THE WARING-GOLDBACH PROBLEM FOR UNLIKE POWERS

by

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Abstract. In this paper, it is proved that with at most $O(N^{\frac{65}{66}})$ exceptions, all even positive integers up to N are expressible in the form $p_2^2 + p_3^3 + p_4^4 + p_5^5$. This improves a recent result $O(N^{\frac{19193}{19200}+\varepsilon})$ due to C.Bauer.

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1. Introduction

In 1951, Roth [14] proved that almost all positive integers n can be written in the form

$$n = m_2^2 + m_3^3 + m_4^4 + m_5^5,$$

where m_k are positive integers. Later in 1953, Prachar [11] improved the above result by proving that

$$n = p_2^2 + p_3^3 + p_4^4 + p_5^5 \tag{1.1}$$

is expressible for almost all positive even integers n. Let $E(N)$ denote the number of even positive integers n up to N that can not be written as (1.1). Then Prachar proved

$$E(N) \ll N(\log N)^{-\frac{30}{47}+\varepsilon}, \tag{1.2}$$

where $\varepsilon > 0$ is arbitrary. In another paper [12], Prachar considered the expression

$$n = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5, \tag{1.3}$$

and showed that all large odd integers n can be written in this way. Recently, Bauer [1] improved (1.2) to

$$E(N) \ll N^{\frac{19193}{19200}+\varepsilon}. \tag{1.4}$$

Here we note that $19193/19200 > 1 - \delta$ with $\delta = 1/2742$.

In this paper, we give the following improvement on (1.4).

Theorem 1. *Let $E(N)$ be defined as above. Let $\delta = 1/66$. Then we have*

$$E(N) \ll N^{1-\delta}.$$

By a standard argument, we deduce from Theorem 1 the second result of Prachar.

Corollary. *All large odd integers n can be expressed as* (1.3).

We prove Theorem 1 by the circle method. A result of this strength needs efforts in two aspects. On one hand, we have to enlarge the major arcs, to which the Siegel-Walfisz theorem does not apply. Usually, one treats the enlarged major arcs by employing the Deuring-Heilbronn phenomenon. Here we attack the major arcs by a different approach, which has been successfully applied in several other occasions, for example [8], [9], and [13]. The key point of this approach is that one can save the factor $r_0^{-\alpha+\varepsilon}$ (see, for example, Lemma 4.2). With this saving, the enlarged major arcs

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can be treated by the classical zero-density estimates (see, for example, Lemma 3.3) and zero-free region for the Dirichlet L-functions (as is defined in (4.19)). Previously, the factor $r_0^{-\alpha+\varepsilon}$ is divided equally to each variable (see, for example, p. 121 in [13]), and this causes some waste. In this paper, we develop an iterative procedure, and hence make full use of the minus power in $r_0^{-\alpha+\varepsilon}$ for each variable. For this, Huxley's zero-density estimate involving arithmetic progressions is applied (see Lemma 3.4). In addition to the iterative procedure, another advantage of our method comes from the skilful handling of $(1+|\gamma|)^{-\xi}$ in the proof of Lemma 4.1, where γ is the imaginary part of a non-trivial zero of the Dirichlet L-functions. Besides, the technique of truncating the summation over q in (4.15) is also crucial in our argument. On the other hand, in handling the minor arcs, we employ result of Kawada and Wooley [7] to give the upper bound estimate for trigonometric sums over primes (see Lemma 2.1). This in combination with invoking of Lemma 5 in Roth [14] leads to a better minor arcs estimate than in Bauer [1].

We prove Theorem 1 in detail in Sections 2-6.

Notation. As usual, $\varphi(n)$ and $\Lambda(n)$ stand for the function of Euler and von Mangoldt, respectively, and $d(n)$ is the divisor function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote respectively a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. We write $e(r)$ for $\exp(i2\pi r)$ and write L for $\log N$ with a large positive integer N. Further, $r \sim R$ means $R < r \le 2R$. The symbols c_j denote unspecified fixed constants. The letters ε and A denote positive constants, which are arbitrarily small and arbitrarily large, respectively. We may not distinguish cA and A for positive constant c. This is also applied to ε.

2. Outline of the method

For large positive integer N, set

$$P = N^{\theta}, \quad Q = NP^{-(1+\eta)}, \quad \text{where} \quad \theta = 2/33 + \eta, \quad \eta = 10^{-5}. \tag{2.1}$$

By Dirichlet's lemma on rational approximations, each $\alpha \in [1/Q, 1+1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \qquad |\lambda| \le 1/qQ \tag{2.2}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a,q) = 1$. We denote by $\mathfrak{M}(q,a)$ the set of α satisfying (2.2), and write $\mathfrak{M}$ for the union of all $\mathfrak{M}(q,a)$ with $1 \le a \le q \le P$. The minor arcs $\mathfrak{m}$ are defined as the complement of $\mathfrak{M}$ in $[1/Q, 1+1/Q]$. It follows from $2P \le Q$ that the major arcs $\mathfrak{M}(q,a)$ are mutually disjoint.

For $k = 2,3,4,5$, let

$$U_k = (N/32)^{\frac{1}{k}}, \tag{2.3}$$

and let

$$r(n) = \sum_{\substack{n = p_2^2 + p_3^3 + p_4^4 + p_5^5 \\ p_k \sim U_k}} (\log p_2) \cdots (\log p_5).$$

Define

$$G_k(\alpha)=\sum_{p\sim U_k}(\log p)e(p^k\alpha),\qquad S_k(\alpha)=\sum_{m\sim U_k}\Lambda(m)e(m^k\alpha). \tag{2.4}$$

Then

$$r(n)=\int_{\frac{1}{Q}}^{1+\frac{1}{Q}}\left(\prod_{k=2}^{5}G_k(\alpha)\right)e(-n\alpha)d\alpha=\int_{\mathfrak{M}}+\int_{\mathfrak{m}}. \tag{2.5}$$

To handle the integral on the major arcs, we need the following.

Theorem 2. *Let $N/2\le n\le N$. Then for all n but at most $O(NP^{-\frac{12}{25}+\varepsilon})$ exceptions, we have*

$$\int_{\mathfrak{M}}\left(\prod_{k=2}^{5}G_k(\alpha)\right)e(-n\alpha)d\alpha=\mathfrak{S}(n)\mathfrak{J}(n)+O\left(N^{\frac{17}{60}}L^{-A}\right). \tag{2.6}$$

Here $\mathfrak{S}(n)$ is the singular series defined by (5.1) *which satisfies*

$$(\log\log n)^{-c^*}\ll\mathfrak{S}(n)\ll d(n),$$

for even integers n and some absolute positive constant c^*; and $\mathfrak{J}(n)$ is defined by (6.2) *and satisfies*

$$N^{\frac{17}{60}}\ll\mathfrak{J}(n)\ll N^{\frac{17}{60}}.$$

To deal with the minor arcs, we make use of the following result of Kawada and Wooley in [7].

