On the Existence of Periodic Solutions for p-Laplacian Generalized Liénard Equation

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Abstract—By employing Mawhin's continuation theorem, the existence of periodic solutions of the p-Laplacian generalized Liénard equation with deviating argument

$$(\varphi_p(x'(t)))' + f(t, x(t))x'(t) + \beta(t)g(x(t - \tau(t))) = e(t)$$

under various assumptions are obtained.

Keywords—periodic solution, Mawhin's continuation theorem, deviating argument.

1. INTRODUCTION

Consider the p-Laplacian generalized Liénard equation with a deviating argument

$$(\varphi_p(x'(t)))' + f(t, x(t))x'(t) + \beta(t)g(x(t - \tau(t))) = e(t), \tag{1.1}$$

where $p > 1$ is a constant; $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$ is a one-dimensional p-Laplacian; $f = f(t, u) \in C(\mathbb{R}^2, \mathbb{R})$ is a periodic function with regard to t with period $T > 0$; and $\beta, g, e, \tau \in C(\mathbb{R}, \mathbb{R})$, where β, τ, e are periodic functions with period T, $e(t) \not\equiv 0$, $\int_0^T e(s)ds = 0$, $\beta(t) > 0$ and $\tau(t) \geq 0$ for $t \in [0, T]$.

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There has been a great deal of work in the literature on such an equation which is used to describe fluid mechanical and nonlinear elastic mechanical phenomena. For example, in [1-3, 6, 10], by using the time maps and the phase plane analysis, the existence of periodic solutions to Eq.(1.1) for $p \neq 2$ and $\tau(t) \equiv 0$ was studied. On the other hand, for $p = 2$, $\tau(t) \not\equiv 0$ and $f(t, x(t))$ being replaced by $f(x(t))$, the existence of $T-$periodic solutions to several second order scalar differential equations were also studied in [5, 7-9]. In [9], S. Ma, Z. Wang and J. Yu studied delay Duffing equations of the type

$$x''(t) + m^2 x(t) + g(x(t - \tau)) = p(t). \tag{1.2}$$

By assuming that

$$\sup_{x \in \mathbb{R}} |g(x)| < \infty, \tag{1.3}$$

several sufficient conditions for the existence of periodic solutions of Eq.(1.2) were established. Recently, S. Lu and W. Ge in [7] discussed the existence of periodic solutions for the second order differential equation with multiple deviating arguments

$$x''(t) + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t) g(x(t - \gamma_j(t))) = p(t). \tag{1.4}$$

In their work, some linear growth condition imposed on $g(x)$ such as

$$\lim_{|x| \to +\infty} \frac{|g(x)|}{|x|} = r \in [0, +\infty). \tag{1.5}$$

was needed.

The main technique of these works [5, 7-9] is to convert the problem into the abstract form $Lx = Nx$, with L being a non-invertible linear operator. Thus the existence of solutions of the problem can be given by Mawhin's continuation theorem [4]. But as far as we are aware of, the corresponding problem of Eq.(1.1) with $p \neq 2$ and $\tau(t) \not\equiv 0$ has never been studied. This is mainly due to the facts that in this situation, on the one hand Mawhin's continuation theorem is not applicable directly since the p-Laplacian $\varphi_p(u) = |u|^{p-2}u$ is not linear with respect to u except when $p = 2$, and on the other hand, the crucial step $\int_0^T f(x(t))x'(t)dt = 0$ which is needed to obtain an *a priori* bound of periodic solutions for Eq.(1.1) is no longer valid.

In this paper, we get around with these difficulties by using some new techniques and translating Eq.(1.1) into a two-dimensional system on which Mawhin's continuation theorem applies. This method can also be used to solve problems for other equations with p-Laplacian.

2. PRELIMINARIES

Let X and Y be real Banach Spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\ L$ is closed in Y and $\dim Ker\ L = \dim(Y/Im\ L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = Ker\ L \oplus X_1$ and $Y = Im\ L \oplus Y_1$ and let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $Ker\ L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be *L-compact* in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact. We first recall the famous Mawhin's continuation theorem.

THEOREM 2.1[4] Suppose that X and Y are Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is $L-$ compact on $\overline{\Omega}$. If

(1)$Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(2)$Nx \notin Im\ L, \forall x \in \partial\Omega \cap Ker\ L$; and

(3)$deg\{JQN, \Omega \cap Ker\ L, 0\} \neq 0$, where $J : Im\ Q \to Ker\ L$ is an isomorphism,

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

The next result is useful in obtaining an *a priori* bound of periodic solutions.

