4.4. **A lemma on orbits of subgroups.** The following lemma, which will be used later in the paper in conjunction with the Bruhat lemma, describes the orbit space of a group on a set in terms of that of a bigger group. We omit the proof since it is straightforward.

Lemma 4.3. *Let P be a group acting (on the left) on a set M and let R be a subgroup of P. Suppose that $\mathcal{O}$ is a subset of M parametrizing the P orbits on M. For each $m \in \mathcal{O}$, let Stab_m be the stabilizer subgroup of P at m. Then every R orbit in M contains a point of the form $p \cdot m$ for a unique $m \in \mathcal{O}$ and some $p \in P$. Two points $p_1 \cdot m$ and $p_2 \cdot m$, where $m \in \mathcal{O}$ and $p_1, p_2 \in P$, are in the same R orbit if and only if p_1 and p_2 are in the same (R, Stab_m) double coset in P.*

5. Proof of Theorem 2.2

5.1. **A special case.** For two arbitrary admissible quadruples $\mathcal{A}$ and $\mathcal{C}$, we will obtain a description of $(R_\mathcal{A}, R_\mathcal{C})$ double coset in $G_1 \times G_2$ by an induction procedure to be presented in §5.2. In this section, we look at the last step in the induction as a special case.

Assume that both a and c have domains Γ_1 and ranges Γ_2. Then $R_\mathcal{A} = K$ and $R_\mathcal{C} = L$. Modular problems with the centers, we can think of both K and L as graphs of isomorphisms from G_1 to G_2.

Denote the centers and the derived subgroups of G_1 and G_2 respectively by Z_1, Z_2, G_1' and G_2'. Recall that $\eta_2 \colon G_1 \times G_2 \to G_2$ is the projection to the second factor and that $K_2 = \eta_2(K)$, $L_2 = \eta_2(L)$. Let $Z(K_2)$ and $Z(L_2)$ be the centers of K_2 and L_2 respectively. It follows from $K_2, L_2 \supset G_2'$ that $Z(K_2) = Z_2 \cap K_2$ and $Z(L_2) = Z_2 \cap L_2$. Set

$$\mathcal{Z}_2 = Z(K_2)Z(L_2) \subset Z_2.$$

For each $s \in Z_2/\mathcal{Z}_2$, fix a representative $\dot{s}$ of s in Z_2. We will also introduce the group

$$J = \{(k_1, l_1) \in G_1 \times G_1 \mid \exists\, g_2 \in G_2 \text{ such that } (k_1, g_2) \in K, (l_1, g_2) \in L\}. \tag{5.1}$$

In terms of compositions of group correspondences in §1.6, $J = \sigma(L) \circ K$, where $\sigma \colon G_1 \times G_2 \to G_2 \times G_1 \colon (g_1, g_2) \mapsto (g_2, g_1)$. By Lemma 3.5, J is a generalized $c^{-1}a$-graph. Let J act on G_1 from the left by

$$(k_1, l_1) \cdot g_1 = k_1 g_1 l_1^{-1}, \qquad (k_1, l_1) \in J,\ g_1 \in G_1.$$

Lemma 5.1. *When both a and c have domains Γ_1 and ranges Γ_2, every $(R_\mathcal{A}, R_\mathcal{C})$ double coset of $G_1 \times G_2$ is of the form*

$$[(g_1, \dot{s})] \text{ for some } g_1 \in G_1,\ s \in Z_2/\mathcal{Z}_2.$$

Two such cosets $[(g_1, \dot{s})]$ and $[(h_1, \dot{t})]$ are the same if and only if $s = t$ and g_1 and h_1 are in the same J orbit on G_1 for the J action on G_1 given in (5.1).

Proof. Using the fact that $G_2 = Z_2 L_2$, we see that every $(R_\mathcal{A}, R_\mathcal{C})$ double coset of $G_1 \times G_2$ is of the form $[(g_1, z)]$ for some $g_1 \in G_1$ and $z \in Z_2$. By writing $z = \dot{s} z_K z_L$ for some $s \in Z_2/\mathcal{Z}_2$, $z_K \in Z(K_2)$, and $z_L \in Z(L_2)$, and by changing g_1 to another element in G_1 if necessary, we see that every $(R_\mathcal{A}, R_\mathcal{C})$ double coset of $G_1 \times G_2$ is of the form $[(g_1, \dot{s})]$ for some $g_1 \in G_1$ and $s \in Z_2/\mathcal{Z}_2$. It is clear that if $g_1, h_1 \in G_1$ are in the same J orbit in G_1, then $[(g_1, \dot{s})] = [(h_1, \dot{s})]$ for any $s \in Z_2/\mathcal{Z}_2$. Assume now that $[(g_1, \dot{s})] = [(h_1, \dot{t})]$ for some $g_1, h_1 \in G_1$ and $s, t \in Z_2/\mathcal{Z}_2$. Then there exist $(k_1, k_2) \in K, (l_1, l_2) \in L$ such that

$$(h_1, \dot{t}) = (k_1, k_2)(g_1, \dot{s})(l_1, l_2)^{-1} = (k_1 g_1 l_1^{-1},\ \dot{s} k_2 l_2^{-1}).$$

We claim that $k_2 l_2^{-1} \in \mathcal{Z}_2$. Indeed, write $k_2 = k_2' z_K$, where $k_2' \in G_2'$ and $z_K \in Z_2$. Since $G_2' \subset K_2$, $z_K \in K_2 \cap Z_2 = Z(K_2)$. Similarly, we can write $l_2^{-1} = l_2' z_L$, where $l_2' \in G_2'$ and $z_L \in Z(L_2)$. Thus $k_2 l_2^{-1} = (k_2' l_2') z_K z_L$. On the other hand, $k_2 l_2^{-1} = \dot{t}\dot{s}^{-1} \in Z_2$, so $k_2' l_2' \in G_2' \cap Z_2 \subset Z(K_2)$. Thus $k_2 l_2^{-1} = (k_2' l_2' z_K) z_L \in \mathcal{Z}_2$. It now follows that $\dot{t} \in \dot{s}\mathcal{Z}_2$, so $s = t$, and $\dot{s} = \dot{t}$. Thus $k_2 = l_2$, and $(k_1, l_1) \in J$. Hence g_1 and h_1 are in the same J orbit in G_1. □

5.2. **The main induction step in the proof of Theorem 2.2.** In this section, we will reduce the classification of $(R_\mathcal{A}, R_\mathcal{C})$ double cosets in $G_1 \times G_2$ to that of similar double cosets in $M_{A_1} \times M_{C_2}$. More precisely, for each pair $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$, we will define two new admissible pairs

$$\mathcal{A}^{\mathrm{new}}(w_1, w_2) \quad \text{and} \quad \mathcal{C}^{\mathrm{new}}(w_1, w_2)$$

for $M_{A_1} \times M_{C_2}$. Denoting by $R_\mathcal{A}^{\mathrm{new}}(w_1, w_2)$ and $R_\mathcal{C}^{\mathrm{new}}(w_1, w_2)$ the subgroups of $M_{A_1} \times M_{C_2}$ defined according to (1.1), corresponding respectively to the admissible pairs $\mathcal{A}^{\mathrm{new}}(w_1, w_2)$ and $\mathcal{C}^{\mathrm{new}}(w_1, w_2)$, we will show that the double coset space $R_\mathcal{A} \backslash G_1 \times G_2 / R_\mathcal{C}$ can be identified with the union over ${}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$ of the double coset spaces

$$R_\mathcal{A}^{\mathrm{new}}(w_1, w_2) \backslash M_{A_1} \times M_{C_2} / R_\mathcal{C}^{\mathrm{new}}(w_1, w_2).$$

Let $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$. Set

$$A_1^{\mathrm{new}}(w_1, w_2) = a^{-1}(A_2 \cap w_2(C_2)), \qquad A_2^{\mathrm{new}}(w_1, w_2) = C_2 \cap w_2^{-1}(A_2) \tag{5.2}$$

$$C_1^{\mathrm{new}}(w_1, w_2) = A_1 \cap w_1(C_1), \qquad C_2^{\mathrm{new}}(w_1, w_2) = c(C_1 \cap w_1^{-1}(A_1)). \tag{5.3}$$

