ON THE VARIETY OF LAGRANGIAN SUBALGEBRAS, II

SAM EVENS AND JIANG-HUA LU

Abstract. Motivated by Drinfeld's theorem on Poisson homogeneous spaces, we study the variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ for a complex semi-simple Lie algebra $\mathfrak{g}$. Let G be the adjoint group of $\mathfrak{g}$. We show that the $(G \times G)$-orbit closures in $\mathcal{L}$ are smooth spherical varieties. We also classify the irreducible components of $\mathcal{L}$ and show that they are smooth. Using some methods of M. Yakimov, we give a new description and proof of Karolinsky's classification of the diagonal G-orbits in $\mathcal{L}$, which, as a special case, recovers the Belavin-Drinfeld classfication of quasi-triangular r-matrices on $\mathfrak{g}$. Furthermore, $\mathcal{L}$ has a canonical Poisson structure, and we compute its rank at each point and describe its symplectic leaf decomposition in terms of intersections of orbits of two subgroups of $G \times G$.

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1. Introduction

Let $\mathfrak{d}$ be a $2n$-dimensional Lie algebra over $k = \mathbb{R}$ or $\mathbb{C}$, together with a symmetric, non-degenerate, and ad-invariant bilinear form $\langle\, , \rangle$. When $k = \mathbb{R}$, we require $\langle\, , \rangle$ to have signature (n, n). A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{d}$ is said to be *Lagrangian* if $\mathfrak{l}$ is maximal isotropic with respect to $\langle\, , \rangle$, i.e., if $\dim_k \mathfrak{l} = n$ and if $\langle x, y \rangle = 0$ for all $x, y \in \mathfrak{l}$. By a *Lagrangian splitting* of $\mathfrak{d}$ we mean a direct sum decomposition $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are two Lagrangian subalgebras of $\mathfrak{d}$. Denote by $\mathcal{L}(\mathfrak{d})$ the set of all Lagrangian subalgebras of $\mathfrak{d}$. It is an algebraic subvariety of the Grassmannian $\mathrm{Gr}(n, \mathfrak{d})$ of n-dimensional subspaces of $\mathfrak{d}$. In [E-L2], we showed that associated to each Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$ there is a Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$, making $\mathcal{L}(\mathfrak{d})$ into a Poisson variety. Moreover, if L_1 and L_2 are the connected subgroups of the adjoint group D of $\mathfrak{d}$ with Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, all the L_1 and L_2-orbits in $\mathcal{L}(\mathfrak{d})$ are Poisson submanifolds of $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$.

The above construction in [E-L2] was motivated by the work of Drinfeld [Dr] on Poisson homogeneous spaces. Indeed, a Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$ of $\mathfrak{d}$ gives rise to the Manin triple $(\mathfrak{d}, \mathfrak{l}_1, \mathfrak{l}_2)$, which in turn defines Poisson structures π_1 and π_2 on the Lie groups L_1 and L_2 respectively, making them into Poisson Lie groups (see [K-S] for details). A Poisson space (M, π) is said to be (L_1, π_1)-homogeneous if L_1 acts on M transitively and if the action map $L_1 \times M \to M$ is a Poisson map. In [Dr], Drinfeld constructed an L_1-equivariant map $M \to \mathcal{L}(\mathfrak{d})$ for every (L_1, π_1)-homogeneous Poisson space (M, π), and he proved (see [Dr] and [E-L2] for more detail) that (L_1, π_1)-homogeneous Poisson spaces correspond to L_1-orbits in $\mathcal{L}(\mathfrak{d})$ in this way. The Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$ is constructed in such a way that the Drinfeld map $M \to \mathcal{L}(\mathfrak{d})$ is a Poisson map. In many cases, the Drinfeld map $M \to \mathcal{L}(\mathfrak{d})$ is a local diffeomorphism onto its image. Thus we can think of L_1-orbits in $\mathcal{L}(\mathfrak{d})$ as models for (L_1, π_1)-homogeneous Poisson spaces. For this reason, it is interesting to study the geometry of the variety $\mathcal{L}(\mathfrak{d})$, the L_1 and L_2-orbits in $\mathcal{L}(\mathfrak{d})$, and the Poisson structures $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$.

There are many examples of Lie algebras $\mathfrak{d}$ with symmetric, non-degenerate, and ad-invariant bilinear forms. The geometry of $\mathcal{L}(\mathfrak{d})$ is different from case to case. Moreover, there can be many Lagrangian splittings for a given $\mathfrak{d}$, resulting in many Poisson structures on $\mathcal{L}(\mathfrak{d})$.

Example 1.1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with Killing form $\ll\, , \gg$. Regard $\mathfrak{g}$ as a real Lie algebra, and let $\langle\, , \rangle$ be the imaginary part of $\ll\, , \gg$. The geometry of $\mathcal{L}(\mathfrak{g})$ in this case was studied in [E-L2]. In particular, we studied the irreducible components of $\mathcal{L}(\mathfrak{g})$ and classified the G-orbits in $\mathcal{L}(\mathfrak{g})$, where G is the adjoint group of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$. Then both $\mathfrak{k}$ and $\mathfrak{a}+\mathfrak{n}$ are Lagrangian subalgebras of $\mathfrak{g}$, so $\mathfrak{g}=\mathfrak{k}+(\mathfrak{a}+\mathfrak{n})$ is an example of a Lagrangian splitting, resulting in a Poisson structure on $\mathcal{L}(\mathfrak{g})$ which we denote by π_0. Many interesting Poisson manifolds appear as G or K-orbits inside $(\mathcal{L}(\mathfrak{g}),\pi_0)$, where K is the connected subgroup of G with Lie algebra $\mathfrak{k}$. Among such Poisson manifolds are the flag manifolds of G and the compact symmetric spaces associated to real forms of G. Detailed studies of the Poisson geometry of these Poisson structures and some applications to Lie theory have been given in [Lu1], [Lu2], [E-L1], and [Ft-L]. For example, the flag manifold X of G consisting of all parabolic subalgebras of $\mathfrak{g}$ of a certain type can be identified with a certain K-orbit in $\mathcal{L}(\mathfrak{g})$. The resulting Poisson structure on X is called the *Bruhat-Poisson structure* because its symplectic leaves are Bruhat cells in X. In [Lu1] and [E-L1], we established connections between the Poisson geometry of the Bruhat-Poisson structure on X and the harmonic forms on X constructed by Kostant [Ko] in 1963, and we gave a Poisson geometric interpretation of the Kostant-Kumar approach [K-K] to Schubert calculus on X.

Example 1.2. Let $\mathfrak{g}$ be any n-dimensional Lie algebra, and let $\mathfrak{d}=\mathfrak{g}\times_{\frac{1}{2}}\mathfrak{g}^*$ be the semi-direct product of $\mathfrak{g}$ and the its dual space $\mathfrak{g}^*$. Then the canonical symmetric product $\langle\, ,\, \rangle$ on $\mathfrak{d}$ defined by

$$\langle x+\xi,\, y+\eta\rangle=\langle x,\eta\rangle+\langle y,\xi\rangle, \qquad x,y\in\mathfrak{g},\, \xi,\eta\in\mathfrak{g}^*$$

is non-degenerate and ad-invariant. When $\mathfrak{g}$ is semi-simple, Lagrangian subalgebras of $\mathfrak{d}$ are not easy to classify (except for low dimensional cases), for, as a sub-problem, one needs to classify all abelian subalgebras of $\mathfrak{g}$. See [K-S], [H-Y], [Ka-St], and the references therein for more detail. The description of the geometry of $\mathcal{L}(\mathfrak{d})$ in this case is an open problem.

In this paper, we will consider the complexification of Example 1.1. Namely, we consider the case where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $\mathfrak{d}=\mathfrak{g}\oplus\mathfrak{g}$ is the direct sum Lie algebra with the bilinear form $\langle\, ,\, \rangle$ given by

$$\langle(x_1,x_2),(y_1,y_2)\rangle = \ll x_1,y_1\gg - \ll x_2,y_2\gg, \qquad x_1,x_2,y_1,y_2\in\mathfrak{g},$$

where $\ll\, ,\, \gg$ is a fixed symmetric, non-degenerate, and ad-invariant bilinear form on $\mathfrak{g}$. The variety of Lagrangian subalgebras of $\mathfrak{d}$ will be denoted by $\mathcal{L}$.

The classification of Lagrangian subalgebras of $\mathfrak{d}$ has been given by Karolinsky [Ka], and Lagrangian splittings of $\mathfrak{g}\oplus\mathfrak{g}$ have been classified by Delorme [De]. In this paper, we establish the first few steps in the study of the Poisson structures on $\mathcal{L}$ defined by Lagrangian splittings of $\mathfrak{g}\oplus\mathfrak{g}$. Namely, we will first describe the geometry of $\mathcal{L}$ in the following terms:

1) the $(G\times G)$-orbits in $\mathcal{L}$ and their closures, where G is the adjoint group of $\mathfrak{g}$;

2) the irreducible components of $\mathcal{L}$;

We will then look at the Poisson structure Π_0 on $\mathcal{L}$ defined by the so-called *standard* Lagrangian splitting $\mathfrak{d}=\mathfrak{g}_\Delta+\mathfrak{g}^*_{\rm st}$, where $\mathfrak{g}_\Delta=\{(x,x): x\in\mathfrak{g}\}$ is the diagonal of $\mathfrak{d}=\mathfrak{g}\oplus\mathfrak{g}$, and $\mathfrak{g}^*_{\rm st}\subset\mathfrak{b}\oplus\mathfrak{b}^-$ with $\mathfrak{b}$ and $\mathfrak{b}^-$ being two opposite Borel subalgebras of $\mathfrak{g}$. Let $G_\Delta=\{(g,g): g\in G\}$ be the diagonal subgroup of $G\times G$. We will study

3) the G_Δ-orbits in $\mathcal{L}$;

4) the symplectic leaf decomposition of $\mathcal{L}$ with respect to Π_0 in terms of the intersections of G_Δ and $(B\times B^-)$-orbits in $\mathcal{L}$, where B and B^- are the Borel subgroups of G with Lie algebras $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively.

The study of the symplectic leaf decomposition of the Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}$ defined by an arbitrary Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ will be carried out in [Lu-Y2], where one first has to classify L-orbits in $\mathcal{L}$, where L is the connected Lie subgroup of $G \times G$ whose Lie algebra is an arbitrary Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$. Such a classification will follow from a general double coset theorem proved in [Lu-Y1]. Since the studies in [Lu-Y1] and [Lu-Y2] are technically involved, we think it worthwhile to treat separately in this paper the important special case of the standard Lagrangian splitting. Moreover, we hope that our study of G_Δ-orbits in $\mathcal{L}$ will find applications outside Poisson geometry. Indeed, as is shown in Section 2.7, the wonderful compactifications of G constructed by De Concini and Procesi [D-P] are closures of special $(G \times G)$-orbits in $\mathcal{L}$. We hope that our classification of G_Δ-orbits in $\mathcal{L}$ will be useful in the study of the conjugacy classes in G and their closures in the wonderful compactifications of G.

We point out that E. Karolinsky has in [Ka] given a classification of G_Δ-orbits in $\mathcal{L}$ in different terms. Our classification is more in line with that of Lagrangian splittings given in [De], and in particular, the Belavin-Drinfeld theorem [B-Dr] on Lagrangian splittings of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}$ follows easily from our classification. Our methods of classifying G_Δ-orbits in $\mathcal{L}$ are adapted from those used in [Y] by Yakimov. In [Lu-Y1] and [Lu-Y2], these methods are also used to classify L-orbits in $\mathcal{L}$, where L is the connected subgroup of $G \times G$ whose Lie algebra is any given Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$.

We now give more details of the results in this paper:

In Section 2, we study $(G \times G)$-orbits in $\mathcal{L}$. Following O. Schiffmann [Sch], we define a *generalized Belavin-Drinfeld triple* (generalized BD-triple) to be a triple (S, T, d), where S and T are two subsets of the set Γ of vertices of the Dynkin diagram of $\mathfrak{g}$, and $d : S \to T$ is an isometry with respect to $\ll\, , \gg$. For a generalized BD-triple (S, T, d) (see Notation 2.12 for detail), let P_S and P_T^- be respectively the standard parabolic subgroups of G of type S and opposite type T with Levi decompositions $P_S = M_S N_S$ and $P_T^- = M_T N_T^-$. Let G_S and G_T be the quotients of M_S and M_T by their centers respectively, and let $\chi_S : M_S \to G_S$ and $\chi_T : M_T \to G_T$ be the natural projections. Denote by $\gamma_d : G_S \to G_T$ the group isomorphism induced by d. We define the subgroup $R_{S,T,d}$ of $P_S \times P_T^-$ by

$$R_{S,T,d} = \{(m_S, m_T) \in M_S \times M_T :\ \gamma_d(\chi_S(m_S)) = \chi_T(m_T)\}(N_S \times N_T^-).$$

We establish the following facts on $(G \times G)$-orbits and their closures in $\mathcal{L}$ (Proposition 2.19, Corollary 2.24, and Proposition 2.27):

1) Every $(G \times G)$-orbit in $\mathcal{L}$ is isomorphic to $(G \times G)/R_{S,T,d}$ for a generalized BD-triple (S, T, d), so there are finitely many $(G \times G)$-orbit types in $\mathcal{L}$, and they correspond bijectively to generalized BD-triples for G; Every $(G \times G)$-orbit in $\mathcal{L}$ is a $(G \times G)$-spherical homogeneous space.

2) When $S = T = \Gamma$, the closure of a $(G \times G)$-orbit of type (S, T, d) is a De Concini-Procesi compactification of G; For an arbitrary generalized BD-triple (S, T, d), the closure of a $(G \times G)$-orbit of type (S, T, d) is a fiber bundle over the flag manifold $G/P_S \times G/P_T^-$ whose fiber is isomorphic to a De Concini-Procesi compactification of G_S. In particular, the closure of every $(G \times G)$-orbit is a smooth $(G \times G)$-spherical variety.

We also study in Section 2 the irreducible components of $\mathcal{L}$. We prove (Corollary 2.29, Theorem 2.31, and Theorem 2.34):

1) The irreducible components of $\mathcal{L}$ are roughly (see Theorem 2.34 for detail) labeled by quadruples (S,T,d,ϵ), where (S,T,d) are generalized BD-triples and $\epsilon \in \{0,1\}$;

2) The irreducible component corresponding to (S,T,d,ϵ) is a fiber bundle over the flag manifold $G/P_S \times G/P_T^-$ whose fiber is isomorphic to the product of a De Concini-Procesi compactification of G_S and a Hermitian symmetric space of a special orthogonal group. In particular, all the irreducible components of $\mathcal{L}$ are smooth;

3) $\mathcal{L}$ has two connected components.

Let again $G_\Delta = \{(g,g) : g \in G\}$ be the diagonal subgroup of $G \times G$. In Section 3, we classify G_Δ-orbits in $\mathcal{L}$, which is equivalent to describing the $(G_\Delta, R_{S,T,d})$-double coset space in $G \times G$ for every generalized Belavin-Drinfeld triple (S,T,d). More precisely, let W_T be the subgroup of the Weyl group W of Γ generated by the elements in T, and let W^T be the set of minimal length representatives in cosets from W/W_T. For each $v \in W^T$, let $\dot{v}$ be a representative of v on G, and let $S(v,d) \subset S$ be the maximal subset of S that is invariant under vd. Let $M_{S(v,d)}$ be the standard Levi subgroup of G defined by $S(v,d)$. Let $R_{\dot{v}}$ be the subgroup of $M_{S(v,d)} \times M_{S(v,d)}$ defined by

$$R_{\dot{v}} = \left(M_{S(v,d)} \times M_{S(v,d)}\right) \cap \left((\mathrm{id} \times \mathrm{Ad}_{\dot{v}})R_{S,T,d}\right),$$

and let $R_{\dot{v}}$ act on $M_{S(v,d)}$ (from the right) by

$$m_1 \cdot (m, m') = (m')^{-1} m_1 m, \qquad m_1 \in M_{S(v,d)},\ (m,m') \in R_{\dot{v}}.$$

We prove (Theorem 3.9) the following statement:

Every $(G_\Delta, R_{S,T,d})$-double coset in $G \times G$ has a representative $(m, \dot{v})$ for some $v \in W^T$ and $m \in M_{S(v,d)}$. Two such cosets through $(m_1, \dot{v}_1)$ and $(m_2, \dot{v}_2)$ coincide if and only if $v_1 = v_2 = v$ and $m_1, m_2 \in M_{S(v,d)}$ are in the same $R_{\dot{v}}$-orbit in $M_{S(v,d)}$.

We also compute the stabilizer subalgebra of $\mathfrak{g}_\Delta$ at every $\mathfrak{l} \in \mathcal{L}$.

In Section 4, we recall the definition of a Poisson structure on $\mathcal{L}$ defined by a Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$. We study the symplectic leaf decomposition of the Poisson structure Π_0 defined by the standard Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}_{\mathrm{st}}^*$. We have (Theorem 4.10 and Theorem 4.19):

1) Every non-empty intersection of a G_Δ-orbit $\mathcal{O}$ and a $(B \times B^-)$-orbit $\mathcal{O}'$ in $\mathcal{L}$ is a regular Poisson manifold with respect to the Poisson structure Π_0;

2) The Cartan subgroup H_Δ of G_Δ, where $H = B \cap B^-$, acts transitively on the set of symplectic leaves in $\mathcal{O} \cap \mathcal{O}'$.

We also compute the rank of Π_0 in Section 4. Thus, the study of symplectic leaves of Π_0 in $\mathcal{L}$ is reduced to the understanding of the intersections of G_Δ and $(B \times B^-)$-orbits in $\mathcal{L}$ as H_Δ-varieties. Since we have classified the G_Δ and $(B \times B^-)$-orbits in $\mathcal{L}$ (in Section 3.3 and Section 2.6 respectively), one would next like to understand when two such orbits intersect and to study the topology of such intersections. The intersections of G_Δ-orbits and $(B \times B^-)$-orbits inside the closed $(G \times G)$-orbits in $\mathcal{L}$ are related to double Bruhat cells in G (see Example 4.14), and Kogan and Zelevinsky [K-Z] have constructed toric charts on some of the symplectic leaves in these closed orbits. It would be interesting to see how their methods can be applied to other symplectic leaves of Π_0.

We point out in Section 4 some interesting Poisson subvarieties $\mathcal{L}$ with respect to the Poisson structures Π_{BD} defined by the Belavin-Drinfeld splittings, i.e., Lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$ that are of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}$ for some $\mathfrak{l} \in \mathcal{L}$. One class of such examples consists of the De Concini-Procesi compactifications of symmetric spaces G/G^σ, where σ is an involutive automorphism of G (Proposition 3.22). Another interesting example is the De Concini-Procesi compactification $Z_1(G)$ of G, the closure of the $(G\times G)$-orbit in $\mathcal{L}$ through $\mathfrak{g}_\Delta$. Conjugacy classes in G and their closures in $Z_1(G)$ are all Poisson subvarieties of $(Z_1(G), \Pi_{BD})$. In particular, the Poisson structure Π_0 restricted to a conjugacy class C in G is non-degenerate precisely on the intersection of C with the open Bruhat cell B^-B (see Corollary 4.11). It will be particularly interesting to compare the Poisson structure Π_0 on the unipotent variety in G with the Kirillov-Kostant structure on the nilpotent cone in $\mathfrak{g}^*$.

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2. The variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$

Throughout this paper, $\mathfrak{g}$ will be a complex semi-simple Lie algebra, and $\ll\, ,\, \gg$ will be a fixed symmetric and non-degenerate ad-invariant bilinear form on $\mathfrak{g}$. We will equip the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the bilinear form

$$\langle (x_1, x_2), (y_1, y_2)\rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \qquad x_1, x_2, y_1, y_2 \in \mathfrak{g}. \tag{2.1}$$

Clearly $\langle\, ,\, \rangle$ is symmetric, non-degenerate, and ad-invariant. By a Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ we mean an n-dimensional complex Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ that is isotropic with respect to $\langle\, ,\, \rangle$.

Notation 2.1. We use $\mathcal{L}$ to denote the variety of all Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$, and we will use $\mathcal{L}_{\text{space}}(\mathfrak{g} \oplus \mathfrak{g})$ to denote the variety of all n-dimensional isotropic *subspaces* of $\mathfrak{g} \oplus \mathfrak{g}$.

Let G be the adjoint group of $\mathfrak{g}$. The group $G \times G$ acts on $\mathcal{L}$ through the adjoint action. In this section, we will classify the $(G \times G)$-orbits and study their closures in $\mathcal{L}$, and we will determine the irreducible components of $\mathcal{L}$. We show that each irreducible component of $\mathcal{L}$ is a fiber bundle with smooth fibers over a generalized flag variety of $G \times G$ and is thus smooth. We show that all $(G\times G)$-orbits in $\mathcal{L}$ and their closures are smooth spherical varieties for $G\times G$. Our results in this section are based on the classification of Lagrangian subalgebras by Karolinsky [Ka].

2.1. **Lagrangian subspaces.** Let U be a finite-dimensional complex vector space with a symmetric and non-degenerate bilinear form $\langle\,,\,\rangle$. A subspace V of U is said to be Lagrangian if V is a maximal isotropic subspace of U with respect to $\langle\,,\,\rangle$. If $\dim U = 2n$ or $2n+1$, then Witt's theorem says that the dimension of a Lagrangian subspace of U is n. The set of Lagrangian subspaces is easily seen to be a closed algebraic subvariety of $\mathrm{Gr}(n, U)$, the Grassmannian of n-dimensional subspaces of U. Denote by $\mathcal{L}_{\mathrm{space}}(U)$ the variety of Lagrangian subspaces of U with respect to $\langle\,,\,\rangle$.

Proposition 2.2 ([A-C-G-H], pp. 102-103). *Assume that* $\dim U = 2n$ *(resp.* $2n+1$*) with* $n > 0$*. Then* $\mathcal{L}_{\mathrm{space}}(U)$ *is a smooth algebraic subvariety of* $\mathrm{Gr}(n, U)$*. It has two (resp. one) connected components, each of which is isomorphic to the generalized flag variety* $SO(2n, \mathbb{C})/P$ *(resp.* $SO(2n+1, \mathbb{C})/P$*) where* P *has Levi factor isomorphic to* $GL(n, \mathbb{C})$*. Moreover,* $\mathcal{L}_{\mathrm{space}}(U)$ *has complex dimension* $\frac{n(n-1)}{2}$ *(resp.* $\frac{n(n+1)}{2}$*). When* $\dim U = 2n$*, two Lagrangian subspaces* V_1 *and* V_2 *are in the same connected component of* $\mathcal{L}_{\mathrm{space}}(U)$ *if and only if* $\dim(V_1) - \dim(V_1 \cap V_2)$ *is even.*

Notation 2.3. In the example of $U = \mathfrak{g} \oplus \mathfrak{g}$ with the bilinear form $\langle\,,\,\rangle$ given in (2.1), we denote by $\mathcal{L}^0$ the intersection of $\mathcal{L}$ with the connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ containing the diagonal of $\mathfrak{g} \oplus \mathfrak{g}$. The intersection of $\mathcal{L}$ with the other connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ will be denoted by $\mathcal{L}^1$.

Let $\mathfrak{h}$ be a Cartan subalgebra, and let $\mathfrak{n}$ be the nilpotent subalgebra of $\mathfrak{g}$ corresponding to a choice of positive roots for $(\mathfrak{g}, \mathfrak{h})$, and let $\mathfrak{n}^-$ be nilpotent subalgebra of $\mathfrak{g}$ defined by the negative roots. For a Lagrangian subspace V of $\mathfrak{h} \oplus \mathfrak{h}$ with respect to $\langle\,,\,\rangle$, let

$$\mathfrak{l}_V = V + \{(x, y) : x \in \mathfrak{n}, y \in \mathfrak{n}^-\} \subset \mathfrak{g} \oplus \mathfrak{g}.$$

Then $\mathfrak{l}_V$ is a Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$. It is easy to see from Proposition 2.2 that $\mathfrak{l}_{V_1}$ and $\mathfrak{l}_{V_2}$ are in the same connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{g} \oplus \mathfrak{g})$ if and only if V_1 and V_2 are in the same connected component of $\mathcal{L}_{\mathrm{space}}(\mathfrak{h} \oplus \mathfrak{h})$. In particular, $\mathcal{L}^1$ is non-empty.

2.2. **Isometries.** We collect some results on automorphisms that will be used in later sections.