THEOREM 2.2[7] Let $0 \leq \alpha \leq T$ be a constant, $s \in C(\mathbb{R}, \mathbb{R})$ be periodic with period T, and $\max_{t\in[0,T]} |s(t)| \leq \alpha$. Then for any $u \in C^1(\mathbb{R}, \mathbb{R})$ which is periodic with period T, we have

$$\int_0^T |u(t) - u(t - s(t))|^2 dt \leq 2\alpha^2 \int_0^T |u'(t)|^2 dt.$$

3. MAIN RESULTS

In order to use Mawhin's continuation theorem to study the existence of $T-$periodic solutions for Eq.(1.1), we rewrite Eq.(1.1) in the following form

$$\begin{cases} x_1'(t) &= \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\ x_2'(t) &= -f(t, x_1(t))\varphi_q(x_2(t)) - \beta(t)g(x_1(t - \tau(t))) + e(t), \end{cases} \tag{3.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a $T-$periodic solution to Eq.(3.1), then $x(t)$ must be a $T-$periodic solution to Eq.(1.1). Thus, the problem of finding a $T-$periodic solution for Eq. (1.1) reduces to finding one for Eq. (3.1).

Now, we set $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t+T) \equiv \phi(t)\}$ with norm $|\phi|_0 = \max_{t\in[0,T]} |\phi(t)|$. It is obvious that $\beta, \tau, e \in C_T$. Set $X = Y = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t) \equiv x(t + T)\}$ with norm $||x|| = \max\{|x_1|_0, |x_2|_0\}$. Clearly, X and Y are Banach spaces. Define

$$L : D(L) = \{x = (x_1(\cdot), x_2(\cdot)) \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t) \equiv x(t + T)\} \subset X \to Y$$

by

$$Lx := x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

and

$$N : X \to Y$$

by

$$Nx := \begin{pmatrix} \varphi_q(x_2) \\ -f(t, x_1(t))\varphi_q(x_2(t)) - \beta(t)g(x_1(t-\tau(t))) + e(t) \end{pmatrix}.$$

It is easy to see that $Ker\ L = \mathbb{R}^2$ and $Im\ L = \{y \in Y : \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q \subset \mathbb{R}^2$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds; \quad Qy = \frac{1}{T}\int_0^T y(s)ds\ ,$$

and let K denote the inverse of $L|_{KerP\cap D(L)}$. Obviously, $KerL = ImQ = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)ds\ , \tag{3.2}$$

where

$$G(t,s) := \begin{cases} \frac{s}{T}, & 0 \le s < t \le T. \\ \frac{s-T}{T}, & 0 \le t \le s \le T. \end{cases}$$

From (3.2), one can easily see that N is $L-$compact on $\overline{\Omega}$, where Ω is an open, bounded subset of X.

For the sake of convenience, we denote by $\beta_1 = \max\limits_{t\in[0,T]} \beta(t)$, $\beta_0 = \min\limits_{t\in[0,T]} \beta(t)$. Obviously $\beta_1 \ge \beta_0 > 0$. Moreover, we list the following assumptions which will be used repeatedly in the sequel.

[H1] There is a constant $r \ge 0$ such that $\lim\limits_{|x|\to+\infty} \sup|\frac{g(x)}{x}| \le r$.

[H2] There is a constant $A > 0$ such that $\mathrm{sgn}(x)g(x) > \frac{|e|_0}{\beta_0}$ for $|x| > A$.

[H3] There is a constant $\sigma > 0$ such that $\inf\limits_{(t,u)\in[0,T]\times\mathbb{R}} |f(t,u)| \ge \sigma > 0$.

[H4] There exist an integer m and a constant $\delta \ge 0$ such that $\max\limits_{t\in[0,T]} |\tau_1(t) - mT| \le \delta$.

[H5] There exists a constant $l > 0$ such that $|g(u) - g(v)| \le l|u - v|$.

THEOREM 3.1 If [H1]-[H3] hold, then Eq.(1.1) has at least a non-constant T-periodic solution if $r < \frac{\sigma}{\beta_1 T}$.

