THE WARING-GOLDBACH PROBLEM UNDER THE HASSE-WEIL HYPOTHESIS

XIUMIN REN [1]

ABSTRACT. In this paper, it is proved that under the HW_3 hypothesis, with at most $O(N^{\frac{52}{53}})$ exceptions, all positive integers up to N satisfying some necessary congruence conditions, are sums of four cubes of primes; and that every sufficiently large odd integer N with $N \not\equiv 0(\mathrm{mod}9)$ is the sum of seven cubes of primes.

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1. INTRODUCTION

It is conjectured that all sufficiently large integers n satisfying some necessary congruence conditions, are the sum of four cubes of primes, i.e.

$$n = p_1^3 + p_2^3 + p_3^3 + p_4^3. \tag{1.1}$$

Such a strong result is out of reach at present. In a recent paper [16], the author has proved that a positive proportion of positive integers can be written as (1.1).

In this paper, we go further to study the representation (1.1) under the assumption of the generalized Riemann hypothesis for the cubic Hasse-Weil L-functions. For a brief description of the hypothesis, we postpone to the next section.

We consider positive integers n satisfying the following conditions

$$2|n, \qquad n \not\equiv \pm 1, \pm 3(\mathrm{mod}9), \qquad n \not\equiv \pm 1(\mathrm{mod}7)\}. \tag{1.2}$$

Let $E(N)$ denote the number of those n up to N that can not be written in the form of (1.1).

Our first result is the following conditional improvement on the result in [16].

Theorem 1. *Assume HW_3. Then*

$$E(N) \ll N^{\frac{52}{53}}.$$

It therefore follows that almost all positive integers n satisfying (1.2) can be represented as (1.1).

Similar argument also gives

Theorem 2. *Assume HW_3. Then every large odd integer N with $N \not\equiv 0(\mathrm{mod}9)$ is the sum of seven cubes of primes,*

$$N = p_1^3 + p_2^3 + \cdots + p_7^3, \tag{1.3}$$

and an asymptotic formula holds for the number of representations.

Our results can be compared with those of Hua [4], which states:

(H1) *Almost all odd integers n in the set*

$$\mathfrak{N} = \{n \geq 1 : n \not\equiv 0, \pm 2(\mathrm{mod}9), n \not\equiv 0(\mathrm{mod}7)\}$$

can be represented as sums of five cubes of primes;

(H2) *All sufficiently large odd integers are the sum of nine cubes of primes.*

We prove our Theorem 1 and 2 by the circle method. To get a result of this strength, we have to deal with rather large major arcs, to which the Siegel-Walfisz theorem does not apply. Usually,

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one treats the enlarged major arcs by employing the Deuring-Heilbroun phenomenon. Here we prove Theorem 1 and 2 by a different approach, which has been successfully applied in several other occasions, for example [10], [11] and [15]. The key point of this approach is that we can save the factor $r_0^{-a+\varepsilon}$ in Lemma 5.2 and (8.5) below. With this saving, our enlarged major arcs can be treated by the classical zero-density estimates (in Lemma 4.3) and zero-free region for the Dirichlet L-functions (defined in (5.16)). Previously, the factor $r_0^{-\alpha+\varepsilon}$ is divided equally to each variable (see for example (5.7) and (5.8) in [15]), and this causes some waste. In this paper, we use an iterative procedure to retrieve the loss. For this, Huxley's zero-density estimate involving arithmetic progressions is applied. Moreover, in handling the minor arcs, we employ Wooley's result in [23] to give the upper bound estimate for trigonometric sums over primes (see Lemma 3.1).

We prove theorem 1 in detail in section 3-7, and give only a sketch of the proof of theorem 2 in section 8.

Notation. As usual, $\varphi(n)$ and $\Lambda(n)$ stand for the function of Euler and von Mangoldt respectively, and $d(n)$ is the divisor function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. We write N for a large positive integer and $L = \log N$. Further $r \sim R$ means $R < r \leq 2R$. If there is no ambiguity, we write $\frac{a}{b} + \theta$ as $a/b + \theta$ or $\theta + a/b$. The same convention will be applied for quotients. The letters ε and A denote positive constants, which are arbitrarily small and arbitrarily large respectively. We may write cA as A for positive constant c. This is also applied to ε.

2. Riemann hypothesis for the cubic Hasse-Weil L-functions

Let $\mathbf{m}, \mathbf{x} \in \mathbb{Z}^6$, consider the linear form

$$\mathbf{mx} = m_1x_1 + m_2x_2 + ... + m_6x_6$$

and the cubic form

$$g(\mathbf{x}) = x_1^3 + ... + x_6^3$$

with discriminant defined by

$$\Delta(\mathbf{m}) = 3\prod \left(m_1^{\frac{3}{2}} + m_2^{\frac{3}{2}} + ... + m_6^{\frac{3}{2}} \right).$$

For each prime p, one can define local factors $L_p(\mathbf{m}; s)$ according as $p \nmid \Delta(\mathbf{m})$ or $p|\Delta(\mathbf{m})$, respectively; this has been done by Bombieri and Swinnerton-Dyer [1], Deligne [2], and Serre [19]. Multiply all these local factors one gets the following global (modified) Hasse-Weil L-function

$$L(\mathbf{m}; s) = \prod_p L_p(\mathbf{m}; s) = \left(\prod_{p\nmid\Delta(\mathbf{m})} L_p(\mathbf{m}; s) \right) \left(\prod_{p|\Delta(\mathbf{m})} L_p(\mathbf{m}; s) \right).$$

For $\sigma > 5/2$, set

$$\xi(\mathbf{m}; s) = (2\pi)^{-5s}\Gamma^5(s)B^{\frac{s}{2}}(\mathbf{m})L(\mathbf{m}; s),$$

where $B(\mathbf{m})$ is the conductor of the L-function.

Then the Riemann hypothesis for the cubic Hasse-Weil L-functions (which we abbreviate as HW_3) states that

(i) $\xi(\mathbf{m};s)$ *is a meromorphic function of finite order that is regular everywhere save possibly for poles at* $s=5/2$ *and* $3/2$.

(ii) $\xi(\mathbf{m};s)$ *satisfies the following functional equation*

$$\xi(\mathbf{m};s)=\omega(\mathbf{m})\xi(\mathbf{m};4-s),$$

where $\omega(\mathbf{m})=\pm1$;

(iii) $\xi(\mathbf{m};s)\neq 0$ *if* $\sigma\neq 2$ *(Riemann Hypothesis).*

Roughly speaking, a Hasse-Weil L-function is the product of local factors, which are defined via the number of solutions of cubic Diophantine equations. The assumption HW_3 asserts that each Hasse-Weil L-function, defined in the way above, has analytic continuation to the whole plane, and all its non-trivial zeros lies on the critical line. Going back to the arithmetic background from the assumption HW_3, Hooley [8] and Heath-Brown [7] have established independently the so-called hypothesis R^* which we record as the following

Lemma 2.1. *Let* $r_3(n)$ *denote the number of representations of* n *as the sum of three non-negative cubes. If the* HW_3 *be assumed, then*

$$\sum_{n\leq X} r_3^2(n)=O(X^{1+\varepsilon}).$$

3. Outline of the method

In order to apply the circle method, for large $N>0$, set

$$P=N^{\theta},\qquad Q=NP^{-1-\eta},\qquad \textit{and}\qquad U=(N/11)^{\frac{1}{3}},\tag{3.1}$$

where $\theta=5/87+\eta$ and $\eta=10^{-4}$. By Dirichlet's lemma on rational approximations, each $\alpha\in(0,1]$ may be written in the form

$$\alpha=\frac{a}{q}+\lambda,\qquad |\lambda|\leq\frac{1}{qQ}\tag{3.2}$$

for some integers a,q with $1\leq a\leq q\leq Q$, and $(a,q)=1$. We denote by $\mathfrak{M}(q,a)$ the set of α satisfying (3.2), and write $\mathfrak{M}$ for the union of all $\mathfrak{M}(q,a)$ with $1\leq a\leq q\leq P$. The minor arcs $\mathfrak{m}$ are defined as the complement of $\mathfrak{M}$ in $(0,1]$. It follows from $2P\leq Q$ that the major arcs $\mathfrak{M}(q,a)$ are mutually disjoint.