Notation 2.4. Throughout this paper, we will fix a Cartan subalgebra $\mathfrak{h}$ and a choice Σ^+ of positive roots in the set Σ of all roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. We will use Γ to denote the set of simple roots in Σ^+. For each $\alpha \in \Sigma$, let $H_\alpha \in \mathfrak{h}$ be such that $\ll H_\alpha, H \gg = \alpha(H)$ for all $H \in \mathfrak{h}$. For each $\alpha \in \Sigma^+$, we fix root vectors $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $\ll E_\alpha, E_{-\alpha} \gg = 1$. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ be the root decomposition for $\mathfrak{g}$.

Let S and T be two subsets of Γ. We are interested in Lie algebra isomorphisms $\mathfrak{g}_S \to \mathfrak{g}_T$ that preserve the restrictions of the bilinear form $\ll\,,\,\gg$ of $\mathfrak{g}$ to $\mathfrak{g}_S$ and $\mathfrak{g}_T$. We will simply refer to this property as preserving $\ll\,,\,\gg$. To describe such isomorphisms, we introduce the following definition.

Definition 2.5. Let S and T be two subsets of Γ. By an isometry from S to T (with respect to the bilinear form $\ll\,,\,\gg$) we mean a bijection $d : S \to T$ such that $\ll d\alpha, d\beta \gg = \ll \alpha, \beta \gg$ for all $\alpha, \beta \in S$, where $\ll \alpha, \beta \gg = \ll H_\alpha, H_\beta \gg$. We use $I(S, T)$ to denote the set of all isometries from S to T. Following [Sch], a triple (S, T, d), where $S, T \subset \Gamma$ and $d \in I(S, T)$, will also be called a *generalized Belavin-Drinfeld (generalized BD-)triple for G.*

Remark 2.6. Note that our definition of $I(S,T)$ depends on our choice of the ad-invariant bilinear form $\ll , \gg$ on $\mathfrak{g}$. On the other hand, for each $S \subset \Gamma$, we can identify S with the vertices of the Dynkin diagram of $\mathfrak{g}_S$ so that there is also the scalar product on elements in S coming from the Killing form B_S of $\mathfrak{g}_S$. For another subset $T \subset \Gamma$, let $I_{\text{Killing}}(S,T)$ denote the set of bijections $d : S \to T$ that preserve the scalar products induced by the Killing forms B_S and B_T. Then it is easy to see that $I(S,T) \subset I_{\text{Killing}}(S,T)$ but $I(S,T)$ is not necessarily equal to $I_{\text{Killing}}(S,T)$. For example, consider the case when $S = T$ consists of exactly two orthogonal simple roots α and β of $\mathfrak{g}$ such that $\ll \alpha, \alpha \gg \neq \ll \beta, \beta \gg$. Then the map $d : S \to T$ that exchanges α and β is in $I_{\text{Killing}}(S,T)$ but not in $I(S,T)$.

Lemma 2.7. *Let S and T be subsets of Γ and let $d \in I(S,T)$. There exists a unique isomorphism $\gamma_d : \mathfrak{g}_S \to \mathfrak{g}_T$ such that*

$$\gamma_d(E_\alpha) = E_{d(\alpha)}, \qquad \gamma_d(H_\alpha) = H_{d(\alpha)} \tag{2.2}$$

for every $\alpha \in S$. Moreover, γ_d preserves $\ll , \gg$, and for every Lie algebra isomorphism $\mu : \mathfrak{g}_S \to \mathfrak{g}_T$ preserving $\ll , \gg$, there is a unique isometry $d \in I(S,T)$ and a unique $g \in G_S$ such that $\mu = \gamma_d \mathrm{Ad}_g$.

Proof. Existence and uniqueness of γ_d is by Theorem 2.108 in [Kn]. For $\alpha \in \Sigma^+$, let $\lambda_\alpha, \mu_\alpha \in \mathbb{C}$ be such that $\gamma_d(E_\alpha) = \lambda_\alpha E_\alpha$ and $\gamma_d(E_{-\alpha}) = \mu_\alpha E_{-\alpha}$. By applying γ_d to the identity $[E_\alpha, E_{-\alpha}] = H_\alpha$ we get $\lambda_\alpha \mu_\alpha = 1$ for every $\alpha \in \Sigma^+$. It follows that γ_d preserves $\ll , \gg$. Now suppose that $\mu : \mathfrak{g}_S \to \mathfrak{g}_T$ is a Lie algebra isomorphism preserving $\ll , \gg$. Let d_1 be any isomorphism from the Dynkin diagram of $\mathfrak{g}_S$ to the Dynkin diagram of $\mathfrak{g}_T$. Let $\gamma_{d_1} : \mathfrak{g}_S \to \mathfrak{g}_T$ be defined as in (2.2). Then $\nu := \gamma_{d_1}^{-1}\mu$ is an automorphism of $\mathfrak{g}_S$. Recall that there is a short exact sequence

$$1 \longrightarrow G_S \longrightarrow \mathrm{Aut}_{\mathfrak{g}_S} \longrightarrow \mathrm{Aut}_S \longrightarrow 1,$$

where $\mathrm{Aut}_{\mathfrak{g}_S}$ is the group of all automorphisms of $\mathfrak{g}_S$, and Aut_S is the group of all automorphisms of the Dynkin diagram of $\mathfrak{g}_S$. Let $d_2 \in \mathrm{Aut}_S$ be the image of ν under the map $\mathrm{Aut}_{\mathfrak{g}_S} \to \mathrm{Aut}_S$ and write $\nu = \gamma_{d_2}\mathrm{Ad}_g$ for some $g \in G_S$. Thus $\mu = \gamma_{d_1}\gamma_{d_2}\mathrm{Ad}_g = \gamma_{d_1 d_2}\mathrm{Ad}_g$. Since μ and Ad_g are isometries of $\ll , \gg$, $\gamma_{d_1 d_2}$ is an isometry of $\ll , \gg$. Thus, $d := d_1 d_2 \in I(S,T)$ is an isometry, and $\mu = \gamma_d \mathrm{Ad}_g$.

Uniqueness of d and g follows from the fact that if $g_0 \in G_S$ preserves a Cartan subalgebra and acts as the identity on all simple root spaces, then g_0 is the identity element.

Q.E.D.

Definition 2.8. For a Lie algebra isomorphism $\mu : \mathfrak{g}_S \to \mathfrak{g}_T$ preserving $\ll , \gg$, we will say that μ is of *type* d for $d \in I(S,T)$ if d is the unique element in $I(S,T)$ such that $\mu = \gamma_d \mathrm{Ad}_g$ for some $g \in G_S$.

2.3. **Karolinsky's classification.** Karolinsky [Ka] has classified the Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ with respect to the bilinear form $\langle\, , \rangle$ given in (2.1). We recall his results now.

Notation 2.9. For a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, let $\mathfrak{n}$ be its nilradical, and let $\mathfrak{m} := \mathfrak{p}/\mathfrak{n}$ be its Levi factor. Let $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{z}$ be the decomposition of $\mathfrak{m}$ into the sum of its derived algebra $[\mathfrak{m}, \mathfrak{m}]$ and its center $\mathfrak{z}$. Recall that $[\mathfrak{m}, \mathfrak{m}]$ is semisimple and that the bilinear form $\ll , \gg$ of $\mathfrak{g}$ induces a well-defined non-degenerate and ad-invariant bilinear form on $\mathfrak{m}$ which we will still denote by $\ll , \gg$. Moreover, $\ll , \gg$ is nondegenerate on $\mathfrak{z}$. If $\mathfrak{p}'$ is another parabolic subalgebra, we denote its nilradical, Levi factor, and center of Levi factor, etc. by $\mathfrak{n}'$, $\mathfrak{m}'$, and $\mathfrak{z}'$, etc..

Let $\mathfrak{p}$ and $\mathfrak{p}'$ be parabolic subalgebras. The bilinear form $\langle\, ,\, \rangle$ is nondegenerate on $\mathfrak{z}\oplus\mathfrak{z}'$. When we speak of Lagrangian subspaces of $\mathfrak{z}\oplus\mathfrak{z}'$, we mean with respect to $\langle\, ,\, \rangle$.

Definition 2.10. A quadruple $(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is called admissible if $\mathfrak{p}$ and $\mathfrak{p}'$ are parabolic subalgebras of $\mathfrak{g}$, $\mu : [\mathfrak{m}, \mathfrak{m}] \to [\mathfrak{m}', \mathfrak{m}']$ is a Lie algebra isomorphism preserving $\ll\, ,\, \gg$, and V is a Lagrangian subspace of $\mathfrak{z}\oplus\mathfrak{z}'$.

If $(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is admissible, set

$$\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V) := \{(x, x') : x \in \mathfrak{p}, x' \in \mathfrak{p}', \mu(x_{[\mathfrak{m},\mathfrak{m}]}) = x'_{[\mathfrak{m}',\mathfrak{m}']}, (x_{\mathfrak{z}}, x'_{\mathfrak{z}'}) \in V\} \subset \mathfrak{g}\oplus\mathfrak{g},$$

where for $x \in \mathfrak{p}$, $x_{[\mathfrak{m},\mathfrak{m}]} \in [\mathfrak{m}, \mathfrak{m}]$ and $x_{\mathfrak{z}} \in \mathfrak{z}$ are respectively the $[\mathfrak{m}, \mathfrak{m}]$- and $\mathfrak{z}$-components of $x + \mathfrak{n} \in \mathfrak{p}/\mathfrak{n} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{z}$. We use similar notation for $x' \in \mathfrak{p}'$.

Theorem 2.11 ([Ka]). *1) $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is a Lagrangian subalgebra if $(\mathfrak{p}, \mathfrak{p}', \mu, V)$ is admissible.*

2) Every Lagrangian subalgebra of $\mathfrak{g}\oplus\mathfrak{g}$ is of the form $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V)$ for some admissible quadruple.

2.4. **Partition of $\mathcal{L}$.** In this subsection, we partition $\mathcal{L}$ into strata and determine the geometry of each stratum. We fix some notation on parabolic subalgebras of $\mathfrak{g}$.

Notation 2.12. Recall the fixed choice of positive roots from Notation 2.4. Set

$$\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha}.$$

A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *standard* if it contains the Borel subalgebra $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}$. For a subset S of Γ, we will use $[S]$ to denote the set of roots in the linear span of S, and we will set

$$\mathfrak{m}_S = \mathfrak{h} + \sum\nolimits_{\alpha\in[S]} \mathfrak{g}_\alpha, \quad \mathfrak{n}_S = \sum\nolimits_{\alpha\in\Sigma^+-[S]} \mathfrak{g}_\alpha, \quad \mathfrak{n}_S^- = \sum_{\alpha\in\Sigma^+-[S]} \mathfrak{g}_{-\alpha}$$

and

$$\mathfrak{p}_S = \mathfrak{m}_S + \mathfrak{n}_S, \quad \mathfrak{p}_S^- = \mathfrak{m}_S + \mathfrak{n}_S^-.$$

We will refer to $\mathfrak{p}_S$ as the standard parabolic subalgebra of $\mathfrak{g}$ defined by S, and we will also refer to $\mathfrak{p}_S^-$ as the *opposite* of $\mathfrak{p}_S$. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. We say that $\mathfrak{p}$ is of the *type* S if $\mathfrak{p}$ is conjugate to $\mathfrak{p}_S$, and we say that $\mathfrak{p}$ is of the *opposite-type* S if $\mathfrak{p}$ is conjugate to $\mathfrak{p}_S^-$. Note that $\mathfrak{p}_S$ is of opposite-type $-w_0[S]$, where w_0 is the long element of the Weyl group. Similarly, $\mathfrak{m}_S$ will be referred to as the standard Levi subalgebra of $\mathfrak{g}$ defined by S. We will further set $\mathfrak{g}_S = [\mathfrak{m}_S, \mathfrak{m}_S]$ and

$$\mathfrak{h}_S = \mathfrak{h}\cap\mathfrak{g}_S = \mathrm{span}_{\mathbb{C}}\{H_\alpha : \alpha \in [S]\}, \quad \mathfrak{z}_S = \{x \in \mathfrak{h} : \alpha(x) = 0, \forall\alpha \in S\}.$$

Then we have the decompositions

$$\mathfrak{p}_S = \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S, \quad \mathfrak{p}_S^- = \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S^-.$$

The connected subgroups of G with Lie algebras $\mathfrak{p}_S, \mathfrak{p}_S^-, \mathfrak{m}_S, \mathfrak{n}_S$ and $\mathfrak{n}_S^-$ will be respectively denoted by P_S, P_S^-, M_S, N_S and N_S^-. Correspondingly we have the group decompositions

$$P_S = M_S N_S, \quad \text{and} \quad P_S^- = M_S N_S^-,$$

and $M_S \cap N_S = \{e\} = M_S \cap N_S^-$. Denote by G_S the adjoint group of $\mathfrak{g}_S$. The adjoint action of M_S on $\mathfrak{m}_S$ leaves $\mathfrak{g}_S$ invariant and induces a natural projection $\chi_S : M_S \to G_S$. We will also use χ_S to denote the map $P_S \to G_S : p_S = m_S n_S \mapsto \chi_S(m_S)$ where $m_S \in M_S$ and $n_S \in N_S$. The similarly defined projection from P_S^- to G_S will also be denoted by χ_S.

Returning to the notation in Notation 2.9, we have

Lemma-Definition 2.13. *Let $(\mathfrak{p}, \mathfrak{p}', \mu)$ be a triple, where $\mathfrak{p}$ and $\mathfrak{p}'$ are parabolic subalgebras of $\mathfrak{g}$, and $\mu : [\mathfrak{m}, \mathfrak{m}] \to [\mathfrak{m}', \mathfrak{m}']$ is a Lie algebra isomorphism preserving $\ll, \gg$. Assume that $\mathfrak{p}$ is of type S and $\mathfrak{p}'$ is of opposite-type T. Let $g_1, g_2 \in G$ be such that $\mathrm{Ad}_{g_1}\mathfrak{p} = \mathfrak{p}_S$ and $\mathrm{Ad}_{g_2}\mathfrak{p}' = \mathfrak{p}_T^-$. Let $\overline{\mathrm{Ad}_{g_1}}$ and $\overline{\mathrm{Ad}_{g_2}}$ be the induced Lie algebra isomorphisms*

$$\overline{\mathrm{Ad}_{g_1}} : [\mathfrak{m}, \mathfrak{m}] \longrightarrow \mathfrak{g}_S, \quad \overline{\mathrm{Ad}_{g_2}} : [\mathfrak{m}', \mathfrak{m}'] \longrightarrow \mathfrak{g}_T,$$

and consider

$$\mu' := \overline{\mathrm{Ad}_{g_2}} \circ \mu \circ (\overline{\mathrm{Ad}_{g_1}})^{-1} : \mathfrak{g}_S \longrightarrow \mathfrak{g}_T.$$

If $\mu' : \mathfrak{g}_S \to \mathfrak{g}_T$ is of type $d \in I(S,T)$ as in Definition 2.8, we will say that the triple $(\mathfrak{p}, \mathfrak{p}', \mu)$ is of type (S, T, d). The type of $(\mathfrak{p}, \mathfrak{p}', \mu)$ is independent of the choice of g_1 and g_2.

Proof. If h_1 and h_2 in G are such that $\mathrm{Ad}_{h_1}\mathfrak{p} = \mathfrak{p}_S$ and $\mathrm{Ad}_{h_2}\mathfrak{p}' = \mathfrak{p}_T^-$, then there exist $p_S \in P_S$ and $p_T^- \in P_T^-$ such that $h_1 = p_S g_1$ and $h_2 = p_T^- g_2$. Thus

$$\mu'' := \overline{\mathrm{Ad}_{h_2}} \circ \mu \circ (\overline{\mathrm{Ad}_{h_1}})^{-1} = \overline{\mathrm{Ad}_{p_T^-}} \circ \mu' \circ (\overline{\mathrm{Ad}_{p_S}})^{-1}.$$

The action of $\overline{\mathrm{Ad}_{p_S}}$ on $\mathfrak{g}_S$ is by definition the adjoint action of $\chi_S(p_S) \in G_S$ on $\mathfrak{g}_S$. Similarly for the action of $\overline{\mathrm{Ad}_{p_T^-}}$ on $\mathfrak{g}_T$. Thus by Definition 2.8, the two maps μ' and μ'' have the same type, so the type of $(\mathfrak{p}, \mathfrak{p}', \mu)$ is well-defined.

Q.E.D.

Remark 2.14. For $S, T \subset \Gamma$ and $d \in I(S,T)$, $(\mathfrak{p}_S, \mathfrak{p}_T, \gamma_d)$ is of type $(S, -w_0(T), w_0 w_0^T d)$, where w_0 is the longest element in the Weyl group W of Γ, and w_0^T is the longest element in the subgroup of W generated by elements in T.

We are now ready to partition $\mathcal{L}$. Recall the definitions of $\mathcal{L}^0$ and $\mathcal{L}^1$ in Notation 2.3.

Definition 2.15. Let $S, T \subset \Gamma, d \in I(S,T)$, and $\epsilon \in \{0, 1\}$. Define $\mathcal{L}^\epsilon(S, T, d)$ to be the set of all Lagrangian subalgebras $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V)$ such that

1) $\mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V) \in \mathcal{L}^\epsilon$;
2) $(\mathfrak{p}, \mathfrak{p}', \mu)$ is of type (S, T, d).

We say that $\mathfrak{l} \in \mathcal{L}$ is of *type* (ϵ, S, T, d) if $\mathfrak{l} \in \mathcal{L}^\epsilon(S, T, d)$.

It is clear that we have a disjoint union

$$\mathcal{L} = \bigcup_{\epsilon \in \{0,1\}} \ \bigcup_{S,T \subset \Gamma, d \in I(S,T)} \mathcal{L}^\epsilon(S, T, d), \tag{2.3}$$

and that each $\mathcal{L}^\epsilon(S, T, d)$ is invariant under $G \times G$. To understand the $(G \times G)$-orbits in $\mathcal{L}^\epsilon(S, T, d)$, we will, for each generalized BD-triple (S, T, d), set

$$\mathfrak{n}_S \oplus \mathfrak{n}_T^- = \{(x, y) : x \in \mathfrak{n}_S, y \in \mathfrak{n}_T^-\} \subset \mathfrak{g} \oplus \mathfrak{g},$$

and for each $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, set

$$\mathfrak{l}_{S,T,d,V} = V + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \{(x, \gamma_d(x)) : x \in \mathfrak{g}_S\} \in \mathcal{L}. \tag{2.4}$$

Lemma-Definition 2.16. *For $V_1, V_2 \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, $\mathfrak{l}_{S,T,d,V_1}$ and $\mathfrak{l}_{S,T,d,V_2}$ are in the same connected component of $\mathcal{L}_{\rm space}(\mathfrak{g}\oplus\mathfrak{g})$ if and only if V_1 and V_2 are in the same connected component of $\mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. For $\epsilon = \{0,1\}$, we define*

$$\mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T) = \{V \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T) : \mathfrak{l}_{S,T,d,V} \in \mathcal{L}^\epsilon\}.$$

Proof. The statement follows from Proposition 2.2 and the fact

$$\dim(\mathfrak{l}_{S,T,d,V_1}) - \dim(\mathfrak{l}_{S,T,d,V_1} \cap \mathfrak{l}_{S,T,d,V_2}) = \dim(V_1) - \dim(V_1 \cap V_2).$$

Q.E.D.

Proposition 2.17. *For any generalized BD-triple (S,T,d) and $\epsilon \in \{0,1\}$, we have the disjoint union*

$$\mathcal{L}^\epsilon(S,T,d) = \bigcup_{V \in \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)} (G \times G) \cdot \mathfrak{l}_{S,T,d,V}. \tag{2.5}$$

Proof. It follows from Definition 2.15 that every $(G \times G)$-orbit in $\mathcal{L}^\epsilon(S,T,d)$ passes through an $\mathfrak{l}_{S,T,d,V}$ for some $V \in \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. If $V_1, V \in \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$ are such that $\mathfrak{l}_{S,T,d,V_1} = \mathrm{Ad}_{(g_1,g_2)}\mathfrak{l}_{S,T,d,V}$, then (g_1,g_2) normalizes $(\mathfrak{n}_S \oplus \mathfrak{n}_T^-)$ so $(g_1,g_2) \in P_S \times P_T^-$, and it follows that $V_1 = V$.

Q.E.D.

2.5. **$(G \times G)$-orbits in $\mathcal{L}$.** The following theorem follows immediately from Proposition 2.17 and the decomposition of $\mathcal{L}$ in (2.3).

Theorem 2.18. *Every $(G \times G)$-orbit in $\mathcal{L}$ passes through an $\mathfrak{l}_{S,T,d,V}$ for a unique quadruple (S,T,d,V), where $S,T \subset \Gamma$, $d \in I(S,T)$ and $V \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$.*

For each $S,T \subset \Gamma$ and $d \in I(S,T)$, let

$$R_{S,T,d} := \{(p_S, p_T^-) \in P_S \times P_T^- : \gamma_d(\chi_S(p_S)) = \chi_T(p_T^-)\} \subset P_S \times P_T^- \tag{2.6}$$

(see Notation 2.12). It is easy to see that the group $R_{S,T,d}$ is the normalizer subgroup in $G \times G$ of $\mathfrak{l}_{S,T,d,V}$ for any $V \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. Thus we have the following proposition.

Proposition 2.19. *Let $S,T \subset \Gamma$, $d \in I(S,T)$, and $V \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$.*

1) The $(G\times G)$-orbit in $\mathcal{L}$ through $\mathfrak{l}_{S,T,d,V}$ is isomorphic to $(G\times G)/R_{S,T,d}$ and it has dimension $n - z$, where $n = \dim \mathfrak{g}$ and $z = \dim \mathfrak{z}_S$.

2) $(G \times G) \cdot \mathfrak{l}_{S,T,d,V}$ fibers over $G/P_S \times G/P_T^-$ with fiber isomorphic to G_S.

Proof. It is routine to check that the stabilizer of $\mathfrak{l}_{S,T,d,V}$ is $R_{S,T,d}$, and the dimensional formula follows. The fiber may be identified with $(P_S \times P_T^-)/R_{S,T,d}$, which may be identified with G_S via the map

$$(p_S, p_T^-) \mapsto \gamma_d^{-1}(\chi_T(p_T^-))(\chi_S(p_S))^{-1}$$

Q.E.D.

Remark 2.20. It follows that $(G\times G)$-orbits in $\mathcal{L}^\epsilon(S,T,d)$ for $\epsilon = 0,1$ have conjugate stabilizers, and there are finitely many conjugacy classes of stabilizers of points in $\mathcal{L}$. Moreover, the number of orbit types for $G\times G$ in $\mathcal{L}$ is exactly the number of generalized BD-triples for G. We will show in Section 2.6 that all $(G \times G)$-orbits in $\mathcal{L}$ (and their closures) are $(G \times G)$-spherical varieties.

The following fact will be used in Section 3.3.

Lemma 2.21. *$R_{S,T,d}$ is connected.*

Proof. The projection map $p : R_{S,T,d} \to P_S$, $(p_S, p_T^-) \mapsto p_S$ is surjective and has fiber $N_T^- \times Z(M_T)$, where $Z(M_T)$ is the center of the group M_T. Clearly N_T^- is connected, and $Z(M_T)$ is connected by Proposition 8.1.4 of [C]. Since P_S is connected, it follows that $R_{S,T,d}$ is also connected.

Q.E.D.

2.6. $(B \times B^-)$**-orbits in $\mathcal{L}$.** Let B and B^- be the Borel subgroups of G with Lie algebras $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$ respectively. In this section, we show that there are finitely many $(B \times B^-)$-orbits in each $(G \times G)$-orbit in $\mathcal{L}$. Recall that a normal variety X with an action of $(G \times G)$ is said to be *spherical* if a Borel subgroup of $G \times G$ has an open orbit on X. Consequently, all $(G \times G)$-orbits in $\mathcal{L}$ are $(G \times G)$-spherical homogeneous spaces. The description of $(B \times B^-)$-orbits in $\mathcal{L}$ in this section will also be used in Section 4 to understand a certain Poisson structure on $\mathcal{L}$.

By Proposition 2.19, every $(G \times G)$-orbit in $\mathcal{L}$ is of the form $(G \times G)/R_{S,T,d}$ for some generalized BD-triple (S, T, d), where $R_{S,T,d}$ is given by (2.6). Thus it is enough to consider $(B \times B^-)$-orbits in $(G \times G)/R_{S,T,d}$ for any given generalized BD-triple (S, T, d).

Let W be the Weyl group of Σ. For a subset $S \subset \Gamma$, we will use W_S to denote the subgroup of W generated by the simple reflections corresponds to the elements in S. We will use W^S to denote the set of minimal length representatives of elements in the cosets in W/W_S. It is well-known that $w \in W^S$ if and only if $w(S) \subset \Sigma^+$. For each $w \in W$, we will also fix a representative $\dot{w}$ of w in G.