We will regard

$$w_2^{-1}a\colon A_1^{\mathrm{new}}(w_1, w_2) \to A_2^{\mathrm{new}}(w_1, w_2) \ \text{ and } \ cw_1^{-1}\colon C_1^{\mathrm{new}}(w_1, w_2) \to C_2^{\mathrm{new}}(w_1, w_2)$$

as partial symmetries from A_1 to C_2. For $i = 1, 2$, let $\dot{w}_i$ be a representative of w_i in the normalizer of H_i in G_i. Define

$$K^{\mathrm{new}}(w_1, w_2) = \left(M_{A_1^{\mathrm{new}}(w_1,w_2)} \times M_{A_2^{\mathrm{new}}(w_1,w_2)}\right) \cap \mathrm{Ad}_{(e,\dot{w}_2)}^{-1} K \tag{5.4}$$

$$L^{\mathrm{new}}(w_1, w_2) = \left(M_{C_1^{\mathrm{new}}(w_1,w_2)} \times M_{C_2^{\mathrm{new}}(w_1,w_2)}\right) \cap \mathrm{Ad}_{(\dot{w}_1,e)} L. \tag{5.5}$$

Then it follows from Lemma 3.4 that

$$\mathcal{A}^{\mathrm{new}}(w_1, w_2) := (w_2^{-1}a,\ K^{\mathrm{new}}(w_1, w_2)) \quad \text{and} \quad \mathcal{C}^{\mathrm{new}}(w_1, w_2) := (cw_1^{-1},\ L^{\mathrm{new}}(w_1, w_2))$$

are admissible pairs for $M_{A_1} \times M_{C_2}$. Let

$$R_\mathcal{A}^{\mathrm{new}}(w_1, w_2) = K^{\mathrm{new}}(w_1, w_2)\left(U^{A_1}_{A_1^{\mathrm{new}}(w_1,w_2)} \times U^{C_2}_{A_2^{\mathrm{new}}(w_1,w_2)}\right)$$

$$R_\mathcal{C}^{\mathrm{new}}(w_1, w_2) = L^{\mathrm{new}}(w_1, w_2)\left(U^{A_1}_{C_1^{\mathrm{new}}(w_1,w_2)} \times U^{C_2}_{C_2^{\mathrm{new}}(w_1,w_2)}\right).$$

Proposition 5.2. (Induction Step): *Every $(R_\mathcal{A}, R_\mathcal{C})$ double coset in $G_1 \times G_2$ is of the form*

$$[(m_{A_1}\dot{w}_1, \dot{w}_2 m_{C_2})] \ \textit{for some } w_i \in {}^{A_i}W_i^{C_i},\ m_{A_1} \in M_{A_1},\ m_{C_2} \in M_{C_2}.$$

Two such cosets $[(m_{A_1}\dot{w}_1, \dot{w}_2 m_{C_2})]$ and $[(m'_{A_1}\dot{w}'_1, \dot{w}'_2 m'_{C_2})]$ coincide if and only if $w'_1 = w_1, w'_2 = w_2$, and (m_{A_1}, m_{C_2}) and (m'_{A_1}, m'_{C_2}) are in the same $(R_\mathcal{A}^{\mathrm{new}}(w_1, w_2), R_\mathcal{C}^{\mathrm{new}}(w_1, w_2))$ double coset of $M_{A_1} \times M_{C_2}$.

Proof. Recall that for $i = 1, 2$,

$$K_i = \eta_i(K) \subset M_{A_1} \quad \text{and} \quad L_i = \eta_i(L) \subset M_{C_i},$$

where $\eta_i \colon G_1 \times G_2 \to G_i$ is the projection to the i'th factor. Let

$$\widetilde{P}_{A_2} = K_2 U_{A_2} \subset P_{A_2}, \qquad P_{\mathcal{A},1} = (M_{A_1} \times \{e\})\, R_{\mathcal{A}} = P_{A_1} \times \widetilde{P}_{A_2}$$
$$\widetilde{P}_{C_1} = L_1 U_{C_1} \subset P_{C_1}, \qquad P_{\mathcal{C},2} = (\{e\} \times M_{C_2})\, R_{\mathcal{C}} = \widetilde{P}_{C_1} \times P_{C_2}.$$

Consider the (left) action of $P_{\mathcal{A},1} \times P_{\mathcal{C},2}$ on $G_1 \times G_2$, defined by

$$(p_{A_1}, \tilde{p}_{A_2}, \tilde{p}_{C_1}, p_{C_2}) \cdot (g_1, g_2) = (p_{A_1} g_1 \tilde{p}_{C_1}^{-1}, \tilde{p}_{A_2} g_2 p_{C_2}^{-1}) \tag{5.6}$$

for $(p_{A_1}, \tilde{p}_{A_2}, \tilde{p}_{C_1}, p_{C_2}) \in P_{\mathcal{A},1} \times P_{\mathcal{C},2}$ and $(g_1, g_2) \in G_1 \times G_2$. The set of orbits for this action coincides with the set of $(P_{\mathcal{A},1}, P_{\mathcal{C},2})$ double cosets of $G_1 \times G_2$, which, by (4.2), consists of the double cosets of the form $P_{\mathcal{A},1}(\dot{w}_1, \dot{w}_2)P_{\mathcal{C},2}$ for $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$. We will apply Lemma 4.3 for $P = P_{\mathcal{A},1} \times P_{\mathcal{C},1}$ and $R = R_{\mathcal{A}} \times R_{\mathcal{C}}$ to classify $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$. For $w_i \in {}^{A_i}W_i^{C_i}$ with $i = 1, 2$, let $\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)} \subset P_{\mathcal{A},1} \times P_{\mathcal{C},2}$ be the stabilizer subgroup of $P_{\mathcal{A},1} \times P_{\mathcal{C},2}$ at $(\dot{w}_1, \dot{w}_2)$. In view of Lemma 4.3, to classify $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$, it suffices to understand the space of double cosets

$$(R_{\mathcal{A}} \times R_{\mathcal{C}}) \backslash P_{\mathcal{A},1} \times P_{\mathcal{C},2} / \mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)} \tag{5.7}$$

for each pair $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$. To this end, consider the projection

$$\pi := \pi_{A_1} \times \pi_{A_2} \times \pi_{C_1} \times \pi_{C_2} \colon P_{\mathcal{A},1} \times P_{C,2} \to M_{A_1} \times K_2 \times L_1 \times M_{C_2}.$$

(See §4 for notation). Since $M_{A_1} \times K_2 \times L_1 \times M_{C_2}$ normalizes $U_{A_1} \times U_{A_2} \times U_{C_1} \times U_{C_2}$, π induces an identification

$$(R_{\mathcal{A}} \times R_{\mathcal{C}}) \backslash P_{\mathcal{A},1} \times P_{\mathcal{C},2} / \mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)} \cong (K \times L) \backslash M_{A_1} \times K_2 \times L_1 \times M_{C_2} / \pi(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)}).$$

Note now that we can identify

$$(K \times L) \backslash (M_{A_1} \times K_2 \times L_1 \times M_{C_2}) \cong (X_1 \times Y_2) \backslash (M_{A_1} \times M_{C_2}), \tag{5.8}$$

where $X_1 = \eta_1(\ker(\eta_2|_K)) \subset Z_{A_1}$ and $Y_2 = \eta_2(\ker(\eta_1|_L)) \subset Z_{C_2}$. Indeed, for

$$(m_{A_1}, k_2, l_1, m_{C_2}) \in M_{A_1} \times K_2 \times L_1 \times M_{C_2},$$

let $k_1 \in K_1$ and $l_2 \in L_2$ be such that $(k_1, k_2) \in K$ and $(l_1, l_2) \in L$. Then

$$(K \times L)(m_{A_1}, k_2, l_1, m_{C_2}) = (K \times L)(k_1^{-1} m_{A_1},\, e,\, e,\, l_2^{-1} m_{C_2}).$$