PROOF. Consider the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1)\ . \tag{3.3}$$

Let $\Omega_1 \in \{x \in X : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1$, then from (3.3), we have

$$\begin{cases} x_1'(t) = \lambda\varphi_q(x_1(t)) = \lambda|x_1(t)|^{q-2}x_1(t) \\ x_2'(t) = -\lambda f(t, x_1(t))\varphi_q(x_2(t)) - \lambda\beta(t)g(x_1(t-\tau_1(t))) + \lambda e(t)\ . \end{cases} \tag{3.4}$$

We first claim that there is a constant $\xi \in R$ such that

$$|x(\xi)| \le A. \tag{3.5}$$

In view of $\int_0^T x_1'(t)dt = 0$, we know that there exist two constants $t_1,\ t_2 \in [0,T]$ such that

$$x_1'(t_1) \ge 0, \qquad x_1'(t_2) \le 0. \tag{3.6}$$

From the first equation of (3.4), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}x_1'(t))$. So

$$x_2(t_1) = \frac{1}{\lambda^{p-1}}|x_1'(t_1)|^{p-2}x_1'(t_1) \geq 0 \ ,$$

$$x_2(t_2) = \frac{1}{\lambda^{p-1}}|x_1'(t_2)|^{p-2}x_1'(t_2) \leq 0 \ .$$

Let $t_3,\ t_4 \in [0,T]$ be, respectively, the maximum point and minimum point of $x_2(t)$. Clearly, we have

$$x_2(t_3) \geq 0, \qquad x_2'(t_3) = 0 \ , \tag{3.7}$$

$$x_2(t_4) \leq 0, \qquad x_2'(t_4) = 0 \ . \tag{3.8}$$

From [H3] and by continuity, f will not change sign for $(t,u) \in [0,T]\times\mathbb{R}$. Without loss of generality, suppose $f(t,u) > 0$ for $(t,u) \in [0,T] \times \mathbb{R}$ and upon substitution of (3.7) into the second equation of (3.4), we have

$$-\lambda\beta(t_3)g(x_1(t_3 - \tau(t_3)) + \lambda e(t) = \lambda f(t, x_1(t_3))\varphi_q(x_2(t_3)) \geq 0 \ ,$$

i.e.,

$$g(x_1(t_3, \tau(t_3))) \leq \frac{e(t_3)}{\beta(t_3)} \leq \frac{|e|_0}{\beta_0} \ . \tag{3.9}$$

From (H2) we see that

$$x_1(t_3 - \tau(t_3)) < A \ . \tag{3.10}$$

Similarly, from (3.8) we have

$$g(x_1(t_4 - \tau(t_4))) \geq \frac{e(t_4)}{\beta(t_4)} \geq -\frac{|e|_0}{\beta_0} \ , \tag{3.11}$$

and again by (H2),

$$x_1(t_4 - \tau(t_4)) > -A \ . \tag{3.12}$$

Case (1) If $x_1(t_3 - \tau(t_3)) \in (-A, A)$, define $\xi = t_3 - \tau(t_3)$. Obviously $|x(\xi)| \leq A$.

Case (2) If $x_1(t_3 - \tau(t_3)) < -A$, from (3.12) and the fact that $x(t)$ is a continuous function in $\mathbb{R}$, there exists a constant ξ between $x_1(t_3 - \tau(t_3))$ and $x_1(t_4 - \tau(t_4))$ such that $|x_1(\xi)| = A$.

This proves (3.5).

Next, in view of $\xi \in \mathbb{R}$, there is an integer k and a constant $t_5 \in [0,T]$ such that $\xi = kT + t_5$, hence $|x_1(\xi)| = |x_1(t_5)| \leq A$. So

$$|x_1|_0 \leq A + \int_0^T |x_1(s)|ds \ . \tag{3.13}$$

Substituting $x_2(t) = \varphi_p(\frac{1}{\lambda}x_1'(t))$ into the second equation of (3.4),

$$[\varphi_p(\frac{1}{\lambda}x_1'(t))]' + \lambda f(t, x_1(t))[\varphi_q(\varphi_p(\frac{1}{\lambda}x_1'(t)))] + \lambda\beta(t)g(x_1(t - \tau_1(t))) = \lambda e(t) \ ,$$

i.e.,

$$[\varphi_p(x_1'(t))]' + \lambda^{p-1}f(t, x_1(t))x_1'(t) + \lambda^p\beta(t)g(x_1(t - \tau_1(t))) = \lambda^p e(t) \ . \tag{3.14}$$

Multiplying both sides of Eq.(3.14) by $x_1'(t)$ and integrating over $[0,T]$, we have

$$\int_0^T f(t,x_1(t))[x_1'(t)]^2dt = -\lambda\int_0^T \beta(t)g(x_1(t-\tau_1(t)))x_1'(t)dt + \lambda\int_0^T e(t)x_1'(t)dt \ . \tag{3.15}$$