The following assertion is similar to Lemma 1.3 in [Sp].

Proposition 2.22. *Let (S, T, d) be an generalized BD-triple for G. Then the $(B \times B^-)$-orbits in $(G \times G)/R_{S,T,d}$ are of the form $Q_{w,v} = (B \times B^-) \cdot (\dot{w}, \dot{v})R_{S,T,d}$, where $w \in W$ and $v \in W^T$. Moreover, $Q_{w,v} = Q_{w_1,v_1}$ if and only if $w = w_1$ and $v = v_1$.*

Proof. Consider the right action of $P_S \times P_T^-$ on $(B \times B^-)\backslash(G \times G)$ by right translations. By the Bruhat decomposition of G, every $(P_S \times P_T^-)$-orbit contains exactly one point of the form $(B \times B^-)(\dot{w}_1, \dot{w}_2)$, where $w_1 \in W^S$ and $w_2 \in W^T$. Denote by $\mathrm{Stab}_{(w_1,w_2)}$ the stabilizer subgroup of $P_S \times P_T^-$ at $(B \times B^-)(\dot{w}_1, \dot{w}_2)$. It is easy to see that

$$\begin{aligned} \mathrm{Stab}_{(w_1,w_2)} &= \left(P_S \cap w_1^{-1}(B)\right) \times \left(P_T^- \cap w_2^{-1}(B^-)\right) \\ &= \left(H(N \cap w_1^{-1}(N))\right) \times \left(H(N^- \cap w_2^{-1}(N^-))\right). \end{aligned}$$

Thus every $R_{S,T,d}$-orbit in $(B \times B^-)\backslash(G \times G)$ is of the form $(B \times B^-)(\dot{w}_1 p_S, \dot{w}_2 p_T^-)$ for a unique pair $(w_1, w_2) \in W^S \times W^T$ and for some $(p_S, p_T^-) \in P_S \times P_T^-$. Two such points $(B \times B^-)(\dot{w}_1 p_S, \dot{w}_2 p_T^-)$ and $(B \times B^-)(\dot{w}_1 q_S, \dot{w}_2 q_T^-)$ are in the same $R_{S,T,d}$-orbit if and only if (p_S, p_T^-) and (q_S, q_T^-) are in the same $(\mathrm{Stab}_{(w_1,w_2)}, R_{S,T,d})$-double coset in $P_S \times P_T^-$. To understand the double coset space $\mathrm{Stab}_{(w_1,w_2)}\backslash P_S \times P_T^-/R_{S,T,d}$, consider the projection $\pi_S : P_S \to M_S$ and $\pi_T : P_T^- \to M_T$ with respect to the decompositions $P_S = M_S N_S$ and $P_T^- = M_T N_T^-$ respectively. Since $M_S \times M_T$

normalizes $N_S \times N_T^-$ in $P_S \times P_T^-$, we see that $\pi_S \times \pi_T : P_S \times P_T^- \to M_S \times M_T$ induces an identification

$$\mathrm{Stab}_{(w_1,w_2)} \backslash P_S \times P_T^- / R_{S,T,d} \cong (\pi_S \times \pi_T)(\mathrm{Stab}_{(w_1,w_2)}) \backslash M_S \times M_T / (\pi_S \times \pi_T)(R_{S,T,d}).$$

Note that $(\pi_S \times \pi_T)(\mathrm{Stab}_{(w_1,w_2)}) = B_S \times B_T^-$, where $B_S = M_S \cap B$ and $B_T^- = M_T \cap B^-$ are Borel subgroups of M_S and M_T respectively. On the other hand, it is easy to see that every $(B_S \times B_T^-, (\pi_S \times \pi_T)(R_{S,T,d}))$-double coset in $M_S \times M_T$ is of the form $(\dot{u}_1, e)$ for a unique $u_1 \in W_S$. By setting $w = w_1 u_1 \in W$ and $v = w_2 \in W^T$ for $(w_1, w_2) \in W^S \times W^T$ and $u_1 \in W_S$, we have proved Proposition 2.22.

Q.E.D.

Corollary 2.23. *Every $(B \times B^-)$-orbit in $\mathcal{L}$ goes through exactly one point of $\mathcal{L}$ of the form $\mathrm{Ad}_{(\dot{w},\dot{v})} \mathfrak{l}_{S,T,d,V}$, where (S,T,d) is a generalized BD-triple, $V \in \mathcal{L}_{space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, and $w \in W$, $v \in W^T$.*

Since each $(G \times G)$-orbit in $\mathcal{L}$ has finitely many $(B \times B^-)$-orbits, at least one of them is open. Thus we have the following corollary.

Corollary 2.24. *All $(G \times G)$-orbits in $\mathcal{L}$ are $(G \times G)$-spherical homogeneous spaces.*

2.7. **The De Concini-Procesi compactifications $Z_d(G)$ of G.** In this section, we will consider the closure in $\mathcal{L}$ of some special $(G \times G)$-orbits. Namely, when $S = T = \Gamma$ and $d \in I(\Gamma, \Gamma)$, we have the graph $\mathfrak{l}_{\gamma_d}$ of γ_d as a point in $\mathcal{L}$:

$$\mathfrak{l}_{\gamma_d} = \{(x, \gamma_d(x)) : x \in \mathfrak{g}\}. \tag{2.7}$$

The $(G \times G)$-orbit in $\mathcal{L}$ through $\mathfrak{l}_{\gamma_d}$ can be identified with G by the map

$$I_d : G \longrightarrow (G \times G) \cdot \mathfrak{l}_{\gamma_d} : \ g \longmapsto \{(x, \gamma_d \mathrm{Ad}_g(x)) : x \in \mathfrak{g}\}. \tag{2.8}$$

The identification I_d is $(G \times G)$-equivariant if we equip G with the action of $G \times G$ given by

$$(g_1, g_2) \cdot g = \gamma_d^{-1}(g_2) g g_1^{-1}. \tag{2.9}$$

Since an orbit of an algebraic group on a variety is open in its closure (see Section 8.3 in [Hu]), the orbit $(G \times G) \cdot \mathfrak{l}_{\gamma_d}$ has the same closure in the Zariski topology and in the classical topology. The closure $\overline{(G \times G) \cdot \mathfrak{l}_{\gamma_d}}$, called a *De Concini-Procesi compactification of* G, is a smooth projective variety of dimension $n = \dim G$ (see [D-P, §6]). We denote this closure by $Z_d(G)$. We note that $Z_d(G) = (\mathrm{id} \times \gamma_d) Z_{\mathrm{id}}(G)$ (but not $(G \times G)$-equivariantly).

It is known [D-P] that $G \times G$ has finitely many orbits in $Z_d(G)$ indexed by subsets of Γ. Indeed, for each $S \subset \Gamma$, let $\mathfrak{l}_{S,d} \in \mathcal{L}$ be given by

$$\mathfrak{l}_{S,d} = \mathfrak{n}_S \oplus \mathfrak{n}_{d(S)}^- + \{(x, \gamma_d(x)) : x \in \mathfrak{m}_S\}. \tag{2.10}$$

Choose $\lambda \in \mathfrak{h}$ such that there exists a one parameter subgroup $e^\lambda : \mathbb{C}^* \to H$ such that $d(e^\lambda)(1) = \lambda$ and $\alpha(\lambda) = 0$ for all $\alpha \in S$ and $\alpha(\lambda) > 0$ for all $\alpha \in \Gamma - S$. Then it is easy to see that

$$\lim_{t \to +\infty} \mathrm{Ad}_{(e^\lambda(t), e)} \mathfrak{l}_{\gamma_d} = \mathfrak{l}_{S,d} \in \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}).$$

Thus $\mathfrak{l}_{S,d} \in Z_d(G)$. It is easy to see that $\mathfrak{l}_{\gamma_d} \in \mathcal{L}^\epsilon(\Gamma, \Gamma, d)$ for $\epsilon = (\dim \mathfrak{h} - \dim \mathfrak{h}^{\gamma_d}) \mod 2$. Thus $\mathfrak{l}_{S,d} \in \mathcal{L}^\epsilon(S, d(S), d|_S)$ for the same value of ϵ.

Theorem 2.25. [D-P] *For every $d \in I(\Gamma, \Gamma)$, $Z_d(G) = \bigcup_{S \subset \Gamma} (G \times G) \cdot \mathfrak{l}_{S,d}$ is a disjoint union.*

2.8. **Closures of $(G\times G)$-orbits in $\mathcal{L}$.** By Theorem 2.18, every $(G\times G)$-orbit in $\mathcal{L}$ passes through an $\mathfrak{l}_{S,T,d,V}$ for a unique quadruple (S,T,d,V), where (S,T,d) is a generalized BD-triple, $V\in\mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S\oplus\mathfrak{z}_T)$, and $\mathfrak{l}_{S,T,d,V}$ is given in (2.4). For each quadruple (S,T,d,V), we will now study the closure of the $(G\times G)$-orbit through $\mathfrak{l}_{S,T,d,V}$ in $\mathrm{Gr}(n,\mathfrak{g}\oplus\mathfrak{g})$. To this end, let $\mathrm{Gr}(m,\mathfrak{g}_S\oplus\mathfrak{g}_T)$ be the Grassmannian of m-dimensional subspaces in $\mathfrak{g}_S\oplus\mathfrak{g}_T$, where $m=\dim\mathfrak{g}_S$. For the Lie algebra isomorphism $\gamma_d:\mathfrak{g}_S\to\mathfrak{g}_T$ given in (2.2), let

$$\mathfrak{l}_{\gamma_d}=\{(x,\,\gamma_d(x)):\,x\in\mathfrak{g}_S\}\in\mathrm{Gr}(m,\mathfrak{g}_S\oplus\mathfrak{g}_T).$$

Definition 2.26. We define $Z_d(G_S)$ to be the closure of the $(G_S\times G_T)$-orbit $(G_S\times G_T)\cdot\mathfrak{l}_{\gamma_d}$ in $\mathrm{Gr}(m,\mathfrak{g}_S\oplus\mathfrak{g}_T)$ through $\mathfrak{l}_{\gamma_d}$.

It is easy to see that $(G_S\times G_T)\cdot\mathfrak{l}_{\gamma_d}$ consists of all $\{(x,\gamma_d\mathrm{Ad}_g(x)):x\in\mathfrak{g}_S\}$ for $g\in G_S$. Thus $Z_d(G_S)$ is the closure of G_S inside $\mathrm{Gr}(m,\mathfrak{g}_S\oplus\mathfrak{g}_T)$ under the embedding

$$G_S\longrightarrow\mathrm{Gr}(m,\mathfrak{g}_S\oplus\mathfrak{g}_T):\;g\longmapsto\{(x,\gamma_d\mathrm{Ad}_gx):\,x\in\mathfrak{g}_S\}. \tag{2.11}$$

Let the group $P_S\times P_T^-$ act on $\mathrm{Gr}(m,\mathfrak{g}_S\oplus\mathfrak{g}_T)$ through the group homomorphism $\chi_S\times\chi_T: P_S\times P_T^-\to G_S\times G_T$, and let $P_S\times P_T^-$ act on G_S by

$$(p_S,p_T^-)\cdot g_S=\gamma_d^{-1}(\chi_T(p_T^-))\,g_S\,(\chi_S(p_S))^{-1},\qquad(p_S,p_T^-)\in P_S\times P_T^-,\,g_S\in G_S \tag{2.12}$$

(see Notation 2.12). Then the embedding in (2.11) is $(P_S\times P_T^-)$-equivariant. In particular, $Z_d(G_S)$ is a $(P_S\times P_T^-)$-equivariant compactification of G_S for the action of $P_S\times P_T^-$ on G_S given in (2.12).

Proposition 2.27. *For every generalized BD-triple (S,T,d) and every $V\in\mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S\oplus\mathfrak{z}_T)$,*

1) the closure $\overline{(G\times G)\cdot\mathfrak{l}_{S,T,d,V}}$ in $\mathrm{Gr}(n,\mathfrak{g}\oplus\mathfrak{g})$ is a smooth subvariety of $\mathrm{Gr}(n,\mathfrak{g}\oplus\mathfrak{g})$ of dimension $n-z$, where $n=\dim\mathfrak{g}$ and $z=\dim\mathfrak{z}_S$, and the map

$$\begin{aligned}\mathbf{a}:\qquad (G\times G)\times_{(P_S\times P_T^-)}Z_d(G_S)&\longrightarrow\overline{(G\times G)\cdot\mathfrak{l}_{S,T,d,V}}\\ [(g_1,g_2),\mathfrak{l}]&\longmapsto\mathrm{Ad}_{(g_1,g_2)}(V+(\mathfrak{n}_S\oplus\mathfrak{n}_T^-)+\mathfrak{l})\end{aligned}$$

is a $(G\times G)$-equivariant isomorphism;

2) $\overline{(G\times G)\cdot\mathfrak{l}_{S,T,d,V}}$ is the finite disjoint union

$$\overline{(G\times G)\cdot\mathfrak{l}_{S,T,d,V}}=\bigcup_{S_1\subset S}(G\times G)\cdot\mathfrak{l}_{S_1,d(S_1),d_1,V_1},$$

where for $S_1\subset S$, we have $d_1=d|_{S_1}$, and

$$V_1=V+\{(x,\gamma_d(x)):\,x\in\mathfrak{h}_S\cap\mathfrak{z}_{S_1}\}\subset\mathfrak{z}_{S_1}\oplus\mathfrak{z}_{T_1}.$$

Proof. Since $G/P_S\times G/P_T^-$ is complete, it follows by standard arguments that the image of $\mathbf{a}$ is closed. Since $(G\times G)\times_{(P_S\times P_T^-)}G_S$ is dense in $(G\times G)\times_{(P_S\times P_T^-)}Z_d(G_S)$ and $\mathbf{a}$ restricts to give an isomorphism from $(G\times G)\times_{(P_S\times P_T^-)}G_S$ to $(G\times G)\cdot\mathfrak{l}_{S,T,d,V}$, it follows that the image of $\mathbf{a}$ is dense in $\overline{(G\times G)\cdot\mathfrak{l}_{S,T,d,V}}$. Hence $\mathbf{a}$ is onto. 2) follows easily from the fact that $\mathbf{a}$ is onto and the description of orbits in $Z_d(G_S)$.

To show that $\mathbf{a}$ is an isomorphism, we note by 2) that if $\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\mu,V)\in\overline{(G\times G)\cdot\mathfrak{l}_{S,T,d,V}}$, then $\mathfrak{p}$ is of type $S_1\subset S$ and $\mathfrak{p}'$ is of opposite type $T_1=d(S_1)\subset T$. For such an $\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\mu,V)$, let

$$\phi(\mathfrak{l}(\mathfrak{p},\mathfrak{p}',\mu,V))=(\rho_S(\mathfrak{p}),\rho_T(\mathfrak{p}'))\in G/P_S\times G/P_T^-,$$

where $\rho_S : G/P_{S_1} \to G/P_S$ and $\rho_T : G/P^-_{T_1} \to G/P^-_T$ are the usual projections between varieties of parabolic subgroups. Then $\phi : \overline{(G \times G) \cdot \mathfrak{l}_{S,T,d,V}} \to G/P_S \times G/P^-_T$ is $(G \times G)$-equivariant and can be shown to be algebraic using the local coordinates in [D-P], §2.3. Moreover, $\phi^{-1}(eP_S, eP^-_T) \cong Z_d(G_S)$. Indeed, if $(g_1, g_2) \notin P_S \times P^-_T$, then it is easy to check using the Bruhat decomposition that $(g_1, g_2) \cdot (V + \mathfrak{n}_S \oplus \mathfrak{n}^-_T + \mathfrak{l})$ does not project to (eP_S, eP^-_T) under ϕ for any $\mathfrak{l} \in Z_d(G_S)$. Now we use Lemma 4 on p. 26 of [Sl] to conclude that $\mathbf{a}$ is an isomorphism.

Q.E.D.

Consider now the case when S and T are the empty set $\emptyset$, so $d = 1$. By Theorem 2.18, every $(G\times G)$-orbit in $\mathcal{L}^0(\emptyset, \emptyset, 1) \cup \mathcal{L}^1(\emptyset, \emptyset, 1)$ goes through a unique Lagrangian subalgebra of the form

$$\mathfrak{l}_V = V + (\mathfrak{n} \oplus \mathfrak{n}^-), \tag{2.13}$$

where $V \in \mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h})$. The following fact follows immediately from Proposition 2.27.

Corollary 2.28. *For every $V \in \mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h})$, the $(G \times G)$-orbit through $\mathfrak{l}_V$ is isomorphic to $G/B \times G/B^-$. These are the only closed $(G \times G)$-orbits in $\mathcal{L}$.*

Corollary 2.29. *$\mathcal{L}$ has two connected components.*

Proof. In Section 2.1, we observed that $\mathcal{L}$ has at least two connected components, namely $\mathcal{L}^0$ and $\mathcal{L}^1$. Since every orbit of an algebraic group on a variety has a closed orbit in its boundary (see Section 8.3 in [Hu]), every point in $\mathcal{L}$ is in the same connected component as $\mathfrak{l}_V$ for some $V \in \mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h})$. Thus $\mathcal{L}$ has at most two connected components.

Q.E.D.

2.9. **The geometry of the strata $\mathcal{L}^\epsilon(S, T, d)$.** For a generalized BD-triple (S, T, d) and for $\epsilon \in \{0, 1\}$, we now determine the geometry of $\mathcal{L}^\epsilon(S, T, d)$. Recall that the group $P_S \times P^-_T$ acts on G_S by (2.12). Let $P_S \times P^-_T$ act trivially on $\mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, and consider the associated bundle

$$(G \times G) \times_{(P_S \times P^-_T)} (G_S \times \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T))$$

over $G/P_S \times G/P^-_T$ and the map

$$\mathbf{a} : \quad (G \times G) \times_{(P_S \times P^-_T)} (G_S \times \mathcal{L}^\epsilon_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)) \longrightarrow \mathcal{L}^\epsilon(S, T, d) \tag{2.14}$$

$$[(g_1, g_2), (g, V)] \longmapsto \mathrm{Ad}_{(g_1, g_2)} \mathfrak{l}_{g,V}, \tag{2.15}$$

where $\mathfrak{l}_{g,V} = V + (\mathfrak{n}_S \oplus \mathfrak{n}^-_T) + \{(x, \gamma_d \mathrm{Ad}_g(x)) : x \in \mathfrak{g}_S\}$ for $g \in G_S$.

Proposition 2.30. *For every $S, T \subset \Gamma, d \in I(S, T)$, and $\epsilon \in \{0, 1\}$, $\mathcal{L}^\epsilon(S, T, d)$ is a smooth connected subvariety of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ of dimension $n + \frac{z(z-3)}{2}$, where $n = \dim \mathfrak{g}$ and $z = \dim \mathfrak{z}_S$, and the map $\mathbf{a}$ in (2.14) is a $(G \times G)$-equivariant isomorphism.*

Proof. Consider the $(G \times G)$-equivariant projection

$$J : \ \mathcal{L}^\epsilon(S, T, d) \longrightarrow G/P_S \times G/P^-_T : \ \mathfrak{l}(\mathfrak{p}, \mathfrak{p}', \mu, V) \longmapsto (\mathfrak{p}, \mathfrak{p}'). \tag{2.16}$$

Let $\mathcal{F}^\epsilon(S, T, d)$ be the fibre of J over the point $(\mathfrak{p}_S, \mathfrak{p}^-_T) \in G/P_S \times G/P^-_T$. By Lemma 4, p. 26 of [Sl], the map

$$(G \times G) \times_{(P_S \times P^-_T)} \mathcal{F}^\epsilon(S, T, d) \longrightarrow \mathcal{L}^\epsilon(S, T, d) : \ [(g_1, g_2), \mathfrak{l}] \longmapsto \mathrm{Ad}_{(g_1, g_2)} \mathfrak{l}$$

is a $(G \times G)$-equivariant isomorphism. By Lemma 2.7,

$$\mathcal{F}^\epsilon(S,T,d) = \{\mathfrak{l}_{g,V} : \; g \in G_S, \; V \in \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)\}.$$

The identification $G_S \times \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T) \to \mathcal{F}^\epsilon(S,T,d)$ given by $(g,V) \mapsto \mathfrak{l}_{g,V}$ is $(P_S \times P_T^-)$-equivariant. It follows that $\mathbf{a}$ is a $(G \times G)$-equivariant isomorphism. The dimension claim now follows from Propositions 2.2 and 2.19. Smoothness and connectedness of $\mathcal{L}^\epsilon(S,T,d)$ follow easily.

Q.E.D.

2.10. **The geometry of the closures $\overline{\mathcal{L}^\epsilon(S,T,d)}$.** In this section, we determine the geometry of the closure of $\mathcal{L}^\epsilon(S,T,d)$ in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ for any generalized BD-triple (S,T,d) and $\epsilon \in \{0,1\}$. The closure is taken in the Zariski topology, and we will show that it is the same as the closure in the classical topology.

Recall that $Z_d(G_S)$ is the closure in $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$ of the embedding of G_S into $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$ given in (2.11). Moreover, $P_S \times P_T^-$ acts on $Z_d(G_S)$ by (2.12). Let $P_S \times P_T^-$ act trivially on $\mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$.

The proof of the following Theorem is quite similar to the proof of Theorem 2.27, and we will omit it.

Theorem 2.31. *For every generalized BD-triple (S,T,d) and every $\epsilon \in \{0,1\}$, the closure $\overline{\mathcal{L}^\epsilon(S,T,d)}$ is a smooth algebraic variety of dimension $n + \frac{z(z-3)}{2}$, where $n = \dim(\mathfrak{g})$, $z = \dim \mathfrak{z}_S$, and the map*

$$\mathbf{a}: \quad (G \times G) \times_{(P_S \times P_T^-)} (Z_d(G_S) \times \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)) \longrightarrow \overline{\mathcal{L}^\epsilon(S,T,d)} \tag{2.17}$$

$$[(g_1,g_2),(\mathfrak{l},V)] \longmapsto \mathrm{Ad}_{(g_1,g_2)}(V + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \mathfrak{l}) \tag{2.18}$$

is a $(G \times G)$-equivariant isomorphism.

Corollary 2.32. *For every generalized BD-triple (S,T,d) and every $\epsilon \in \{0,1\}$, we have a disjoint union*

$$\overline{\mathcal{L}^\epsilon(S,T,d)} = \bigcup_{V \in \mathcal{L}^\epsilon_{space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)} \; \bigcup_{S_1 \subset S} (G \times G) \cdot \mathfrak{l}_{S_1, d(S_1), d, V_1(V,S_1)}, \tag{2.19}$$

where for $S_1 \subset S$ and $V \in \mathcal{L}^\epsilon_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$,

$$V_1(V,S_1) = V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S \cap \mathfrak{z}_{S_1}\} \subset \mathfrak{z}_{S_1} \oplus \mathfrak{z}_{d(S_1)}.$$

Remark 2.33. 1). Since $Z_d(G_S)$ is also the closure in the classical topology of the $(G_S \times G_T)$-orbit through $\mathfrak{l}_{\gamma_d}$ inside $\mathrm{Gr}(m, \mathfrak{g}_S \oplus \mathfrak{g}_T)$, the $\mathcal{L}^\epsilon(S,T,d)$'s also have the same closures in the two topologies of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$.

2). Since G_S is a Zariski open subvariety of $Z_d(G_S)$, $Z_d(G_S) - G_S$ is an algebraic variety of dimension strictly lower than $m = \dim G_S$. It follows from the proof of Theorem 2.31 that $\overline{\mathcal{L}^\epsilon(S,T,d)} - \mathcal{L}^\epsilon(S,T,d)$ is of strictly lower dimension than $\overline{\mathcal{L}^\epsilon(S,T,d)}$.

2.11. **Irreducible components of $\mathcal{L}$.** We can now determine the irreducible components of $\mathcal{L}$. Since $\overline{\mathcal{L}^\epsilon(S,T,d)}$ is smooth and connected, it is a closed irreducible subvariety of $\mathcal{L}$. Since we have the finite union

$$\mathcal{L} = \bigcup_{\epsilon\in\{0,1\}} \bigcup_{S,T\subset\Gamma, d\in I(S,T)} \overline{\mathcal{L}^\epsilon(S,T,d)},$$

the irreducible components of $\mathcal{L}$ are those $\overline{\mathcal{L}^\epsilon(S,T,d)}$'s not properly contained in some other such set.