Lemma 2.1. *Let $U\ge 2$ be a real number. Suppose that α is a real number, and that there exist integers a and q satisfying*

$$(a,q)=1,\quad 1\le q\le U^2,\quad \text{and}\quad |q\alpha-a|\le U^{-2}. \tag{2.7}$$

Then one has

$$\sum_{U<p\le 2U}(\log p)e(\alpha p^4)\ll U^{\frac{31}{32}+\varepsilon}+\frac{q^{\varepsilon}\omega(q)^{\frac12}U(\log U)^5}{(1+U^4|\alpha-a/q|)^{\frac12}},$$

where $\omega(q)$ is a multiplicative function defined by

$$\omega(p^{4u+v})=\begin{cases}4\,p^{-u-\frac12}, & u\ge 0,\quad \text{and}\quad v=1;\\ p^{-u-1}, & u\ge 0,\quad \text{and}\quad 2\le v\le 4.\end{cases}$$

Proof of Theorem 1. By Bessel's inequality, we have

$$\sum_{N/2<n\le N}\left|\int_{\mathfrak{m}}\left(\prod_{k=2}^{5}G_k(\alpha)\right)e(-n\alpha)\,d\alpha\right|^2\ll\sup_{\alpha\in\mathfrak{m}}|G_4(\alpha)|^2\int_0^1|G_2(\alpha)G_3(\alpha)G_5(\alpha)|^2\,d\alpha. \tag{2.8}$$

By Dirichlet's lemma on rational approximations, for each $\alpha\in\mathfrak{m}$, there exist positive integers a and q satisfying (2.7). Since $\alpha\in\mathfrak{m}$, we conclude that $q>P$, or $q\le P$ but $|\alpha q-a|>1/Q$. So by Lemma 2.1, we have

$$\sup_{\alpha\in\mathfrak{m}}|G_4(\alpha)|\ll U_4^{\frac{31}{32}+\varepsilon}+U_4^{1+\varepsilon}P^{-\frac18}\ll U_4^{1+\varepsilon}P^{-\frac18}. \tag{2.9}$$

Moreover, by Lemma 5(b) in Roth [14], one has

$$\int_0^1 |G_2(\alpha)G_3(\alpha)G_5(\alpha)|^2\,d\alpha \ll N^{\frac{16}{15}+\varepsilon}. \tag{2.10}$$

Inserting (2.9) and (2.10) into (2.8), the right-hand side of (2.8) becomes $O(N^{\frac{47}{30}+\varepsilon}P^{-\frac{1}{4}})$. Therefore, for even integers $n \in [N/2, N]$ with at most $O(NP^{-\frac{1}{4}+\varepsilon}) = O(N^{\frac{65}{66}})$ exceptions, one has the estimate

$$\left|\int_{\mathfrak{m}} \prod_{k=2}^{5} G_k(\alpha)e(-n\alpha)d\alpha\right| \ll N^{\frac{17}{60}-\varepsilon}.$$

This in combination with Theorem 2 and (2.5) shows that for those unexceptional n, the expression (1.1) holds and the number of expressions satisfies

$$r(n) = \mathfrak{S}(n)\mathfrak{J}(n) + O(N^{\frac{17}{60}}L^{-A}).$$

The assertion of Theorem 1 now follows by summing over dyadic intervals. □

3. An explicit expression

The purpose of this section is to establish in Lemma 3.2 an explicit expression for the left-hand side of (2.6).

For χ mod q and χ^0 mod q, define

$$C_k(\chi, a) = \sum_{m=1}^{q} \overline{\chi}(m)e\left(\frac{am^k}{q}\right), \qquad C_k(q,a) = C_k(\chi^0, a). \tag{3.1}$$

Then Vinogradov's estimate (see for example [16], Ch.VI, problem 14b(α)) gives

$$|C_k(\chi, a)| \le 2q^{\frac{1}{2}}d^{t_k}(q) \quad \text{with} \quad t_k = \log k/\log 2. \tag{3.2}$$

Define

$$\Phi_k(\lambda) = \int_{U_k}^{2U_k} e(\lambda u^k)du, \qquad \Psi_k(\lambda,\rho) = \int_{U_k}^{2U_k} u^{\rho-1}e(\lambda u^k)du, \tag{3.3}$$

and write

$$V_k(\lambda) = \frac{C_k(q,a)}{\varphi(q)}\Phi_k(\lambda), \qquad W_k(\lambda, T) = -\frac{1}{\varphi(q)}\sum_{\chi \bmod q} C_k(\chi, a)\sum_{|\gamma|\le T}\Psi_k(\lambda,\rho), \tag{3.4}$$

where $\sum_{|\gamma|\le T}$ denotes summation with respect to non-trivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ for $|\gamma| \le T$ and $0 < \beta < 1$. For $k = 2, 3, 4, 5$, we set

$$T_k = P^{\alpha_k}, \quad \text{where} \quad \alpha_2 = \alpha_3 = 3/2 + 2\eta, \quad \alpha_4 = \alpha_5 = 1 + 2\eta. \tag{3.5}$$

Writing $W_k(\lambda) = W_k(\lambda, T_k)$ for simplicity, we have the following.

Lemma 3.1. *For $\alpha = a/q + \lambda$ with $(a,q) = 1$, we have*

$$G_k(\alpha) = V_k(\lambda) + W_k(\lambda) + R_k(\lambda),$$

where

$$R_k(\lambda) \ll q^{\frac{1}{2}+\varepsilon}\frac{U_k}{T_k}(1 + |\lambda|N)L^2.$$

Proof. By (2.4), we have

$$|S_k(\alpha) - G_k(\alpha)| \ll \sum_{h=2}^{\infty} \sum_{p^h \sim U_k} \log p \ll U_k^{\frac{1}{2}} L \ll \frac{U_k}{T_k}. \tag{3.6}$$

Introducing Dirichlet characters, $S_k(\alpha)$ can be rewritten as (see [2], §26, (2))

$$S_k\left(\frac{a}{q} + \lambda\right) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C_k(\chi, a) \sum_{m \sim U_k} \Lambda(m)\chi(m)e\left(\lambda m^k\right) + O\left(L^2\right). \tag{3.7}$$