It is easy to see that the map

$$(K \times L)(m_{A_1}, k_2, l_1, m_{C_2}) \mapsto (X_1 \times Y_2)(k_1^{-1} m_{A_1},\, l_2^{-1} m_{C_2})$$

gives a well-defined identification of the two spaces in (5.8), and that the right translation of $(m_{A_1}, k_2, l_1, m_{C_2})$ on $(K \times L) \backslash (M_{A_1} \times K_2 \times L_1 \times M_{C_2})$ becomes the following map on $(X_1 \times Y_2) \backslash (M_{A_1} \times M_{C_2})$:

$$(X_1 \times Y_2)(m_1, m_2) \overset{(m_{A_1}, k_2, l_1, m_{C_2})}{\longmapsto} (X_1 \times Y_2)(k_1^{-1} m_1 m_{A_1},\, l_2^{-1} m_2 m_{C_2}). \tag{5.9}$$

Thus, every $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset in $G_1 \times G_2$ is of the form $[(m_{A_1} \dot{w}_1, \dot{w}_2 m_{C_2}^{-1})]$ for a unique pair $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$ and for some $(m_{A_1}, m_{C_2}) \in M_{A_1} \times M_{C_2}$. Two such double cosets $[(m_{A_1} \dot{w}_1, \dot{w}_2 m_{C_2}^{-1})]$ and $[(l_{A_1} \dot{w}_1, \dot{w}_2 l_{C_2}^{-1})]$ coincide if and only if $(X_1 \times Y_2)(m_{A_1}, m_{C_2})$ and $(X_1 \times Y_2)(l_{A_1}, nl_{C_2})$ are in the same $\pi(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)})$ orbit for the action of $\pi(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)})$ on $(X_1 \times Y_2) \backslash (M_{A_1} \times M_{C_2})$ given in (5.9). Consider the map

$$\nu \colon M_{A_1} \times M_{C_2} \to (X_1 \times Y_2) \backslash (M_{A_1} \times M_{C_2}) \colon \nu(m_{A_1}, m_{C_2}) = (X_1 \times Y_2)(m_{A_1}, m_{C_2}^{-1}).$$

The proof of Proposition 5.2 will be complete once we prove the following fact about ν: for every $(m_{A_1} \times m_{C_2}) \in M_{A_1} \times M_{C_2}$,

(5.10)
$$\nu^{-1}\left((X_1 \times Y_2)(m_{A_1}, m_{C_2}^{-1}) \cdot \pi(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)})\right) = R_{\mathcal{A}}^{\mathrm{new}}(w_1, w_2)(m_{A_1}, m_{C_2}) R_{\mathcal{C}}^{\mathrm{new}}(w_1, w_2),$$

where again the action of $\pi(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)})$ on $(X_1 \times Y_2)\backslash(M_{A_1} \times M_{C_2})$ is given in (5.9). To prove this property of ν, we set

$$D_1 = A_1 \cap w_1(C_1),\ D_2 = A_2 \cap w_2(C_2),\ E_1 = C_1 \cap w_1^{-1}(A_1),\ E_2 = C_2 \cap w_2^{-1}(A_2).$$

Then we know from 1) of Lemma 4.2 that $\pi(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)})$ consists precisely of all the elements of the form

$$(\mathrm{Ad}_{\dot{w}_1}(l_1) u_{D_1}^{A_1},\ k_2 u_{D_2}^{A_2},\ l_1 u_{E_1}^{C_1},\ \mathrm{Ad}_{\dot{w}_2}^{-1}(k_2) u_{E_2}^{C_2}),$$

where $u_{D_i}^{A_i} \in U_{D_i}^{A_i}$, $u_{E_i}^{C_i} \in U_{E_i}^{C_i}$ for $i = 1, 2$, $l_1 \in L_1 \cap M_{E_1}$, and $k_2 \in K_2 \cap M_{D_2}$. It is then easy to see from the definition of the action in (5.9) and from the properties of the groups K and L as stated in 2) and 3) of Lemma 3.4 that (5.10) holds. □

5.3. **Proof of Theorem 2.2.** Fix two elements $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$. According to Proposition 4.1 and (4.5), they can be uniquely decomposed as

$$v_1 = u_1 w_1,\ w_1 \in {}^{A_1}W_1^{C_1},\ u_1 \in W_{A_1}^{A_1 \cap w_1(C_1)}; \tag{5.11}$$

$$v_2 = w_2 u_2,\ w_2 \in {}^{A_2}W_2^{C_2},\ u_2 \in {}^{w_2^{-1}(A_2) \cap C_2}W_{C_2}. \tag{5.12}$$

Recall the definition of $A_1^{\mathrm{new}}(w_1, w_2)$ in (5.2). Set

(5.13)
$$A_1^{\mathrm{new}}(w_1, w_2)(u_1, u_2) = \{\alpha \in A_1^{\mathrm{new}}(w_1, w_2)\ |(u_1(cw_1^{-1})^{-1} u_2^{-1}(w_2^{-1}a))^n \alpha \text{ is well defined}$$
$$\text{and is in } A_1^{\mathrm{new}}(w_1, w_2) \text{ for } n = 0, 1, \ldots\}.$$

Recall that $A_1(v_1, v_2)$ is defined in (2.1).

Lemma 5.3. *Let $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$ be decomposed as $v_1 = u_1 w_1$ and $v_2 = w_2 u_2$ as in* (5.11)-(5.12). *Then $A_1^{\mathrm{new}}(w_1, w_2)(u_1, u_2) = A_1(v_1, v_2)$.*

Proof. Note that the composition of maps in (5.13) is nothing but

$$(u_1 w_1) c^{-1} (u_2^{-1} w_2^{-1}) a = v_1 c^{-1} v_2^{-1} a.$$

Thus if $\alpha \in A_1^{\mathrm{new}}(w_1, w_2)(u_1, u_2)$ then $(v_1 f^{-1} v_2^{-1} d)^n \alpha$ is well-defined and is a root of $A_1^{\mathrm{new}}(w_1, w_2) \subset A_1$. Therefore $\alpha \in A_1(v_1, v_2)$. Next we prove the opposite inclusion. First we show that if $\beta \in A_1$ and $c^{-1} v_2^{-1} a(\beta)$ is well-defined, then $\beta \in A_1^{\mathrm{new}}(w_1, w_2)$. The assumptions imply that $a(\beta) \in A_2$ and $u_2^{-1} w_2^{-1} a(\beta) \in C_2$. Since $u_2 \in W_{C_2}$, we get that $w_2^{-1} a(\beta) \in [C_2]$. Because $w_2 \in {}^{A_2}W_2^{C_2}$, we further deduce that $w_2^{-1} a(\beta) \in [C_2]^+$. Now $a(\beta) \in A_2 \cap w_2([C_2]^+)$ and due to (4.6), $a(\beta) \in A_2 \cap w_2(C_2)$, so $\beta \in A_1^{\mathrm{new}}(w_1, w_2)$. If $\alpha \in A_1(v_1, v_2)$ then $(c^{-1} v_2^{-1} a)(v_1 c^{-1} v_2^{-1} a)^n(\alpha)$ needs to be well defined for all $n = 0, 1, \ldots$ so $\alpha \in A_1^{\mathrm{new}}(w_1, w_2)$ and $(v_1 c^{-1} v_2^{-1} a)^n(\alpha)$ is well defined and is in $A_1^{\mathrm{new}}(w_1, w_2)$ for all $n = 1, 2, \ldots$ Therefore $\alpha \in A_1^{\mathrm{new}}(w_1, w_2)(u_1, u_2)$. □

Recall the groups $K(v_1, v_2)$ and $L(v_1, v_2)$ given in (2.4) and (2.5). It is easy to see from Lemma 5.1 that Theorem 2.2 is equivalent to the following proposition.