Theorem 2.34. *$\overline{\mathcal{L}^\epsilon(S,T,d)}$ is an irreducible component of $\mathcal{L}$ unless $|\Gamma - S| = 1$, $T = d_1(S)$ for some $d_1 \in I(\Gamma,\Gamma)$, $d = d_1|_S$, and $\epsilon = (\dim\mathfrak{h} - \dim\mathfrak{h}^{\gamma_{d_1}}) \mod 2$.*

Proof. When (S,T,d,ϵ) are as described in the proposition, the set $\mathcal{L}^\epsilon(S,T,d)$ consists of a single $(G\times G)$-orbit because $\dim\mathfrak{z}_S = 1$, and this single $(G\times G)$-orbit lies in $Z_{d_1}(G)$ by Theorem 2.25. We need to show that this is the only nontrivial case when the closure $\overline{\mathcal{L}^\epsilon(S,T,d)}$ is contained in another $\overline{\mathcal{L}^\epsilon(S_1,T_1,d_1)}$.

Assume that $\mathcal{L}^\epsilon(S,T,d)$ is in the boundary of $\overline{\mathcal{L}^\epsilon(S_1,T_1,d_1)}$. Then by Corollary 2.32, $S\subset S_1$ and $T\subset T_1$. By Remark 2.33, $\dim\mathcal{L}^\epsilon(S,T,d) < \dim\mathcal{L}^\epsilon(S_1,T_1,d_1)$, and thus

$$\frac{\dim(\mathfrak{z}_S)(\dim(\mathfrak{z}_S)-3)}{2} < \frac{\dim(\mathfrak{z}_{S_1})(\dim(\mathfrak{z}_{S_1})-3)}{2}$$

by the dimension formula in Proposition 2.30. Since $S\subset S_1$, so $\dim(\mathfrak{z}_S)\geq\dim(\mathfrak{z}_{S_1})$, these two inequalities imply that $\dim(\mathfrak{z}_{S_1}) = 0$ and $\dim(\mathfrak{z}_S) = 1$ or 2. In particular, $S_1 = T_1 = \Gamma$, so $\epsilon = (\dim\mathfrak{h} - \dim\mathfrak{h}^{\gamma_{d_1}}) \mod 2$, and $\overline{\mathcal{L}^\epsilon(S_1,T_1,d_1)} = Z_{d_1}(G)$.

If $\dim(\mathfrak{z}_S) = 2$, $\mathcal{L}^\epsilon(S,T,d)$ contains infinitely many $(G\times G)$-orbits by Theorem 2.18 and Proposition 2.2. Since $Z_{d_1}(G)$ has only finitely many $(G\times G)$-orbits, $\mathcal{L}^\epsilon(S,T,d)$ can not be contained in $Z_{d_1}(G)$.

Assume now $\dim(\mathfrak{z}_S) = 1$. Then we know by Proposition 2.30 that $\mathcal{L}^\epsilon(S,T,d)$ is a single $(G\times G)$-orbit. By the description of all $(G\times G)$-orbits in $Z_{d_1}(G)$ given in Theorem 2.25, we see that we must have T and d as described in the proposition.

Q.E.D.

Example 2.35. Let $\mathfrak{g} = \mathfrak{s}l(2,\mathbb{C})$. Then $\mathcal{L}$ has two irreducible components, one being the De Concini-Procesi compactification $Z_{\mathrm{id}}(G)$ of $G = PSL(2,C)$ which is isomorphic to $\mathbb{C}P^3$ (see [D-P]), and the other being isomorphic to $\mathbb{C}P^1\times\mathbb{C}P^1$, the closed $(G\times G)$-orbit through the Lagrangian subalgebra $\mathfrak{h}_\Delta + (\mathfrak{n}\oplus\mathfrak{n}^-)$, where $\mathfrak{h}$ consists of diagonal elements in $\mathfrak{s}l(2,C)$, $\mathfrak{h}_\Delta$ is the diagonal of $\mathfrak{h}\oplus\mathfrak{h}$, and $\mathfrak{n}$ and $\mathfrak{n}^-$ are respectively the nilpotent subalgebras of $\mathfrak{s}l(2,\mathbb{C})$ consisting of strictly upper and lower triangular elements in $\mathfrak{s}l(2,\mathbb{C})$.

For $\mathfrak{g} = \mathfrak{s}l(3,\mathbb{C})$, there are four irreducible components $Z_{\mathrm{id}}(G), Z_{d_1}(G), C_1$ and C_2, where $Z_{\mathrm{id}}(G)$ and $Z_{d_1}(G)$ are the two De Concini-Procesi compactifications of $G = PSL(3,\mathbb{C})$ corresponding to the identity and the non-trivial automorphism of the Dynkin diagram of $\mathfrak{s}l(3,\mathbb{C})$, and C_1 and C_2 are the two components $\mathcal{L}_0(\emptyset,\emptyset,d)$ and $\mathcal{L}_1(\emptyset,\emptyset,d)$. Both C_1 and C_2 have dimension 7. Moreover, $Z_{\mathrm{id}}(G)\cap C_1$ is a 6-dimensional closed $(G\times G)$-orbit, and so is $Z_{d_1}(G)\cap C_2$.

3. Classification of G_Δ-orbits in $\mathcal{L}$

3.1. **Some results on Weyl groups and generalized BD-triples.** In this section, we discuss some results on Weyl groups in relation to generalized BD-triples. We will use these results in Sections 3.2 and 3.3, to determine the G_Δ-orbits in $\mathcal{L}$, where $G_\Delta = \{(g,g) : g \in G\}$. We first fix some notation.

Notation 3.1. Let W be the Weyl group of Γ. If F is a subset of Γ, we let W_F denote the subgroup of W generated by elements in F. If E and F are two subsets of Γ, then G has the Bruhat decomposition

$$G = \coprod_{[w] \in W_E \backslash W / W_F} P_E w P_F^-. \tag{3.1}$$

It is also well-known (see Proposition 2.7.3 of [C] or Lemma 4.3 of [Y]) that each coset from $W_E \backslash W / W_F$ has a unique minimal length representative w with the property that

$$w^{-1}(E) \subset \Sigma^+ \quad \text{and} \quad w(F) \subset \Sigma^+. \tag{3.2}$$

Let ${}^E W^F$ be the set of minimal length representatives for double cosets from $W_E \backslash W / W_F$. When E is the empty set $\emptyset$, we set $W^F = {}^\emptyset W^F$. If E_1 and E_2 are two subsets of F, the set of minimal length representatives in W_F for the double cosets from $W_{E_1} \backslash W_F / W_{E_2}$ will be denoted by ${}^{E_1}(W_F)^{E_2}$. If $u \in {}^{E_1}(W_F)^{E_2}$ and $v \in {}^{E_1'}(W_{F'})^{E_2'}$ are two such minimal length representatives, we can regard both $u \in W_F$ and $v \in W_{F'}$ as elements in W, and by uv we will mean their product in W.

Definition 3.2. Let (S, T, d) be a generalized BD-triple in Γ. For $v \in W^T$, regard vd as a map $S \to \Delta$. We define $S(v, d) \subset S$ to be the largest subset in S that is invariant under vd. In other words,

$$S(v, d) = \{\alpha \in S : (vd)^n \alpha \in S, \forall \, \text{integer } n \geq 1\}. \tag{3.3}$$

We will show that every G_Δ-orbit in $\mathcal{L}$ gives rise to a unique generalized BD-triple (S, T, d) and $v \in W^T$, and we will classify G_Δ-orbits in $\mathcal{L}$ in terms of twisted conjugacy classes in $M_{S(v,d)}$.

We first have the following lemma which follows directly from Proposition 2.7.5 of [C] or Lemma 4.3 of [Y].

Lemma 3.3. *1) Let $w \in {}^S W^T$. Then $uw \in W^T$ for every $u \in (W_S)^{S \cap w(T)}$;*

2) Every $v \in W^T$ has a unique decomposition $v = uw$ where $w \in {}^S W^T$ and $u \in (W_S)^{S \cap w(T)}$. Moreover, $l(v) = l(u) + l(w)$.

For each $w \in {}^S W^T$, set

$$T_w = S \cap w(T), \qquad S_w = d^{-1}(T \cap w^{-1}(S)). \tag{3.4}$$

Since $S_w, T_w \subset S$, we can regard (S_w, T_w, wd) as a generalized BD-triple in S. Let $S_w(u, wd)$ be the largest subset of S_w that is invariant under uwd, i.e.,

$$S_w(u, wd) = \{\alpha \in S_w : (uwd)^n \alpha \in S_w \ \forall n \geq 1\}. \tag{3.5}$$

Lemma 3.4. *For $v \in W^T$ and $v = uw$ as in Lemma 3.3, we have $S_w(u, wd) = S(v, d)$.*

Proof. Clearly $S_w(u, wd) \subset S(v, d)$. Suppose now that $\alpha \in S(v, d)$. To show that $\alpha \in S_w(u, wd)$, we show $wd\alpha \in S$. Since $\alpha \in S(v, d)$, $vd\alpha \in S$, so $wd\alpha \in u^{-1}S \subset [S]$. Since $w(T) \subset \Sigma^+$, $wd\alpha \in [S] \cap \Sigma^+$. To show that $wd\alpha \in S$, suppose that $wd\alpha = \beta_1 + \beta_2$ with

$\beta_1, \beta_2 \in [S] \cap \Sigma^+$. Then $d\alpha = w^{-1}\beta_1 + w^{-1}\beta_2$. But $w^{-1}\beta_1, w^{-1}\beta_2 \in w^{-1}([S] \cap \Sigma^+) \subset \Sigma^+$. This contradicts to the fact that $d\alpha \in T$ is a simple root. Thus $wd\alpha \in S$. It follows easily that $\alpha \in S_w$, and hence $\alpha \in S_w(u, wd)$.

Q.E.D.

Notation 3.5. Consider a sequence of quadruples (S_i, T_i, d_i, w_i) indexed by $i \in \mathbb{N}$ such that:

1) (S_i, T_i, d_i) is a generalized BD-triple;
2) $w_i \in {}^{S_i}(W_{S_{i-1}})^{T_i}$ (we set $S_0 = \Gamma$);
3) $T_{i+1} = S_i \cap w_i(T_i)$, $d_{i+1} = w_i d_i$, and $S_{i+1} = d_{i+1}^{-1}(T_{i+1})$.

Since $S_{i+1} \subset S_i$, there exists some k such that $S_k = S_{k+1}$. Let $v = w_k w_{k-1} \cdots w_1$. Also note that if $(S_1, T_1, d_1, w_1) = (S, T, d, w)$, then $S_2 = S_w$, $T_2 = T_w$, and $d_2 = wd$.

Proposition 3.6. *Let the sequence $\{(S_i, T_i, d_i, w_i) : i \in \mathbb{N}\}$ be as in Notation 3.5, and assume that $(S_1, T_1, d_1) = (S, T, d)$. Then*

1) $S_{k+1} = T_{k+1} = S_k$.
2) $v \in W^T$.
3) $S(v, d) = S_{k+1}$ and $d_{k+1} = vd$.

Proof. For 1), since $S_k = S_{k+1}$, it follows that the cardinalities of S_k and $T_{k+1} = S_k \cap w_k(T_k)$ coincide. In particular, $S_k = w_k(T_k)$, so $T_{k+1} = S_k$. Let u_{k+1} be the identity and let $u_i := w_k w_{k-1} \cdots w_i$. For 2), use decreasing induction to show that $u_i \in (W_{S_{i-1}})^{T_i}$. The case $i = k+1$ is clear, and the inductive step follows from Lemma 3.3 (1). Repeated application of Lemma 3.4 gives

$$S(v, d) = S_2(u_2, d_2) = S_3(u_3, d_3) = \cdots = S_{k+1}(u_{k+1}, d_{k+1}).$$

Since u_{k+1} is the identity and $d_{k+1} = w_k d_k : S_{k+1} \to T_{k+1}$ is a self-map by (1), it follows that $S_{k+1}(u_{k+1}, d_{k+1}) = S_{k+1}$, which gives the first part of 3), and the remaining part follows easily.

Q.E.D.

Example 3.7. Let $\mathfrak{g} = \mathfrak{s}l(n+1, \mathbb{C})$ with the simple roots labeled as $\alpha_1, \alpha_2, \ldots, \alpha_n$. let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$, $T = \{\alpha_2, \alpha_3, \ldots, \alpha_n\}$, and $d : S \to T : d(\alpha_j) = \alpha_{j+1}$ for $j = 1, 2, \ldots, n-1$. The triple (S, T, d) is related to the Cremmer-Gervais Lie bialgebra structure on $\mathfrak{g}$ (see [Cr-G]). We take all $w_i = 1$, the identity element in the Weyl group. Then $k = n$ and moreover, $S_i = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-i}\}$, $T_i = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-i+1}\}$ for $1 \leq i \leq n-1$, and $S_n = T_n = \emptyset$.

3.2. **A double coset theorem.** By Theorem 2.18, to describe the G_Δ-orbits in $\mathcal{L}$, it is enough to describe G_Δ-orbits in $(G \times G)/R_{S,T,d}$ for all generalized BD-triples (S, T, d) for Γ, where $R_{S,T,d}$ is given by (2.6). In this section, we will prove a double coset theorem which will allow us to describe the G_Δ-orbits in $\mathcal{L}$. The method we use is adapted from [Y], and a more general double coset theorem is proved in [Lu-Y2], which, as special cases, gives a classification of $(R_{S_1,T_1,d_1}, R_{S_2,T_2,d_2})$-double cosets in $G \times G$ for two BD-triples (S_1, T_1, d_1) and (S_2, T_2, d_2).

For the rest of this section, we assume that G is a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$, not necessarily of adjoint type. We use the same notation as in Notation 2.12 and Notation 3.1 for various subalgebras of $\mathfrak{g}$ and subgroups of G and for elements in the Weyl group. We now define a class of subgroups R of $G \times G$ that are slightly more general than the subgroups $R_{S,T,d}$, and we will prove a theorem on (G_Δ, R)-double cosets in $G \times G$ for such a subgroup R.

Definition 3.8. Let (S, T, d) be a generalized BD-triple in Γ. Denote the centers of the Levi subgroups M_S and M_T by Z_S and Z_T respectively. If C_S is a subgroup of Z_S and C_T is a subgroup of Z_T, we let

$$j_S : M_S \longrightarrow M_S/C_S \quad \text{and} \quad j_T : M_T \longrightarrow M_T/C_T$$

be the projections. By a (S, T, d)-admissible subgroup of $G \times G$ we mean a subgroup $R = R(C_S, C_T, \theta_d)$ of the form

$$R(C_S, C_T, \theta_d) = \{(m, m') \in M_S \times M_T : \theta_d(j_S(m)) = j_T(m')\}(N_S \times N_T^-), \tag{3.6}$$

where C_S is a subgroup of Z_S, C_T is a subgroup of Z_T, and $\theta_d : M_S/C_S \longrightarrow M_T/C_T$ is a group isomorphism that maps the one-dimensional unipotent subgroup of M_S/C_S defined by α to the corresponding subgroup of M_T/C_T defined by $d\alpha$ for each $\alpha \in [S]$. .

Clearly $R(Z_S, Z_T, \gamma_d) = R_{S,T,d}$. Let R be any (S, T, d)-admissible subgroup of $G \times G$. Recall that the subset $S(v, d)$ of S for $v \in W^T$ is defined in (3.3). If $\dot{v}$ is a representative of v in G, set

$$R_{\dot{v}} = \{(m_1, m_1') \in M_{S(v,d)} \times M_{S(v,d)} : \theta_d(j_S(m_1)) = j_T(\dot{v}^{-1} m_1' \dot{v})\} \tag{3.7}$$

$$= (M_{S(v,d)} \times M_{S(v,d)}) \cap ((\mathrm{id} \times \mathrm{Ad}_{\dot{v}})R), \tag{3.8}$$

where $\mathrm{Ad}_{\dot{v}} : G \to G : g \mapsto \dot{v} g \dot{v}^{-1}$. Let $R_{\dot{v}}$ act on $M_{S(v,d)}$ (from the right) by

$$m \cdot (m_1, m_1') = (m_1')^{-1} m m_1, \qquad m \in M_{S(v,d)}, \ (m_1, m_1') \in R_{\dot{v}}. \tag{3.9}$$

For $(g_1, g_2) \in G \times G$, we will use $[g_1, g_2]$ to denote the double coset $G_\Delta(g_1, g_2)R$ in $G \times G$.

Theorem 3.9. *Let (S, T, d) be a generalized BD-triple, and let $R = R(C_S, C_T, \theta_d)$ be a (S, T, d)-admissible subgroup of $G \times G$ as given in (3.6). For $v \in W^T$, let $S(v, d) \subset S$ be given in (3.3), and let $\dot{v}$ be a fixed representative of v in G. Then*

1) every (G_Δ, R)-double coset in $G \times G$ is of the form $[m, \dot{v}]$ for some $v \in W^T$ and $m \in M_{S(v,d)}$;

2) Two double cosets $[m_1, \dot{v}_1]$ and $[m_2, \dot{v}_2]$ in 1) coincide if and only if $v_1 = v_2 = v$ and m_1 and m_2 are in the same $R_{\dot{v}}$-orbit in $M_{S(v,d)}$, where we use the $R_{\dot{v}}$ action in (3.9).

Remark 3.10. If $v \in W^T$ and if $\hat{v}$ is another representative of G, then $\hat{v} = h_v \dot{v}$ for some $h_v \in H$, where H is the Cartan subgroup of G with Lie algebra $\mathfrak{h}$. It is easy to see that $(m, m') \in R_{\hat{v}}$ if and only if $(m, h_v^{-1} m' h_v) \in R_{\dot{v}}$. It follows that if Theorem 3.9 holds for a particular set $\{\dot{v} : v \in W^T\}$, then it holds for every such set.

We will present the main induction step in the proof of Theorem 3.9 in a lemma. To this end, recall that every $w \in {}^S W^T$ gives rise to the generalized BD-triple (S_w, T_w, wd) in S given in (3.4). For each $w \in {}^S W^T$, fix a representative $\dot{w}$ in G, and set

$$N_{S_w}^S = N_{S_w} \cap M_S, \quad \text{and} \quad N_{T_w}^{S,-} = N_{T_w}^- \cap M_S.$$

Define

$$R_{\dot{w}}^S = \{(m, m') \in M_{S_w} \times M_{T_w} : \theta_d(j_S(m)) = j_T(\dot{w}^{-1} m' \dot{w})\} \left(N_{S_w}^S \times N_{T_w}^{S,-}\right) \tag{3.10}$$

$$= ((M_{S_w} \times M_{T_w}) \cap ((\mathrm{id} \times \mathrm{Ad}_{\dot{w}})R)) \left(N_{S_w}^S \times N_{T_w}^{S,-}\right). \tag{3.11}$$

Note that $R^S_{\dot w}$ is an (S_w, T_w, wd)-admissible subgroup of $M_S \times M_S$ defined by the subgroup C_S of Z_{S_w}, the subgroup $w(C_T)$ of Z_{T_w} and the group isomorphism

$$\gamma_{\dot w d} \stackrel{\text{def}}{=} \mathrm{Ad}_{\dot w}\theta_d :\ M_{S_w}/C_S \longrightarrow M_{T_w}/w(C_T).$$

Lemma 3.11. *1) Every (G_Δ, R)-double coset in $(G \times G)$ is of the form $[m, m'\dot w]$ for a unique $w \in {}^SW^T$ and some $m \in M_S$.*

2) $[m_1, m_1'\dot w] = [m_2, m_2'\dot w]$, where $w \in {}^SW^T$ and $(m_1, m_1'), (m_2, m_2') \in M_S \times M_S$, if and only if (m_1, m_1') and (m_2, m_2') are in the same $((M_S)_\Delta, R^S_{\dot w})$-double coset in $M_S \times M_S$.

Proof. Consider the right action of $P_S \times P_T^-$ on $G_\Delta \backslash (G \times G)$ by right translations. By the Bruhat decomposition $G = \bigcup_{w \in {}^SW^T} P_S w P_T^-$, the $(P_S \times P_T^-)$-orbits are parameterized by the set $\{G_\Delta(e, \dot w) : w \in {}^SW^T\}$. Let $w \in {}^SW^T$. The stabilizer subgroup of $P_S \times P_T^-$ at $G_\Delta(e, \dot w)$ is $P_S \cap (\dot w P_T^- \dot w^{-1})$ considered as a subgroup of $P_S \times P_T^-$ via the embedding

$$P_S \cap (\dot w P_T^- \dot w^{-1}) \longrightarrow P_S \times P_T^- :\ p_S \longmapsto (p_S,\ \dot w^{-1} p_S \dot w). \tag{3.12}$$

Thus the set of R-orbits in $G_\Delta \backslash (G \times G)$ can be identified with the disjoint union over $w \in {}^SW^T$ of the spaces of R-orbits in $P_S \cap (\dot w P_T^- \dot w^{-1}) \backslash (P_S \times P_T^-)$. Thus we get an injective map

$$(P_S \cap \dot w P_T^- \dot w^{-1}) \backslash P_S \times P_T^- / R \longrightarrow G_\Delta \backslash G \times G / R \tag{3.13}$$

given by $(P_S \cap \dot w P_T^- \dot w^{-1})(p_S, p_T^-)R \to [p_S, \dot w p_T^-]$.

We will complete the proof by identifying

$$(P_S \cap (\dot w P_T^- \dot w^{-1}) \backslash P_S \times P_T^- / R \ \cong\ (M_S)_\Delta \backslash M_S \times M_S / R^S_{\dot w} \tag{3.14}$$

through a series of steps. Let $\pi_S : P_S \to M_S$ be the projection with respect to the decomposition $P_S = M_S N_S$. Similarly, we have the projection $\pi_T : P_T^- \to M_T$. Then the projection $\pi_S \times \pi_T : P_S \times P_T^- \to M_S \times M_T$ gives an identification

$$P_S \cap (\dot w P_T^- \dot w^{-1}) \backslash P_S \times P_T^- / R \longrightarrow R_1 \backslash M_S \times M_T / R_2, \tag{3.15}$$

where

$$R_1 = (\pi_S \times \pi_T)(P_S \cap (\dot w P_T^- \dot w^{-1})), \qquad R_2 = (M_S \times M_T) \cap R.$$

Now since the projection from $(M_S \times M_T) \cap R$ to M_T is onto with kernel $(C_S \times \{e\})$, the map

$$\phi_w :\ (M_S \times M_T)/R_2 \longrightarrow (M_S \times M_S)/(M_S)_\Delta (C_S \times \{e\})$$

that maps $(m_S, m_T)R_2$ to $(m_S', m_S)((M_S)_\Delta(C_S \times \{e\}))$ is a well-defined bijection, where for $m_T \in M_T$, m_S' is any element in M_S such that $(m_S', m_T) \in R_2$. Thus ϕ_w induces an identification

$$\psi_w :\ R_1 \backslash M_S \times M_T / R_2 \longrightarrow R_3 \backslash M_S \times M_S / ((M_S)_\Delta (C_S \times \{e\})), \tag{3.16}$$

where

$$R_3 \stackrel{\text{def}}{=} \{(m_S', m_S) \in M_S \times M_S : \exists m_T \in M_T \,\text{such that}\, (m_S, m_T) \in R_1, (m_S', m_T) \in R_2\}.$$

By Theorem 2.8.7 of [C], we have the decomposition of $P_S \cap (\dot w P_T^- \dot w^{-1})$ as

$$P_S \cap (\dot w P_T^- \dot w^{-1}) = M_S \cap \mathrm{Ad}_{\dot w}(M_T)(M_S \cap \mathrm{Ad}_{\dot w}(N_T^-))(N_S \cap \mathrm{Ad}_{\dot w}(M_T))(N_S \cap \mathrm{Ad}_{\dot w}(N_T^-)). \tag{3.17}$$

We note that $M_S \cap \mathrm{Ad}_{\dot w}(M_T) = M_{S \cap w(T)}$, $M_S \cap \mathrm{Ad}_{\dot w}(N_T^-) = N^{S,-}_{S \cap w(T)} = M_S \cap N^-_{S \cap w(T)}$, and $N_S \cap \mathrm{Ad}_{\dot w}(M_T) = N^T_{T \cap w^{-1}(S)} = M_T \cap N_{T \cap w^{-1}(S)}$. These identities are easily verified at the level of Lie algebras using the identity $[S] \cap [w(T)] = [S \cap w(T)]$ and follow for groups since all these groups are connected. Thus

$$R_1 = \left\{(m, \mathrm{Ad}_{\dot{w}^{-1}}(m)) : m \in M_{S\cap w(T)}\right\} \left(N^{S,-}_{S\cap w(T)} \times N^T_{T\cap w^{-1}(S)}\right).$$

Therefore $(m'_S, m_S) \in R_3$ if and only if there exist $n \in N^{S,-}_{S\cap w(T)}$, $n_1 \in N^T_{T\cap w^{-1}(S)}$, and $m \in M_{S\cap w(T)}$ such that

$$m_S = mn, \quad (m'_S, \mathrm{Ad}_{\dot{w}^{-1}}(m)n_1) \in R_2.$$

Recall that $T_w = S \cap w(T)$ and $S_w = d^{-1}(T \cap w^{-1}(S))$. It follows now from the definition of R that $(m'_S, m_S) \in R_3$ if and only if there exist $m' \in M_{S_w}, m \in M_{T_w}$, $n \in N^{S,-}_{T_w}$, and $n' \in N^S_{S_w} = M_S \cap N_{S_w}$ such that

$$m_S = mn, \quad m'_S = m'n', \quad (m', \mathrm{Ad}^{-1}_{\dot{w}}(m)) \in R.$$

Thus R_3 is precisely the group $R^S_{\dot{w}}$ as given in (3.10). Moreover, since C_S lies is the center of M_S and since $C_S \times \{e\} \subset R^S_{\dot{w}}$, the (right) action of $C_S \times \{e\}$ on $R^S_{\dot{w}}\backslash(M_S \times M_S)$ is trivial. Thus we have

$$\begin{aligned} R_3\backslash M_S \times M_S/((M_S)_\Delta(C_S \times \{e\})) &\cong R^S_{\dot{w}}\backslash M_S \times M_S/((M_S)_\Delta(C_S \times \{e\})) \\ &\cong R^S_{\dot{w}}\backslash M_S \times M_S/(M_S)_\Delta \\ &\cong (M_S)_\Delta\backslash M_S \times M_S/R^S_{\dot{w}}, \end{aligned}$$

where the last identification is induced by the inverse map of $M_S \times M_S$.