Now we apply the explicit formula (see [2], §17, (9)-(10); §19, (4)-(9))

$$\sum_{m \le x} \chi(m)\Lambda(m) = \delta_\chi x - \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x(\log qxT)^2}{T} + (\log qx)^2\right),$$

where $\delta_\chi = 1$ or 0 according as $\chi = \chi^0$ or not. The inner sum in (3.7) therefore becomes

$$\int_{U_k}^{2U_k} e(\lambda u^k) d\left\{\sum_{m \le u} \chi(m)\Lambda(m)\right\} = \delta_\chi \Phi_k(\lambda) - \sum_{|\gamma| \le T} \Psi_k(\lambda, \rho) + \int_{U_k}^{2U_k} e(\lambda u^k) d\bar{r}(u),$$

where $\bar{r}(x)$ is the O-term in the above explicit formula. The last integral is then

$$\ll |\bar{r}(2U_k)| + |\bar{r}(U_k)| + U_k^k |\lambda| \max_{u \sim U_k} |\bar{r}(u)| \ll \frac{U_k}{T}(1 + |\lambda|N)L^2.$$

By making use of (3.2) and letting $T = T_k$, we thus get

$$S_k(\alpha) = V_k(\lambda) + W_k(\lambda) + O\left(q^{\frac{1}{2}+\varepsilon} \frac{U_k}{T_k}(1 + |\lambda|N)L^2\right).$$

This together with (3.6) finishes the proof of Lemma 3.1. □

Now we state the main result of this section.

Lemma 3.2. *Let $N/2 < n \le N$. Then we have*

$$\int_{\mathfrak{M}} \left\{\prod_{k=2}^{5} G_k(\alpha)\right\} e(-n\alpha) d\alpha$$

$$= \sum_{q \le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left\{\prod_{k=2}^{5} (V_k(\lambda) + W_k(\lambda))\right\} e(-n\lambda)\, d\lambda + O(N^{\frac{17}{60}} L^{-A}). \tag{3.8}$$

To prove this result, we need the following lemmas on zero-density estimates.

Let $N(\sigma, T, \chi)$ denote the number of zeros of $L(s, \chi)$ in the region $\sigma \le \mathrm{Re}s \le 1$, $|\mathrm{Im}s| \le T$. Let

$$N(\sigma, T, q) = \sum_{\chi \bmod q} N(\sigma, T, \chi), \qquad N^*(\sigma, T, X, d) = \sum_{\substack{q \le X \\ q \equiv 0 (\bmod d)}} \sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi),$$

and write $N^*(\sigma, T, X) = N^*(\sigma, T, X, 1)$, where * means that the summation is restricted to primitive characters χ mod q.

Lemma 3.3. *Let $1/2 \le \sigma < 1$. Then we have*

$$N(\sigma, T, q) \ll (qT)^{(\frac{12}{5}+\varepsilon)(1-\sigma)},$$

and
$$N^*(\sigma,T,X)\ll\left(X^2T^{\frac{6}{5}}\right)^{(A(\sigma)+\varepsilon)(1-\sigma)},$$
where
$$A(\sigma)=\begin{cases}20/9, & 3/4\le\sigma\le 1;\\ 5/(3-\sigma), & 1/2\le\sigma<3/4.\end{cases}$$
In particular $A(\sigma)\le 20/9$ *holds for all* $\sigma\in[1/2,1]$.

Proof. The first conclusion follows easily from (1.1) of Huxley [5] and Theorem 1 of Jutila [6]. The second result is an immediate consequence of Theorem 2 in Heath-Brown [3]. □

Lemma 3.4. *Let* $T\ge 1$ *and* $1\le d\le X$. *Then for* $1/2\le\sigma\le 1-\eta$,
$$N^*(\sigma,T,X,d)\ll\left(X^2T/d\right)^{(\frac{12}{5}+\varepsilon)(1-\sigma)};\tag{3.9}$$
and for $1-\eta<\sigma\le 1$,
$$N^*(\sigma,T,X,d)\ll\left(X^3T^2\right)^{(1+\varepsilon)(1-\sigma)}.\tag{3.10}$$

Proof. For $1/2\le\sigma<1-\eta$, (3.9) follows easily from (1.1) of Huxley [5]. For $1-\eta\le\sigma\le 1$, the left-hand side of (3.10) is bounded by $N^*(\sigma,T,X)$, which admits the following estimate (see p. 54 in [6])
$$N^*(\sigma,T,X)\ll\left(X^3T^2\right)^{(1+\varepsilon)(1-\sigma)}. \qquad \square$$

Proof of Lemma 3.2. We have
$$\begin{aligned}&\int_{\mathfrak{M}}\left\{\prod_{k=2}^{5}G_k(\alpha)\right\}e(-n\alpha)\,d\alpha\\ &=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{an}{q}\right)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left\{\prod_{k=2}^{5}G_k\left(\frac{a}{q}+\lambda\right)\right\}e(-n\lambda)\,d\lambda.\end{aligned}$$

So by Lemma 3.1, the difference between the two main terms in (3.8) is
$$\ll\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left\{|I_1(\lambda)|+|I_2(\lambda)|+|I_3(\lambda)|+|I_4(\lambda)|\right\}d\lambda,$$

where
$$I_1(\lambda)=\left\{\prod_{k=2}^{4}\left(V_k(\lambda)+W_k(\lambda)\right)\right\}R_5(\lambda),\tag{3.11}$$
$$I_2(\lambda)=\left\{\prod_{k=2}^{3}\left(V_k(\lambda)+W_k(\lambda)\right)\right\}R_4(\lambda)G_5\left(\frac{a}{q}+\lambda\right),\tag{3.12}$$
$$I_3(\lambda)=\{V_2(\lambda)+W_2(\lambda)\}\,R_3(\lambda)\prod_{k=4}^{5}G_k\left(\frac{a}{q}+\lambda\right),$$
$$I_4(\lambda)=R_2(\lambda)\prod_{k=3}^{5}G_k\left(\frac{a}{q}+\lambda\right).$$

Therefore to prove Lemma 3.2, we only need to prove

$$\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|I_j(\lambda)|\,d\lambda \ll N^{\frac{17}{60}}L^{-A},\quad \text{for}\quad j=1,\,2,\,3,\,4. \tag{3.13}$$

To this end, we first establish the following estimates: For $\lambda\in\mathbb{R}$, and $q\in\mathbb{N}$,

$$V_k(\lambda)\ll q^{-\frac{1}{2}+\varepsilon}\frac{U_k}{1+|\lambda|N},\quad \text{for}\ \ k=2,\,...,\,5; \tag{3.14}$$

and, for $|\lambda|\le 1/qQ$ with $1\le q\le P$,

$$W_k(\lambda)\ll q^{-\frac{1}{2}+\varepsilon}L^2\begin{cases}\frac{U_k}{\sqrt{1+|\lambda|N}}, & \text{for}\quad k=2,\,3,\,4;\\ \frac{U_5}{(1+|\lambda|N)^{1/5}}, & \text{for}\quad k=5.\end{cases} \tag{3.15}$$