Define

$$S(\alpha)=\sum_{m\sim U}\Lambda(m)e(m^3\alpha),\quad \textit{and}\quad G(\alpha)=\sum_{p\sim U}(\log p)e(p^3\alpha),\tag{3.3}$$

where $e(r)=\exp(i2\pi r)$. Let

$$r(n)=\sum_{\substack{n=p_1^3+\ldots+p_4^3\\ p_j\sim U}}\log p_1\cdots\log p_4.$$

Then

$$r(n)=\int_0^1 G^4(\alpha)e(-n\alpha)d\alpha=\left\{\int_{\mathfrak{M}}+\int_{\mathfrak{m}}\right\}G^4(\alpha)e(-n\alpha)d\alpha.$$

To handle the integral on the major arcs, we need to obtain the following

Theorem 3. *For $N/2 < n \le N$, we have*

$$\int_{\mathfrak{M}} S^4(\alpha)e(-n\alpha)d\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O(UL^{-A}). \tag{3.4}$$

Here $\mathfrak{S}(n)$ is the singular series defined as in (5.3) and satisfies $\mathfrak{S}(n) \gg (\log\log n)^{-c^*}$ for some positive absolute constant c^*. $\mathfrak{J}(n) = \mathfrak{J}(n; 1, ..., 1)$ is defined as in (5.10) and satisfies

$$U \ll \mathfrak{J}(n) \ll U. \tag{3.5}$$

To deal with the minor arcs, we will make use of the following result of Wooley [23].

Lemma 3.1. *Let $U \ge 2$. Suppose that α is a real number, and that there exist integers a and q satisfying*

$$(a, q) = 1, \quad 1 \le q \le U^{\frac{3}{2}} \quad \text{and} \quad |q\alpha - a| \le U^{-\frac{3}{2}}. \tag{3.6}$$

Then one has

$$\sum_{U<p\le 2U} (\log p)e(p^3\alpha) \ll U^{\frac{23}{24}+\varepsilon} + \frac{q^{\varepsilon}\omega(q)^{\frac{1}{2}}U(\log U)^5}{(1+U^3|\alpha - a/q|)^{\frac{1}{2}}},$$

where $\omega(q)$ is a multiplicative function defined by

$$\omega(p^{3u+v}) = \begin{cases} 3p^{-u-\frac{1}{2}}, & \text{when} \quad u \ge 0, \quad \text{and} \quad v = 1; \\ p^{-u-1}, & \text{when} \quad u \ge 0, \quad \text{and} \quad v = 2, 3. \end{cases}$$

Proof of Theorem 1. On noting that the measure of the major arcs $\mathfrak{M}$ is $O(PQ^{-1})$, and moreover that

$$|S(\alpha) - G(\alpha)| \le \sum_{h=2}^{\infty} \sum_{U<p^h\le 2U} \log p \ll U^{\frac{1}{2}}\log U,$$

one finds from the trivial estimate $|S(\alpha)| \ll U$ that

$$\left| \int_{\mathfrak{M}} S(\alpha)^4 e(-n\alpha)d\alpha - \int_{\mathfrak{M}} G(\alpha)^4 e(-n\alpha)d\alpha \right| \ll U^{\frac{7}{2}+\varepsilon}PQ^{-1} \ll U^{\frac{3}{4}}.$$

This along with Theorem 3 gives

$$\int_{\mathfrak{M}} G^4(\alpha)e(-n\alpha)d\alpha = \mathfrak{S}(n)\mathfrak{J}(n) + O(UL^{-A}). \tag{3.7}$$

Now by Bessel's inequality, we have

$$\sum_{N/2<n\le N} \left| \int_{\mathfrak{m}} G(\alpha)^4 e(-n\alpha)d\alpha \right|^2 \ll \sup_{\alpha\in\mathfrak{m}} |G(\alpha)|^2 \int_0^1 |G(\alpha)|^6 d\alpha. \tag{3.8}$$

By Dirichlet's lemma on rational approximations, for each $\alpha \in \mathfrak{m}$, there exist positive integers a and q satisfying (3.6). Since $\alpha \in \mathfrak{m}$, we deduce that $q > P$ or $q \le P$ but $|\alpha q - a| > Q^{-1}$. Hence by Lemma 3.1, and on noting that $q^{-\frac{1}{2}} \le \omega(q) \le q^{-\frac{1}{3}}$, we have

$$\sup_{\alpha\in\mathfrak{m}} |G(\alpha)| \ll U^{\frac{23}{24}+\varepsilon} + U^{1+\varepsilon}P^{-\frac{1}{6}} \ll U^{1+\varepsilon}P^{-\frac{1}{6}}. \tag{3.9}$$

While by Lemma 2.1, one easily derives that

$$\int_0^1 |G(\alpha)|^6 d\alpha \ll U^{3+\varepsilon}. \tag{3.10}$$

Inserting (3.9) and (3.10) into (3.8), then the right-hand side of (3.8) becomes $O(U^{5+\varepsilon}P^{-\frac{1}{3}})$. Therefore with at most $O(NP^{-\frac{1}{3}+\varepsilon}) = O(N^{\frac{52}{53}})$ exceptions, for all integers $N/2 < n \le N$ satisfying (1.2), one has

$$\left|\int_{\mathfrak{m}} G(\alpha)^4 e(-n\alpha)d\alpha\right| \ll U^{1-\varepsilon}.$$

This along with (3.7) shows that for those unexceptional n, the expression (1.1) holds and the number of expressions satisfies

$$r(n) = \mathfrak{S}(n)\mathfrak{J}(n) + O(UL^{-A}).$$

Now the assertion of Theorem 1 follows by summing over dyadic intervals. □

The following Sections 4-7 are devoted to the proof of Theorem 3.

4. An explicit expression

The purpose of this section is to establish in Lemma 4.2 an explicit expression for the left-hand side of (3.4).