Proposition 5.4. *Any $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double coset of $G_1 \times G_2$ is of the form $[(m_1 \dot{v}_1, \dot{v}_2 m_2)]$ for some $v_1 \in W_1^{C_1}$, $v_2 \in {}^{A_2}W_2$, $m_1 \in M_{A_1(v_1, v_2)}$, and $m_2 \in M_{C_2(v_1, v_2)}$. Two such cosets $[(m_1 \dot{v}_1, \dot{v}_2 m_2)]$ and $[(l_1 \dot{v}_1', \dot{v}_2' l_2)]$ coincide if and only $v_i' = v_i$ for $i = 1, 2$, and (m_1, m_2) and (l_1, l_2) are in the same $(K(v_1, v_2), L(v_1, v_2))$ double coset in $M_{A_1(v_1, v_2)} \times M_{C_2(v_1, v_2)}$.*

Proof of Proposition 5.4. It is not hard to see that by repeatedly using Proposition 5.2 and by using Lemma 5.3, we can reduce the description of $(R_{\mathcal{A}}, R_{\mathcal{C}})$ double cosets in $G_1 \times G_2$ to that of the special cases treated in Lemma 5.1. Proposition 5.4 holds trivially for such cases. Thus it suffices to prove Proposition 5.4 for $G_1 \times G_2$ by assuming that it holds $M_{A_1} \times M_{C_2}$.

For $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$, set

$$\mathbf{D}(w_1, w_2) = R_{\mathcal{A}}^{\text{new}}(w_1, w_2) \backslash M_{A_1} \times M_{C_2} / R_{\mathcal{C}}^{\text{new}}(w_1, w_2),$$

and we use $[(m_{A_1}, m_{C_2})]'$ to denote the point in $\mathbf{D}(w_1, w_2)$ defined by $(m_{A_1}, m_{C_2}) \in M_{A_1} \times M_{C_2}$. Then by Proposition 5.2, we have the disjoint union

$$G_1 \times G_2 = \bigcup_{(w_1,w_2)} \bigcup_{[(m_{A_1}, m_{C_2})]' \in \mathbf{D}(w_1,w_2)} [(m_{A_1}\dot{w}_1, \dot{w}_2 m_{C_2}],$$

where $(w_1, w_2) \in {}^{A_1}W_1^{C_1} \times {}^{A_2}W_2^{C_2}$. For such a pair (w_1, w_2), and for $u_1 \in W_{A_1}^{A_1 \cap w_1(C_1)}$ and $u_2 \in {}^{C_2 \cap w_2^{-1}(A_2)}W_{C_2}$, set $v_1 = u_1 w_1$ and $v_2 = w_2 u_2$. Then we know from Proposition 4.1 and (4.5) that $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, and that every $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$ are of this form. Choose representatives $\hat{u}_1$ and $\hat{u}_2$ of u_1 and u_2 in the normalizers of H_1 and H_2 in M_{A_1} and M_{C_2} respectively such that $\dot{v}_1 = \hat{u}_1 \dot{w}_1$ and $\dot{v}_2 = \dot{w}_2 \hat{u}_2$. Set

$$\mathbf{D}(v_1, v_2) = K(v_1, v_2) \backslash M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)} / L(v_1, v_2),$$

and let $[(m_1, m_2)]_{\text{fin}}$ denote the point in $\mathbf{D}(v_1, v_2)$ defined by $(m_1, m_2) \in M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)})$. Then by applying Proposition 5.4 to $\mathbf{D}(w_1, w_2)$ and by using Lemma 5.3, we know that

$$\mathbf{D}(w_1, w_2) = \bigcup_{(u_1,u_2)} \bigcup_{[(m_1,m_2)]_{\text{fin}} \in \mathbf{D}(v_1,v_2)} [(m_1\hat{u}_1, \hat{u}_2 m_2)]'$$

is a disjoint union, where $(u_1, u_2) \in W_{A_1}^{A_1 \cap w_1(C_1)} \times {}^{C_2 \cap w_2^{-1}(A_2)}W_{C_2}$. Thus we have

$$G_1 \times G_2 = \bigcup_{v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2} \bigcup_{[(m_1,m_2)]_{\text{fin}} \in \mathbf{D}(v_1,v_2)} [(m_1\dot{v}_1, \dot{v}_2 m_2)]$$

as a disjoint union. □

6. Proofs of Theorem 2.5 and Theorem 2.7

We will keep the notation as in §2.3 and §2.4.

6.1. **Proof of Theorem 2.5.**

Lemma 6.1. *For $q = (m_1\dot{v}_1, \dot{v}_2 s_2)$, where $(v_1, v_2) \in W_1^{C_1} \times {}^{A_2}W_2$, $m_1 \in M_{A_1(v_1,v_2)}$ and $s_2 \in Z_{C_2(v_1,v_2)}$, let*

$$\begin{aligned}
&(6.1) \qquad \operatorname{Stab}_M(q) &= \;& K(v_1, v_2) \cap \operatorname{Ad}_{(m_1,e)} L(v_1, v_2) \\
&(6.2) &= \;& \left(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}\right) \cap \operatorname{Ad}_{(e,\dot{v}_2)}^{-1} K \cap \operatorname{Ad}_{(m_1\dot{v}_1,e)} L \\
&(6.3) \qquad \operatorname{Stab}_U(q) &= \;& \operatorname{Stab}(q) \cap \left((U_{A_1(v_1,v_2)} \cap v_1(U_1)) \times (U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2))\right).
\end{aligned}$$

Then $\operatorname{Stab}(q) = \operatorname{Stab}_M(q) \operatorname{Stab}_U(q)$.

Proof. By Proposition 4.1 and (4.5), we can write $v_1 = u_1 w_1$ and $v_2 = w_2 u_2$, where $w_1 \in {}^{A_1}W_1^{C_1}$, $u_1 \in W_{A_1}^{A_1 \cap w_1(C_1)}$, $w_2 \in {}^{A_2}W_1^{C_2}$ and $u_2 \in {}^{C_2 \cap w_2^{-1}(A_2)}W_{C_2}$. We will first give a description of $\operatorname{Stab}(q)$ in terms of the groups $K^{\text{new}}(w_1, w_2)$ and $L^{\text{new}}(w_1, w_2)$ in (5.4) and (5.5). Set $\dot{u}_1 = \dot{v}_1 \dot{w}_1^{-1} \in M_{A_1}$ and $\dot{u}_2 = \dot{w}_2^{-1} \dot{v}_2 \in M_{C_2}$.

Let $(p_1, p_2) \in \operatorname{Stab}(q)$. Then $\operatorname{Ad}_{\dot{v}_2 s_2}(p_2) \in P_{A_2} \cap w_2(P_{C_2})$ because $\dot{u}_2 s_2 \in P_{C_2}$. By 1) of Lemma 4.2, $\operatorname{Ad}_{\dot{v}_2 s_2}(p_2) \in P_{A_2 \cap w_2(C_2)}$. It follows from $(p_1, \operatorname{Ad}_{\dot{v}_2 s_2}(p_2)) \in R_{\mathcal{A}}$ and (3.7) that $p_1 \in P_{a^{-1}(A_2 \cap w_2(C_2))}$. Hence

$$(p_1, \operatorname{Ad}_{\dot{v}_2 s_2}(p_2)) \in R_{\mathcal{A}} \cap \left(P_{a^{-1}(A_2 \cap w_2(C_2))} \times (P_{A_2} \cap w_2(P_{C_2})\right).$$

By 1) of Lemma 4.2 and 3) of Lemma 3.4, we see that $(p_1, \mathrm{Ad}_{\dot{v}_2 s_2}(p_2))$ belongs to the group

$$\left((M_{a^{-1}(A_2\cap w_2(C_2))} \times M_{A_2\cap w_2(C_2)}) \cap K\right) \left(U_{a^{-1}(A_2\cap w_2(C_2))} \times w_2(U_{C_2\cap w_2^{-1}(A_2)})\right).$$

Thus

$$(p_1, \mathrm{Ad}_{\dot{u}_2 s_2}(p_2)) \in K^{\mathrm{new}}(w_1,w_2)\left(U_{A_1^{\mathrm{new}}(w_1,w_2)} \times U_{A_2^{\mathrm{new}}(w_1,w_2)}\right),$$

where $A_1^{\mathrm{new}}(w_1,w_2)$ and $A_2^{\mathrm{new}}(w_1,w_2)$ are given in (5.2). Similarly, with $C_1^{\mathrm{new}}(w_1,w_2)$ and $C_2^{\mathrm{new}}(w_1,w_2)$ given in (5.3), one shows that

$$(\mathrm{Ad}_{m_1\dot{u}_1}^{-1}(p_1), p_2) \in L^{\mathrm{new}}(w_1,w_2)\left(U_{C_1^{\mathrm{new}}(w_1,w_2)} \times U_{C_2^{\mathrm{new}}(w_1,w_2)}\right).$$