One can easily obtain (3.14) from (3.3) and (3.4) by using (3.2) and the following obvious estimate

$$\Phi_k(\lambda)=\frac{1}{k}\int_{U_k^k}^{(2U_k)^k}u^{\frac{1}{k}-1}e(\lambda u)du\ll U_k\min\left\{1,\frac{1}{|\lambda|U_k^k}\right\}. \tag{3.16}$$

To prove (3.15), we apply Lemmas 4.3 and 4.5 in Titchmarsh [15], to get

$$\begin{aligned}\Psi_k(\lambda,\rho) &= \frac{1}{k}\int_{U_k^k}^{(2U_k)^k}u^{\frac{\beta}{k}-1}e\left(\lambda u+\frac{\gamma}{2k\pi}\log u\right)du\\ &\ll U_k^{\beta-k}\min\left\{U_k^k,\frac{U_k^k}{\min\limits_{U_k^k\le u\le(2U_k)^k}|\gamma+2k\pi\lambda u|},\frac{U_k^k}{\sqrt{|\gamma|}}\right\}.\end{aligned} \tag{3.17}$$

Let T_k be as defined in (3.5). Since

$$\min_{U_k^k\le u\le(2U_k)^k}|\gamma+2k\pi\lambda u|\gg\begin{cases}|\lambda|U_k^k, & \text{if }|\gamma|\le k\pi|\lambda|U_k^k;\\ |\gamma|, & \text{if }|\gamma|>4k\pi|\lambda|(2U_k)^k,\end{cases}$$

one derives from (3.17) and (2.3) that

$$\Psi_k(\lambda,\rho)\ll\begin{cases}\frac{U_k^\beta}{\sqrt{1+|\lambda|N}}, & \text{for}\quad |\gamma|\le 20\pi|\lambda|N;\\ \frac{U_k^\beta}{1+|\gamma|}, & \text{for}\quad 20\pi|\lambda|N<|\gamma|\le T_k.\end{cases} \tag{3.18}$$

Therefore by applying (3.2), we get from (3.4) that

$$\begin{aligned}W_k(\lambda) &\ll q^{-\frac{1}{2}+\varepsilon}U_k\left\{\frac{1}{\sqrt{1+|\lambda|N}}\sum_{\chi\bmod q}\sum_{|\gamma|\le 20\pi|\lambda|N}U_k^{\beta-1}\right.\\ &\qquad\left.+L\max_{20\pi|\lambda|N<R\le T_k/2}(1+R)^{-1}\sum_{\chi\bmod q}\sum_{|\gamma|\le 2R}U_k^{\beta-1}\right\}.\end{aligned} \tag{3.19}$$

By integrating by parts, the first double sum in (3.19) is estimated as

$$\begin{aligned} &= -\int_{\frac{1}{2}}^{1} U_k^{\sigma-1} dN(\sigma, 20\pi|\lambda|N, q) + O\left(U_k^{-\frac{1}{2}} N(1/2, 20\pi|\lambda|N, q)\right) \\ &\ll (\log U_k) \int_{\frac{1}{2}}^{1} U_k^{\sigma-1} N(\sigma, 20\pi|\lambda|N, q) d\sigma + U_k^{-\frac{1}{2}} N(1/2, 20\pi|\lambda|N, q). \end{aligned} \tag{3.20}$$

Making use of Lemma 3.3, we see that the right-hand side of (3.20) is

$$\ll L\left\{1 + (q|\lambda|N)^{\frac{6}{5}+\varepsilon} U_k^{-\frac{1}{2}}\right\} \ll L\left\{1 + P^{\frac{6}{5}(1+\eta)+\varepsilon} U_k^{-\frac{1}{2}}\right\} \ll L,$$

on recalling (2.1) and (2.3).

Similarly, the second quantity in the bracket of (3.19) is

$$\begin{aligned} &\ll \frac{L^2}{(1+|\lambda|N)^{\eta_k}} \max_{1<R\le T_k/2} R^{\eta_k-1}\left\{1 + (qR)^{\frac{6}{5}+\varepsilon} U_k^{-\frac{1}{2}}\right\} \\ &\ll \frac{L^2}{(1+|\lambda|N)^{\eta_k}}\left\{1 + P^{\frac{6}{5}+\varepsilon} T_k^{\frac{1}{5}+\eta_k+\varepsilon} U_k^{-\frac{1}{2}}\right\} \\ &\ll \frac{L^2}{(1+|\lambda|N)^{\eta_k}} \end{aligned}$$

provided that $\eta_2 = \eta_3 = \eta_4 = 1/2$, and $\eta_5 = 1/5$. This proves (3.15).

Now we come to prove (3.13). By (3.14) and (3.15), we have for $|\lambda| \le 1/qQ$ with $1 \le q \le P$,

$$V_k(\lambda) + W_k(\lambda) \ll q^{-\frac{1}{2}+\varepsilon} L^2 \begin{cases} \frac{U_k}{\sqrt{1+|\lambda|N}}, & \text{for} \quad k = 2,\ 3,\ 4; \\ \frac{U_5}{(1+|\lambda|N)^{1/5}}, & \text{for} \quad k = 5. \end{cases} \tag{3.21}$$

Thus, by (3.11) and Lemma 3.1, we get

$$|I_1(\lambda)| \ll q^{-1+\varepsilon} T_5^{-1} L^8 \frac{N^{\frac{77}{60}}}{\sqrt{1+|\lambda|N}},$$

and therefore,

$$\sum_{q\le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |I_1(\lambda)|\, d\lambda \ll N^{\frac{17}{60}} T_5^{-1} L^8 P^{\frac{1}{2}(1+\eta)} \sum_{q\le P} q^{-\frac{1}{2}+\varepsilon} \ll N^{\frac{17}{60}} L^{-A},$$

on noting $T_5 = P^{1+2\eta}$ in the last step.