Combining the above identification with the identifications in (3.15)-(3.16) and with the inclusion of (3.13), we get a well-defined injective map

$$(M_S)_\Delta\backslash M_S \times M_S/R^S_{\dot{w}} \longrightarrow G_\Delta\backslash G \times G/R$$

which is given by

$$(M_S)_\Delta(m, m')R^S_{\dot{w}} \to [((m')^{-1}, \dot{w}\theta_d(m^{-1}))] = [(m')^{-1}m, \dot{w}] = [m, m'\dot{w}].$$

This finishes the proof of Lemma 3.11.

Q.E.D.

Proof of Theorem 3.9. By Lemma 3.11, each (G_Δ, R) double coset in $G \times G$ determines some $w \in {}^SW^T$ and a double coset $[m, m']_1 \in (M_S)_\Delta\backslash M_S \times M_S/R^S_{\dot{w}}$. Let $S_0 = \Gamma$, $S_1 = S$, $T_1 = T$, $w_1 = w$, and $d_1 = d$. By successively applying Lemma 3.11 to a sequence of smaller subgroups, we obtain a sequence of quadruples (S_i, T_i, d_i, w_i) as in Notation 3.5, as well as a double coset in

$$(M_{S_i})_\Delta\backslash M_{S_i} \times M_{S_i}/R_i,$$

where R_i is the subgroup of $M_{S_i} \times M_{S_i}$ defined analogously to $R^S_{\dot{w}}$.

Let k be minimal such that $S_{k+1} = S_k$. It follows that ${}^{S_{k+1}}(W_{S_k})^{T_{k+1}}$ is the trivial group, so $w_{k+1} = e$ is the identity. As in Notation 3.5, let $v = w_k \cdots w_1 \in W^T$. By Proposition 3.6, $S_{k+1} = S(v, d)$, and it follows that each double coset is of the form $[m, m'\dot{v}]$ for $m \in M_{S(v,d)}$. It follows from definitions that $R_{k+1} = R_{\dot{v}}$, and thus double cosets in $(M_{S_{k+1}})_\Delta\backslash M_{S_{k+1}} \times M_{S_{k+1}}/R_{k+1}$ coincide with double cosets in $(M_{S(v,d)})_\Delta\backslash M_{S(v,d)} \times M_{S(v,d)}/R_{\dot{v}}$. It is easy to see that the map

$$(M_{S(v,d)})_\Delta\backslash M_{S(v,d)} \times M_{S(v,d)}/R_{\dot{v}} \longrightarrow M_{S(v,d)}/R_{\dot{v}} : \ [m, m'] \longmapsto [{m'}^{-1}m]$$

is a bijection. This proves Theorem 3.9.

Q.E.D.

3.3. G_Δ**-orbits in** $\mathcal{L}$**.** Recall from Theorem 2.18 that every $(G\times G)$-orbit in $\mathcal{L}$ passes through exactly one point of the form $\mathfrak{l}_{S,T,d,V}$ given in (2.4), where (S,T,d) is a generalized BD-triple, and $V\in\mathcal{L}_{\text{space}}(\mathfrak{z}_S\oplus\mathfrak{z}_T)$. Recall also from Proposition 2.19 that the stabilizer subgroup of $G\times G$ at $\mathfrak{l}_{S,T,d,V}$ is $R_{S,T,d}$ given in (2.6). Thus to describe the space of G-orbits in $\mathcal{L}$, it is enough to describe the space of G_Δ-orbits in $(G\times G)\cdot\mathfrak{l}_{S,T,d,V}\cong(G\times G)/R_{S,T,d}$, which are the same as $(G_\Delta,R_{S,T,d})$-double cosets in $G\times G$.

Notation 3.12. For a generalized BD-triple (S,T,d), $V\in\mathcal{L}_{\text{space}}(\mathfrak{z}_S\oplus\mathfrak{z}_T)$, $m\in M_{S(v,d)}$, $v\in W^T$, and $\dot v\in G$ a fixed representative of v in G, set

$$\mathfrak{l}_{S,T,d,V,\dot v,m}=\mathrm{Ad}_{(m,\dot v)}\mathfrak{l}_{S,T,d,V}, \tag{3.18}$$

where $\mathfrak{l}_{S,T,d,V}$ is given in (2.4). Define

$$R_{\dot v}=\{(m_1,m_1')\in M_{S(v,d)}\times M_{S(v,d)}:\gamma_d(\chi_S(m_1))=\chi_T(\dot v^{-1}m_1'\dot v)\}, \tag{3.19}$$

and let $R_{\dot v}$ act on $M_{S(v,d)}$ (from the right) by

$$m\cdot(m_1,m_1')=(m_1')^{-1}mm_1,\quad m\in M_{S(v,d)},\ (m_1,m_1')\in R_{\dot v}. \tag{3.20}$$

As an immediate corollary of Theorem 3.9, we have

Corollary 3.13. *Every $G\cong G_\Delta$-orbit in $\mathcal{L}$ passes through an $\mathfrak{l}_{S,T,d,V,\dot v,m}$ for a unique generalized BD-triple (S,T,d), a unique $V\in\mathcal{L}_{\text{space}}(\mathfrak{z}_S\oplus\mathfrak{z}_T)$, a unique $v\in W^T$, and some $m\in M_{S(v,d)}$; Two such Lagrangian subalgebras $\mathfrak{l}_{S,T,d,V,v,m_1}$ and $\mathfrak{l}_{S,T,d,V,v,m_2}$ are in the same G_Δ-orbit if and only if m_1 and m_2 are in the same $R_{\dot v}$-orbits in $M_{S(v,d)}$.*

3.4. **Normalizer subalgebras of $\mathfrak{g}_\Delta$ at $\mathfrak{l}\in\mathcal{L}$.** For a Lagrangian subalgebra $\mathfrak{l}=\mathfrak{l}_{S,T,d,V,\dot v,m}$ in Corollary 3.13, we now compute its normalizer subalgebra $\mathfrak{n}(\mathfrak{l})\subset\mathfrak{g}\cong\mathfrak{g}_\Delta=\{(x,x):x\in\mathfrak{g}\}$. Introduce the map

$$\phi:=\mathrm{Ad}_{\dot v}\gamma_d\chi_S\mathrm{Ad}_m^{-1}:\ \mathfrak{p}_S\longrightarrow\mathfrak{g}.$$

Consider the standard parabolic subalgebra $\mathfrak{p}_{S(v,d)}$ and its decomposition $\mathfrak{p}_{S(v,d)}=\mathfrak{z}_{S(v,d)}+\mathfrak{g}_{S(v,d)}+\mathfrak{n}_{S(v,d)}$ (see Notation 2.12).

Lemma 3.14. *The map $\phi=\mathrm{Ad}_{\dot v}\gamma_d\chi_S\mathrm{Ad}_m^{-1}$ leaves each of $\mathfrak{z}_{S(v,d)}$, $\mathfrak{g}_{S(v,d)}$, and $\mathfrak{n}_{S(v,d)}$ invariant. Moreover, $\phi:\mathfrak{n}_{S(v,d)}\to\mathfrak{n}_{S(v,d)}$ is nilpotent.*

Proof. Let $x\in\mathfrak{z}_{S(v,d)}$. Then $\phi(x)=\mathrm{Ad}_{\dot v}\gamma_d\chi_S(x)\in\mathfrak{h}$. For every $\alpha\in S(v,d)$, since $(vd)^{-1}\alpha\in S(v,d)$, we have

$$\alpha(\phi(x))=((vd)^{-1}\alpha)(\chi_S(x))=((vd)^{-1}\alpha)(x)=0.$$

Thus $\phi(x)\in\mathfrak{z}_{S(v,d)}$, so $\mathfrak{z}_{S(v,d)}$ is ϕ-invariant. Since both $\mathrm{Ad}_{\dot v}\gamma_d$ and Ad_m^{-1} leave $\mathfrak{g}_{S(v,d)}$ invariant, we see that $\phi|_{\mathfrak{g}_{S(v,d)}}=\mathrm{Ad}_{\dot v}\gamma_d\mathrm{Ad}_m^{-1}$ leaves $\mathfrak{g}_{S(v,d)}$ invariant.

It remains to show that $\mathfrak{n}_{S(v,d)}$ is ϕ-invariant and that $\phi:\mathfrak{n}_{S(v,d)}\to\mathfrak{n}_{S(v,d)}$ is nilpotent. Decompose $\mathfrak{n}_{S(v,d)}$ as $\mathfrak{n}_{S(v,d)}=\mathfrak{n}_S+\mathfrak{n}_{S(v,d)}^S$, where $\mathfrak{n}_{S(v,d)}^S=\oplus_{\alpha\in[S]-[S(v,d)]}\mathfrak{g}_\alpha$. Then $\phi(\mathfrak{n}_S)=0$ and $\phi|_{\mathfrak{n}_{S(v,d)}^S}=\mathrm{Ad}_{\dot v}\gamma_d\mathrm{Ad}_m^{-1}$, and $\mathrm{Ad}_{\dot v}\gamma_d\mathrm{Ad}_m^{-1}(\mathfrak{n}_{S(v,d)}^S)\subset\mathfrak{n}_{S(v,d)}$. Indeed, $\mathrm{Ad}_m^{-1}(\mathfrak{n}_{S(v,d)}^S)\subset\mathfrak{n}_{S(v,d)}^S$ and $\mathrm{Ad}_{\dot v}\gamma_d(\mathfrak{n}_{S(v,d)}^S)\subset\mathfrak{n}_{S(v,d)}$ since $v\in W^T$. Thus $\mathfrak{n}_{S(v,d)}$ is ϕ-invariant.

To show that $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent, set $\Sigma_0^+ = \Sigma^+ - [S]$, and for $j \geq 1$, set

$$\begin{aligned} \Sigma_j^+ &= \{\alpha \in \Sigma^+ : \alpha \in [S], \cdots, (vd)^{j-1}(\alpha) \in [S], (vd)^j(\alpha) \notin [S]\} \\ &= \{\alpha \in [S] \cap \Sigma^+ : vd(\alpha) \in \Sigma_{j-1}^+\} \end{aligned}$$

Then $\Sigma^+ - [S(v,d)] = \bigcup_{j\geq 0} \Sigma_j^+$ is a finite disjoint union. Let $K \geq 0$ be an integer such that $\Sigma_j^+ = \emptyset$ for $j > K$. For $0 \leq j \leq K$, set

$$\mathfrak{n}_j = \oplus_{\alpha\in\Sigma_j^+} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}_0 = \mathfrak{n}_S$, and

$$\mathfrak{n}_{S(v,d)} = \mathfrak{n}_0 + \mathfrak{n}_1 + \cdots + \mathfrak{n}_K$$

is a direct sum. We claim that $[\mathfrak{m}_{S(v,d)}, \mathfrak{n}_j] \subset \mathfrak{n}_j$ each $j \geq 0$. Indeed, for $\alpha \in \Sigma_j^+$, since $\alpha \notin [S(v,d)]$, $\alpha + \beta \neq 0$ for any $\beta \in [S(v,d)]$. Thus to prove the claim, it is enough to show the following statement for every $j \geq 0$:

$$\alpha \in \Sigma_j^+, \beta \in [S(v,d)], \alpha + \beta \in \Sigma \implies \alpha + \beta \in \Sigma_j^+. \tag{3.21}$$

We prove (3.21) by induction on j. When $j = 0$, since $\beta \in [S]$ and $\alpha \in \Sigma^+ - [S]$, $\alpha + \beta \in \Sigma^+ - [S] = \Sigma_0^+$. Now let $j \geq 1$ and assume that (3.21) holds for $j-1$. Let $\alpha \in \Sigma_j^+$ and $\beta \in [S(v,d)]$ be such that $\alpha + \beta$ is a root. Then $vd(\alpha) \in \Sigma_{j-1}^+$, $vd(\beta) \in [S(v,d)]$ and $vd(\alpha) + vd(\beta) = vd(\alpha + \beta)$ is a root. Thus $vd(\alpha + \beta) \in \Sigma_{j-1}^+$. It follows that $\alpha + \beta \in \Sigma_j^+$. We therefore have proved the claim that $[\mathfrak{m}_{S(v,d)}, \mathfrak{n}_j] \subset \mathfrak{n}_j$ for each $j \geq 0$. It follows that

$$\mathrm{Ad}_m \mathfrak{n}_j = \mathfrak{n}_j, \quad \forall j \geq 0.$$

By setting $\mathfrak{n}_{-1} = 0$, we also see from the definitions that $\mathrm{Ad}_{\dot{v}} \gamma_d \chi_S(\mathfrak{n}_j) \subset \mathfrak{n}_{j-1}$ for every $j \geq 0$. Thus we have

$$\phi(\mathfrak{n}_j) \subset \mathfrak{n}_{j-1}, \quad \forall j \geq 0.$$

It thus follows that $\mathfrak{n}_{S(v,d)}$ is ϕ-invariant and that $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent.

Q.E.D.

Let again $\phi = \mathrm{Ad}_{\dot{v}} \gamma_d \chi_S \mathrm{Ad}_m^{-1}$ and note that $\phi(\mathfrak{p}_{S(v,d)}) \subset \mathfrak{p}_{S(v,d)}$ by Lemma 3.14. Since $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent, we can define

$$\psi := (1 - \phi)^{-1} = 1 + \phi + \phi^2 + \phi^3 + \cdots : \ \mathfrak{n}_{S(v,d)} \longrightarrow \mathfrak{n}_{S(v,d)}.$$

Let $\Sigma_v^+ = \{\alpha \in \Sigma^+ : v^{-1}\alpha \in \Sigma^-\}$. Since $v([T] \cap \Sigma^+) \subset \Sigma^+$, it follows that $\Sigma_v^+ \subset \Sigma^+ - [S(v,d)]$. Let

$$\mathfrak{n}_v = \oplus_{\alpha\in\Sigma_v^+} \mathfrak{g}_\alpha = \mathfrak{n} \cap \mathrm{Ad}_{\dot{v}}(\mathfrak{n}^-).$$

Then $\mathfrak{n}_v \subset \mathfrak{n}_{S(v,d)}$.

Theorem 3.15. *The normalizer subalgebra* $\mathfrak{n}(\mathfrak{l})$ *in* $\mathfrak{g}_\Delta \cong \mathfrak{g}$ *of* $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$ *in (3.18) is*

$$\mathfrak{n}(\mathfrak{l}) = \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^\phi_{S(v,d)} + \psi(\mathfrak{n}_v),$$

where $\mathfrak{g}^\phi_{S(v,d)}$ *is the fixed point set of* $\phi|_{\mathfrak{g}_{S(v,d)}} = \mathrm{Ad}_{\dot{v}} \gamma_d \mathrm{Ad}_m^{-1}$ *in* $\mathfrak{g}_{S(v,d)}$*, and*

$$\mathfrak{z}'_{S(v,d)} = \{z \in \mathfrak{z}_{S(v,d)} : z - \phi(z) \in \mathrm{Ad}_{\dot{v}} \mathfrak{z}_T\} = \{z \in \mathfrak{z}_{S(v,d)} : \gamma_d \chi_S(z) = \chi_T(\mathrm{Ad}_{\dot{v}}^{-1} z)\}.$$

Proof. Set $\mathfrak{n}(\mathfrak{l})_\Delta = \{(x,x) : x \in \mathfrak{n}(\mathfrak{l})\}$. Since the normalizer subgroup of $\mathfrak{l}_{S,T,d,v}$ in $G \times G$ is $R_{S,T,d}$, we have

$$\mathfrak{n}(\mathfrak{l})_\Delta = \mathfrak{g}_\Delta \cap \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}_{S,T,d},$$

where

$$\mathfrak{r}_{S,T,d} = (\mathfrak{z}_S \oplus \mathfrak{z}_T) + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \{(x, \gamma_d(x)) : x \in \mathfrak{g}_S\}$$

is the Lie algebra of $R_{S,T,d}$. Thus

$$\mathfrak{n}(\mathfrak{l}) = \{x \in \mathfrak{g} : (\mathrm{Ad}_m^{-1}x,\, \mathrm{Ad}_{\dot{v}}^{-1}x) \in \mathfrak{r}_{S,T,d}\}.$$

It follows that $x \in \mathfrak{n}(\mathfrak{l})$ if and only if $x \in \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$ and $\gamma_d\chi_S(\mathrm{Ad}_m^{-1}(x)) = \chi_T(\mathrm{Ad}_{\dot{v}}^{-1}x)$, which is equivalent to $\mathrm{Ad}_{\dot{v}}^{-1}x - \gamma_d\chi_S(\mathrm{Ad}_m^{-1}(x)) \in \mathfrak{z}_T + \mathfrak{n}_T^-$, or

$$x - \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S(\mathrm{Ad}_m^{-1}(x)) \in \mathrm{Ad}_{\dot{v}}(\mathfrak{z}_T + \mathfrak{n}_T^-). \tag{3.22}$$

Recall that the map χ_S is the projection from $\mathfrak{p}_S \to \mathfrak{g}_S$ with respect to the decomposition $\mathfrak{p}_S = \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S$. We will also use χ_S to denote the projection $\mathfrak{g} \to \mathfrak{g}_S$ with respect to the decomposition $\mathfrak{g} = \mathfrak{n}_S^- + \mathfrak{z}_S + \mathfrak{g}_S + \mathfrak{n}_S$, so $\mathrm{Ad}_{\dot{v}}\gamma_d\chi_S(\mathrm{Ad}_m^{-1}x)$ is defined for all $x \in \mathfrak{g}$. Let $\mathfrak{c}$ be the set of all $x \in \mathfrak{g}$ satisfying (3.22). We will first determine $\mathfrak{c}$ and then determine $\mathfrak{c} \cap \big(\mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-\big)$.

Set again $\phi = \mathrm{Ad}_{\dot{v}}\gamma_d\chi_S\mathrm{Ad}_m^{-1} : \mathfrak{g} \to \mathfrak{g}$, and consider the decomposition

$$\mathfrak{g} = \mathfrak{n}_{S(v,d)}^- + \mathfrak{m}_{S(v,d)} + \mathfrak{n}_{S(v,d)}. \tag{3.23}$$

By Lemma 3.14, both $\mathfrak{m}_{S(v,d)}$ and $\mathfrak{n}_{S(v,d)}$ are invariant under ϕ and $\phi : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)}$ is nilpotent. Arguments similar to those in the proof of Lemma 3.14 show that $\mathfrak{n}_{S(v,d)}^-$ is also invariant under ϕ and that $\phi : \mathfrak{n}_{S(v,d)}^- \to \mathfrak{n}_{S(v,d)}^-$ is nilpotent. Note that since $v^{-1}\alpha \notin [T]$ implies that $\alpha \notin S(v,d)$, we have

$$\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- = \big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n}^- + \big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n} \;\subset\; \mathfrak{n}_{S(v,d)}^- + \mathfrak{n}_{S(v,d)}.$$

Moreover, it is easy to see that $\big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n} = \big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n}_{S(v,d)} = \mathfrak{n}_v$, so

$$\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- = \mathfrak{n}_v + \big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n}_{S(v,d)}^-. \tag{3.24}$$

Now let $x \in \mathfrak{g}$ and write $x = x_- + x_0 + x_+$, where $x_- \in \mathfrak{n}_{S(v,d)}^-$, $x_0 \in \mathfrak{m}_{S(v,d)}$, and $x_+ \in \mathfrak{n}_{S(v,d)}$. Then it follows from (3.24) that $x \in \mathfrak{c}$, i.e., x satisfies (3.22), if and only if

$$\begin{cases} x_0 - \phi(x_0) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T \\ x_+ - \phi(x_+) \in \mathfrak{n}_v \\ x_- - \phi(x_-) \in \big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n}_{S(v,d)}^-. \end{cases} \tag{3.25}$$

For $x_0 \in \mathfrak{m}_{S(v,d)}$, write $x_0 = z_0 + y_0$, where $z_0 \in \mathfrak{z}_{S(v,d)}$ and $y_0 \in \mathfrak{g}_{S(v,d)}$. Since both $\mathfrak{z}_{S(v,d)}$ and $\mathfrak{g}_{S(v,d)}$ are ϕ-invariant, and since $\mathrm{Ad}_{\dot{v}}\mathfrak{z}_T \subset \mathfrak{z}_{S(v,d)}$, $x_0 - \phi(x_0) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T$ if and only if $z_0 - \phi(z_0) \in \mathrm{Ad}_{\dot{v}}\mathfrak{z}_T$ and $y_0 - \phi(y_0) = 0$, which implies that $x_0 \in \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)}$. Recall that $\psi = (1-\phi)^{-1}$ on $\mathfrak{n}_{S(v,d)}$, and note that the same formula defines ψ on $\mathfrak{n}_{S(v,d)}^-$. Thus, $x_+ - \phi(x_+) \in \mathfrak{n}_v$ if and only if $x_+ \in \psi(\mathfrak{n}_v)$. Similarly, $x_- - \phi(x_-) \in \mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- \cap \mathfrak{n}_{S(v,d)}^-$ if and only if $x_- \in \psi(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- \cap \mathfrak{n}_{S(v,d)}^-)$. Thus,

$$\mathfrak{c} = \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v) + \psi\left(\big(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\big) \cap \mathfrak{n}_{S(v,d)}^-\right)$$

as a direct sum.

We now determine $\mathfrak{c} \cap \left(\mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-\right)$. Since $S(v,d) \subset S \cap v(T)$ and $\mathfrak{m}_{S(v,d)} \subset \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$, we have

$$\mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} \subset \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-.$$

On the other hand, it follows from definitions that $\psi(\mathfrak{n}_v) \subset \mathfrak{n}_{S(v,d)} \subset \mathfrak{n} \subset \mathfrak{p}_S$ and

$$\psi(\mathfrak{n}_v) \subset (\mathfrak{n}_v + \phi(\mathfrak{n}_{S(v,d)})) \subset \mathrm{Ad}_{\dot{v}}(\mathfrak{n}^- + \mathfrak{m}_T) = \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-,$$

so $\psi(\mathfrak{n}_v) \subset \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$. Thus

$$\mathfrak{c} \cap \left(\mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-\right) = \mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v) + \mathfrak{c}_1,$$

where $\mathfrak{c}_1 = \psi\left(\left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n}^-_{S(v,d)}\right) \cap \mathfrak{p}_S \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$. The theorem follows once we show $\mathfrak{c}_1 = 0$.