It is now clear by Lemma 5.3 that one can show inductively that

$$\begin{aligned}(p_1, \mathrm{Ad}_{s_2}(p_2)) &\in K(v_1,v_2)\left(U_{A_1(v_1,v_2)} \times U_{C_2(v_1,v_2)}\right)\\ (\mathrm{Ad}_{m_1}^{-1}(p_1), p_2) &\in L(v_1,v_2)\left(U_{A_1(v_1,v_2)} \times U_{C_2(v_1,v_2)}\right).\end{aligned}$$

Thus

$$(p_1,p_2) \in \left(K(v_1,v_2) \cap \mathrm{Ad}_{(m_1,e)}L(v_1,v_2)\right)\left(U_{A_1(v_1,v_2)} \times U_{C_2(v_1,v_2)}\right). \tag{6.4}$$

It is easy to see that

$$\mathrm{Stab}_M(q) := K(v_1,v_2) \cap \mathrm{Ad}_{(m_1,e)}L(v_1,v_2) \subset \mathrm{Stab}(q)$$

and that (6.2) holds. Thus, if we set

$$\mathrm{Stab}_U(q) = \mathrm{Stab}(q) \cap \left(U_{A_1(v_1,v_2)} \times U_{C_2(v_1,v_2)}\right),$$

it remains to show that $\mathrm{Stab}_U(q)$ is also given as in (6.3). We note that if $(p_1,p_2) \in \mathrm{Stab}_U(q)$, then $\mathrm{Ad}_{m_1}^{-1}p_1 \in v_1(P_{C_1})$. On the other hand, $\mathrm{Ad}_{m_1}^{-1}(p_1) \in P_{A_1(v_1,v_2)}$ by (6.4). Thus $\mathrm{Ad}_{m_1}^{-1}(p_1) \in P_{A_1(v_1,v_2)} \cap v_1(P_{C_1})$, so $p_1 \in P_{A_1(v_1,v_2)} \cap v_1(P_{C_1})$. Now $v_1 \in {}^{A_1(v_1,v_2)}W_1^{C_1}$ because $v_1^{-1}(A_1(v_1,v_2)) \subset C_1 \subset \Delta_1^+$, Thus by (4.10) in Lemma 4.2, $p_1 \in U_{A_1(v_1,v_2)} \cap v_1(U_1)$. Similarly, $p_2 \in U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)$. Thus $\mathrm{Stab}_U(q)$ is given by (6.3). □

Recall that $J_{m_1}(v_1,v_2)$ denotes the stabilizer subgroup of $J(v_1,v_2)$ at $m_1 \in M_{A_1(v_1,v_2)}$ for the action given in (2.8). The group $\mathrm{Stab}_M(q)$ consists of all pairs $(k_1,k_2) \in K(v_1,v_2)$ such that $(\mathrm{Ad}_{m_1}^{-1}k_1, k_2) \in L(v_1,v_2)$. Firstly this implies that the group $(\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K)$ lies in the center of $\mathrm{Stab}_M(q) = K(v_1,v_2) \cap \mathrm{Ad}_{(m_1,e)}L(v_1,v_2)$. Secondly, for $(k_1,k_2) \in \mathrm{Stab}_M(q)$ we have that $(k_1, \mathrm{Ad}_{m_1}^{-1}k_1) \in J_{m_1}(v_1,v_2)$ because of (2.7). Therefore

$$\mu\colon \mathrm{Stab}_M(q) \to J_{m_1}(v_1,v_2), \quad \mu(k_1,k_2) = (k_1, \mathrm{Ad}_{m_1}^{-1}k_1)$$

is a well defined homomorphism. The following Lemma describes explicitly $\mathrm{Stab}_M(q)$. Its proof is straightforward and we will omit it.

Lemma 6.2. *The following is a group exact sequence*

$$\{e\} \to (\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K) \hookrightarrow \mathrm{Stab}_M(q) \to J_{m_1}(v_1,v_2) \to \{e\}.$$

The next Lemma will be used to describe the structure of $\mathrm{Stab}_U(q)$, stated in Theorem 2.5.

Lemma 6.3. *The maps σ_1 and σ_2 given by* (2.12)-(2.13) *are well defined endomorphisms of $U_{A_1(v_1,v_2)} \cap v_1(U_1)$ and $U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)$, respectively. Moreover for some sufficiently large integer k, $\sigma_i^{k+1} \equiv e$ for $i = 1, 2$.*

Proof. We will only prove the statements for σ_1 as the ones for σ_2 are analogous. First we show that

$$\tilde{\sigma}_1 = \mathrm{Ad}_{\dot{v}_1}\theta_c^{-1}\pi_{C_2}\mathrm{Ad}_{\dot{v}_2 s_2}^{-1}\theta_a\pi_{A_1}$$

is a well defined endomorphism of $U_{A_1(v_1,v_2)} \cap v_1(U_1)$ and $\tilde{\sigma}_1^{k+1} \equiv e$ for some sufficiently large integer k. For a given subset $\Phi \subset \Delta_1^+$ with the property

$$\alpha, \beta \in \Phi \ \text{ and } \ \alpha + \beta \in \Delta_1^+ \Rightarrow \alpha + \beta \in \Phi \tag{6.5}$$

define

$$U_1^\Phi = \prod_{\alpha \in \Phi} U_1^\alpha.$$

Recall that (6.5) implies that this product does not depend on the order in which it is taken and moreover U_1^Φ is an algebraic subgroup of U_1.

Extend by linearity $a \colon A_1 \to A_2$ and $c \colon C_1 \to C_2$ to bijections $a \colon [A_1]^+ \to [A_2]^+$ and $c \colon [C_1]^+ \to [C_2]^+$. For a partial map $f \colon \Delta_1 \to \Delta_1$ and a subset $Y \subset \Delta_1$ denote $f(Y) := f(Y \cap \mathrm{Dom}(f))$, where $\mathrm{Dom}(f)$ is the domain of f. Denote $d = v_1 c^{-1} v_2^{-1} a$ which will be treated as a partial bijection from Δ_1 to Δ_1. Set

$$\Phi^{(0)} = \left(\Delta_1^+ - [A_1(v_1,v_2)]^+\right) \cap v_1(\Delta_1^+) \ \text{ and inductively } \ \Phi^{(i+1)} = d(\Phi^{(i)}), \ i \geq 0.$$

Since the partial map d preserves addition, all sets $\Phi^{(i)}$ satisfy the property (6.5) and thus

$$U_1^{(i)} := U_1^{\Phi^{(i)}}$$

are algebraic subgroups of U_1. Moreover $\Phi^{(i+1)} \subset \Phi^{(i)}$ for $i \geq 0$ because $v_1 \in W_1^{C_1}$ and for some positive integer k, $\Phi^{(k+1)} = \emptyset$ because all powers d^i are well defined on a root $\alpha \in \Delta_1$ only if $\alpha \in [A_1(v_1,v_2)]$. Thus we have the decreasing sequence of unipotent groups

$$U_{A_1(v_1,v_2)} \cap v_1(U_1) = U_1^{(0)} \supset U^{(1)} \supset \ldots \supset U^{(k)} \supset U^{(k+1)} = \{e\}. \tag{6.6}$$

By direct computation one checks that for $\alpha \in \Phi^{(0)}$, $\tilde{\sigma}_1(U_1^\alpha) = U_1^{d(\alpha)}$ if $d(\alpha)$ is defined and $\tilde{\sigma}_1(U_1^\alpha) = \{e\}$ otherwise. As a consequence, $\tilde{\sigma}_1^i(U_{A_1(v_1,v_2)} \cap v_1(U_1)) = U_1^{(i)}$ for all $i \geq 1$. Now (6.6) immediately implies that $\tilde{\sigma}_1$ is a well defined endomorphism of $U_{A_1(v_1,v_2)} \cap v_1(U_1)$ and $\tilde{\sigma}_1^{k+1} \equiv e$.