For χ mod q and χ^0 mod q, define

$$C(\chi, a) = \sum_{m=1}^{q} \overline{\chi}(m) e\left(\frac{am^3}{q}\right) \qquad \textit{and} \qquad C(q,a) = C(\chi^0, a). \tag{4.1}$$

Then Vinogradov's estimate (see for example [22], Ch.VI, problem 14b(α)) gives

$$C(\chi, a) \le 2q^{\frac{1}{2}} d^2(q). \tag{4.2}$$

Define

$$\Phi(\lambda) = \int_U^{2U} e(\lambda u^3)du \qquad \textit{and} \qquad \Psi(\lambda, \rho) = \int_U^{2U} u^{\rho-1} e(\lambda u^3)du. \tag{4.3}$$

Now we give the following formula for $S(\alpha)$ with $\alpha \in \mathfrak{M}$.

Lemma 4.1. *For $\alpha = a/q + \lambda \in \mathfrak{M}$, we have*

$$S(\alpha) = S_1(\lambda) + S_2(\lambda) + S_3(\lambda),$$

where

$$S_1(\lambda) = \frac{C(q,a)}{\varphi(q)}\Phi(\lambda), \qquad S_2(\lambda) = -\frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\chi, a) \sum_{|\gamma| \le T} \Psi(\lambda, \rho),$$

and

$$S_3(\lambda) = O\left\{ q^{\frac{1}{2}+\varepsilon} \frac{U}{T}(1 + |\lambda|U^3)L^2 \right\}.$$

Here $\rho = \beta + i\gamma$ denotes a non-trivial zero (possibly the Siegel zero) of the Dirichlet L-function $L(s, \chi)$.

Proof. See Lemma 3.1 in Ren [17]. □

Define for $j = 0, 1, ..., 4$,

$$I_j = \binom{4}{j} \sum_{q \le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\infty}^{+\infty} S_1^{4-j}(\lambda) S_2^j(\lambda) e(-n\lambda) d\lambda. \tag{4.4}$$

Now we state the main result of this section.

Lemma 4.2. *Let $T = P^{\frac{5}{4}(1+2\eta)}$. Then we have*

$$\int_{\mathfrak{M}} S^4(\alpha) e(-n\alpha) d\alpha = \sum_{j=0}^{4} I_j + O(UL^{-A}).$$

To prove this result, we need the following lemmas on zero-density estimates.

Let $N(\sigma, T, \chi)$ denote the number of zeros of $L(s, \chi)$ in the region $\sigma \le \mathrm{Re}s \le 1$, $|\mathrm{Im}s| \le T$. Define

$$N(\sigma, T, q) = \sum_{\chi \bmod q} N(\sigma, T, \chi), \qquad N^*(\sigma, T, X, d) = \sum_{\substack{q \le X \\ q \equiv 0(\bmod d)}} \sum_{\chi \bmod q}{}^{*} N(\sigma, T, \chi),$$

where * means that the summation is restricted to primitive characters χ mod q. Write $N^*(\sigma, T, X) = N^*(\sigma, T, X, 1)$.

Lemma 4.3. *Let $\eta = 10^{-4}$ and $T \ge 1$. Then for $1 \le d \le X$,*

$$N^*(\sigma, T, X, d) \ll \begin{cases} (X^2 T/d)^{(\frac{12}{5}+\varepsilon)(1-\sigma)}, & 1/2 \le \sigma \le 1 - \eta; \\ (X^3 T^2)^{(1+\varepsilon)(1-\sigma)}, & 1 - \eta < \sigma \le 1. \end{cases} \tag{4.5}$$

In particular

$$N^*(\sigma, T, X) \ll (X^2 T)^{\left(\frac{12}{5}+\varepsilon\right)(1-\sigma)}$$

holds for $1/2 \le \sigma \le 1$.

Proof. For $1/2 \le \sigma < 1 - \eta$, (4.5) follows easily from (1.1) of Huxley [5]. For $1 - \eta \le \sigma \le 1$, the left-hand side of (4.5) is bounded up by $N^*(\sigma, T, X)$, which admits the following estimate

$$N^*(\sigma, T, X) \ll \left(X^3 T^2\right)^{(1+\varepsilon)(1-\sigma)}.$$

This can be found in Jutila [9]. This proves Lemma 4.3 □

Proof of Lemma 4.2. At first, we have

$$S_1(\lambda), \quad S_2(\lambda) \quad \ll q^{-\frac{1}{2}+\varepsilon} \min(U, |\lambda|^{-\frac{1}{3}}) L^2, \tag{4.6}$$

by (3.6) in Ren [17]. Next we show that, on substituting $S(\alpha)$ by $S_1(\lambda) + S_2(\lambda)$ in the integral in Lemma 4.2, the resulting error is acceptable, i.e.

$$\int_{\mathfrak{M}} S^4(\alpha) e(-n\alpha) d\alpha - \sum_{q \le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \{S_1(\lambda) + S_2(\lambda)\}^4 e(-n\lambda) d\lambda \ll UL^{-A}. \tag{4.7}$$

By Hölder's inequality, the left-hand side of (4.7) is bounded by

$$\begin{aligned}
&\sum_{\substack{i+j+k=4\\k\ge1}}\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_1|^i|S_2|^j|S_3|^k d\lambda\\
&\ll \sum_{\substack{i+j+k=4\\k\ge1}}\sum_{q\le P} q\bigg\{\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_1|^4d\lambda\bigg\}^{\frac{i}{4}}\bigg\{\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_2|^4d\lambda\bigg\}^{\frac{j}{4}}\bigg\{\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_3|^4d\lambda\bigg\}^{\frac{k}{4}}. \qquad (4.8)
\end{aligned}$$

By (4.6), one has

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_1|^4d\lambda \ll q^{-2+\varepsilon}L^8\bigg\{\int_0^{U^{-3}}U^4d\lambda+\int_{U^{-3}}^{\infty}\lambda^{-\frac{4}{3}}d\lambda\bigg\}\ll q^{-2+\varepsilon}UL^8.$$

And analogously

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_2|^4d\lambda \ll q^{-2+\varepsilon}UL^8.$$

Moreover, by Lemma 4.1,

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_3|^4d\lambda \ll \frac{U^4}{T^4}q^{2+\varepsilon}L^8\left(\int_0^{U^{-3}}d\lambda+\int_{U^{-3}}^{\frac{1}{qQ}}(\lambda U^3)^4d\lambda\right)\ll \frac{U}{T^4}q^{-3+\varepsilon}P^{5(1+\eta)}L^8.$$

Inserting these estimates into (4.8), we get

$$\begin{aligned}
&\sum_{\substack{i+j+k=4\\k\ge1}}\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|S_1|^i|S_2|^j|S_3|^k d\lambda\\
&\ll UL^8\sum_{1\le k\le4}T^{-k}P^{\frac{5k}{4}(1+\eta)}\sum_{q\le P}q^{-\frac{k}{4}-1+\varepsilon}\ll UL^8\sum_{1\le k\le 4}T^{-k}P^{\frac{5k}{4}(1+\eta)}\ll UL^{-A},
\end{aligned}$$

on choosing $T=P^{\frac{5}{4}(1+2\eta)}$. This proves (4.7).