Next, by (3.12) and Lemma 3.1, we have

$$\begin{aligned} |I_2(\lambda)| \ll &\left\{\prod_{k=2}^{3} (V_k(\lambda) + W_k(\lambda))\right\} (V_5(\lambda) + W_5(\lambda))\, R_4(\lambda) \\ &+ \left\{\prod_{k=2}^{3} (V_k(\lambda) + W_k(\lambda))\right\} R_4(\lambda) R_5(\lambda). \end{aligned}$$

On using (3.21), the last expression is

$$\ll q^{-1+\varepsilon}T_4^{-1}L^8\frac{N^{\frac{77}{60}}}{(1+|\lambda|N)^{1/5}}+q^{\varepsilon}T_4^{-1}T_5^{-1}L^8N^{\frac{77}{60}}(1+|\lambda|N).$$

Thus, on recalling $T_4=T_5=P^{1+2\eta}$, we get

$$\begin{aligned}&\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|I_2(\lambda)|\,d\lambda\\ &\ll N^{\frac{17}{60}}L^8\left\{T_4^{-1}P^{\frac{4}{5}(1+\eta)}\sum_{q\le P}q^{-\frac{4}{5}+\varepsilon}+T_4^{-1}T_5^{-1}P^{2(1+\eta)}\sum_{q\le P}q^{-1+\varepsilon}\right\}\\ &\ll N^{\frac{17}{60}}L^{-A}.\end{aligned}$$

Similar arguments also show that

$$\begin{aligned}|I_3(\lambda)| &\ll q^{-1+\varepsilon}T_3^{-1}L^8\frac{N^{\frac{77}{60}}}{(1+|\lambda|N)^{1/5}}+q^{\varepsilon}T_3^{-1}T_4^{-1}L^8N^{\frac{77}{60}}(1+|\lambda|N)^{\frac{13}{10}}\\ &\quad+q^{1+\varepsilon}T_3^{-1}T_4^{-1}T_5^{-1}L^8N^{\frac{77}{60}}(1+|\lambda|N)^{\frac{5}{2}},\end{aligned}$$

and since $T_3=P^{3/2+2\eta}$,

$$\begin{aligned}&\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|I_3(\lambda)|\,d\lambda\\ &\ll N^{\frac{17}{60}}L^8\left\{T_3^{-1}P^{1+\eta}+T_3^{-1}T_4^{-1}P^{\frac{23}{10}(1+\eta)}+T_3^{-1}T_4^{-1}T_5^{-1}P^{\frac{7}{2}(1+\eta)}\right\}\\ &\ll N^{\frac{17}{60}}L^{-A}.\end{aligned}$$

Finally, we have analogously

$$\begin{aligned}&\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}|I_4(\lambda)|\,d\lambda\\ &\ll N^{\frac{17}{60}}L^8\Big\{T_2^{-1}P^{1+\eta}+T_2^{-1}T_4^{-1}P^{\frac{23}{10}(1+\eta)}+T_2^{-1}T_4^{-1}T_5^{-1}P^{\frac{7}{2}(1+\eta)}\\ &\quad+T_2^{-1}T_3^{-1}T_4^{-1}P^{\frac{19}{5}(1+\eta)}+T_2^{-1}T_3^{-1}T_4^{-1}T_5^{-1}P^{5(1+\eta)}\Big\}\\ &\ll N^{\frac{17}{60}}L^{-A},\end{aligned}$$

on noting $T_2=P^{3/2+2\eta}$. This proves (3.13), and thus finishes the proof of Lemma 3.2. □

Define $\Delta(\lambda)$ by

$$\prod_{k=2}^{5}(V_k(\lambda)+W_k(\lambda))=\prod_{k=2}^{5}V_k(\lambda)+\Delta(\lambda), \tag{3.22}$$

and let

$$I=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{an}{q}\right)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left(\prod_{k=2}^{5}V_k(\lambda)\right)e(-n\lambda)\,d\lambda, \tag{3.23}$$

$$J=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{an}{q}\right)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\Delta(\lambda)e(-n\lambda)\,d\lambda.$$

Then by Lemma 3.2, we have

$$\int_{\mathfrak{M}}\left(\prod_{k=2}^{5}G_k(\alpha)\right)e(-n\alpha)\,d\alpha=I+J+O\left(N^{\frac{17}{60}}L^{-A}\right). \tag{3.24}$$

In the following sections, we will prove that I gives the main term while J contributes to the error term.

4. Estimation of J

Lemma 4.1. *For all but $O(NP^{-\frac{12}{25}+\varepsilon})$ exceptional integers $n\in[N/2,N]$, we have*

$$J\ll N^{\frac{17}{60}}L^{-A}.$$

To prove Lemma 4.1, we need some preparations.

Let $C_k(\chi,a)$ and $C_k(q,a)$ be as defined in (3.1). For characters χ_j mod q, we define

$$B(n,q,\chi_2,...,\chi_5)=\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{an}{q}\right)\prod_{k=2}^{5}C_k(\chi_k,a). \tag{4.1}$$

Lemma 4.2. *Let χ_j be primitive characters* $\bmod\, r_j$ *with $j=2,...,5$, and χ^0 be the principal character* $\bmod q$. *Denote $r_0=[r_2,...,r_5]$, the least common multiple of $r_2,...,r_5$. Then for any real number $x\ge 2$ and positive integer n up to N, one has*

$$\sum_{\substack{q\le x\\ r_0|q}}\frac{1}{\varphi^4(q)}|B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|\ll r_0^{-1+\varepsilon}\log^{c_0}x. \tag{4.2}$$

Proof. By (3.2), one has

$$B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)\ll\sum_{\substack{a=1\\(a,q)=1}}^{q}\prod_{j=2}^{5}|C(\chi_j\chi^0,a)|\ll q^3d^7(q).$$

So the left-hand side of (4.2) is

$$\ll\sum_{\substack{q\le x\\ r_0|q}}\frac{q^3d^7(q)}{\varphi^4(q)}\ll r_0^{-1+\varepsilon}\log x\sum_{q\le x}\frac{d^7(q)}{q}\ll r_0^{-1+\varepsilon}\log^{c_0}x. \qquad\square$$

Lemma 4.3. *Let $1 < Y^{3+\eta} < U \le N$, $1 \le T \le N$ and $g \ge 1$. Then for $6\eta \le \alpha \le 12/5$ and $3\eta \le \xi < 6/5$, we have*

$$\sum_{r \le Y} [g, r]^{-\alpha} \sum_{\chi \bmod r} {}^{*} \sum_{|\gamma| \le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}} \ll g^{-\alpha} d(g) G(\alpha, \xi, Y, T, U) L^3,$$

where

$$G(\alpha, \xi, Y, T, U) = 1 + U^{-\frac{1}{2}} Y^{\frac{12}{5}+\varepsilon-\min(\alpha, \frac{6}{5})} T^{\frac{6}{5}-\xi+\varepsilon}.$$