For notational simplicity, we set $\mathfrak{n}' = \left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-\right) \cap \mathfrak{n}^-_{S(v,d)}$. It suffices to show that $\psi(\mathfrak{n}') \cap \mathfrak{p}_S = 0$. Since $\psi(\mathfrak{n}') \subset \mathfrak{n}^-_{S(v,d)}$, it suffices to show that $\psi(\mathfrak{n}') \cap \mathfrak{n}^{S,-}_{S(v,d)} = 0$, where $\mathfrak{n}^{S,-}_{S(v,d)} = \mathfrak{m}_S \cap \mathfrak{n}^-_{S(v,d)}$. Since

$$\psi(\mathfrak{n}') \cap \mathfrak{n}^{S,-}_{S(v,d)} = \psi\left(\mathfrak{n}' \cap (1-\phi)\left(\mathfrak{n}^{S,-}_{S(v,d)}\right)\right) = \psi\left(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- \cap (1-\phi)\left(\mathfrak{n}^{S,-}_{S(v,d)}\right)\right),$$

we only need to show that $\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- \cap (1-\phi)\left(\mathfrak{n}^{S,-}_{S(v,d)}\right) = 0$. Let

$$A_1 = \{\alpha \in \Sigma^- : \alpha \notin vd[S]\},$$

and for $i \geq 1$,

$$A_{i+1} = vd(A_i \cap [S]).$$

It is easy to see inductively that $A_i \subset \Sigma^-$ for $i \geq 1$. Set $\mathfrak{s}_i = \sum_{\alpha \in A_i} \mathfrak{g}_\alpha \subset \mathfrak{n}^-$ for $i \geq 1$. It is easy to see that $A_i \cap A_j = \emptyset$ if $i \neq j$ and it follows that $\mathfrak{s}_i \cap \mathfrak{s}_j = 0$ if $i \neq j$. Moreover, since $[S] - [S(v,d)] = \cup_{i \geq 1}(A_i \cap [S])$, we have

$$\mathfrak{n}^{S,-}_{S(v,d)} = \oplus_i(\mathfrak{s}_j \cap \mathfrak{g}_S).$$

A proof similar to that of (3.21) in the proof of Lemma 3.14 shows that $M_{S(v,d)}$ preserves $\mathfrak{s}_i$ for each $i \geq 1$. It follows easily that ϕ maps $\mathfrak{s}_i \cap \mathfrak{g}_S$ injectively into $\mathfrak{s}_{i+1}$ for all $i \geq 1$. If $x \in \mathfrak{n}^{S,-}_{S(v,d)}$, x nonzero, write

$$x = x_1 + \ldots x_k, \; x_j \in \mathfrak{s}_j \cap \mathfrak{g}_S, \text{ with } x_k \neq 0.$$

Then

$$(1-\phi)(x) = x_1 + (x_2 - \phi(x_1)) + \cdots + (x_k - \phi(x_{k-1})) - \phi(x_k).$$

Here $x_1 \in \mathfrak{s}_1$, $x_i - \phi(x_{i-1}) \in \mathfrak{s}_i$, and $\phi(x_k) \in \mathfrak{s}_{k+1}$. Note that $\phi(x_k) \neq 0$ since ϕ is injective on $\mathfrak{s}_k$. Since $\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-$ is a sum of its root spaces and the $\mathfrak{s}_i$'s have trivial intersections, $(1-\phi)(x) \in \mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-$ implies $\phi(x_k) \in \mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-$. But $\phi(x_k) \in \mathrm{Ad}_{\dot{v}}(\mathfrak{g}_T)$ and $\mathrm{Ad}_{\dot{v}}(\mathfrak{g}_T) \cap \mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^- = 0$, so $\phi(x_k) = 0$. Thus, $x_k = 0$, so $x = 0$ and it follows that $(\mathrm{Ad}_{\dot{v}}\mathfrak{n}_T^-) \cap (1-\phi)\mathfrak{n}^-(S_v) = 0$. This proves that $\mathfrak{c}_1 = 0$, and the Theorem follows.

Q.E.D.

Remark 3.16. Theorem 3.15 implies that $\mathfrak{n}(\mathfrak{l}) \subset \mathfrak{p}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^-$. Since $v \in {}^{S(v,d)}W^T$ and since $S(v,d) \subset v(T)$ we have the direct sum decomposition

(3.26) $$\mathfrak{p}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{p}_T^- = \mathfrak{m}_{S(v,d)} + \mathfrak{n}_v + \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T.$$

Note that $\mathfrak{n}_v + \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T$ is a subalgebra with $\mathfrak{n}_v$ as an ideal. Note also that $\phi(\mathfrak{n}_{S(v,d)}) \subset \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T$. If we define

$$\tilde{\psi} = \psi - 1 = \phi\psi = \phi + \phi^2 + \cdots : \mathfrak{n}_{S(v,d)} \to \mathfrak{n}_{S(v,d)} \cap \mathrm{Ad}_{\dot{v}}\mathfrak{m}_T,$$

then $\psi(\mathfrak{n}_v) = \{x + \tilde{\psi}(x) : x \in \mathfrak{n}_v\}$. The fact that $\psi(\mathfrak{n}_v)$ is a Lie subalgebra of $\mathfrak{g}$ implies that

$$[x, y]_{\mathrm{new}} := [x, y] + [x, \tilde{\psi}(y)] + [\tilde{\psi}(x), y], \qquad \forall x, y \in \mathfrak{n}_v,$$

is a new Lie bracket on $\mathfrak{n}_v$, and that $\tilde{\psi} : (\mathfrak{n}_v, [\,,\,]_{\mathrm{new}}) \to (\mathfrak{n}_v, [\,,\,])$ is a Lie algebra homomorphism.

3.5. **Intersections of $\mathfrak{g}_\Delta$ with an arbitrary $\mathfrak{l} \in \mathcal{L}$.** In this section, we compute the intersection of $\mathfrak{g}_\Delta$ with an arbitrary Lagrangian subalgebra $\mathfrak{l}$ of $\mathfrak{g} \oplus \mathfrak{g}$. By Corollary 3.13, it is enough to assume that $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$ as given in (3.18).

Proposition 3.17. *For the Lagrangian subalgebra $\mathfrak{l} = \mathfrak{l}_{S,T,d,V,\dot{v},m}$ as given in (3.18), let the notation be as in Theorem 3.15. Then we have*

$$\mathfrak{g}_\Delta \cap \mathfrak{l}_{S,T,d,V,\dot{v},m} = \mathrm{Ad}_{(m,\dot{v})}V' + \left(\mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta,$$

where

$$V' = \{(z, v^{-1}z) : z \in \mathfrak{z}'_{S(v,d)}\} \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}).$$

Proof. By Theorem 3.15,

$$\mathfrak{g}_\Delta \cap \mathfrak{l} \subset \mathfrak{n}(\mathfrak{l}) = \left(\mathfrak{z}'_{S(v,d)} + \mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta.$$

Since $\left(\mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta \subset \mathfrak{l}$, we see that

$$\mathfrak{g}_\Delta \cap \mathfrak{l} = \left((\mathfrak{z}'_{S(v,d)})_\Delta \cap \mathfrak{l}\right) + \left(\mathfrak{g}^{\phi}_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta,$$

and

$$\left(\mathfrak{z}'_{S(v,d)}\right)_\Delta \cap \mathfrak{l} = \left(\mathfrak{z}'_{S(v,d)}\right)_\Delta \cap \mathfrak{l} \cap (\mathfrak{h} \oplus \mathfrak{h}) = \mathrm{Ad}_{(m,\dot{v})}V'.$$

Q.E.D.

Recall that a Belavin-Drinfeld triple [B-Dr] for $\mathfrak{g}$ is a triple (S, T, d), where $S, T \subset \Gamma$, $d \in I(S,T)$, and $S(1, d) = \emptyset$, where 1 is the identity element in the Weyl group W.

Definition 3.18. By a *Belavin-Drinfeld system* we mean a quadruple (S, T, d, V), where (S, T, d) is a Belavin-Drinfeld triple, and V is a Lagrangian subspace of $\mathfrak{z}_S \oplus \mathfrak{z}_T$ such that

$$\mathfrak{h}_\Delta \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}) = 0.$$

We now show that a theorem of Belavin and Drinfeld [B-Dr] follows easily from Proposition 3.17.

Corollary 3.19. *[Belavin-Drinfeld] A Lagrangian subalgebra $\mathfrak{l}$ of $\mathfrak{g} \oplus \mathfrak{g}$ has trivial intersection with $\mathfrak{g}_\Delta$ if and only if $\mathfrak{l}$ is G_Δ-conjugate to a Lagrangian subalgebra of the form $\mathfrak{l}_{S,T,d,V}$, where (S, T, d, V) is a Belavin-Drinfeld system.*

Proof. With the same notation as that in Theorem 3.15 and Proposition 3.17, it is enough to determine those $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ such that $\mathfrak{g}_\Delta \cap \mathfrak{l}_{S,T,d,V,\dot{v},m} = 0$. Suppose that $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ has this property. Since $\dim \psi(\mathfrak{n}_v) = l(v)$, the length of v, and since every automorphism of a semi-simple Lie algebra has fixed point set of dimension at least one [Wi], $v = 1$ and $S(1,d) = \emptyset$. In this case, V' as in Proposition 3.17 is given by

$$V' = \mathfrak{h}_\Delta \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}),$$

so $\mathfrak{h}_\Delta \cap (V + \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}) = 0$, and we have

$$\mathfrak{l}_{S,T,d,V,\dot{v},m} = \mathrm{Ad}_{(m,\dot{v})} \mathfrak{l}_{S,T,d,V}$$

for some $m \in H$ and $\dot{v} \in H$. Note that in this case

$$R_{\dot{v}} = \{(h_1, h_2) \in H \times H : \gamma_d(\chi_S(h_1)) = \chi_T(h_2)\}$$

and $R_{\dot{v}}$ acts on H from the right by $h \cdot (h_1, h_2) = h h_1 h_2^{-1}$, where $h \in H$ and $(h_1, h_2) \in R_{\dot{v}}$. Consider the map

$$\mathbf{m} : \ R_{\dot{v}} \longrightarrow H : \ (h_1, h_2) \longmapsto h_1 h_2^{-1}.$$

The assumption that $V' = 0$ implies that the dimension of the kernel of the differential of $\mathbf{m}$ is less than or equal to $\dim(\mathfrak{z}_T)$. It follows easily that the differential of $\mathbf{m}$ is onto, thus $\mathbf{m}$ is onto. Thus, by Corollary 3.13, $\mathfrak{l}_{S,T,d,V,\dot{v},m}$ is in the G_Δ-orbit of $\mathfrak{l}_{S,T,d,V}$.

Q.E.D.

3.6. **Examples of smooth G_Δ-orbit closures in $\mathcal{L}$.** The closure of a G_Δ-orbit in $\mathcal{L}$ is in general not necessarily smooth. In this section, we look at two cases for which such a closure is smooth.

Proposition 3.20. *If $\mathfrak{l}$ is a Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ such that $\mathfrak{g}_\Delta \cap \mathfrak{l} = 0$, then the closure of the G_Δ-orbit $G_\Delta \cdot \mathfrak{l}$ is the same as the closure of the $(G \times G)$-orbit $(G \times G) \cdot \mathfrak{l}$ which is smooth.*

Proof. We only need to show that $G_\Delta \cdot \mathfrak{l}$ and $(G \times G) \cdot \mathfrak{l}$ have the same dimension. By the Belavin-Drinfeld theorem, we may assume that $\mathfrak{l} = \mathfrak{l}_{S,T,d,V}$, where (S,T,d,V) is a Belavin-Drinfeld system. In particular,

$$\mathfrak{g}_\Delta \cap \mathfrak{r}_{S,T,d} = \mathfrak{h}_\Delta \cap ((\mathfrak{z}_S \oplus \mathfrak{z}_T) + V_S)),$$

where $V_S = \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}$. For a subspace A of $\mathfrak{h} \oplus \mathfrak{h}$, let

$$A^\perp = \{(x,y) \in \mathfrak{h} \oplus \mathfrak{h} : \langle (x,y), (x_1,y_1) \rangle = 0 \, \forall (x_1, y_1) \in A\}.$$

Then

$$(\mathfrak{h}_\Delta \cap ((\mathfrak{z}_S \oplus \mathfrak{z}_T) + V_S)))^\perp = \mathfrak{h}_\Delta + V_S.$$

Since $\mathfrak{h}_\Delta \cap V_S = 0$, we see that $\dim(\mathfrak{h}_\Delta + V_S) = \dim \mathfrak{h} + \dim \mathfrak{h}_S$, so

$$\dim(\mathfrak{h}_\Delta \cap ((\mathfrak{z}_S \oplus \mathfrak{z}_T) + V_S))) = \dim \mathfrak{z}_S.$$

Thus $\dim(G_\Delta \cdot \mathfrak{l}) = \dim \mathfrak{g} - \dim \mathfrak{z}_S = \dim((G \times G) \cdot \mathfrak{l})$ by Proposition 2.19.

Q.E.D.

We now show that the De Concini-Procesi compactifications of complex symmetric spaces of G can be embedded into $\mathcal{L}$ as closures of some G_Δ-orbits in $\mathcal{L}$.

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an involution with lift σ to G, and let $\mathfrak{g}^\sigma$ and G^σ be the fixed subalgebra and subgroup of σ. Let again $\mathfrak{l}_\sigma \in \mathcal{L}$ be the graph of σ. The orbit $G_\Delta \cdot \mathfrak{l}_\sigma$ may be identified with the complex symmetric space G/G^σ. Let $\overline{G_\Delta \cdot \mathfrak{l}_\sigma}$ be the closure of $G_\Delta \cdot \mathfrak{l}_\sigma$ in $\mathcal{L}$. We will show that $\overline{G_\Delta \cdot \mathfrak{l}_\sigma}$ may be identified with the De Concini-Procesi compactification of G/G^σ, which may be defined as follows. Let $\dim(\mathfrak{g}^\sigma) = m$, so $\mathfrak{g}^\sigma \in \mathrm{Gr}(m, \mathfrak{g})$. Then $G \cdot \mathfrak{g}^\sigma \cong G/G^\sigma$, and $X_\sigma := \overline{G \cdot \mathfrak{g}^\sigma}$ is the De Concini–Procesi compactification. It is known to be smooth with finitely many G-orbits [D-P].

We recall some basic results about involutions. Choose a σ-stable maximal split Cartan subalgebra $\mathfrak{h}_s$ of $\mathfrak{g}$, i.e., a σ-stable Cartan subalgebra $\mathfrak{h}_s$ such that $\mathfrak{h}_s^{-\sigma}$ has maximal dimension. There is an induced action of σ on the roots of $\mathfrak{h}_s$ in $\mathfrak{g}$, and there is a positive root system $\Sigma^+(\mathfrak{h}_s)$ for $\mathfrak{h}_s$ with the property that if $\alpha \in \Sigma^+(\mathfrak{h}_s)$, then either $\sigma(\alpha) = \alpha$ and $\sigma|_{\mathfrak{g}_\alpha} = \mathrm{id}$, or $\sigma(\alpha) \notin \Sigma^+(\mathfrak{h}_s)$. A weight $\lambda \in \mathfrak{h}_s^*$ is called a regular special dominant weight if λ is nonnegative on roots in $\Sigma^+(\mathfrak{h}_s)$, $\sigma(\lambda) = -\lambda$, and $\lambda(H_\alpha) = 0$ for α simple implies that $\sigma(\alpha) = \alpha$. If λ and μ are weights, we say $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\alpha \in \Sigma^+(\mathfrak{h}_s), n_\alpha \geq 0} n_\alpha \alpha$. For a weight μ, let $\overline{\mu} = \frac{1}{2}(\mu - \sigma(\mu))$.

Lemma 3.21. *[De Concini-Procesi,* [D-P]*, Lemmas 4.1 and 6.1] Let V be a representation of G, and suppose there exists a vector $v \in V$ such that G^σ is the stabilizer of the line through v. Suppose that when we decompose v into a sum of weight vectors for $\mathfrak{h}_s$, $v = v_\lambda + \sum v_i$ where v_λ has regular special dominant weight λ and each v_i has weight μ_i where $\lambda \geq \overline{\mu_i}$. Let $[v]$ be class of v in* Proj(V) *and let X' be the closure of $G \cdot [v]$ in* Proj(V)*. Then $X' \cong X_\sigma$.*

Proposition 3.22. *There is a G-equivariant isomorphism $\overline{G_\Delta \cdot \mathfrak{l}_\sigma} \cong X_\sigma$.*

Proof. To apply the Lemma 3.21, let $n = \dim(\mathfrak{g})$ and consider the diagonal action of G on $V = \wedge^n(\mathfrak{g} \oplus \mathfrak{g})$ and the vector $v_\sigma = \wedge^n(\mathfrak{l}_\sigma)$. In order to represent v_σ as a sum of weight vectors in $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$, we choose a basis. Let $U_1, \dots, U_l$ be a basis of $\mathfrak{h}_s$. Let $\beta_1, \dots, \beta_s$ be the roots of $\Sigma^+(\mathfrak{h}_s)$ such that $\sigma(\beta_i) = \beta_i$, and let $\alpha_1, \dots, \alpha_t$ be the other roots in $\Sigma^+(\mathfrak{h}_s)$. For each root α, choose a root vector X_α. Then

$$\{(U_i, \sigma(U_i)) | i = 1, \dots, l\} \cup \{(X_{\pm\beta_i}, X_{\pm\beta_i}) | i = 1, \cdots, s\} \cup \{(X_{\pm\alpha_j}, \sigma(X_{\pm\alpha_j})) | i = 1, \cdots t\}$$

is clearly a basis of $\mathfrak{l}_\sigma$. Now v_σ is the wedge of the vectors $(Y_i, \sigma(Y_i))$ as Y_i runs through the above basis, and v_σ contains the summand

$$u : \bigwedge_{i=1,\dots,l} (U_i, \sigma(U_i)) \bigwedge_{i=1,\dots,s} (X_{\beta_i}, 0) \wedge (X_{-\beta_i}, 0) \bigwedge_{j=1,\dots,t} (X_{\alpha_i}, 0) \wedge (0, \sigma(X_{-\alpha_i})).$$

It is easy to see that u is a weight vector for the diagonal Cartan subalgebra with weight $\nu := \sum_{i=1,\dots,t} \alpha_i - \sigma(\alpha_i)$, and $\nu = 2 \sum_{i=1,\dots,t} \alpha_i$ on the subspace $\mathfrak{h}_s^{-\sigma}$. Thus, ν is a regular special dominant weight by Lemma 6.1 in [D-P]. Moreover, the other weight vectors appearing in v_σ have weights ψ such that $\overline{\psi}$ is of the form $\nu - \sum_{n_\alpha \geq 0, \alpha \in \Sigma^+(\mathfrak{h}_s)} n_\alpha \alpha$. Thus, by Lemma 3.21, $\overline{G \cdot v_\sigma} \cong X_\sigma$.

Note that using the Plucker embedding of $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) \hookrightarrow$ Proj(V), we can identify $\overline{G \cdot v_\sigma}$ with $\overline{G_\Delta \cdot \mathfrak{l}_\sigma}$. Thus, $\overline{G_\Delta \cdot \mathfrak{l}_\sigma} \cong X_\sigma$.

Q.E.D.

Remark 3.23. Let d be the automorphism of the Dynkin diagram of $\mathfrak{g}$ such that $\sigma = \gamma_d \mathrm{Ad}_{g_0}$ for some g_0. Consider the embedding

$$G/G^\sigma \longrightarrow G: \; gG^\sigma \longmapsto \gamma_d^{-1}(g) g_0 g^{-1},$$

which in turn gives an embedding of G/G^σ into the De Concini-Procesi compactification Z_d of G. Proposition 3.22 then says that the closure of G/G^σ in Z_d is isomorphic to the De Concini-Procesi compactification of G/G^σ.

4. The Poisson structure Π_0 on $\mathcal{L}$

By a *Lagrangian splitting* of $\mathfrak{g} \oplus \mathfrak{g}$ we mean a decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$. In this section, we will recall the definition of the Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}$ associated to a Lagrangian splitting of $\mathfrak{g} \oplus \mathfrak{g}$. For the Poisson structure Π_0 determined by the so-called standard splitting (see Definition 4.4), we will study its symplectic leaf decomposition in terms of intersections of G_Δ and $(B \times B^-)$-orbits. We will also point out some interesting Poisson submanifold/varieties of $\mathcal{L}$ with respect to the Poisson structures defined by the Belavin-Drinfeld splittings (Definition 4.4). A review of some basic facts on Poisson Lie groups is given in Section 4.1. Details of most of these facts can be found in [K-S].

4.1. **Poisson Lie groups and Lagrangian splittings.** Recall that a Poisson bi-vector field π_L on a Lie group L is said to be *multiplicative* if the map $m : L \times L \to L : (l_1, l_2) \mapsto l_1 l_2$ is a Poisson map with respect to π_L. A Poisson Lie group is a pair (L, π_L), where L is a Lie group and π_L is a multiplicative Poisson bi-vector field on L. An action $\sigma : L \times P \to P$ of a Poisson Lie group (L, π_L) on a Poisson manifold (P, π_P) is said to be Poisson if σ is a Poisson map, where $L \times P$ is equipped with the product Poisson structure $\pi_L \oplus \pi_P$. A Poisson homogeneous space of (L, π_L) is a Poisson manifold (P, π_P) with a transitive Poisson action by (L, π_L). We now recall the relations between Poisson Lie groups and Lagrangian splittings (or Manin triples).

Assume that $\mathfrak{d}$ is a $2n$-dimensional Lie algebra over a field of characteristic 0, and assume that $\langle \, , \, \rangle$ is a symmetric, non-degenerate, and ad-invariant bilinear form on $\mathfrak{d}$. By a Lagrangian subalgebra of $\mathfrak{d}$ we mean an n-dimensional Lie subalgebra of $\mathfrak{d}$ that is also isotropic with respect to $\langle \, , \, \rangle$. By a *Lagrangian splitting* of $\mathfrak{d}$ we mean a decomposition $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$, where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Lagrangian subalgebras of $\mathfrak{d}$. The triple $(\mathfrak{d}, \mathfrak{l}_1, \mathfrak{l}_2)$ is also called a Manin triple.

Assume that $(\mathfrak{d}, \mathfrak{l}_1, \mathfrak{l}_2)$ is a Manin triple. Define

$$(4.1) \qquad \delta_1 : \; \mathfrak{l}_1 \longrightarrow \wedge^2 \mathfrak{l}_1 : \; \langle \delta_1(x_1), y_2 \wedge z_2 \rangle = \langle x_1, [y_2, z_2] \rangle, \quad \forall x_1 \in \mathfrak{l}_1, y_2, z_2 \in \mathfrak{l}_2.$$

Let D be the adjoint group of $\mathfrak{d}$, and let L_1 be the connected subgroup of D with Lie algebra $\mathfrak{l}_1$. Then there is a unique multiplicative Poisson bivector field π_{L_1} on L_1 whose linearization at the identity element e of L_1 is δ_1, i.e.,

$$(L_{\tilde{x}} \pi_{L_1})(e) = \delta_1(x), \qquad x_1 \in \mathfrak{l}_1,$$

where for $x_1 \in \mathfrak{l}_1$, $\tilde{x}_1$ is any vector field on L_1 with $\tilde{x}_1(e) = x_1$, and $L_{\tilde{x}_1}$ denotes the Lie derivative by $\tilde{x}_1$. By changing the roles of $\mathfrak{l}_1$ and $\mathfrak{l}_2$, we also have a multiplicative Poisson bi-vector field π_{L_2} on the connected subgroup L_2 of D whose Lie algebra is $\mathfrak{l}_2$.

Denote by $\mathcal{L}(\mathfrak{d})$ the set of all Lagrangian subalgebras of $\mathfrak{d}$. Then $\mathcal{L}(\mathfrak{d})$ is an algebraic subvariety of the Grassmannian $\mathrm{Gr}(n, \mathfrak{d})$ of n-dimensional subspaces of $\mathfrak{d}$. It is shown in [E-L2] that every Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$ of $\mathfrak{d}$ defines a Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{d})$. Indeed, if $\{x_j\}$

is a basis for $\mathfrak{l}_1$, and if $\{\xi_j\}$ is the basis for $\mathfrak{l}_2$ such that $\langle x_j, \xi_k\rangle = \delta_{jk}$ for $1 \leq j, k \leq n = \dim \mathfrak{g}$, we set

$$R = \frac{1}{2}\sum_{j=1}^{n}(\xi_j \wedge x_j) \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g}). \tag{4.2}$$

The action of D on the Grassmannian $\mathrm{Gr}(n, \mathfrak{d})$ through the adjoint action defines a Lie algebra anti-homomorphism κ from $\mathfrak{d}$ to the space of vector fields on $\mathrm{Gr}(n, \mathfrak{d})$. We will also use κ to denote the induced map from $\wedge^2\mathfrak{d}$ to the space of bi-vector fields on $\mathrm{Gr}(n, \mathfrak{d})$. Set

$$\Pi_{\mathfrak{l}_1,\mathfrak{l}_2} = \kappa(R) = \frac{1}{2}\sum_{j=1}^{n}(\kappa(\xi_j) \wedge \kappa(x_j)).$$

Note that $\mathcal{L}(\mathfrak{d}) \subset \mathrm{Gr}(n, \mathfrak{d})$ is D-invariant, so $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ restricts to a bi-vector field on $\mathcal{L}(\mathfrak{d})$. Let L_1 and L_2 act on $\mathcal{L}(\mathfrak{d})$ as subgroups of D.