Since $\sigma_1 = \mathrm{Ad}_{m_1} \tilde{\sigma}_1$ the same will be true for σ_1 once we show that $M_{A_1(v_1,v_2)}$ normalizes all groups $U_1^{(i)}$. To show this recall that for a subset $\Phi \subset \Delta_1^+$ satisfying (6.5), the unipotent group $U_1^\Phi \subset U_1$ is normalized by M_{D_1} (for $D_1 \subset \Gamma_1$) if and only

$$\alpha \in \Phi, \ \beta \in [D_1] \ \text{ and } \ \alpha + \beta \in \Delta_1^+ \Rightarrow \alpha + \beta \in \Phi.$$

By induction on $i \geq 0$ one easily checks that this property is satisfied by $D_1 = A_1(v_1,v_2)$ and $\Phi = \Phi^{(i)}$ because $A_1(v_1,v_2)$ is d-stable and $\Phi^{(i)} = d(\Phi^{(i-1)})$. $\square$

To complete the *proof of Theorem 2.5,* it remains to deduce the formula for $\mathrm{Stab}_U(q)$. Since $v_1 \in {}^{A_1(v_1,v_2)}W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2^{C_2(v_1,v_2)}$, it follows from (4.9) and (4.10) that we have decompositions

$$U_{A_1(v_1,v_2)} \cap v_1(U_1) \ = \ (U_1 \cap v_1(U_{C_1})) \ v_1 \left(U^{C_1}_{c^{-1}(C_2(v_1,v_2))}\right) \tag{6.7}$$

$$U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2) \ = \ \left(U_2 \cap v_2^{-1}(U_{A_2})\right) \ v_2^{-1} \left(U^{A_2}_{a(A_1(v_1,v_2))}\right). \tag{6.8}$$

Both groups on the right hand side of (6.7) are invariant under Ad_m for any $m \in M_{A_1(v_1,v_2)}$ because $\mathrm{Ad}_{\dot{v}_1}^{-1} M_{A_1(v_1,v_2)} = M_{c^{-1}(C_2(v_1,v_2))} \subset M_{C_1}$. Note also that

$$\phi_1(U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)) \subset v_1 \left(U^{C_1}_{c^{-1}(C_2(v_1,v_2))}\right) \subset U_{A_1(v_1,v_2)} \cap v_1(U_1) \tag{6.9}$$

$$\phi_2(U_{A_1(v_1,v_2)} \cap v_1(U_1)) \subset v_2^{-1} \left(U^{A_2}_{a(A_1(v_1,v_2))}\right) \subset U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2). \tag{6.10}$$

Let now

$$(n_1 u_1, n_2 u_2) \in \left(U_{A_1(v_1,v_2)} \cap v_1(U_1)\right) \times \left(U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)\right)$$

be decomposed according to (6.7) and (6.8). Then $(n_1u_1, n_2u_2) \in \mathrm{Stab}(q)$ if and only if

$$\begin{cases} \mathrm{Ad}^{-1}_{\dot{v}_2 s_2}\theta_a \pi_{A_1}(n_1u_1) = u_2 \\ \mathrm{Ad}^{-1}_{m_1\dot{v}_1}(u_1) = \theta_c^{-1}\pi_{C_2}(n_2u_2) \end{cases}, \qquad \text{or} \qquad \begin{cases} \phi_2(n_1u_1) = u_2 \\ u_1 = \phi_1(n_2u_2) \end{cases}. \tag{6.11}$$

It follows that

$$u_1 = \phi_1(n_2)\sigma_1(n_1u_1) \quad \text{and} \quad u_2 = \phi_2(n_1)\sigma_2(n_2u_2). \tag{6.12}$$

Let k be such that $\sigma_i^{k+1} \equiv e$ for $i = 1, 2$. Using the fact that σ_1 and σ_2 are group homomorphisms and by iterating (6.12), we get

$$u_1 = \psi_1(n_1, n_2) \quad \text{and} \quad u_2 = \psi_2(n_1, n_2), \tag{6.13}$$

where ψ_1 and ψ_2 are respectively given in (2.16) and (2.17). Conversely, it is easy to see that any (n_1u_1, n_2u_2) of the form in (6.13) for $(n_1, n_2) \in (U_1 \cap v_1(U_{C_1}) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right)$ satisfies (6.11). This completes the proof of Theorem 2.5. □

Remark 6.4. The Decompositions in (6.7) and (6.8) are semi-direct products because $v_1\left(U^{C_1}_{c^{-1}(C_2(v_1,v_2))}\right)$ normalizes $U_1 \cap v_1(U_{C_1})$ and $v_2^{-1}\left(U^{A_2}_{a(A_1(v_1,v_2))}\right)$ normalizes $U_2 \cap v_2^{-1}(U_{A_2})$. On the other hand, because of (6.9) and (6.10), the subgroup

$$\mathrm{Stab}_U(q) \subset (U_{A_1(v_1,v_2)} \cap v_1(U_1)) \times (U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2))$$

can be regarded as the "graph"

$$\mathrm{Stab}_U(q) = \{n\psi(n) \mid \; n \in (U_1 \cap v_1(U_{C_1})) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right)\}$$

of the map

$$\psi = \psi_1 \times \psi_2 \colon \; (U_1 \cap v_1(U_{C_1})) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right) \to v_1\left(U^{C_1}_{c^{-1}(C_2(v_1,v_2))}\right) \times v_2^{-1}\left(U^{A_2}_{a(A_1(v_1,v_2))}\right).$$

We will show in §7 that the map ψ defines a solution to the set-theoretical quantum Yang–Baxter equation on

$$\mathrm{Stab}_U(q) \;\cong\; (U_1 \cap v_1(U_{C_1})) \times (U_2 \cap v_2^{-1}(U_{A_2})),$$

(as algebraic varieties).

6.2. **Proof of Theorem 2.7.** By Lemmas 3.2 and 3.4, we know that

$$\begin{aligned} \dim R_{\mathcal{A}} &= \dim U_{A_1} + \dim U_{A_2} + \dim K \\ &= \dim P_{A_1} - \dim M_{A_1(v_1,v_2)} + \dim U_{A_2} + \dim K(v_1, v_2) \quad \text{and} \\ \dim R_{\mathcal{C}} &= \dim U_{C_1} + \dim U_{C_2} + \dim L \\ &= \dim U_{C_1} + \dim P_{C_2} - \dim M_{C_2(v_1,v_2)} + \dim L(v_1, v_2). \end{aligned}$$

The second equalities above follow from the fact that in the setting of Theorem 2.7 for any subset D_1 of A_1

$$\dim K - \dim K \cap (M_{D_1} \times M_{a(D_1)}) = \dim M_{A_1} - \dim M_{D_1}.$$

This is easily deduced from the description of a-graphs given in Lemma 3.2. The second part of Lemma 4.2 and the part of Theorem 2.5 on $\mathrm{Stab}_U(q)$ imply

$$\begin{aligned} \dim \mathrm{Stab}_U(q) &= \dim U_1 \cap v_1(U_{C_1}) + \dim U_2 \cap v_2^{-1}(U_{A_2}) \\ &= \dim U_{C_1} + \dim U_{A_2} - l(v_1) - l(v_2). \end{aligned}$$

Applying Theorem 2.5 for the structure of $\mathrm{Stab}(q)$, we get

$$\begin{aligned} &\dim[(m_1\dot{v}_1, \dot{v}_2\dot{s})] = \dim R_{\mathcal{A}} + \dim R_{\mathcal{C}} - \dim \mathrm{Stab}(q) \\ &= l(v_1) + l(v_2) + \dim P_{A_1} - \dim M_{A_1(v_1,v_2)} + \dim P_{C_2} - \dim M_{C_2(v_1,v_2)} \\ &+ \dim K(v_1, v_2) + \dim L(v_1, v_2) - \dim J_{m_1}(v_1, v_2) - \dim(\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K). \end{aligned}$$

Since $\dim J(v_1, v_2).m_1 = \dim J(v_1, v_2) - \dim J_{m_1}(v_1, v_2)$, to complete the proof of Theorem 2.7 it only remains to show that