In particular, for $k = 2$, 3, 4, and $\rho \le 6/7$, we have

$$G(1-\varepsilon, \xi_k, P^{\rho}, T_k, U_k) \ll 1, \tag{4.3}$$

and

$$G(6/5-\varepsilon, \xi_k, P, T_k, U_k) \ll 1, \tag{4.4}$$

where

$$\xi_2 = 5\eta, \quad \xi_3 = 1/5 - 6\eta, \quad \xi_4 = 2/5. \tag{4.5}$$

Proof. We have $[g, r] = gr(g, r)^{-1}$, and so

$$\sum_{r \le Y} [g, r]^{-\alpha} \sum_{\chi \bmod r} {}^{*} \sum_{|\gamma| \le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}} \le g^{-\alpha} \sum_{\substack{d \le Y \\ d|g}} d^{\alpha} \sum_{\substack{r \le Y \\ d|r}} r^{-\alpha} \sum_{\chi \bmod r} {}^{*} \sum_{|\gamma| \le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}}. \tag{4.6}$$

Moreover

$$\begin{aligned} &\sum_{\substack{r \le Y \\ d|r}} r^{-\alpha} \sum_{\chi \bmod r} {}^{*} \sum_{|\gamma| \le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}} \\ &\ll (\log Y)(\log T) \max_{d \le R \le Y/2} R^{-\alpha} \max_{1 \le \Gamma \le T/2} (1+\Gamma)^{-\xi} \sum_{\substack{r \sim R \\ d|r}} \sum_{\chi \bmod r} {}^{*} \sum_{|\gamma| \sim \Gamma} U^{\beta-1}. \end{aligned} \tag{4.7}$$

By integrating by parts, one has

$$\begin{aligned} &\sum_{\substack{r \le 2R \\ d|r}} \sum_{\chi \bmod r} {}^{*} \sum_{|\gamma| \le 2\Gamma} U^{\beta-1} \\ &= -\int_{\frac{1}{2}}^{1} U^{\sigma-1} dN^*(\sigma, 2\Gamma, 2R, d) + O\left(U^{-\frac{1}{2}} N^*(1/2, 2\Gamma, 2R, d)\right) \\ &\ll L \left\{ U^{-\frac{1}{2}} N^*(1/2, 2\Gamma, 2R, d) + \int_{\frac{1}{2}}^{1} U^{\sigma-1} N^*(\sigma, 2\Gamma, 2R, d) d\sigma \right\}. \end{aligned}$$

An application of Lemma 3.4 gives

$$N^*(1/2, 2\Gamma, 2R, d) \ll \left\{R^2 \Gamma / d\right\}^{\frac{6}{5}+\varepsilon},$$

$$\int_{\frac{1}{2}}^{1-\eta} U^{\sigma-1}N^*(\sigma,2\Gamma,2R,d)d\sigma \ll \left\{U^{-1}(R^2\Gamma/d)^{\frac{12}{5}+\varepsilon}\right\}^{\frac{1}{2}} + \left\{U^{-1}(R^2\Gamma/d)^{\frac{12}{5}+\varepsilon}\right\}^{\eta}$$
$$< U^{-\frac{1}{2}}(R^2\Gamma/d)^{\frac{6}{5}+\varepsilon} + U^{-\eta}\Gamma^{3\eta}(R^2/d)^{3\eta},$$

and

$$\int_{1-\eta}^{1} U^{\sigma-1}N^*(\sigma,2\Gamma,2R,d)d\sigma \ll 1 + \left\{U^{-1}(R^3\Gamma^2)^{1+\varepsilon}\right\}^{\eta} < 1 + U^{-\eta}\Gamma^{3\eta}R^{3(1+\varepsilon)\eta}.$$

Therefore

$$\sum_{\substack{r\le 2R\\ d|r}} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le 2\Gamma} U^{\beta-1}$$
$$\ll L\left\{1 + U^{-\frac{1}{2}}(R^2\Gamma/d)^{\frac{6}{5}+\varepsilon} + U^{-\eta}\Gamma^{3\eta}(R^2/d)^{3\eta} + U^{-\eta}\Gamma^{3\eta}R^{3(1+\varepsilon)\eta}\right\},$$

and (4.7) thus becomes

$$\sum_{\substack{r\le Y\\ d|r}} r^{-\alpha} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}}$$
$$\ll L^3\left\{d^{-\alpha} + U^{-\frac{1}{2}}Y^{\frac{12}{5}-\alpha+\varepsilon}T^{\frac{6}{5}-\xi+\varepsilon}d^{-\frac{6}{5}} + U^{-\eta}d^{3(1+\varepsilon)\eta-\alpha}\right\},$$

since $6\eta \le \alpha \le 12/5$ and $3\eta \le \xi \le 6/5$. Putting this in (4.6) and noticing $Y^{3+\eta} < U$, we get the desired inequality.

To check (4.3) and (4.4), we observe that when $\rho \le 6/7$,

$$G(1-\varepsilon,\xi_k,P^{\rho},T_k,U_k) \le G(1-\varepsilon,\xi_k,P^{\frac{6}{7}},T_k,U_k) \le G(6/5-\varepsilon,\xi_k,P,T_k,U_k),$$

so we only need to check (4.4), which is an easy exercise. □

Lemma 4.4. *Let $1 < U < N$, $2 \le 2X < Y < N$ and $1 < T \le N$. Then for $\alpha \ge 0$ and $\xi \ge 0$, we have*

$$\sum_{X<r\le Y} r^{-\alpha} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}} \ll H(\alpha,\xi,X,Y,T,U)L^3,$$

where

$$H(\alpha,\xi,X,Y,T,U) = X^{-\alpha} + \max_{1/2\le\sigma\le 3/4} U^{\sigma-1}Y^{\max\{0,\frac{10(1-\sigma)}{3-\sigma}-\alpha+\varepsilon\}}T^{\max\{0,\frac{6(1-\sigma)}{3-\sigma}-\xi+\varepsilon\}}.$$

In particular, for $k = 2,...,5$, and $\rho \le 4/5$, one has

$$H(1-\varepsilon,\xi_k,X,P^{\rho},T_k,U_k) \ll X^{-1+\varepsilon} + N^{-\eta}, \tag{4.8}$$

and

$$H(6/5-\varepsilon,\xi_k,X,P,T_k,U_k) \ll X^{-\frac{6}{5}+\varepsilon} + N^{-\eta}, \tag{4.9}$$

where for $k = 2,\ 3,\ 4$, ξ_k are defined by (4.5), and $\xi_5 = 2/5$.