It follows from [H3] that

$$\begin{aligned}
&\sigma\textstyle\int_0^T |x_1'(t)|^2dt\\
&\le \textstyle\int_0^T |f(t,x_1(t))|[x_1'(t)]^2dt\\
&= |\textstyle\int_0^T f(t,x_1(t))[x_1'(t)]^2dt|\\
&\le |\textstyle\int_0^T \beta(t)g(x_1(t-\tau_1(t)))x_1'(t)dt| + |\int_0^T e(t)x_1'(t)dt|\\
&\le \textstyle\int_0^T |\beta(t)g(x_1(t-\tau_1(t)))x_1'(t)|dt + \int_0^T |e(t)x_1'(t)|dt\\
&\le \beta_1\textstyle\int_0^T |g(x_1(t-\tau_1(t)))||x_1'(t)|dt + |e|_0\int_0^T |x_1'(t)|dt \ .
\end{aligned} \tag{3.16}$$

For $\varepsilon = \frac{1}{2}(\frac{\sigma}{\beta_1 T} - r)$, by [H1] there is a constant $A_1 > 0$ such that

$$g(x_1(t-\tau_1(t))) \le (r+\varepsilon)|x_1(t-\tau(t))| \quad \text{for} \quad |x_1(t-\tau(t))| \ge A_1 \ . \tag{3.17}$$

Define

$$E_1 = \{t\in[0,T]||x_1(t-\tau(t))| < A_1\}, \qquad E_2 = \{t\in[0,T]||x_1(t-\tau(t))| \ge A_1\} \ .$$

Then (3.16) can be transformed into

$$\begin{aligned}
&\sigma\textstyle\int_0^T |x_1'(t)|^2dt\\
&\le \beta_1\textstyle\int_{E_1} |g(x_1(t-\tau_1(t)))||x_1'(t)|dt + \beta_1\int_{E_2} |g(x_1(t-\tau_1(t)))||x_1'(t)|dt + |e|_0\int_0^T |x_1'(t)|dt\\
&\le [\beta_1 g_{A_2} + |e|_0]\textstyle\int_0^T |x_1'(t)|dt + \beta_1(r+\varepsilon)|x|_0\int_0^T |x_1'(t)|dt\\
&= [\beta_1 g_{A_1} + |e|_0]\textstyle\int_0^T |x_1'(t)|dt + \beta_1(r+\varepsilon)[A + \int_0^T |x_1'(t)|dt]\int_0^T |x_1'(t)|dt\\
&\le T^{\frac{1}{2}}[\beta_1 g_{A_2} + |e|_0 + \beta_1(r+\varepsilon)A](\textstyle\int_0^T |x_1'(t)|^2dt)^{\frac{1}{2}} + \beta_1(r+\varepsilon)T\int_0^T |x_1'(t)|^2dt \ ,
\end{aligned}$$

i.e.,

$$[\sigma - \beta_1(r+\varepsilon)T]\int_0^T |x_1'(t)|^2dt \le c_3(\int_0^T |x_1'(t)|^2dt)^{\frac{1}{2}} \ , \tag{3.18}$$

where $g_{A_1} := \max\limits_{|u|\le A_1} |g(u)|$ and $c_3 := T^{\frac{1}{2}}[\beta_1 g_{A_1} + |e|_0 + \beta_1(r+\varepsilon)A]$. In view of $r < \frac{\sigma}{\beta_1 T}$ and $\varepsilon = \frac{1}{2}(\frac{\sigma}{\beta_1 T} - r)$, it is easy to see that $\sigma - \beta_1(r+\varepsilon)T = \frac{1}{2}(\sigma - \beta_1 Tr) > 0$. So from (3.18) we have

$$\int_0^T |x_1'(t)|^2dt \le (\frac{2c_3}{\sigma - \beta_1 Tr})^2$$

and so

$$\int_0^T |x_1'(t)|dt \le T^{\frac{1}{2}}(\int_0^T |x_1'(t)|^2dt)^{\frac{1}{2}} = \frac{2T^{\frac{1}{2}}c_3}{\sigma - \beta_1 Tr} := A_2 \ . \tag{3.19}$$