Proposition 4.1. [E-L2] *For any Lagrangian splitting $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$, the bi-vector field $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ is a Poisson structure on $\mathcal{L}(\mathfrak{d})$ with the properties that*

1) the actions of (L_1, π_{L_1}) and (L_2, π_{L_2}) on $(\mathcal{L}(\mathfrak{d}), \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$ are Poisson;

2) all L_1 and L_2-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$, and are thus Poisson homogeneous spaces of (L_1, π_{L_1}) and (L_2, π_{L_2}) respectively. Moreover, their Zariski closures are Poisson subvarieties.

Remark 4.2. It is clear from the definition of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ that $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ is tangent to every D-orbit in $\mathcal{L}(\mathfrak{d})$. Thus every D-orbit in $\mathcal{L}(\mathfrak{d})$ is a Poisson submanifold of $(\mathcal{L}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$, and the closure of every D-orbit in $\mathcal{L}(\mathfrak{d})$ is a Poisson subvariety of $(\mathcal{L}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$. This property also follows from 2) of Proposition 4.1.

The rank of the Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ can be computed as in the following Lemma 4.3. A version of Lemma 4.3 first appeared in [E-L2], and a generalization of Lemma 4.3 can be found in [Lu-Y2].

Lemma 4.3. *Let $\mathfrak{d} = \mathfrak{l}_1 + \mathfrak{l}_2$ be a Lagrangian splitting. For $\mathfrak{l} \in \mathcal{L}(\mathfrak{d})$, let $\mathfrak{n}(\mathfrak{l})$ be the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{d}$, and let $\mathfrak{n}_1(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l}) \cap \mathfrak{l}_1$. Let $\mathfrak{n}(\mathfrak{l})^\perp = \{x \in \mathfrak{d} : \langle x, y\rangle = 0 \; \forall y \in \mathfrak{n}(\mathfrak{l})\}$. Set*

$$\mathcal{T}(\mathfrak{l}) = \mathfrak{n}_1(\mathfrak{l}) + \mathfrak{n}(\mathfrak{l})^\perp \subset \mathfrak{d}.$$

Then $\mathcal{T}(\mathfrak{l})$ is a Lagrangian subalgebra of $\mathfrak{d}$, and the rank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ at $\mathfrak{l}$ is equal to $\dim(L_1 \cdot \mathfrak{l}) - \dim(\mathfrak{l}_2 \cap \mathcal{T}(\mathfrak{l}))$*, where $L_1 \cdot \mathfrak{l}$ is the L_1-orbit in $\mathcal{L}$ through $\mathfrak{l}$.*

Proof. In Theorem 2.21 of [E-L2], we showed that $\mathcal{T}_1(\mathfrak{l}) := \mathfrak{n}_1(\mathfrak{l}) + \mathfrak{n}_1(\mathfrak{l})^\perp \cap \mathfrak{l}$ is a Lagrangian subalgebra (in fact, it is the Lagrangian subalgebra associated to the Poisson homogeneous space $(L_1 \cdot \mathfrak{l}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$ at $\mathfrak{l}$ by Drinfeld [Dr]). We show $\mathcal{T}(\mathfrak{l}) = \mathcal{T}_1(\mathfrak{l})$. Clearly, $\mathcal{T}_1(\mathfrak{l}) \subset \mathfrak{n}(\mathfrak{l})$, so since $\mathcal{T}_1(\mathfrak{l})$ is Lagrangian, $\mathfrak{n}(\mathfrak{l})^\perp \subset \mathcal{T}_1(\mathfrak{l})$. Thus, $\mathcal{T}(\mathfrak{l}) \subset \mathcal{T}_1(\mathfrak{l})$. Since $\mathfrak{n}(\mathfrak{l})$ is co-isotropic and $\mathfrak{l}_1$ is Lagrangian, it follows easily that $\mathcal{T}(\mathfrak{l})$ is co-isotropic, so $\mathcal{T}(\mathfrak{l}) = \mathcal{T}_1(\mathfrak{l})$.

To compute the rank of the symplectic leaf $\mathcal{E}_\mathfrak{l}$ at $\mathfrak{l}$, we identify $T_\mathfrak{l}(L_1 \cdot \mathfrak{l}) \cong \mathfrak{l}_1/\mathfrak{n}_1(\mathfrak{l})$, $T_\mathfrak{l}(D \cdot \mathfrak{l}) \cong \mathfrak{d}/\mathfrak{n}(\mathfrak{l})$, and $T_\mathfrak{l}^*(D \cdot \mathfrak{l}) \cong \mathfrak{n}(\mathfrak{l})^\perp$. The Poisson tensor $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}(\mathfrak{l}) \in \wedge^2(T_\mathfrak{l}(D \cdot \mathfrak{l}))$ induces a linear map $\tilde{A} : T_\mathfrak{l}^*(D \cdot \mathfrak{l}) \to T_\mathfrak{l}(D \cdot \mathfrak{l})$ via $\tilde{A}(\lambda)(\mu) = \Pi_{\mathfrak{l}_1,\mathfrak{l}_2}(\lambda, \mu)$ for $\lambda, \mu \in T_\mathfrak{l}^*(D \cdot \mathfrak{l})$. Under the above identifications, $\tilde{A}$ corresponds to a linear map $A : \mathfrak{n}(\mathfrak{l})^\perp \to \mathfrak{d}/\mathfrak{n}(\mathfrak{l})$. In the proof of Theorem 2.18

in [E-L2], we show that $A(X_1+X_2) = X_1 + \mathfrak{n}(\mathfrak{l})$ for $X_1 + X_2 \in \mathfrak{n}(\mathfrak{l})^\perp$, $X_1 \in \mathfrak{l}_1$, $X_2 \in \mathfrak{l}_2$. Thus, A factors through $\mathfrak{l}_1/\mathfrak{n}_1(\mathfrak{l}) \subset \mathfrak{d}/\mathfrak{n}(\mathfrak{l})$. By construction, $T_{\mathfrak{l}}\mathcal{E}_{\mathfrak{l}}$ is the image of A, so it follows that

$$T_{\mathfrak{l}}\mathcal{E}_{\mathfrak{l}} = q \circ p(\mathfrak{n}(\mathfrak{l})^\perp) = q \circ p(\mathcal{T}(\mathfrak{l})). \tag{4.3}$$

The dimension formula follows easily.

Q.E.D.

4.2. **The standard Lagrangian splitting and the Belavin-Drinfeld splittings.** We now return to the semi-simple Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the bilinear form $\langle \, , \rangle$ given in (2.1). Lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$ up to conjugation by elements in $G \times G$ have been classified by P. Delorme [De].

A study of the Poisson structures $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ defined by arbitrary Lagrangian splittings $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ will be carried out in [Lu-Y2]. More precisely, let $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$ be respectively the normalizer subgroups of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ in $G \times G$. Then both $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$ are conjugate to subgroups of $G \times G$ of the type $R_{S,T,d}$ in (2.6). By Proposition 4.1, all $N(\mathfrak{l}_1)$-orbits and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$. It will be shown in [Lu-Y2] that every non-empty intersection of an $N(\mathfrak{l}_1)$-orbit and an $N(\mathfrak{l}_2)$-orbit in $\mathcal{L}$ is a regular Poisson manifold with respect to $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$. Thus the study of the symplectic leaves of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ is reduced to the study of intersections of $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}$. To classify $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}$, we need first to classify double cosets in $G \times G$ by two groups of the type $R_{S,T,d}$. Such a classification will be given in [Lu-Y1]. Using the classification of $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}$, the rank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ at every point in $\mathcal{L}$ will be computed in [Lu-Y2].

Definition 4.4. By the *standard Lagrangian splitting* of $\mathfrak{g} \oplus \mathfrak{g}$ we mean the splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}_{\mathrm{st}}^*$, where

$$\mathfrak{g}_{\mathrm{st}}^* = \mathfrak{h}_{-\Delta} + (\mathfrak{n} \oplus \mathfrak{n}^-).$$

We will denote by Π_0 the Poisson structure on $\mathcal{L}$ determined by the standard Lagrangian splitting. The multiplicative Poisson structure on G defined by the standard splitting will be denoted by π_0. By a *Belavin-Drinfeld splitting* of $\mathfrak{g} \oplus \mathfrak{g}$ we will mean a splitting of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{S,T,d,V}$, where (S,T,d,V) is a Belavin-Drinfeld system (Definition 3.18). When a Belavin-Drinfeld splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{S,T,d,V}$ is fixed, we will set $\mathfrak{l}_{BD} = \mathfrak{l}_{S,T,d,V}$, the Poisson structure on $\mathcal{L}$ defined by the splitting will be denoted by Π_{BD}, and the corresponding multiplicative Poisson structure on the group $G \cong G_\Delta$ will be denoted by π_{BD}.

Note that when $S = T = \emptyset$ and $V = \mathfrak{h}_{-\Delta} = \{(x - x) : x \in \mathfrak{h}\}$, the Belavin-Drinfeld splitting becomes the standard Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}_{\mathrm{st}}^*$. In Section 4.3, we will compute the rank of Π_0. As a consequence, we will see that every non-empty intersection of a G_Δ-orbit and a $(B \times B^-)$-orbit in $\mathcal{L}$ is a regular Poisson submanifold with respect to Π_0, and the group $H_\Delta = \{(h,h) : h \in H\}$ acts transitively on the set of symplectic leaves in any such intersection. Thus the study of symplectic leaves of Π_0 becomes the study of the G_Δ and the $(B \times B^-)$-orbits in $\mathcal{L}$ as H_Δ-varieties.

We will now point out some interesting Poisson submanifolds of $(\mathcal{L}, \Pi_{BD})$. We first state a consequence of Remark 4.2 and Proposition 2.27, which holds for any Lagrangian splitting of $\mathfrak{g} \oplus \mathfrak{g}$.

Proposition 4.5. *Every $(G \times G)$-orbit in $\mathcal{L}$ is a Poisson submanifold of $(\mathcal{L}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$ for any Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$. Consequently, every $(G \times G)$-orbit closure in $\mathcal{L}$ is a smooth Poisson subvariety of $(\mathcal{L}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$.*

Example 4.6. Fix a diagram automorphism d and consider the embedding of G into $\mathcal{L}$ as the $(G \times G)$-orbit through $\mathfrak{l}_{\gamma_d}$:

$$G \hookrightarrow \mathcal{L}: \;\; g \longmapsto \{(x, \gamma_d \mathrm{Ad}_g(x)) : x \in \mathfrak{g}\}. \tag{4.4}$$

Then by Proposition 4.5, every Lagrangian splitting of $\mathfrak{g} \oplus \mathfrak{g}$ gives rise to a Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on G which extends to the closure $Z_d(G)$ of G in $\mathcal{L}$. Recall from Section 2.7 that $Z_d(G)$ is a De Concini-Procesi compactification of G. Under the embedding (4.4), the G_Δ-action on $\mathcal{L}$ becomes the following action of G on itself

$$G \times G \longrightarrow G: \; (h, g) \longmapsto \gamma_d^{-1}(h) g h^{-1}. \tag{4.5}$$

We will refer to the action in (4.5) as the *d-twisted conjugation* action of G on itself and refer to its orbits as the *d-twisted conjugacy classes* of G.

For a Belavin-Drinfeld splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{BD}$ and a diagram automorphism d, the restriction of Π_{BD} to $G \hookrightarrow \mathcal{L}$ (via (4.4)) has the following properties by Proposition 4.1.

Proposition 4.7. *For a Belavin-Drinfeld splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{BD}$ and a diagram automorphism d, embed G into $\mathcal{L}$ via (4.4) and regard Π_{BD} as a Poisson structure on G and on $Z_d(G)$. Then*

1) the d-twisted conjugation action of (G, π_{BD}) on (G, Π_{BD}) in (4.5) is Poisson;

2) every d-twisted conjugacy class in G is a Poisson submanifold of (G, Π_{BD}) and is thus a Poisson homogeneous space of (G, π_{BD}), and the closure of a d-twisted conjugacy class in G is a Poisson subvariety of $(Z_d(G), \Pi_{BD})$.

Example 4.8. Let σ be an involutive automorphism of $\mathfrak{g}$. Write $\sigma = \gamma_d \circ \mathrm{Ad}_g$ for a diagram automorphism d and $g \in G$. By results from Section 3.6, the De Concini-Procesi compactification X_σ of the complex symmetric space G/G^σ is isomorphic to the closure of the G_Δ-orbit in $\mathcal{L}$ through the point $g = \{(x, \sigma(x)) : x \in \mathfrak{g}\}$ of $\mathcal{L}$. Consequently, for every Belavin-Drinfeld splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{BD}$, the restriction of Π_{BD} to $G/G^\sigma \hookrightarrow \mathcal{L}$ is a Poisson structure on G/G^σ that extends smoothly to X_σ. Moreover, the action of G on (X_σ, Π_{BD}), which is the extension of the action of G on G/G^σ by left translations, is Poisson for the Poisson Lie group (G, π_{BD}) determined by the given Belavin-Drinfeld splitting.

Remark 4.9. Let L_{BD} be the connected Lie subgroup of $G \times G$ with Lie algebra $\mathfrak{l}_{BD}$. By Section 4.1, the splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{l}_{BD}$ induces a multiplicative Poisson structure $\pi_{L_{BD}}$ on L_{BD}. The pair $(L_{BD}, \pi_{L_{BD}})$ is called a dual Poisson Lie group [K-S] of (G, π_G). The restriction to L_{BD} of the map $F : G \times G \to G : (g_1, g_2) \to g_2 g_1^{-1}$ is a local diffeomorphism from L_{BD} to an open subset U of G containing the identity element. The Poisson structure Π_{BD} on G can be regarded as an extension of $\pi_{L_{BD}}$ on U to G. Symplectic leaves of (G, π_{BD}) and $(L_{BD}, \pi_{L_{BD}})$ have been classified by Yakimov [Y] and Kogan and Zelevinsky [K-Z].

4.3. **The rank of the Poisson structure Π_0.** Recall from Definition 4.4 that Π_0 is the Poisson structure on $\mathcal{L}$ defined by the standard Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}^*_{\rm st}$, where $\mathfrak{g}^*_{\rm st} = \mathfrak{h}_{-\Delta} + (\mathfrak{n} \oplus \mathfrak{n}^-)$. In this section, we will compute the rank of Π_0 on $\mathcal{L}$.

Let $\mathcal{O}$ be a G_Δ-orbit in $\mathcal{L}$ and $\mathcal{O}'$ a $(B \times B^-)$-orbit in $\mathcal{L}$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. Since $(\mathfrak{b} \oplus \mathfrak{b}^-) + \mathfrak{g}_\Delta = \mathfrak{g} \oplus \mathfrak{g}$, $\mathcal{O}$ and $\mathcal{O}'$ intersect transversally in their $(G \times G)$-orbit. Moreover, since both $\mathcal{O}$ and $\mathcal{O}'$ are Poisson submanifolds for Π_0, the intersection $\mathcal{O} \cap \mathcal{O}'$ is a Poisson submanifold of $(\mathcal{L}, \Pi_0)$. Thus, it is enough to compute the rank of Π_0 as a Poisson structure in the intersection $\mathcal{O} \cap \mathcal{O}'$. By Theorem 2.18, there exists a generalized Belavin-Drinfeld triple (S, T, d) and $V \in \mathcal{L}_{\rm space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$ such that $\mathcal{O}, \mathcal{O}' \subset (G \times G) \cdot \mathfrak{l}_{S,T,d,V}$ with $\mathfrak{l}_{S,T,d,V}$ given in (2.4). By Corollaries 2.23 and 3.13, there exist $w \in W$, $v, v_1 \in W^T$, and $m \in M_{S(v,d)}$ such that

$$\mathcal{O} = G_\Delta \cdot \mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V} \tag{4.6}$$

$$\mathcal{O}' = (B \times B^-) \cdot \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,V} \tag{4.7}$$

where $\dot{w}, \dot{v}$ and $\dot{v}_1$ are representatives of w, v, and v_1 in G respectively. Set

$$X_{S,T,d,v} = \{(z, v^{-1}z) : z \in \mathfrak{z}_{S(v,d)}, \gamma_d(\chi_S(z)) = \chi_T(v^{-1}z)\} + V_S \subset \mathfrak{h} \oplus \mathfrak{h} \tag{4.8}$$

with $V_S = \{(x, \gamma_d(x)) : x \in \mathfrak{h}_S\}$. One can show directly that $X_{S,T,d,v}$ is a Lagrangian subspace of $\mathfrak{h} \oplus \mathfrak{h}$.

Theorem 4.10. *Let $\mathcal{O}$ and $\mathcal{O}'$ be as in (4.6) and (4.7), and suppose that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. The rank of Π_0 at every $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$ is equal to*

$$\dim(\mathcal{O} \cap \mathcal{O}') - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}),$$

where $X_{S,T,d,v}$ is given in (4.8). In particular, the intersection $\mathcal{O} \cap \mathcal{O}'$ is a regular Poisson submanifold of Π_0

Proof. Let $\mathfrak{l}$ be an arbitrary point in $\mathcal{L}$. Let $\mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l})$ be the normalizer of $\mathfrak{l}$ in $\mathfrak{g} \oplus \mathfrak{g}$, and let

$$\mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l})^\perp = \{(y, z) \in \mathfrak{g} \oplus \mathfrak{g} : \langle (y, z), \mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l}) \rangle = 0\}.$$

Set

$$\mathcal{T}(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l})_\Delta + \mathfrak{n}_{\mathfrak{g}\oplus\mathfrak{g}}(\mathfrak{l})^\perp. \tag{4.9}$$

By Lemma 4.3, the rank of Π_0 at $\mathfrak{l}$ is equal to $\dim(G_\Delta \cdot \mathfrak{l}) - \dim(\mathfrak{g}^*_{\rm st} \cap \mathcal{T}(\mathfrak{l}))$.

Let now $\mathfrak{l} = \mathrm{Ad}_{(g,g)}\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V} \in \mathcal{O}$, where $g \in G$. It is easy to see from (4.9) that $\mathcal{T}(\mathfrak{l}) = \mathrm{Ad}_{(g,g)}\mathcal{T}(\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V})$. Let $\mathfrak{r}'_{S,T,d} = (\mathfrak{n}_S \oplus \mathfrak{n}^-_T) + \{(x, \gamma_d(x)) : x \in \mathfrak{g}_S\}$, and let

$$\mathfrak{l}_{S,T,d,v} = X_{S,T,d,v} + \mathfrak{r}'_{S,T,d}. \tag{4.10}$$

By (4.9) and Theorem 3.15,

$$\begin{aligned} \mathcal{T}(\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}) &= \mathfrak{g}_\Delta \cap \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}_{S,T,d} + \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}'_{S,T,d} \\ &= (\mathfrak{z}'_{S(v,d)})_\Delta + \left(\mathfrak{g}^\phi_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta + \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}'_{S,T,d}. \end{aligned}$$

Since $\mathrm{Ad}^{-1}_{(m,\dot{v})}\left(\mathfrak{g}^\phi_{S(v,d)} + \psi(\mathfrak{n}_v)\right)_\Delta \subset \mathfrak{r}'_{S,T,d}$, we have

$$\begin{aligned} \mathcal{T}(\mathrm{Ad}_{(m,\dot{v})}\mathfrak{l}_{S,T,d,V}) &= \mathrm{Ad}_{(m,\dot{v})}\left(\mathrm{Ad}^{-1}_{(m,\dot{v})}(\mathfrak{z}'_{S(v,d)})_\Delta + \mathfrak{r}'_{S,T,d}\right) \\ &= \mathrm{Ad}_{(m,\dot{v})}(\mathfrak{l}_{S,T,d,v}) \subset \mathrm{Ad}_{(m,\dot{v})}\mathfrak{r}_{S,T,d}. \end{aligned}$$

Thus the rank of Π_0 at $\mathfrak{l}$ is equal to

$$\mathrm{Rank}_{\Pi_0}(\mathfrak{l}) = \dim \mathcal{O} - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}).$$

Let

$$\delta = \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d}) - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}).$$

Then

$$\mathrm{Rank}_{\Pi_0}(\mathfrak{l}) = \dim \mathcal{O} + \delta - \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d}).$$

Since

$$\dim \mathcal{O}' = \dim(\mathfrak{b} \oplus \mathfrak{b}^-) - \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d}),$$

we have

$$\mathrm{Rank}_{\Pi_0}(\mathfrak{l}) = \dim \mathcal{O} + \dim \mathcal{O}' + \delta - \dim(\mathfrak{b} \oplus \mathfrak{b}^-) = \dim \mathcal{O} + \dim \mathcal{O}' + \delta - 2\dim \mathfrak{b}.$$

Since $\mathcal{O}$ and $\mathcal{O}'$ intersect transversally at $\mathfrak{l}$ inside the $(G\times G)$-orbit through $\mathfrak{l}$, and since $\dim(G \times G)\cdot \mathfrak{l} = \dim \mathfrak{g} - \dim \mathfrak{z}_S$ by Proposition 2.19, we have

$$\begin{aligned} \mathrm{Rank}_{\Pi_0}(\mathfrak{l}) &= \dim(\mathcal{O} \cap \mathcal{O}') + \dim((G \times G)\cdot \mathfrak{l}) + \delta - 2\dim \mathfrak{b} \\ &= \dim(\mathcal{O} \cap \mathcal{O}') - (\dim \mathfrak{z}_S + \dim \mathfrak{h}) + \delta. \end{aligned}$$

It remains to compute δ. Since $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$, there exist $r \in R_{S,T,d}$ and $(b, b^-) \in B \times B^-$ such that $(gm, g\dot{v}) = (b, b^-)(\dot{w}, \dot{v}_1)r$. Thus, using $\mathrm{Ad}_{(b,b^-)}(\mathfrak{b} \oplus \mathfrak{b}^-) = \mathfrak{b} \oplus \mathfrak{b}^-$ and $\mathrm{Ad}_{(b,b^-)}\mathfrak{g}_{\mathrm{st}^*} = \mathfrak{g}_{\mathrm{st}^*}$, we have

$$\delta = \dim((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{r}_{S,T,d}) - \dim(\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}).$$

Set

$$Y = (\mathfrak{n} \oplus \mathfrak{n}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\left((\mathfrak{n}_S \oplus \mathfrak{n}_T^-) + \mathrm{span}_{\mathbb{C}}\{(E_\alpha, \gamma_d(E_\alpha)) : \alpha \in [S]\}\right).$$

Then

$$(\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{r}_{S,T,d} = (w, v_1)(\mathfrak{z}_S \oplus \mathfrak{z}_T + V_S) + Y.$$

Since $Y \subset \mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}$, we have

$$\mathfrak{g}_{\mathrm{st}}^* \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v} = Y + \mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}.$$

Thus

$$\begin{aligned} \delta &= \dim(\mathfrak{z}_S \oplus \mathfrak{z}_T + V_S) - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}) \\ &= \dim \mathfrak{z}_S + \dim \mathfrak{h} - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}). \end{aligned}$$

Thus the rank of Π_0 at $\mathfrak{l}$ is equal to

$$\dim(\mathcal{O} \cap \mathcal{O}') - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}).$$

In particular, $\mathcal{O} \cap \mathcal{O}'$ is a regular Poisson manifold for Π_0.

Q.E.D.

Corollary 4.11. *Equip G with the Poisson structure Π_0 via the embedding of G into $\mathcal{L}$ in (4.4) for $d = 1$. Let C be a conjugacy class of in G and let $w \in W$ be such that $C \cap (B^-wB) \neq \emptyset$. Then the rank of Π_0 at every point in $C \cap (B^-wB)$ is*

$$\dim C - l(w) - \dim(\mathfrak{h}^{-w}),$$

where $l(w)$ is the length of w, and $\mathfrak{h}^{-w} = \{x \in \mathfrak{h} : w(x) = -x\}$. In particular, $C \cap B^-B$ is an open dense leaf for C, and Π_0 is degenerate on the complement of $B^-B \cap C$ in C.

Proof. By Proposition 4.7, each conjugacy class of G is a Poisson submanifold of (G, Π_0). By the Bruhat decomposition

$$C = \cup_{w \in W} (C \cap (B^- wB)).$$

Since $B^- B$ is open in G and $C \cap B \neq \emptyset$ (Theorem 1 on P. 69 of [St]), it follows that $C \cap B^- B$ is open and dense in C. The rank formula follows from Theorem 4.10, and it follows easily that $C \cap B^- B$ is a symplectic leaf and Π_0 is degenerate on $C \cap (B^- wB)$ if $w \neq e$.