Proof. First we have

$$\sum_{X<r\le Y} r^{-\alpha} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le T} \frac{U^{\beta-1}}{(1+|\gamma|)^{\xi}}$$
$$\ll (\log Y)(\log T) \max_{X\le R\le Y/2} R^{-\alpha} \max_{1\le\Gamma<T/2} (1+\Gamma)^{-\xi} \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le 2\Gamma} U^{\beta-1}. \tag{4.10}$$

By integrating by parts, we have

$$\sum_{r\le 2R} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le 2\Gamma} U^{\beta-1}$$
$$\ll L\left\{ U^{-\frac{1}{2}} N^{*}\left(1/2, 2\Gamma, 2R\right) + \int_{\frac{1}{2}}^{1} U^{\sigma-1} N^{*}(\sigma, 2\Gamma, 2R) d\sigma \right\}.$$

By Lemma 3.3, one has

$$N^{*}\left(1/2, 2\Gamma, 2R\right) \ll \left(R^2 \Gamma^{\frac{6}{5}}\right)^{(1+\varepsilon)},$$

and

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \ll \max_{\frac{1}{2}\le\sigma\le\frac{3}{4}} U^{\sigma-1}\left(R^2\Gamma^{\frac{6}{5}}\right)^{\frac{5(1-\sigma)}{3-\sigma}+\varepsilon}, \quad \int_{\frac{3}{4}}^{1} \ll 1 + U^{-\frac{1}{4}}\left(R^2\Gamma^{\frac{6}{5}}\right)^{\frac{5}{9}+\varepsilon}.$$

Therefore

$$\sum_{r\le 2R} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma|\le 2\Gamma} U^{\beta-1} \ll L\left\{1 + \max_{\frac{1}{2}\le\sigma\le\frac{3}{4}} U^{\sigma-1}\left(R^2\Gamma^{\frac{6}{5}}\right)^{\frac{5(1-\sigma)}{3-\sigma}+\varepsilon}\right\}.$$

By inserting this estimate into (4.10), one yields the desired inequality.

To prove (4.8) and (4.9), we note that for $\rho \le 4/5$,

$$H(1-\varepsilon, \xi_k, X, P^{\rho}, T_k, U_k) \le H(1-\varepsilon, \xi_k, X, P^{\frac{4}{5}}, T_k, U_k),$$

and also

$$H\left(6/5-\varepsilon, \xi_k, X, P, T_k, U_k\right) \le H(1-2\varepsilon, \xi_k, X, P^{\frac{4}{5}}, T_k, U_k),$$

because for $\sigma \ge 1/2$,

$$\frac{10(1-\sigma)}{3-\sigma} - \frac{6}{5} \le \frac{4}{5}\left\{\frac{10(1-\sigma)}{3-\sigma} - 1\right\}.$$

So it remains to check (4.8) for $\rho = 4/5$. We have

$$H(1-\varepsilon, \xi_k, X, P^{\frac{4}{5}}, T_k, U_k)$$
$$= X^{-\alpha} + \max_{1/2\le\sigma\le 3/4} U_k^{\sigma-1} P^{\frac{8(1-\sigma)}{3-\sigma}-\frac{4}{5}+\varepsilon} T_k^{\frac{6(1-\sigma)}{3-\sigma}-\xi_k+\varepsilon}$$
$$\ll X^{-\alpha} + N^{\varepsilon} \max_{1/2\le\sigma\le 3/4} N^{f_k(\sigma)},$$

where

$$f_k(\sigma) = \frac{\sigma-1}{k} + \left\{\frac{8(1-\sigma)}{3-\sigma} - \frac{4}{5} + \alpha_k\left(\frac{6(1-\sigma)}{3-\sigma} - \xi_k\right)\right\}\theta.$$

By checking the sign of $f_k'(\sigma)$, we find that for $k=2,\ ...,5$,

$$\max_{1/2\le\sigma\le 3/4} f_k(\sigma) < -10^{-3}.$$

This finishes the proof of Lemma 4.4. $\Box$

Proof of Lemma 4.1. We see from (3.22) that $\Delta(\lambda)$ consists of fifteen terms of the form $\prod_{k=2}^5 Z_k(\lambda)$ with $Z_k(\lambda)=V_k(\lambda)$ or $W_k(\lambda)$ except the case $\prod_{k=2}^5 Z_k(\lambda)=\prod_{k=2}^5 V_k(\lambda)$. So if we can prove that for all $n\in[N/2,N]$ but at most $O(NP^{-\frac{12}{25}+\varepsilon})$ exceptions,

$$\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left(\prod_{k=2}^5 Z_k(\lambda)\right)e(-n\lambda)d\lambda \ll N^{\frac{17}{60}}L^{-A} \tag{4.11}$$

holds for all the above fifteen terms, then Lemma 4.1 follows. We will deal with the most complicated case $\prod_{k=2}^5 Z_k(\lambda)=\prod_{k=2}^5 W_k(\lambda)$ in detail, and give only a sketch for the other cases since the proofs are similar and allow better ranges of θ.

Denote by $J_w(n)$ the left-hand side of (4.11) with $\prod_{k=2}^5 Z_k(\lambda)=\prod_{k=2}^5 W_k(\lambda)$. Then by (3.4) and (4.1), one has

$$J_w(n)=\sum_{q\le P}\sum_{\chi_2 \bmod q}\cdots\sum_{\chi_5 \bmod q}\frac{B(n,q,\chi_2,...,\chi_5)}{\varphi^4(q)}\sum_{|\gamma_2|\le T_2}\cdots\sum_{|\gamma_5|\le T_5}\mathfrak{I}(n;\rho_2,...,\rho_5), \tag{4.12}$$

where

$$\mathfrak{I}(n;\rho_2,...,\rho_5)=\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left\{\prod_{k=2}^5\Psi_k(\lambda,\rho_k)\right\}e(-n\lambda)d\lambda. \tag{4.13}$$

Now we give an upper estimate for $\mathfrak{I}(n;\rho_2,...,\rho_5)$.

By (3.18), we have for $k=2,...,5$,

$$\Psi_k(\lambda,\beta+i\gamma)\ll\frac{U_k^{\beta}}{(1+|\gamma|)^{\xi_k}(1+|\lambda|N)^{1/2-\xi_k}}, \tag{4.14}$$

where ξ_k is as defined in Lemmas 4.3 and 4.4. Therefore

$$|\mathfrak{I}(n;\rho_2,...,\rho_5)|\ll\frac{U_2^{\beta_2}}{(1+|\gamma_2|)^{\xi_2}}\cdots\frac{U_5^{\beta_5}}{(1+|\gamma_5|)^{\xi_5}}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\frac{d\lambda}{(1+|\lambda|N)^{1+\eta}}\ll N^{\frac{17}{60}}\prod_{k=2}^5\frac{U_k^{\beta_k-1}}{(1+|\gamma_k|)^{\xi_k}}.$$