Finally we extend the interval of integration in the second integral in (4.7) to $(-\infty,+\infty)$, then by (4.6), the resulting error is bounded by

$$\begin{aligned}
&\sum_{q\le P}\sum_{a=1}^{q}\int_{\frac{1}{qQ}}^{\infty}|S_1(\lambda)+S_2(\lambda)|^4d\lambda \ll L^8\sum_{q\le P}q^{-1+\varepsilon}\int_{\frac{1}{qQ}}^{\infty}|\lambda|^{-\frac{4}{3}}d\lambda\\
&\ll L^8Q^{\frac{1}{3}}\sum_{q\le P}q^{-\frac{2}{3}+\varepsilon}\ll L^8Q^{\frac{1}{3}}P^{\frac{1}{3}+\varepsilon}\ll UL^{-A},
\end{aligned}$$

on recalling (3.1). Therefore (4.7) becomes

$$\int_{\mathfrak{M}}S^4(\alpha)e(-n\alpha)d\alpha=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{an}{q}\right)\int_{-\infty}^{+\infty}\{S_1(\lambda)+S_2(\lambda)\}^4e(-n\lambda)d\lambda+O(UL^{-A}).$$

This proves Lemma 4.2. □

5. Estimation of I_1, I_2, I_3, I_4

The purpose of this section is to establish the following Lemmas 5.1 and 5.2.

Lemma 5.1. *Let I_j be as defined in (4.4). Then for $j = 1, ..., 4$, we have*

$$I_j \ll UL^{-A}. \tag{5.1}$$

We need some more notations. Let $C(\chi, a)$ and $C(q, a)$ be as defined in (4.1). For $\chi_j \bmod q$, define

$$B(n, q, \chi_1, ..., \chi_4) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) C(\chi_1, a) \cdots C(\chi_4, a), \qquad B(n, q) = B(n, q, \chi_1^0, ..., \chi_4^0), \tag{5.2}$$

and

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{5.3}$$

This $\mathfrak{S}(n)$ is the singular series appearing in Theorem 3.

Lemma 5.2. *For $j = 1, ..., 4$, let χ_j be primitive characters* $\bmod r_j$ *and χ^0 the principal character* $\bmod q$. *Write $r_0 = [r_1, ..., r_4]$, the least common multiple of $r_1, ..., r_4$. Then for all positive integers n up to N, one has*

$$\sum_{\substack{q \le P \\ r_0 | q}} \frac{1}{\varphi^4(q)} |B(n, q, \chi_1\chi^0, ..., \chi_4\chi^0)| \ll r_0^{-1+\varepsilon} L^{260}. \tag{5.4}$$

Proof. By (4.2), one has

$$B(n, q, \chi_1\chi^0, ..., \chi_4\chi^0) \ll \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{j=1}^{4} |C(\chi_j\chi^0, a)| \ll q^3 d^8(q).$$

Thus

$$\sum_{\substack{q \le P \\ r_0 | q}} \frac{1}{\varphi^4(q)} |B(n, q, \chi_1\chi^0, ..., \chi_4\chi^0)| \ll \sum_{\substack{q \le P \\ r_0 | q}} \frac{q^3 d^8(q)}{\varphi^4(q)} \ll r_0^{-1+\varepsilon} L^4 \sum_{q \le Y} \frac{d^8(q)}{q}.$$

Now (5.4) follows from Lemma 4.2 of Pan [13]. □

Lemma 5.3. *Let $X \ge 1$. Assume U and P be as defined in (3.1) and T be as in Lemma 4.2. Then for positive integer $g \ge 1$, we have*

$$\sum_{X \le r \le P} [g, r]^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta - 1} \ll g^{-1+\varepsilon} d(g) G(X) L^2,$$

where

$$G(X) = \begin{cases} X^{-1+\varepsilon} + N^{-\eta}, & g = 1; \\ 1, & g > 1. \end{cases}$$

Proof. We first consider the case $g > 1$. Note that $[g, r] = gr(g, r)^{-1}$, hence

$$\sum_{X \le r \le P} [g, r]^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1} \le g^{-1+\varepsilon} \sum_{\substack{d \le P \\ d|g}} d^{1-\varepsilon} \sum_{\substack{X \le r \le P \\ d|r}} r^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1}. \tag{5.5}$$

Now we have

$$\sum_{\substack{X \le r \le P \\ d|r}} r^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1} \ll \log P \max_{\max(d, X) \le R \le P/2} R^{-1+\varepsilon} \sum_{\substack{r \sim R \\ d|r}} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1}. \tag{5.6}$$

Integrating by parts, we get

$$\begin{aligned} &\sum_{\substack{r \le 2R \\ d|r}} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1} \\ &= U^{-\frac{1}{2}} N^{*}(1/2, T, 2R, d) + \log U \left(\int_{\frac{1}{2}}^{1-\eta} + \int_{1-\eta}^{1} \right) U^{\sigma-1} N^{*}(\sigma, T, 2R, d) d\sigma. \end{aligned} \tag{5.7}$$

By Lemma 4.3, we have

$$N^{*}(1/2, T, 2R, d) \ll \left(R^2 T/d\right)^{\frac{6}{5}+\varepsilon},$$

$$\int_{\frac{1}{2}}^{1-\eta} U^{\sigma-1} N^{*}(\sigma, T, 2R, d) d\sigma \ll U^{-\frac{1}{2}} \left(R^2 T/d\right)^{\frac{6}{5}+\varepsilon} + \left(U^{-1} (R^2 T/d)^{\frac{12}{5}+\varepsilon}\right)^{\eta},$$

and

$$\int_{1-\eta}^{1} U^{\sigma-1} N^{*}(\sigma, T, 2R, d) d\sigma \ll 1 + \left(U^{-1} (R^3 T^2)^{1+\varepsilon}\right)^{\eta}.$$

Putting these estimates in (5.7), then by (5.6) we come to the estimate

$$\sum_{\substack{X \le r \le P \\ d|r}} r^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1} \ll L^2 \left(d^{-1+\varepsilon} + d^{-\frac{6}{5}-\varepsilon} N^{-\eta}\right). \tag{5.8}$$

Here we have used the supposition that $\theta = 5/87 + \eta$. Now the desired estimate for $g > 1$ follows easily from (5.8) and (5.5).

If $g = 1$, then instead of (5.5) and (5.6), we have

$$\sum_{X \le r \le P} r^{-1+\varepsilon} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1} \ll \log P \max_{X \le R \le P/2} R^{-1+\varepsilon} \sum_{r \sim R} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1}. \tag{5.9}$$

By letting $d = 1$ in (5.7) and then making use of Lemma 4.3, we obtain

$$\sum_{r \sim R} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T} U^{\beta-1} \ll L \left(1 + U^{-\frac{1}{2}} \left(R^2 T\right)^{\frac{6}{5}+\varepsilon}\right).$$

Putting this in (5.9) we get the desired result for $g = 1$. □

To prove Lemma 5.1, we will use the following lemma, which is a modification of Lemma 4.7 in Liu and Tsang [12].

Lemma 5.4. *Let ρ_j be any complex numbers with $0 < \mathrm{Re}\rho_j \le 1,\ j = 1, ..., 4$. Then*

$$\begin{aligned}\int_{-\infty}^{+\infty} e(-n\lambda) \prod_{j=1}^{4} \Psi(\lambda, \rho_j) d\lambda &= \frac{1}{3^4} \int_{\mathfrak{D}} u_1^{\frac{\rho_1}{3}-1} \cdots u_4^{\frac{\rho_4}{3}-1} du_1 du_2 du_3 \\ &=: \mathfrak{J}(n; \rho_1, ..., \rho_4), \end{aligned} \tag{5.10}$$

where

$$\mathfrak{D} = \{(u_1, ..., u_3) : U^3 \le u_1, ..., u_4 \le 8U^3\} \tag{5.11}$$

with $u_4 = n - u_1 - u_2 - u_3$.