Hence

$$|x_1|_0 \le A + \int_0^T |x_1'(t)|dt \le A + A_2 := M_1 \ . \tag{3.20}$$

By the first equation of (3.4), we have

$$\int_0^T |x_2(s)|^{q-2}x_2(s)ds = 0 \ , \tag{3.21}$$

which implies that there is a constant $t_2 \in [0,T]$ such that $x_2(t_2) = 0$. So

$$|x_2|_0 \le \int_0^T |x_2'(s)|ds \ . \tag{3.22}$$

On the other hand, taking absolute value and integrating over $[0,T]$ on both sides of the second equation of (3.4), we obtain

$$\begin{aligned} \int_0^T |x_2'(s)|ds &\le \int_0^T |f(t,x_1(t))||x_1'(t)|dt + \lambda \int_0^T |\beta(t)g(x_1(t-\tau_1(t)))|dt + \lambda \int_0^T |e(t)|dt \\ &\le f_{M_1} A_2 T + \beta_1(g_{M_1}T) + |e|_1 \ , \end{aligned}$$

where $f_{M_1} := \max\limits_{t\in[0,T],|u|\le M_1} f(t,u)$, $g_{M_1} := \max\limits_{|u|\le M_1} |g(u)|$ and $|e|_1 := \int_0^T |e(t)|dt$. So from (3.22), we have

$$|x_2|_0 \le f_{M_1} A_2 T + \beta_1(g_{M_1}T) + |e|_1 := M_2 \ . \tag{3.23}$$

Let $\Omega_2 := \{x \in Ker\ L : Nx \in Im\ L\}$. If $x \in \Omega_2$, then $x \in Ker\ L$ and $QNx = 0$. From assumption $\int_0^T e(t)dt = 0$ we see that

$$\begin{cases} |x_2|^{q-2}x_2 = 0 \\ g(x_1) = 0 \ . \end{cases} \tag{3.24}$$

So

$$|x_1| \le A \le M_1, \qquad x_2 = 0 \le M_2 \ . \tag{3.25}$$

Let $\Omega = \{x = (x_1,x_2)^\top \in X : |x_1|_0 < N_1, |x_2|_0 < N_2\}$, where N_1 and N_2 are constants with $N_1 > M_1$, $N_2 > M_2$ and $(N_2)^q > A\overline{\beta}g_A$, where $g_A := \max\limits_{|u|\le A} |g(u)|$ and $\overline{\beta} := \frac{1}{T}\int_0^T \beta(t)dt$. Then $\overline{\Omega_1} \subset \Omega$, $\overline{\Omega_2} \subset \Omega$. From (3.20), (3.22) and (3.25), it is obvious that conditions (1) and (2) of Theorem 2.1 are satisfied.

Next, we claim that condition (3) of Theorem A is also satisfied. For this, define the isomorphism $J : Im\ Q \to Ker\ L$ by $J(x_1,x_2) := (-x_2,x_1)$ and let $H(v,\mu) := \mu v + (1-\mu)JQNv,\ \ (v,\mu) \in \Omega \times [0,1]$. By simple calculation, we obtain, for $(x,\mu) \in \partial(\Omega \cap KerL) \times [0,1]$,

$$x^\top H(x,\mu) = \mu(x_1^2 + x_2^2) + (1-\mu)(\overline{\beta}x_1 g(x_1) + |x_2|^q) > 0 \ .$$

Hence

$$\begin{aligned} & deg\{JQN, \Omega \cap KerL, 0\} = deg\{H(x,0), \Omega \cap KerL, 0\} \\ = \ & deg\{H(x,1), \Omega \cap KerL, 0\} = deg\{I, \Omega \cap KerL, 0\} \\ \ne \ & 0 \ , \end{aligned}$$

and so condition (3) of Theorem 2.1 is satisfied.

Therefore, by Theorem 2.1, we conclude that equation

$$Lx = Nx$$

has a solution $x(t) = (x_1(t),x_2(t))^\top$ on $\overline{\Omega}$, i.e., Eq.(1.1) has a $T-$periodic solution $x_1(t)$ with $|x_1|_0 \le M_1$.

Finally, observe that $x_1(t)$ is not a constant. For if not, it follows from (3.14) that $e(t) = c\beta(t) \geq c$ which will contradict to $e(t) \not\equiv 0$ and $\int_0^T e(s)ds = 0$. This completes the proof of Theorem 3.1.

THEOREM 3.2 If (H2)-(H5) hold, then equation (1.1) has a non-constant T-periodic solution if $\sqrt{2}\beta_1 l\delta < \sigma$.