(6.14)
$$\dim J(v_1, v_2) = \dim K(v_1, v_2) + \dim L(v_1, v_2) - \dim M'_{C_2(v_1,v_2)}$$
$$- \dim \eta_2 \left(KL \cap (v_1^{-1} Z_{A_1(v_1,v_2)} \times v_2 Z_{C_2(v_1,v_2)})\right) - \dim(\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K).$$

Define

$$Q = \{(k_1, k_2, l_1, l_2) \in K(v_1, v_2) \times L(v_1, v_2) \mid k_2 = l_2\} \subset K(v_1, v_2) \times L(v_1, v_2).$$

In the two cases of Theorem 2.7 Q is respectively an algebraic/Lie subgroup of $K(v_1, v_2) \times L(v_1, v_2)$. By definition $J(v_1, v_2)$ is the image of Q under $(k_1, k_2, l_1, l_2) \mapsto (k_1, l_1)$, in particular $J(v_1, v_2)$ is also an algebraic/Lie group. It is easy to see that the kernel of this homomorphism is $(\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K)$. Therefore

$$\dim J(v_1, v_2) = \dim Q - \dim(\ker \eta_1|_L) \cap (\mathrm{Id} \times v_2^{-1})(\ker \eta_1|_K). \tag{6.15}$$

The homogeneous space $(K(v_1, v_2)\times L(v_1, v_2))/Q$ is isomorphic to $\eta_2(K(v_1, v_2))\eta_2(L(v_1, v_2)) \subset M_{C_2(v_1,v_2)}$ by $(k_1, k_2, l_1, l_2)Q \mapsto k_2 l_2^{-1}$. Moreover from Lemma 3.2 one obtains

$$\eta_2(K(v_1, v_2))\eta_2(L(v_1, v_2)) = M'_{C_2(v_1,v_2)}\eta_2 \left(K(v_1, v_2)L(v_1, v_2) \cap (Z_{A_1(v_1,v_2)} \times Z_{C_2(v_1,v_2)})\right)$$

and

$$\begin{aligned}\dim Q = \dim K(v_1, v_2) &+ \dim L(v_1, v_2) - \dim M'_{C_2(v_1,v_2)} \\ &- \dim \eta_2 \left(K(v_1, v_2)L(v_1, v_2) \cap (Z_{A_1(v_1,v_2)} \times Z_{C_2(v_1,v_2)})\right).\end{aligned} \tag{6.16}$$

Finally substituting (6.15) in (6.16) and using that the projections of $K(v_1, v_2)L(v_1, v_2) \cap (Z_{A_1(v_1,v_2)} \times Z_{C_2(v_1,v_2)})$ and $KL \cap (v_1^{-1} Z_{A_1(v_1,v_2)} \times v_2 Z_{C_2(v_1,v_2)})$ under η_2 have the same dimensions lead to (6.14). □

7. Solutions to the set-theoretical quantum Yang–Baxter equation

Recall that for a set V, the set-theoretical quantum Yang–Baxter equation for an invertible map $T\colon V \times V \to V \times V$ is

$$T^{12}T^{13}T^{23} = T^{23}T^{13}T^{12}, \tag{7.1}$$

where for $i, j \in \{1, 2, 3\}$, T^{ij} denotes the map from $V \times V \times V$ to itself that has T act on the (i, j)'th components.

For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, we will set, for notational simplicity,

$$N_{v_1,v_2} = (U_1 \cap v_1(U_{C_1})) \times \left(U_2 \cap v_2^{-1}(U_{A_2})\right)$$
$$Q_{v_1,v_2} = v_1 \left(U^{C_1}_{c^{-1}(C_2(v_1,v_2))}\right) \times v_2^{-1} \left(U^{A_2}_{a(A_1(v_1,v_2))}\right).$$

Then Q_{v_1,v_2} normalizes N_{v_1,v_2}, and

$$N_{v_1,v_2}Q_{v_1,v_2} = (U_{A_1(v_1,v_2)} \cap v_1(U_1)) \times (U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2))$$

Recall from Remark 6.4 that for a point $q = (\dot{v}_1 m, \dot{v}_2 s_2) \in G\times G$, where, $\dot{v}_i$ is a representative of v_i in the normalizer of H_i in G_i for $i = 1, 2$, $m \in M_{A_1(v_1,v_2)}$, and $s_2 \in Z_{C_2(v_1,v_2)}$, we have

$$\mathrm{Stab}_U(q) = \{n\psi(n) \mid n \in N_{v_1,v_2}\} \subset (U_{A_1(v_1,v_2)} \cap v_1(U_1)) \times (U_{C_2(v_1,v_2)} \cap v_2^{-1}(U_2)),$$

where $\psi = \psi_1 \times \psi_2\colon N_{v_1,v_2} \to Q_{v_1,v_2}$ and ψ_1 and ψ_2 are respectively given in (2.16) and (2.17). In this section, we show that the map ψ defines a solution to the set-theoretical quantum Yang–Baxter equation on N_{v_1,v_2}.

As is true for any group [6], the map

$$T_0\colon N_{v_1,v_2} \times N_{v_1,v_2} \to N_{v_1,v_2} \times N_{v_1,v_2} \colon (n,\ n') \mapsto (n',\ (n')^{-1}nn') \tag{7.2}$$

satisfies the set-theoretical quantum Yang–Baxter equation (7.1). We now show that a twist of T_0 using the map ψ is also a solution to (7.1).

Let σ and F be the maps from $N_{v_1,v_2} \times N_{v_1,v_2}$ to itself given respectively by

$$\sigma(n, n') = (n', n), \tag{7.3}$$

$$F(n, n') = (n,\ \psi(n)n'\psi(n)^{-1}). \tag{7.4}$$

Define

$$T = (\sigma F \sigma)^{-1} T_0\, F \colon N_{v_1,v_2} \times N_{v_1,v_2} \to N_{v_1,v_2} \times N_{v_1,v_2}. \tag{7.5}$$

Proposition 7.1. *The map T is a solution to the set-theoretical quantum Yang–Baxter equation* (7.1) *on N_{v_1,v_2}.*

Proposition 7.1 is a special case of the following general fact.

Lemma 7.2. *Assume that U is a group and that Q and N are subgroups of U such that $N \cap Q = \{e\}$ and that Q normalizes N. Suppose that $\psi \colon N \to Q$ is a map such that*

$$S := \{n\psi(n) \mid n \in N\}$$

is a subgroup of U. Let T_0, σ, F, and T be the maps from $N \times N$ to itself given by replacing N_{v_1,v_2} by N in (7.2), (7.3), (7.4), *and* (7.5) *respectively. Then T is a solution to the set-theoretical quantum Yang–Baxter Equation* (7.1) *on N.*

Proof. Since S is a subgroup of U, the map

$$\widetilde{\psi} \colon S \to Q \colon n\psi(n) \mapsto \psi(n)$$

is a group homomorphism. Thus the map

$$S \times N \to N \colon (n\psi(n), n') \mapsto (n\psi(n)) \cdot n' := \psi(n)n'\psi(n)^{-1}$$

defines a left action of S on N by group automorphisms. Moreover, it is clear from the definition that the map $\pi \colon S \to N \colon n\psi(n) \mapsto n$ is a bijective 1-cocycle on S with coefficients in N, i.e., π is bijective and that

$$\pi(s_1 s_2) = \pi(s_1)(s_1 \cdot \pi(s_2)), \qquad s_1, s_2 \in S.$$

By [19, Theorem 6], the map T is a solution to (7.1). □

Remark 7.3. Although for any integer $m \geq 2$, the action of the Braid group B_m on $N^{\times m}$ induced by T is isomorphic to the one induced by T_0 [19, Theorem 6], the fact that T, as the twisting of T_0 by F, still satisfies the quantum Yang–Baxter Equation (7.1) is non-trivial. When N is abelian, the fact that T satisfies (7.1) is also proved in [9].