Now we establish the main results of this section.

Proof of Lemma 5.1. We treat the case $j = 4$ in (5.1) in detail. In other cases, the proofs for (5.1) are similar and better ranges of θ (in (3.1)) available, so we will only give a sketch.

By (4.4), Lmma 4.1 and (5.2), one has

$$\begin{aligned} I_4 &= \sum_{q \le P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\infty}^{+\infty} \left\{ \sum_{\chi \bmod q} C(\chi, a) \sum_{|\gamma| \le T} \Psi(\lambda, \rho) \right\}^4 e(-n\lambda) d\lambda \\ &= \sum_{q \le P} \sum_{\chi_1 \bmod q} \cdots \sum_{\chi_4 \bmod q} \frac{B(n, q, \chi_1, ..., \chi_4)}{\varphi^4(q)} \sum_{|\gamma_1| \le T} \cdots \sum_{|\gamma_4| \le T} \mathfrak{J}(n; \rho_1, ..., \rho_4), \end{aligned}$$

where $\mathfrak{J}(n; \rho_1, ..., \rho_4)$ is defined by (5.10) with $\mathfrak{D}$ defined by (5.11). Now we recall that if a primitive character $\chi \bmod r$ induces a character $\psi \bmod k$, then $r | k$ and $\psi = \chi\chi^0$, where χ^0 is the principal character modulo k. Collecting all contributions made by an individual primitive character, one obtains

$$\begin{aligned} I_4 &= \sum_{r_1 \le P} \cdots \sum_{r_4 \le P} \sideset{}{^*}\sum_{\chi_1 \bmod r_1} \cdots \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \sum_{|\gamma_1| \le T} \cdots \sum_{|\gamma_4| \le T} \mathfrak{J}(n; \rho_1, ..., \rho_4) \\ &\quad \times \sum_{\substack{q \le P \\ r_0 | q}} \frac{B(n, q, \chi_1\chi^0, ..., \chi_4\chi^0)}{\varphi^4(q)}. \end{aligned} \tag{5.12}$$

Now we come to an upper bound estimate for $\mathfrak{J}(n; \rho_1, ..., \rho_4)$. By definition,

$$\mathfrak{J}(n; \rho_1, ..., \rho_4) = \frac{1}{81} \int_{U^3}^{8U^3} u_1^{\frac{\rho_1}{3}-1} du_1 \int_{U^3}^{8U^3} u_2^{\frac{\rho_2}{3}-1} du_2 \int_{\max(U^3, x-8U^3)}^{\min(8U^3, x-U^3)} u_3^{\frac{\rho_3}{3}-1} (x - u_3)^{\frac{\rho_4}{3}-1} du_3,$$

where we have written

$$x = n - u_1 - u_2.$$

By (3.1), one has $x \le 9U^3$, hence $\min(8U^3, x - U^3) = x - U^3$ and $\max(U^3, x - 8U^3) = U^3$. If $x \le 2U^3$, then $x - U^3 \le U^3$, and thus the innermost integral is 0. For $2U^3 < x \le 9U^3$, we make the substitution $u_3 = xu$, then the innermost integral is bounded by

$$x^{\frac{\beta_3}{3}+\frac{\beta_4}{3}-1} \int_{\frac{U^3}{x}}^{1-\frac{U^3}{x}} u^{\frac{\beta_3}{3}-1} (1-u)^{\frac{\beta_4}{3}-1} du \ll U^{\beta_3+\beta_4-3} \int_{\frac{1}{9}}^{\frac{8}{9}} u^{-1}(1-u)^{-1} du \ll U^{\beta_3+\beta_4-3}.$$

Estimating the other integrals in $\mathfrak{J}(n;\rho_1,...,\rho_4)$ trivially, one gets the following bound:

$$\mathfrak{J}(n;\rho_1,...,\rho_4) \ll UU^{\beta_1+...+\beta_4-4}. \tag{5.13}$$

Inserting (5.13) into (5.12), one yields

$$\begin{aligned} I_4 \ll\ & U \sum_{r_1\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\sum_{|\gamma_1|\le T} U^{\beta_1-1}\cdots\sum_{r_4\le P}\sum_{\chi_4 \bmod r_4}{}^{*}\sum_{|\gamma_4|\le T} U^{\beta_4-1}\sum_{\substack{q\le P\\ r_0|q}}\frac{B(n,q,\chi_1\chi^0,...,\chi_4\chi^0)}{\varphi^4(q)} \\ =:\ & UJ, \end{aligned} \tag{5.14}$$

say. In the following, we will prove that $J \ll L^{-A}$.

By Lemma 5.2, one has

$$J \ll L^{260}\sum_{r_1\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\sum_{|\gamma_1|\le T} U^{\beta_1-1}\cdots\sum_{r_4\le P}\sum_{\chi_4 \bmod r_4}{}^{*}\sum_{|\gamma_4|\le T} U^{\beta_4-1}[r_1,r_2,r_3,r_4]^{-1+\varepsilon}.$$

Note that $[r_1,r_2,r_3,r_4]=[[r_1,r_2,r_3],r_4]$, so by Lemma 5.3,

$$\sum_{r_4\le P}\sum_{\chi_4 \bmod r_4}{}^{*}\sum_{|\gamma_4|\le T} U^{\beta_4-1}[r_1,r_2,r_3,r_4]^{-1+\varepsilon} \ll [r_1,r_2,r_3]^{-1+\varepsilon}L^2.$$

Repeating the above process for triple sums over $(r_k,\chi_k,|\gamma|_k)$ for $k=3$ and $k=2$ successively, one finally achieves

$$J \ll L^{266}\sum_{r_1\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\sum_{|\gamma_1|\le T} U^{\beta_1-1}r_1^{-1+\varepsilon} \ll L^{266}(J_1+J_2), \tag{5.15}$$

where J_1 and J_2 represent the contributions from those with $r_1\le L^B$ and those with $L^B<r_1\le P$, respectively with $B=2A+4$.

By Lemma 5.3, one sees

$$J_2 \ll L^2\left(L^{-B(1-\varepsilon)}+N^{-\eta}\right) \ll L^{-A}.$$

Now we turn to J_1. By Satz VIII.6.2 of Prachar [14], there exists a positive constant c_1 such that $\prod_{\chi \bmod q} L(s,\chi)$ is zero-free in the region

$$\sigma \ge 1-c_1/\max\{\log q,\ \log^{\frac{4}{5}} N\}, \qquad |t|\le N, \tag{5.16}$$

except for the possible Siegel zero. But by Siegel's theorem (see [3],§21), the Siegel zero does not exist in this situation, since $q\le L^B$. Let $\eta(N)=c_1\log^{-\frac{4}{5}}N$. Then by Lemma 4.3,

$$\begin{aligned} J_1 \ll\ & \sum_{r\le L^B}\sum_{\chi \bmod r}{}^{*}\sum_{|\gamma|\le T} U^{\beta-1} \ll L\max_{\frac{1}{2}\le\sigma\le 1-\eta(N)}\left(L^{2B}T\right)^{(\frac{12}{5}+\varepsilon)(1-\sigma)}U^{\sigma-1} \\ \ll\ & L\max_{\frac{1}{2}\le\sigma\le 1-\eta(N)} N^{\{3\theta-\frac{1}{3}+\eta\}(1-\sigma)} \ll N^{-\frac{\eta(N)}{7}} \\ \ll\ & \exp\{-c_2L^{\frac{1}{5}}\} \ll L^{-A}. \end{aligned}$$

This proves the desired estimate for J_1, hence for J.