Q.E.D.

Remark 4.12. Let $d = 1$ in Corollary 4.11. Then any unipotent conjugacy class (and its closure in $Z_1(G)$) has an induced Poisson structure Π_0 with an open symplectic leaf, although the structure is not symplectic unless the orbit is a single point. Since the unipotent variety is isomorphic to the nilpotent cone in $\mathfrak{g}^*$, it follows that every nilpotent orbit in $\mathfrak{g}^*$ has an induced Poisson structure with the same properties. It would be quite interesting to compare this structure with the Kirillov-Kostant symplectic structure.

Remark 4.13. Let σ be an involutive automorphism as in Example 4.8. Then the G_Δ orbit through σ does not have an open symplectic leaf if σ is not inner. The leaves of maximal rank have dimension $\dim(G/G^\sigma) - \dim(\mathfrak{h}^{-\gamma_d})$.

Example 4.14. Consider the closed $(G \times G)$-orbit through a Lagrangian subalgebra of the form $V + (\mathfrak{n} \oplus \mathfrak{n}^-)$, where V is any Lagrangian subspace of $\mathfrak{h} \oplus \mathfrak{h}$. Such an orbit can be identified with $G/B \times G/B^-$, so we can regard Π_0 as a Poisson structure on $G/B \times G/B^-$. Let $\mathcal{O}$ be a G_Δ-orbit and let $\mathcal{O}'$ be a $(B \times B^-)$-orbit in $G/B \times G/B^-$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. By the Bruhat decomposition of G, there are elements $w, u, v \in W$ such that

$$\mathcal{O} = G_\Delta \cdot (B, wB^-), \qquad \mathcal{O}' = (B \times B^-) \cdot (uB, vB^-).$$

The stabilizer subgroup of $G_\Delta \cong G$ at the point $(B, \dot{w}B^-) \in G/B \times G/B^-$ is $B \cap w(B^-)$. Identify $\mathcal{O} \cong G/(B \cap w(B^-))$, and let

$$p: \ G \longrightarrow \mathcal{O} \ \cong \ G/(B \cap w(B^-))$$

be the projection. It is then easy to see that $\mathcal{O} \cap \mathcal{O}' = p(G_w^{u,v}) \subset \mathcal{O}$, where

$$G_w^{u,v} = (BuB) \cap (B^- vB^- w^{-1}).$$

We will refer to $G_w^{u,v}$ as the *shifted double Bruhat cell* in G determined by u, v and w. Note that $B \cap w(B^-)$ acts freely on $G_w^{u,v}$ by right multiplications, so

$$\mathcal{O} \cap \mathcal{O}' \ \cong \ G_w^{u,v}/(B \cap w(B^-)).$$

Since $\dim \mathcal{O} = \dim \mathfrak{g} - \dim \mathfrak{h} - l(w)$ and $\dim \mathcal{O}' = l(u) + l(v)$, we have

$$\dim(\mathcal{O} \cap \mathcal{O}') = \dim \mathcal{O} + \dim \mathcal{O}' - \dim(G/B \times G/B^-) = l(u) + l(v) - l(w),$$

and

$$\dim G_w^{u,v} = l(u) + l(v) + \dim \mathfrak{h}.$$

By Theorem 4.10, the rank of Π_0 at every point of $\mathcal{O} \cap \mathcal{O}'$ is equal to

$$l(u) + l(v) - l(w) - \dim \mathfrak{h}^{-u^{-1}vw^{-1}},$$

where $\mathfrak{h}^{-u^{-1}vw^{-1}} = \{x \in \mathfrak{h} : u^{-1}vw^{-1}x = -x\}$. When $w = 1$, we have $\mathcal{O} \cong G/H$, and $\mathcal{O} \cap \mathcal{O}' \cong G^{u,v}/H$, where $G^{u,v} = G_1^{u,v}$ is the double Bruhat cell in G determined by u and v. The set $G^{u,v}/H$ is called a *reduced double Bruhat cell* in [Z]. In [K-Z], Kogan and Zelevinsky constructed toric charts on symplectic leaves of Π_0 in $\mathcal{O} \cap \mathcal{O}'$ (for the case when $w = 1$) by using the so-called *twisted minors* that are developed in [Fm-Z], and they also constructed integrable systems on the symplectic leaves. It would very interesting to generalize Kogan-Zelevinsky construction to all symplectic leaves of Π_0 in $G/B \times G/B^-$.

4.4. **The action of H_Δ on the set of symplectic leaves of the Poisson structure Π_0.**

Proposition 4.15. *Let D be a connected complex algebraic group with connected algebraic subgroups A and C. Suppose there exists a connected algebraic subgroup $C_1 \subset C$ such that the multiplication morphism $A \times C_1 \to D$ is an isomorphism to a connected open set U of D. Let X be a homogeneous space for D such that the stabilizer in D of a point in X is connected. Then any nonempty intersection of an A-orbit in X with a C-orbit in X is smooth and connected.*

Proof. Let $A \cdot x \cap C \cdot x$ be a nonempty intersection of orbits in X, and note that this intersection is smooth since the hypotheses imply that the orbits intersect transversely. We show there is a fiber bundle $\pi : V \to U$, with fiber $\pi^{-1}(e) \cong A \cdot x \cap C \cdot x$ over the identity and V connected, that is trivial in the Zariski topology. This implies the connectedness of the intersection, and hence the proposition. The proof is inspired by the proof of Kleiman's transversality theorem.

Let $Y = C \cdot x$ and $Z = A \cdot x$. Let $h : D \times Y \to X$ be the action map and let $i : Z \to X$ be the obvious embedding. Let

$$W = (D \times Y) \times_X Z$$

be the fiber product. Then h is a smooth fiber bundle (see the proof of 10.8 in [Ha]) and the fibers $h^{-1}(x)$ are connected. For the second claim, note that $h^{-1}(x) = \{(d, c \cdot x) : dc \cdot x = x\}$ and $\psi : h^{-1}(x) \to D_x \cdot C$ given by $\psi(d, c \cdot x) = d$ is an isomorphism. Since D_x and C are connected, the claim follows. Thus, the induced morphism from $W \to Z$ also has connected fibers. Since Z is connected, it follows that W is connected. Moreover, W is smooth (again by the proof of 10.8 in [Ha]), so W is irreducible.

Let $\pi : W \to D \times Y \to D$ be the composition of the induced fiber product map with projection to the first factor. Since $\pi^{-1}(U)$ is open in W, it is smooth and irreducible, and thus connected. Note also that $\pi^{-1}(e) \cong Y \cap Z$. It remains to show that $\pi : \pi^{-1}(U) \to U$ is a trivial fiber bundle. We define a free left A action and a free right C_1 action on W by the formulas

$$a \cdot (d, y, z) = (ad, y, a \cdot z)$$
$$c \cdot (d, y, z) = (dc, c^{-1} \cdot y, z)$$
$$a \in A, c \in C, d \in D, y \in Y, z \in Z$$

A and C_1 have the obvious free left and right multiplication actions on U, and $\pi : \pi^{-1}(U) \to U$ is equivariant for these actions. It follows that the morphism

$$\phi : A \times C_1 \times (A \cdot x \cap C \cdot x) \to \pi^{-1}(U),\ (a, c, v) \mapsto (ac, c^{-1} \cdot v, a \cdot v),\ a \in A, c \in C_1, v \in Y \cap Z$$

is a bijection, and hence is an isomorphism since $\pi^{-1}(U)$ is smooth. This implies the fiber bundle is trivial.

Q.E.D.

Remark 4.16. We thank Michel Brion for suggesting this approach. We also remark that the Proposition 4.15 is false as stated if we only assume that $A \cdot C$ is open in D. For example, let $A = G_\Delta$, and let

$$C = \{(nh, h^{-1}n^-) : n \in N, h \in H, n^- \in N^-\}$$

be the connected subgroup of $D = G \times G$ corresponding to $\mathfrak{g}_{\rm st}^*$. Let $X = D$ and let D act on X by left translation. Then the intersection of the A-orbit and the C-orbit through the identity element of D is $A \cap C$ which is disconnected.

Proposition 4.17. *The intersection of any G_Δ-orbit and any $(B \times B^-)$-orbit in $\mathcal{L}$ is either empty or a smooth connected subvariety of $\mathcal{L}$.*

Proof. This is a consequence of Proposition 4.15. Indeed, we take $A = G_\Delta$, $C = B \times B^-$, $D = G \times G$, and $C_1 = B \times N^-$. The fact that the stabilizer of a point in $\mathcal{L}$ is connected follows from Lemma 2.21.

Q.E.D.

Let now H be the Cartan subgroup of G with Lie algebra $\mathfrak{h}$, and let $H_\Delta = \{(h,h) : h \in H\}$. For every G_Δ-orbit $\mathcal{O}$ and every $(B \times B^-)$-orbit $\mathcal{O}'$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, H_Δ clearly leaves $\mathcal{O} \cap \mathcal{O}'$ invariant. It is easy to show that the element $R \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ given in (4.2) is invariant under $\mathrm{Ad}_{(h,h)}$ for every $h \in H$. Thus the Poisson structure Π_0 on $\mathcal{L}$ is H_Δ-invariant. In particular, for every $h \in H$, $\mathrm{Ad}_{(h,h)}\mathcal{E}$ is a symplectic leaf of Π_0 in $\mathcal{O} \cap \mathcal{O}'$ if $\mathcal{E}$ is.

Lemma 4.18. *Let $\mathcal{O}$ be a G_Δ-orbit and $\mathcal{O}'$ a $(B \times B^-)$-orbit in $\mathcal{L}$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$. Let $\mathcal{E}$ be any symplectic leaf of Π_0 in $\mathcal{O} \cap \mathcal{O}'$. Then the map*

$$\sigma : \ H \times \mathcal{E} \longrightarrow \mathcal{O} \cap \mathcal{O}' : \ (h, \mathfrak{l}) \longmapsto \mathrm{Ad}_{(h,h)}\mathfrak{l}$$

is a submersion.

Proof. Let e be the identity element of H and let $\mathfrak{l} \in \mathcal{E}$. It is enough to show that

$$\dim \ker \sigma_*(e, \mathfrak{l}) = \dim \mathfrak{h} + \dim \mathcal{E}_\mathfrak{l} - \dim \mathcal{O} \cap \mathcal{O}',$$

where $\sigma_*(e,\mathfrak{l}) : \mathfrak{h} \times T_\mathfrak{l}\mathcal{E} \to T_\mathfrak{l}(\mathcal{O} \cap \mathcal{O}')$ is the differential of σ at $(e, \mathfrak{l})$.

We may assume that $\mathcal{O}$ and $\mathcal{O}'$ are respectively given by (4.6) and (4.7), and that

$$\mathfrak{l} = \mathrm{Ad}_{(gm, g\dot{v})}\mathfrak{l}_{S,T,d,V} = \mathrm{Ad}_{(b\dot{w}, b^-\dot{v}_1)}\mathfrak{l}_{S,T,d,V}$$

for some $g \in G$ and $(b, b^-) \in B \times B^-$. By Theorem 4.10, it is enough to show that

$$\dim(\ker \sigma_*(e, \mathfrak{l})) = \dim \mathfrak{h} - \dim(\mathfrak{h}_{-\Delta} \cap (w, v_1)X_{S,T,d,v}),$$

where $X_{S,T,d,v}$ is given in (4.8). Identify the tangent space of $\mathcal{O}$ at $\mathfrak{l}$ as

$$T_\mathfrak{l}\mathcal{O} \cong \mathfrak{g}_\Delta/(\mathfrak{g}_\Delta \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d})$$

and let $q : \mathfrak{g}_\Delta \to \mathfrak{g}_\Delta/(\mathfrak{g}_\Delta \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{r}_{S,T,d})$ be the projection. Let $p : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}_\Delta$ be the projection with respect to the decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_\Delta + \mathfrak{g}_{\rm st}^*$. By (4.3) and the computation of $\mathcal{T}(\mathfrak{l})$ in the proof of Theorem 4.10, the tangent space of $\mathcal{E}$ at $\mathfrak{l}$ is given by

$$T_\mathfrak{l}\mathcal{E} = (q \circ p)\left(\mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}\right),$$

where $\mathfrak{l}_{S,T,d,v}$ is given in (4.10). For $x \in \mathfrak{h}$, let κ_x be the vector field on $\mathcal{O} \cap \mathcal{O}'$ that generates the action of $\mathrm{Ad}_{(\exp tx, \exp tx)}$. Then

$$\ker \sigma_*(e, \mathfrak{l}) \cong \{x \in \mathfrak{h} : \kappa_x(\mathfrak{l}) \in T_\mathfrak{l}\mathcal{E}\}.$$

Let $x \in \mathfrak{h}$. If $\kappa_x(\mathfrak{l}) \in T_\mathfrak{l}\mathcal{E}$, then there exists $y \in \mathfrak{g}$ and $(y_1, y_2) \in \mathfrak{g}_{\rm st}^*$ with $(y + y_1, y + y_2) \in \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}$ such that

$$(x-y, x-y) \in \mathfrak{g}_\Delta \cap \mathrm{Ad}_{(gm, g\dot{v})}\mathfrak{r}_{S,T,d} \subset \mathfrak{g}_\Delta \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}.$$

It follows that

$$(x+y_1, x+y_2) \in (\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}.$$

Let $r \in R_{S,T,d}$ be such that $(gm, g\dot{v}) = (b\dot{w}, b^-\dot{v}_1)r$. Then

$$(\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v} = \mathrm{Ad}_{(b,b^-)}\left((\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}\right).$$

Thus there exists $(y_1', y_2') \in \mathfrak{g}_{\mathrm{st}}^*$ such that

$$(x+y_1', x+y_2') \in (\mathfrak{b} \oplus \mathfrak{b}^-) \cap \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v}.$$

If $(y', -y')$ is the $\mathfrak{h}_{-\Delta}$-component of $(y_1', y_2') \in \mathfrak{g}_{\mathrm{st}}^*$, we see that

$$(x+y', x-y') \in (w, v_1)X_{S,T,d,v}.$$

Thus $(x,x) \in p((w,v_1)X_{S,T,d,v})$, where we are also using p to denote the projection $\mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h}_\Delta$ with respect to the decomposition $\mathfrak{h} \oplus \mathfrak{h} = \mathfrak{h}_\Delta + \mathfrak{h}_{-\Delta}$. Conversely, if $x \in \mathfrak{h}$ is such that $(x,x) \in p((w,v_1)X_{S,T,d,v})$, then there exists $y' \in \mathfrak{h}$ such that

$$(x+y', x-y') \in (w,v_1)X_{S,T,d,v} \subset \mathrm{Ad}_{(\dot{w},\dot{v}_1)}\mathfrak{l}_{S,T,d,v},$$

and thus

$$\mathrm{Ad}_{(b,b^-)}(x+y', x-y') \in \mathrm{Ad}_{(gm,g\dot{v})}\mathfrak{l}_{S,T,d,v}.$$

Since $\mathrm{Ad}_{(b,b^-)}(x+y', x-y') = (x+y', x-y') \mathrm{mod}(\mathfrak{n} \oplus \mathfrak{n}^-)$, we see that

$$p(\mathrm{Ad}_{(b,b^-)}(x+y', x-y')) = (x,x),$$

so $\kappa_x(\mathfrak{l}) \in T_\mathfrak{l}\mathcal{E}$. Thus we have shown that

$$\ker \sigma_*(e, \mathfrak{l}) \cong \{x \in \mathfrak{h} : (x,x) \in p((w,v_1)X_{S,T,d,v})\}.$$

Hence,

$$\dim(\ker \sigma_*(e,\mathfrak{l})) = \dim \mathfrak{h} - \dim(\mathfrak{h}_{-\Delta} \cap (w,v_1)X_{S,T,d,v}).$$

The lemma now follows from Theorem 4.10.

Q.E.D.

Theorem 4.19. *For every G_Δ-orbit $\mathcal{O}$ and every $(B \times B^-)$-orbit $\mathcal{O}'$ such that $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, H_Δ acts transitively on the set of symplectic leaves of Π_0 in $\mathcal{O} \cap \mathcal{O}'$.*

Proof. For $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$, let $\mathcal{E}_\mathfrak{l}$ be the symplectic leaf of Π_0 through $\mathfrak{l}$, and let

$$\mathcal{F}_\mathfrak{l} = \bigcup_{h \in H} \mathrm{Ad}_{(h,h)}\mathcal{E}_\mathfrak{l} \subset \mathcal{O} \cap \mathcal{O}'.$$

Then it is easy to see that either $\mathcal{F}_\mathfrak{l} \cap \mathcal{F}_{\mathfrak{l}'} = \emptyset$ or $\mathcal{F}_\mathfrak{l} = \mathcal{F}_{\mathfrak{l}'}$ for any $\mathfrak{l}, \mathfrak{l}' \in \mathcal{O} \cap \mathcal{O}'$. It follows from Lemma 4.18 that $\mathcal{F}_\mathfrak{l}$ is open in $\mathcal{O} \cap \mathcal{O}'$ for every $\mathfrak{l}$. Since $\mathcal{O} \cap \mathcal{O}'$ is connected by Proposition 4.17, $\mathcal{O} \cap \mathcal{O}' = \mathcal{F}_\mathfrak{l}$ for every $\mathfrak{l} \in \mathcal{O} \cap \mathcal{O}'$.

Q.E.D.

Remark 4.20. When $\mathcal{O}$ and $\mathcal{O}'$ are respectively given as in (4.6) and (4.7), we take any subspace $\mathfrak{h}_1$ of $\mathfrak{h}$ such that $(\mathfrak{h}_1)_\Delta$ is transversal to $p((w,v_1)X_{\mathcal{S},v})$ in $\mathfrak{h}_\Delta$ and such that the connected subgroup H_1 of H with Lie algebra $\mathfrak{h}_1$ is closed in H. By the proofs of Lemma 4.18 and Theorem 4.19, the subtorus H_1 already acts transitively on the set of symplectic leaves of Π_0 in $\mathcal{O} \cap \mathcal{O}'$.

5. Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{h}$

Let again $\mathfrak{g}$ be a complex semi-simple Lie algebra with Killing form $\ll\,,\,\gg$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. In this section, we will consider the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$, together with the symmetric, non-degenerate, and ad-invariant bilinear form

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \ll x_1, x_2 \gg - \ll y_1, y_2 \gg, \qquad x_1, x_2 \in \mathfrak{g}, y_1, y_2 \in \mathfrak{h}. \tag{5.1}$$

We wish to describe the variety $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{h}$ with respect to $\langle\,,\,\rangle$. We can describe all such Lagrangian subalgebras by using a theorem of Delorme [De].

Definition 5.1. [De] Let $\mathfrak{m}$ be a complex reductive Lie algebra with simple factors $\mathfrak{m}_i, i \in I$. A complex linear involution σ of $\mathfrak{m}$ is called an f-involution if σ does not preserve any $\mathfrak{m}_i$.

Theorem 5.2. [De] *Let $\mathfrak{u}$ be a complex reductive Lie algebra with a symmetric, non-degenerate, and ad-invariant bilinear form β.*

1). Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{u}$ with Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$, and decompose $\mathfrak{m}$ into $\mathfrak{m} = \overline{\mathfrak{m}} + \mathfrak{z}$, where $\overline{\mathfrak{m}}$ is its semisimple part and $\mathfrak{z}$ its center. Let σ be an f-involution of $\overline{\mathfrak{m}}$ such that $\overline{\mathfrak{m}}^\sigma$ is a Lagrangian subalgebra of $\overline{\mathfrak{m}}$ with respect to the restriction of β, and let V be a Lagrangian subspace of $\mathfrak{z}$ with respect to the restriction of β. Then $\mathfrak{l}(\mathfrak{p}, \sigma, V) := \overline{\mathfrak{m}}^\sigma \oplus V \oplus \mathfrak{n}$ is a Lagrangian subalgebra of $\mathfrak{u}$ with respect to β.

2). Every Lagrangian subalgebra of $\mathfrak{u}$ is $\mathfrak{l}(\mathfrak{p}, \sigma, V)$ for some $\mathfrak{p}$, σ, and V as in 1).

Proposition 5.3. *Every Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$ with respect to $\langle\,,\,\rangle$ given in (5.1) is of the form $\mathfrak{n} + V$, where $\mathfrak{n}$ is the nilradical of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, V is a Lagrangian subspace of $\mathfrak{h} \oplus \mathfrak{h}$, and*

$$\mathfrak{n} + V = \{(x + y_1, y_2) : \, x \in \mathfrak{n}, (y_1, y_2) \in V\}.$$

Proof. Applying Delorme's theorem to our case of $\mathfrak{u} = \mathfrak{g} \oplus \mathfrak{h}$ and $\langle\,,\,\rangle$ as the bilinear form β, every Lagrangian subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$ is of the form

$$\mathfrak{l} = \{(x + y_1, y_2) : \, x \in \overline{\mathfrak{m}}^\sigma + \mathfrak{n}, \, (y_1, y_2) \in V\}$$

for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n} = \overline{\mathfrak{m}} + \mathfrak{z} + \mathfrak{n}$, an f-involution σ on $\overline{\mathfrak{m}}$, and a Lagrangian subspace V of $\mathfrak{z} \oplus \mathfrak{h}$. we will now show that if $\overline{\mathfrak{m}} \neq 0$ and if σ is an f-involution of $\overline{\mathfrak{m}}$, then $\overline{\mathfrak{m}}^\sigma$ is not an isotropic subspace of $\overline{\mathfrak{m}}$ for the restriction of the Killing form $\ll\,,\,\gg$ of $\mathfrak{g}$ to $\overline{\mathfrak{m}}$. It follows that $\mathfrak{p}$ must be Borel, which gives Proposition 5.3.

Assume that $\overline{\mathfrak{m}} \neq 0$. Let $\mathfrak{m}_i$ be a simple factor of $\overline{\mathfrak{m}}$. Then since $\mathfrak{m}_i$ is simple, it has a unique nondegenerate invariant form up to scalar multiplications. Hence the Killing form $\ll\,,\,\gg$ of $\mathfrak{g}$ restricts to a scalar multiple of the Killing form of $\mathfrak{m}_i$. Recall that the Killing form on a maximal compact subalgebra of a semisimple Lie algebra is negative definite. It follows that the Killing form of $\mathfrak{g}$ restricts to a nonzero positive scalar multiple of the Killing form on $\mathfrak{m}_i$. Suppose that σ is an involution of $\overline{\mathfrak{m}}$ mapping $\mathfrak{m}_i$ to $\mathfrak{m}_j$ with $i \neq j$. Then σ is an isometry with respect to the Killing form of $\mathfrak{m}_i$ and the Killing form of $\mathfrak{m}_j$. Thus, there exists a nonzero positive scalar μ such that

$$\ll \sigma(x), \sigma(y) \gg = \mu \ll x, y \gg, \quad \forall\, x, y \in \mathfrak{m}_i. \tag{5.2}$$

The fixed point set $\overline{\mathfrak{m}}^\sigma$ contains the subspace $\{x + \sigma(x) : x \in \mathfrak{m}_i\}$. Let x be a nonzero element of a maximal compact subalgebra of $\mathfrak{m}_i$. Then $\ll x + \sigma(x), x + \sigma(x) \gg = (1 + \mu) \ll x, x, \gg \neq 0$. Thus $\overline{\mathfrak{m}}^\sigma$ cannot be isotropic with respect to $\ll\, , \gg$.

Q.E.D.

Now let G be the adjoint group of $\mathfrak{g}$, and let B be the Borel subgroup of G corresponding to a Borel subalgebra $\mathfrak{b}$.

Theorem 5.4. *The variety $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ is isomorphic to the trivial fiber bundle over G/B with fibre $\mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h}, \langle\, , \rangle)$. In particular, $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ is smooth with two disjoint irreducible components, corresponding to the two connected components of $\mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h}, \langle\, , \rangle)$.*

Proof. Identify G/B with the variety of all Borel subalgebras of $\mathfrak{g}$. We map $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{h})$ to G/B by mapping a Lagrangian algebra $\mathfrak{l} = \mathfrak{n} + V$ to the unique Borel subalgebra with nilradical $\mathfrak{n}$. The fiber over $\mathfrak{n}$ may be identified with $\mathcal{L}_{\text{space}}(\mathfrak{h} \oplus \mathfrak{h}, \langle\, , \rangle)$. The claim about connected components follows from the fact the bundle is trivial.

Q.E.D.

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Department of Mathematics, University of Notre Dame, Notre Dame, 46556
Department of Mathematics, University of Hong Kong, Pokfulam Road, Hong Kong

E-mail address: evens.1@nd.edu, jhlu@maths.hku.hk