8. The groups $R_{\mathcal{A}}$ are spherical subgroups of $G_1 \times G_2$

Recall that a subgroup R of a reductive group G is called spherical if R has finitely many orbits on the flag variety G/B where B is a Borel subgroup of G. In this section, we fix an admissible pair $\mathcal{A} = (a, K)$ for $G_1 \times G_2$. Let A_1 and A_2 be the domain and the range of a respectively. For $i = 1, 2$, we fix a representative $\dot{v}_i$ in the normalizer of H_i in G_i for each $v_i \in W_i$. The main result in this section is:

Proposition 8.1. *Let $C_1 \subset \Gamma_1$ and $C_2 \subset \Gamma_2$ be arbitrary. Then every $(R_{\mathcal{A}}, P_{C_1} \times P_{C_2})$ double coset in $G_1 \times G_2$ contains exactly one point of the form $(\dot{v}_1, \dot{v}_2)$, where $v_1 \in {}^{A_1}W_1^{C_1}$, $v_2 \in {}^{a(A_1 \cap v_1(C_1))}W_2^{C_2}$.*

As an immediate corollary of Proposition 8.1 we obtain:

Corollary 8.2. *The group $R_{\mathcal{A}}$ is a spherical subgroup of $G_1 \times G_2$.*

We will first prove an auxiliary Lemma which can be also viewed as a special case of Proposition 8.1. Recall that for $D_i \subset A_i \subset \Gamma_i$, $P_{D_i}^{A_i} = P_{D_i} \cap M_{A_i}$ denotes the standard parabolic subgroup of M_{A_i} determined by D_i.

Lemma 8.3. *For any subsets $D_1 \subset A_1$ and $D_2 \subset A_2$, every $(K, P_{D_1}^{A_1} \times P_{D_2}^{A_2})$ double coset in $M_{A_1} \times M_{A_2}$ contains exactly one point of the form $(e, \dot{w})$, where $w \in {}^{a(D_1)}W_{A_2}^{D_2}$.*

Proof. Let $H_{A_i} = H_i \cap M_{A_i}$ for $i = 1, 2$. Since $\eta_1(K) \supset M'_{A_1}$ and $P_{D_1}^{A_1} \supset H_{A_1}$, every $(K, P_{D_1}^{A_1} \times P_{D_2}^{A_2})$ double coset in $M_{A_1} \times M_{A_2}$ is of the form $[(e, m)]$ for some $m \in M_{A_2}$. Let $\widetilde{P}_{a(D_1)}^{A_2} = P_{a(D_1)} \cap \eta_2(K)$. Since $H_{A_2}\widetilde{P}_{a(D_1)}^{A_2} = P_{a(D_1)}^{A_2}$ the Bruhat decomposition of M_{A_2}, see (4.2), implies

$$M_{A_2} = \coprod_{w \in {}^{a(D_1)}W_{A_2}^{D_2}} \widetilde{P}_{a(D_1)}^{A_2} \dot{w} P_{D_2}^{A_2}.$$

For $m \in M_{A_2}$, write $m = q_2 \dot{w} p_2$ for $w \in {}^{a(D_1)}W_{A_2}^{D_2}$, $q_2 \in \widetilde{P}_{a(D_1)}^{A_2}$, and $p_2 \in P_{D_2}^{A_2}$. Then by Part 3) of Lemma 3.4, we know that there exists $q_1 \in P_{D_1}^{A_1}$ such that $(q_1, q_2) \in K$. Thus

$$[(e, m)] = [(e, q_2 \dot{w} p_2)] = [(q_1^{-1}, \dot{w} p_2)] = [(e, \dot{w})].$$

It is easy to see that if $w, w' \in {}^{a(D_1)}W_{A_2}^{D_2}$ are such that $[(e, \dot{w})] = [(e, \dot{w}')]$, then $w = w'$ which completes the proof of the lemma. $\square$

Proof of Proposition 8.1. For the left action of $P_{A_1} \times P_{A_2}$ on $(G_1 \times G_2)/(P_{C_1} \times P_{C_2})$, we know from the Bruhat decomposition that every $P_{A_1} \times P_{A_2}$ orbit passes through a point $(\dot{w}_1, \dot{w}_2)(P_{C_1} \times P_{C_2})$ for a unique $w_1 \in {}^{A_1}W^{C_1}$ and a unique $w_2 \in {}^{A_2}W^{C_2}$. Denote by $\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)}$ the stabilizer subgroup of $P_{A_1} \times P_{A_2}$ at the point $(\dot{w}_1, \dot{w}_2)(P_{C_1} \times P_{C_2})$. Then by Lemma 4.3, every $R_{\mathcal{A}}$-orbit in $(G_1 \times G_2)/(P_{C_1} \times P_{C_2})$ contains a point of the form $(p_1 \dot{w}_1, p_2 \dot{w}_2)(P_{C_1} \times P_{C_2})$ for a unique $w_1 \in {}^{A_1}W^{C_1}$, a unique $w_2 \in {}^{A_2}W^{C_2}$, and for some $(p_1, p_2) \in P_{A_1} \times P_{A_2}$, and two such points $(p_1 \dot{w}_1, p_2 \dot{w}_2)(P_{C_1} \times P_{C_2})$ and $(q_1 \dot{w}_1, q_2 \dot{w}_2)(P_{C_1} \times P_{C_2})$ are in the same $R_{\mathcal{A}}$-orbit if and only if (p_1, p_2) and (q_1, q_2) are in the same $R_{\mathcal{A}}$-orbit on $(P_{A_1} \times P_{A_2})/\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)}$. Thus we need to understand the double coset spaces $R_{\mathcal{A}} \backslash (P_{A_1} \times P_{A_2})/\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)}$.

Fix now $w_1 \in {}^{A_1}W^{C_1}$ and $w_2 \in {}^{A_2}W^{C_2}$. Then it is easy to see that

$$\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)} = (P_{A_1} \cap w_1(P_{C_1})) \times (P_{A_2} \cap w_2(P_{C_2})).$$

On the other hand, recall that for $i = 1, 2$, $\pi_{A_i} : P_{A_i} \to M_{A_i}$ is the projection with respect to the decomposition $P_{A_i} = M_{A_i} N_{A_i}$. The projection $\pi_{A_1} \times \pi_{A_2} : P_{A_1} \times P_{A_2} \to M_{A_1} \times M_{A_2}$ induces an isomorphism between the quotient spaces $R_{\mathcal{A}} \backslash (P_{A_1} \times P_{A_2})$ and $K \backslash (M_{A_1} \times M_{A_2})$. We know from 1) of Lemma 4.2 that

$$(\pi_{A_1} \times \pi_{A_2})(\mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)}) = P_{A_1 \cap w_1(C_1)}^{A_1} \times P_{A_2 \cap w_2(C_2)}^{A_2}.$$

Since $M_{A_1} \times M_{A_2}$ normalizes $N_{A_1} \times N_{A_2}$, we see that the projection $\pi_{A_1} \times \pi_{A_2}$ induces an isomorphism of double coset spaces

$$R_{\mathcal{A}} \backslash P_{A_1} \times P_{A_2} / \mathrm{Stab}_{(\dot{w}_1, \dot{w}_2)} \;\cong\; K \backslash M_{A_1} \times M_{A_2} / (P_{A_1 \cap w_1(C_1)}^{A_1} \times P_{A_2 \cap w_2(C_2)}^{A_2}).$$

By Lemma 8.3, every $(K, P_{A_1 \cap w_1(C_1)}^{A_1} \times P_{A_2 \cap w_2(C_2)}^{A_2})$-double coset in $M_{A_1} \times M_{A_2}$ contains exactly one point of the form $(e, \dot{w})$ for some $w \in {}^{a(A_1 \cap w_1(C_1))}W_{A_2}^{A_2 \cap w_2(C_2)}$. Thus every $R_{\mathcal{A}}$-orbit in $(G_1 \times G_2)/(P_{C_1} \times P_{C_2})$ contains a unique element of the form $(\dot{w}_1, \dot{w}\dot{w}_2)$, where $w_1 \in {}^{A_1}W^{C_1}$, $w_2 \in {}^{A_2}W^{C_2}$, and $w \in {}^{a(A_1 \cap w_1(C_1))}W_{A_2}^{A_2 \cap w_2(C_2)}$. Let $v_1 = w_2$ and $v_2 = w w_2$. Proposition 8.1 now follows from Proposition 4.1. $\square$

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