To conclude the proof, we need to sketch how to estimate $I_1, ..., I_3$. As an example, we only consider I_3. By arguments similar to those leading to (5.12), we have

$$\begin{aligned} I_3 &= 4\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{4} e\left(-\frac{an}{q}\right)\int_{-\infty}^{\infty} S_1(\lambda)S_2^3(\lambda)e(-n\lambda)d\lambda \\ &\ll \sum_{r_1\le P}\sum_{r_2\le P}\sum_{r_3\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\cdots\sum_{\chi_3 \bmod r_3}{}^{*}\sum_{|\gamma_1|\le T}\cdots\sum_{|\gamma_3|\le T}|\mathfrak{J}(n;\rho_1,...,\rho_3,1)| \\ &\quad\times\sum_{q\le P}\frac{B(n,q,\chi_1,...,\chi_3,\chi^0)}{\varphi^4(q)}, \end{aligned}$$

Now instead of (5.13), we have

$$\mathfrak{J}(n;\rho_1,\rho_2,\rho_3,1) \ll UU^{\beta_1+...+\beta_3-3}.$$

So I_3 can be estimated as

$$\begin{aligned} I_3 &\ll U\sum_{r_1\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\sum_{|\gamma_1|\le T}U^{\beta_1-1}\cdots\sum_{r_3\le P}\sum_{\chi_3 \bmod r_3}{}^{*}\sum_{|\gamma_3|\le T}U^{\beta_3-1}\sum_{q\le P}\frac{B(n,q,\chi_1,...,\chi_3,\chi^0)}{\varphi^4(q)} \\ &=: UK, \end{aligned}$$

say. By the same approach as used in estimating J above, we will obtain $K \ll L^{-A}$ for $\theta = 5/87+\eta$. This proves $I_3 \ll L^{-A}$. The proof of Lemma 5.1 is thus finished. □

6. The singular series

The main result of this section is the following:

Lemma 6.1. *We have*

$$\sum_{q\ge x}|A(n,q)| = O(x^{-\frac{1}{4}+\varepsilon}d(n)),$$

so the singular series defined in (5.3) is absolutely convergent. Moreover, we have

$$\mathfrak{S}(n) \gg (\log\log n)^{-c_*}$$

for some positive constant c^*.

To prove Lemma 6.1, we need the following lemmas.

Lemma 6.2. *If $(q_1, q_2) = 1$, then $A(n, q_1q_2) = A(n, q_1)A(n, q_2)$.*

For the proof of this lemma, one is referred to Lemma 8.1 of Hua [4].

Lemma 6.3. *Let*

$$\gamma = \begin{cases} 1, & p \ne 3; \\ 2, & p = 3. \end{cases}$$

Then we have $A(n, p^t) = 0$ whenever $t > \gamma$.

This is an easy consequence of Lemma 6.2 and Lemma 17 in Roth [18].

Lemma 6.4. *For $p \equiv 2(\mathrm{mod} 3)$, we have*

$$A(n,p) = \begin{cases} (p-1)^{-3}, & p|n; \\ -(p-1)^{-4}, & p\nmid n. \end{cases}$$

While for $p \equiv 1(\bmod 3)$, we have when $p \geq 13$,

$$|A(n,p)| < \begin{cases} 12p^{-1}, & p|n; \\ 30p^{-\frac{3}{2}}, & p \nmid n. \end{cases}$$

Proof. By (5.3), we have

$$\begin{aligned} A(n,p) &= \frac{1}{\varphi^4(p)} \sum_{a=1}^{p-1} C^4(p,a) e\left(-\frac{an}{p}\right) \\ &= \frac{1}{\varphi^4(p)} \sum_{a=1}^{p-1} (S(p,a)-1)^4 \, e\left(-\frac{an}{p}\right), \end{aligned} \tag{6.1}$$

where

$$S(p,a) = \sum_{m=1}^{p} e\left(\frac{am^3}{p}\right).$$

If $p \equiv 2(\bmod 3)$, then $S(p,a) = 0$, by Lemma 4.3 of Vaughan [21], and hence

$$A(n,p) = \begin{cases} (p-1)^{-3}, & p|n; \\ -(p-1)^{-4}, & p \nmid n. \end{cases}$$

This proves the first part of Lemma 6.4.

If $p \equiv 1(\bmod 3)$, then the same lemma of Vaughan [21] gives

$$S(p,a) = \bar{\chi}(a)\tau(\chi) + \chi(a)\tau(\bar{\chi}), \tag{6.2}$$

where χ and its conjugate $\bar{\chi}$ are non-principle characters mod$p$ such that χ^3 is the principal character. $\tau(\chi)$ is the Gauss sum defined by

$$\tau(\chi) = \sum_{a=1}^{p} \chi(a) e\left(\frac{a}{p}\right).$$

Recall that $|\tau(\chi)| = p^{\frac{1}{2}}$. Since $\chi(-1) = \chi^3(-1) = 1$, we also have $\tau(\bar{\chi}) = \overline{\tau(\chi)}$.

For $j = 0, 1, ..., 4$, write

$$E_j = \sum_{a=1}^{p-1} S^j(p,a) e\left(-\frac{an}{p}\right).$$

Then by (6.1), one has

$$\varphi^4(p) A(n,p) = E_4 - 4E_3 + 6E_2 - 4E_1 + E_0. \tag{6.3}$$

Now we go to estimate each E_j for $j = 0, 1, ..., 4$ separately.

At first it is easy to derive the following

$$E_0 = \begin{cases} p-1, & p|n; \\ -1, & p \nmid n. \end{cases}$$

Next we have

$$
\begin{aligned}
E_1 &= \sum_{a=1}^{p-1} \left(\bar{\chi}(a)\tau(\chi) + \chi(a)\tau(\bar{\chi})\right) e\left(-\frac{an}{p}\right) \\
&= \tau(\chi)\sum_{a=1}^{p-1} \bar{\chi}(a) e\left(-\frac{an}{p}\right) + \tau(\bar{\chi})\sum_{a=1}^{p-1} \chi(a) e\left(-\frac{an}{p}\right) \\
&= \begin{cases} 0, & p|n; \\ p(\chi(n)+\bar{\chi}(n)), & p\nmid n. \end{cases}
\end{aligned}
$$