PROOF. Let Ω_1 be defined as in Theorem 3.1. If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1$, then from the proof of Theorem 3.1 we see that

$$[\varphi_p(x_1'(t))]' + \lambda^{p-1}f(t, x_1(t))x_1'(t) + \lambda^p\beta(t)g(x_1(t-\tau_1(t))) = \lambda^p e(t) \ , \tag{3.26}$$

and

$$|x_1|_0 \leq A + \int_0^T |x_1'(s)|ds \ . \tag{3.27}$$

We claim that $|x_1|_0$ is bounded.

Multiplying both sides of Eq.(3.26) by $x_1'(t)$ and integrating over $[0, T]$, we have

$$\int_0^T f(t, x_1(t))(x_1'(t))^2 dt + \lambda \int_0^T \beta(t)g(x_1(t-\tau_1(t)))x_1'(t)dt = \lambda \int_0^T e(t)x_1'(t)dt \ . \tag{3.28}$$

By (3.28) and (H3),

$$\begin{aligned}
&\sigma \textstyle\int_0^T |x_1(t)|^2 dt \\
&\leq \textstyle\int_0^T |f(t, x_1(t))|(x_1'(t))^2 dt \\
&= |\textstyle\int_0^T f(t, x_1(t))(x_1'(t))^2 dt| \\
&\leq \textstyle\int_0^T \beta(t)|g(x_1(t-\tau_1(t)))x_1'(t)|dt + |\int_0^T e(t)x_1'(t)dt| \\
&\leq \beta_1|\textstyle\int_0^T [g(x_1(t-\tau_1(t))) - g(x_1(t))]x_1'(t)dt + \int_0^T g(x_1(t))x_1'(t)dt| + |\int_0^T e(t)x_1'(t)dt| \ .
\end{aligned} \tag{3.29}$$

Considering $\int_0^T g(x_1(t))x_1'(t)dt = 0$ and by assumption (H5), we have from (3.29) that

$$\begin{aligned}
&\sigma \textstyle\int_0^T |x_1(t)|^2 dt \\
&\leq \beta_1|\textstyle\int_0^T [g(x_1(t-\tau_1(t))) - g(x_1(t))]x_1'(t)dt| + |\int_0^T e(t)x_1'(t)dt| \\
&\leq \beta_1 l \textstyle\int_0^T |x_1(t-\tau_1(t)) - x_1(t)||x_1'(t)|dt + |\int_0^T e(t)x_1'(t)dt| \\
&\leq \beta_1 l(\textstyle\int_0^T |x_1(t-\tau_1(t)) - x_1(t)|^2 dt)^{\frac{1}{2}}(\int_0^T |x_1(t)|^2 dt)^{\frac{1}{2}} + (\int_0^T |x_1(t)|^2 dt)^{\frac{1}{2}}(\int_0^T |e(t)|^2 dt)^{\frac{1}{2}} \ .
\end{aligned} \tag{3.30}$$

By (H4), and applying Theorem 2.2, we obtain

$$(\int_0^T |x_1(t-\tau_1(t)) - x_1(t)|^2 dt)^{\frac{1}{2}} = (\int_0^T |x_1(t-\tau_1(t)+mT) - x_1(t)|^2 dt)^{\frac{1}{2}} \leq \sqrt{2}\delta(\int_0^T |x_1(t)|^2 dt)^{\frac{1}{2}} \ . \tag{3.31}$$

Substituting (3.31) into (3.29) yields

$$(\sigma - \sqrt{2}\beta_1 l\delta)(\int_0^T |x_1(t)|^2 dt) \leq (\int_0^T |x_1(t)|^2 dt)^{\frac{1}{2}}(\int_0^T |e(t)|^2 dt)^{\frac{1}{2}} \ . \tag{3.32}$$

As $\sqrt{2}\beta_1 l\delta < \sigma$, we obtain

$$(\int_0^T |x_1(t)|^2 dt)^{\frac{1}{2}} \leq \frac{(\int_0^T |e(t)|^2 dt)^{\frac{1}{2}}}{\sigma - \sqrt{2}\beta_1 l\delta} \ . \tag{3.33}$$

Hence (3.27) can be transformed into

$$|x_1|_0 \leq A + \int_0^T |x_1'(t)|dt \leq A + T^{\frac{1}{2}}(\int_0^T |x_1(t)|^2 dt)^{\frac{1}{2}} \leq A + \frac{T^{\frac{1}{2}}(\int_0^T |e(t)|^2 dt)^{\frac{1}{2}}}{\sigma - \sqrt{2}\beta_1 l\delta}\ .$$

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1.

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