By noting $\bar{\chi}^2 = \chi$ and $\chi^2 = \bar{\chi}$, we see that

$$
\begin{aligned}
E_2 &= \sum_{a=1}^{p-1} \left(\bar{\chi}(a)\tau(\chi) + \chi(a)\tau(\bar{\chi})\right)^2 e\left(-\frac{an}{p}\right) \\
&= \tau^2(\chi)\sum_{a=1}^{p-1} \chi(a) e\left(-\frac{an}{p}\right) + 2\tau(\chi)\tau(\bar{\chi})\sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) + \tau^2(\bar{\chi})\sum_{a=1}^{p-1} \bar{\chi}(a) e\left(-\frac{an}{p}\right) \\
&= \begin{cases} 2p(p-1), & p|n; \\ \bar{\chi}(n)\tau^3(\chi) - 2p + \chi(n)\tau^3(\bar{\chi}), & p\nmid n. \end{cases}
\end{aligned}
$$

Also,

$$
\begin{aligned}
E_3 &= \sum_{a=1}^{p-1} \left(\bar{\chi}(a)\tau(\chi) + \chi(a)\tau(\bar{\chi})\right)^3 e\left(-\frac{an}{p}\right) \\
&= \tau^3(\chi)\sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) + 3\tau^2(\chi)\tau(\bar{\chi})\sum_{a=1}^{p-1} \bar{\chi}(a) e\left(-\frac{an}{p}\right) \\
&\quad +3\tau(\chi)\tau^2(\bar{\chi})\sum_{a=1}^{p-1} \chi(a) e\left(-\frac{an}{p}\right) + \tau^3(\bar{\chi})\sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) \\
&= \begin{cases} (p-1)\left(\tau^3(\chi)+\tau^3(\bar{\chi})\right), & p|n; \\ -\left(\tau^3(\chi)+\tau^3(\bar{\chi})\right) + 3p^2(\chi(n)+\bar{\chi}(n)), & p\nmid n. \end{cases}
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
E_4 &= \sum_{a=1}^{p-1} \left(\bar{\chi}(a)\tau(\chi) + \chi(a)\tau(\bar{\chi})\right)^4 e\left(-\frac{an}{p}\right) \\
&= \tau^4(\chi)\sum_{a=1}^{p-1} \bar{\chi}(a) e\left(-\frac{an}{p}\right) + 4\tau^3(\chi)\tau(\bar{\chi})\sum_{a=1}^{p-1} \chi(a) e\left(-\frac{an}{p}\right) + 6|\tau(\chi)|^4\sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) \\
&+ \quad 4\tau(\chi)\tau^3(\bar{\chi})\sum_{a=1}^{p-1} \bar{\chi}(a) e\left(-\frac{an}{p}\right) + \tau^4(\bar{\chi})\sum_{a=1}^{p-1} \chi(a) e\left(-\frac{an}{p}\right) \\
&= \begin{cases} 6(p-1)p^2, & p|n; \\ p\tau^3(\chi)\left(\chi(n)+4\bar{\chi}(n)\right) - 6p^2 + p\tau^3(\bar{\chi})(\bar{\chi}(n)+4\chi(n)), & p\nmid n. \end{cases}
\end{aligned}
$$

Collecting the above estimates, we get from (6.3)

$$\varphi^4(p)\,|A(n,p)| \ll \begin{cases} \left(6p^2+8p^{\frac{3}{2}}+12p+1\right)(p-1), & p|n; \\ 10p^{\frac{5}{2}}+30p^2+20p^{\frac{3}{2}}+20p+1, & p\nmid n. \end{cases}$$

And hence for $p\geq 13$,

$$|A(n,p)| \ll \begin{cases} 12p^{-1}, & p|n; \\ 30p^{-\frac{3}{2}}, & p\nmid n. \end{cases} \tag{6.4}$$

This finishes the proof of Lemma 6.4. □

Proof of Lemma 6.1. The first part of Lemma 6.1 can be proved in the same way as Lemma 18 in Roth [18].

To prove the second part, write

$$Y_p = \begin{cases} 1+A(n,p), & p\neq 3; \\ 1+A(n,p)+A(n,p^2), & p=3. \end{cases}$$

Then by Lemma 6.2, we have

$$\mathfrak{S}(n) = \prod Y_p = \left(\prod_{2\leq p\leq 11} Y_p\right)\left(\prod_{p\geq 13} Y_p\right). \tag{6.5}$$

For $p=2,5,11$, we have $Y_p>0$, by Lemma 6.4. For $p=3,7$, we have

$$Y_3=\varphi(9)^{-3}N(n,3^2), \qquad Y_7=\varphi(7)^{-3}N(n,7),$$

where for $j=1,2$, $N(n,p^j)$ denote the number of solutions of the equation

$$m_1^3+m_2^3+m_3^3+m_4^3=n(\mathrm{mod}p^j), \quad 1\leq m_i<p^j, \quad (m_i,p)=1.$$

By Lemma 8.12 of Hua [4], we have $N(n,9)\geq 1$ for $n\not\equiv \pm1,\pm3(\mathrm{mod}9)$, and hence $Y_3>0$. Counting the number of solutions of the equation

$$m_1^3+\cdots+m_4^3=n(\mathrm{mod}7), \qquad m_i=1,2,...,6,$$

we get $Y_7>0$ for $n\not\equiv\pm1(\bmod 7)$. Collecting these estimates we get $\prod_{2\leq p\leq 11} Y_p>0$ for n satisfying (1,2). This along with (6.4) and (6.5) shows that

$$\mathfrak{S}(n) \gg \left(\prod_{\substack{p\geq 13\\ p\nmid n}}\left(1-30p^{-\frac{3}{2}}\right)\right)\left(\prod_{\substack{p\geq 13\\ p|n}}\left(1-12p^{-1}\right)\right) \gg \prod_{\substack{p\geq 13\\ p|n}}\left(1-12p^{-1}\right) \gg (\log\log n)^{-c_*}.$$

This finish the proof of Lemma 6.1. □

7. Estimation of I_0 and the proof of Theorem 3

Lemma 7.1. *With the notation of Theorem 3, we have*

$$I_0=\mathfrak{S}(n)\mathfrak{J}(n)+O(UL^{-A}),$$

where $\mathfrak{J}(n)=\mathfrak{J}(n;1,1,1,1)$ is defined as in (5.10) and satisfies $U\ll\mathfrak{J}(n)\ll U$.

Proof. By definition,

$$I_0 = \sum_{q \le P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^4(q,a) e\left(-\frac{an}{q}\right) \int_{-\infty}^{+\infty} \Phi^4(\lambda) e(-n\lambda) d\lambda.$$

Using Lemma 5.4, we get

$$I_0 = \sum_{q \le P} \frac{B(n,q)}{\varphi^4(q)} \mathfrak{J}(n; 1, ..., 1) = \mathfrak{J}(n) \sum_{q \le P} A(n,q), \tag{7.1}$$

where we recall that $\mathfrak{J}(n) = \mathfrak{J}(n; 1, ..., 1)$. Now by Lemma 6.1, we have

$$\sum_{q \le P} A(n,q) = \mathfrak{S}(n) + O(P^{-\frac{1}{4}+\varepsilon} d(n)),$$

so (7.1) becomes

$$I_0 = \mathfrak{S}(n)\mathfrak{J}(n) + O(UL^{-A}).$$

Here in the O-term we have used the estimate

$$U \ll \mathfrak{J}(n) \ll U, \tag{7.2}$$

which will be established now.

We first note that the second inequality in (7.2) is a consequence of (5.13). To bound $\mathfrak{J}(n)$ from below, we define the set

$$\mathfrak{D}^* = \left\{ (u_1, ..., u_3) : U^3 \le u_1, ..., u_3 \le 3U^3/2 \right\}.$$

For $(u_1, ..., u_3) \in \mathfrak{D}^*$, one easily derives from $U^3 = N/11$ and $N/2 < n \le N$ that

$$U^3 < u_4 = n - u_1 - ... - u_3 \le 8U^3.$$

Thus $\mathfrak{D}^*$ is a subset of $\mathfrak{D}$ in (5.11), and consequently,

$$\mathfrak{J}(n) \ge \frac{1}{3^4} \int_{\mathfrak{D}^*} u_1^{-\frac{2}{3}} u_2^{-\frac{2}{3}} u_3^{-\frac{2}{3}} u_4^{-\frac{2}{3}} du_1 \cdots du_3 \gg U.$$

This proves (7.2), and hence Lemma 7.1. □

Proof of Theorem 3. The absolute convergence and positivity of $\mathfrak{S}(n)$ have been proved in Lemma 6.1. Other assertions of Theorem 3 follow from Lemmas 4.2, 5.1, and 7.1. □

8. Proof of Theorem 2

Let $G(\alpha)$ be defined by (3.3) with U defined by (3.1). Define

$$r^*(N) = \sum_{\substack{N = p_1^3 + \ldots + p_7^3 \\ p_j \sim U}} \log p_1 \cdots \log p_7.$$

Then

$$r^*(N) = \int_0^1 G^7(\alpha) e(-N\alpha) d\alpha = \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} G^7(\alpha) e(-N\alpha) d\alpha, \tag{8.1}$$

where $\mathfrak{M}$ and $\mathfrak{m}$ is the major arcs and minor arcs defined in section 3 with P, Q defined in (3.1).

Making use of (3.9) and (3.10), the minor arcs can be estimated as

$$\left|\int_{\mathfrak{m}} G^7(\alpha)e(-N\alpha)d\alpha\right| \ll \sup_{\alpha\in\mathfrak{m}}|G(\alpha)|\int_0^1 |G^6(\alpha)|d\alpha \ll U^{4+\varepsilon}P^{-\frac{1}{6}} \ll U^4L^{-A}. \tag{8.2}$$

So we remain to estimate the major arcs.

Let $S_j(\lambda)$ with $j=1,2$ be as in Lemma 4.1. For $j=0,1,...,7$, write

$$I_j^* = \binom{7}{j}\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aN}{q}\right)\int_{-\infty}^{+\infty} S_1^{7-j}(\lambda)S_2^j(\lambda)e(-N\lambda)d\lambda.$$

Then by arguments similar to those as used in Section 4, we obtain

$$\int_{\mathfrak{M}} G^7(\alpha)e(-N\alpha)d\alpha = \sum_{j=0}^{7} I_j^* + O(U^4L^{-A}). \tag{8.3}$$

Besides, we can prove that for $j=1,2...,7$,

$$I_j^* \ll U^4L^{-A}. \tag{8.4}$$

Since the method used in proving (8.4) is similar to that used in proving Lemma 5.1, so we only give a sketch for $j=7$.

Firstly, instead of Lemma 5.2, we now have

$$\sum_{\substack{q\le x\\ \bar{r_0}|q}} \frac{1}{\varphi^7(q)}|B^*(N,q,\chi_1\chi^0,...,\chi_7\chi^0)| \ll \bar{r_0}^{-\frac{5}{2}+\varepsilon}, \tag{8.5}$$

where $\bar{r_0}=[r_1,r_2,...,r_7]$ and

$$B^*(N,q,\chi_1,...,\chi_7) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aN}{q}\right)C(\chi_1,a)\cdots C(\chi_7,a).$$

The anology of Lemma 5.3 is

$$\sum_{X\le r\le P}[g,r]^{-\frac{5}{2}+\varepsilon}\sum_{\chi \bmod r}{}^{*}\sum_{|\gamma|\le T} U^{\beta-1} \ll g^{-\frac{5}{2}+\varepsilon}d(g)G(X)L^2,$$

where

$$G(X) = \begin{cases} X^{-\frac{5}{2}+\varepsilon}+N^{-\eta}, & g=1;\\ 1, & g>1.\end{cases}$$

Next, by an argument similar to that leading to (5.14), we will obtain

$$\begin{aligned} I_7^* &\ll U^4\sum_{r_1\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\sum_{|\gamma_1|\le T}U^{\beta_1-1}\cdots\sum_{r_7\le P}\sum_{\chi_7 \bmod r_7}{}^{*}\sum_{|\gamma_7|\le T}U^{\beta_7-1}\bar{r_0}^{-\frac{5}{2}+\varepsilon}\\ &=: U^4\mathcal{L}, \end{aligned}$$

say. By similar method as used in estimating J in Section 5, we get $\mathcal{L} \ll L^{-A}$ for $\theta = 5/87+\eta$, and hence (8.4) follows.

Now we go to estimate I_0^*. Write

$$B^*(N,q) = B(N,q,\chi_1^0,...,\chi_7^0), \qquad A^*(N,q) = \frac{B^*(N,q)}{\varphi^7(q)},$$

and

$$\mathfrak{S}^*(N) = \sum_{q=1}^{\infty} A^*(N,q).$$

Then by Lemma 8.10 and 8.12 of Hua [4], the above series is convergent absolutely and for all odd integer N with $N \not\equiv 0(\mathrm{mod}9)$,

$$\mathfrak{S}^*(N) \geq c > 0, \tag{8.6}$$

where c is an absolute constant. Also by making use of (4.2) we see that

$$\sum_{q\leq P} A^*(N,q) = \mathfrak{S}^*(N) + O(P^{-\frac{3}{2}+\varepsilon}). \tag{8.7}$$

Define

$$J^*(n;\rho_1,...,\rho_7) = \int_{\mathfrak{D}} u_1^{\frac{\rho_1}{3}-1} u_2^{\frac{\rho_2}{3}-1} \cdots u_6^{\frac{\rho_6}{3}-1} (N-u_1-...-u_6)^{\frac{\rho_7}{3}-1} du_1 du_2 \cdots du_6,$$

where

$$\mathfrak{D} = \{(u_1,...,u_6) : U^3 \leq u_1,...,u_6, N-u_1-...-u_6 \leq 8U^3\}.$$

And write

$$J^*(N) = J^*(N;1,\cdots,1).$$

Then by analogous arguments as in section 7, we get

$$I_0^* = J^*(N)\sum_{q\leq P} A^*(N,q) = \mathfrak{S}^*(N)J^*(N) + O(U^4L^{-A}) \tag{8.8}$$

for $\theta = 5/87 + \eta$. Here in the O term we have used the estimate

$$U^4 \ll J^*(N) \ll U^4 \tag{8.9}$$

which can be established in a similar way as (7.2).

Theorem 2 now follows from (8.1)-(8.4), (8.8) and (8.9). □

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Permanent Address:	Current Address: [2]
School of Mathematics	Department of Mathematics
Shandong University	The University of Hong Kong
Jinan, Shandong 250100	Pokfulam,
China	Hong Kong
	`xmren@submaths.hku.hk`

[2]Please use this address.