Now we recall that if a primitive character $\chi \bmod r$ induces a character $\psi \bmod t$, then $r|t$ and $\psi=\chi\chi^0$, where χ^0 is the principal character modulo t. Collecting all contributions made by an individual primitive character, we get

$$\begin{aligned}J_w(n) &\ll N^{\frac{17}{60}}\sum_{r_5\le P}\ \sideset{}{^*}\sum_{\chi_5 \bmod r_5}\sum_{|\gamma_5|\le T_5}\frac{U_5^{\beta_5-1}}{(1+|\gamma_5|)^{\xi_5}}\cdots\sum_{r_2\le P}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\sum_{|\gamma_2|\le T_2}\frac{U_2^{\beta_2-1}}{(1+|\gamma_2|)^{\xi_2}}\\ &\quad\times\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|}{\varphi^4(q)} =: N^{\frac{17}{60}}(J_{w1}(n)+J_{w2}(n)),\end{aligned} \tag{4.15}$$

where $J_{w1}(n)$ and $J_{w2}(n)$ denote the contributions from $r_0 \le P^{\frac{4}{5}}$ and $P^{\frac{4}{5}} < r_0 \le P$, respectively. Here $r_0 = [r_2, r_3, r_4, r_5]$. So we are reduced to proving that for all $n \in [N/2, N]$ but at most $O(NP^{-\frac{12}{25}})$ exceptions

$$J_{w1}(n), \quad J_{w2}(n) \ll L^{-A}. \tag{4.16}$$

We first consider $J_{w1}(n)$. By Lemma 4.2, one has

$$J_{w1}(n) \ll L^{c_0} \sum_{r_5 \le P^{\frac{4}{5}}} \sum_{\chi_5 \bmod r_5}{}^{*} \sum_{|\gamma_5| \le T_5} \frac{U_5^{\beta_5 - 1}}{(1+|\gamma_5|)^{\xi_5}} \cdots \sum_{r_2 \le P^{\frac{4}{5}}} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{|\gamma_2| \le T_2} \frac{U_2^{\beta_2 - 1}}{(1+|\gamma_2|)^{\xi_2}} r_0^{-1+\varepsilon}.$$

Note that $r_0 = [r_5, r_4, r_3, r_2] = [[r_5, r_4, r_3], r_2]$, so by Lemma 4.3, the last triple sum is

$$\ll [r_5, r_4, r_3]^{-1+2\varepsilon} G(1-\varepsilon, \xi_2, P^{\frac{4}{5}}, T_2, U_2) L^3 \ll [r_5, r_4, r_3]^{-1+2\varepsilon} L^3.$$

Repeating the above process for the triple sums over $(r_k, \chi_k, |\gamma_k|)$ for $k = 3$ and $k = 4$ successively, one finally achieves

$$\begin{aligned} J_{w1}(n) &\ll L^{c_0+9} \sum_{r_5 \le P^{\frac{4}{5}}} r_5^{-1+\varepsilon} \sum_{\chi_5 \bmod r_5}{}^{*} \sum_{|\gamma_5| \le T_5} \frac{U_5^{\beta_5-1}}{(1+|\gamma_5|)^{\xi_5}} \\ &\ll L^{c_0+9} \{ J_{w1}^{*} + J_{w1}^{**} \}, \end{aligned} \tag{4.17}$$

where J_{w1}^{*} and J_{w1}^{**} represent the contributions from those with $r_5 \le L^B$ and those with $L^B < r_5 \le P^{\frac{4}{5}}$, respectively for $B = 2A$. Applying Lemma 4.4 to J_{w1}^{**}, one has

$$J_{w1}^{**} \ll L^3 H(1-\varepsilon, \xi_5, L^B, P^{\frac{4}{5}}, T_5, U_5) \ll L^{-A}. \tag{4.18}$$

Now we turn to J_{w1}^{*}. By Satz VIII.6.2 of Prachar [10], there exists a positive constant c_1 such that $\prod_{\chi \bmod q} L(s,\chi)$ is zero-free in the region

$$\sigma \ge 1 - c_1 / \max\{\log q,\ \log^{\frac{4}{5}} N\}, \qquad |t| \le N, \tag{4.19}$$

except for the possible Siegel zero. But by Siegel's theorem (see [2], §21), the Siegel zero does not exist in this situation, since $q \le L^B$. Let $\eta(N) = c_1 \log^{-\frac{4}{5}} N$. Then by Lemma 3.3,

$$\begin{aligned} J_{w1}^{*} &\ll \sum_{r \le L^B} \sum_{\chi \bmod r}{}^{*} \sum_{|\gamma| \le T_5} U_5^{\beta-1} \ll L \max_{1/2 \le \sigma \le 1-\eta(N)} \left(L^{2B} T_5^{\frac{6}{5}} \right)^{\left(\frac{20}{9}+\varepsilon\right)(1-\sigma)} U_5^{\sigma-1} \\ &\ll L \max_{1/2 \le \sigma \le 1-\eta(N)} N^{\{\frac{8}{3}\theta - \frac{1}{5} + \eta\}(1-\sigma)} \ll L N^{-\frac{\eta(N)}{40}} \ll L^{-A}. \end{aligned} \tag{4.20}$$

Inserting (4.18) and (4.20) into (4.17), we have proved that

$$J_{w1}(n) \ll L^{-A}, \quad \text{for} \quad \text{all} \quad n \in [N/2, N].$$

Now it remains to estimate $J_{w2}(n)$. By Cauchy's inequality, one has

$$\begin{aligned}
&J_{w2}^2(n)\\
&\ll \sum_{r_5\le P}\sum_{\chi_5 \bmod r_5}{}^{*}\sum_{|\gamma_5|\le T_5}\frac{U_5^{\beta_5-1}}{(1+|\gamma_5|)^{\xi_5}}\cdots\sum_{r_2\le P}\sum_{\chi_2 \bmod r_2}{}^{*}\sum_{|\gamma_2|\le T_2}\frac{U_2^{\beta_2-1}}{(1+|\gamma_2|)^{\xi_2}}\sum_{\substack{q\le P\\ P^{\frac45}<r_0|q}}\varphi^{-\frac32}(q)\\
&\quad\times\sum_{r_5\le P}\sum_{\chi_5 \bmod r_5}{}^{*}\sum_{|\gamma_5|\le T_5}\frac{U_5^{\beta_5-1}}{(1+|\gamma_5|)^{\xi_5}}\cdots\sum_{r_2\le P}\sum_{\chi_2 \bmod r_2}{}^{*}\sum_{|\gamma_2|\le T_2}\frac{U_2^{\beta_2-1}}{(1+|\gamma_2|)^{\xi_2}}\\
&\quad\sum_{\substack{q\le P\\ P^{\frac45}<r_0|q}}\varphi^{-\frac{13}{2}}(q)|B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|^2.
\end{aligned}$$

We have for $r_0>P^{\frac45}$,

$$\sum_{\substack{q\le P\\ r_0|q}}\varphi^{-\frac32}(q)\ll r_0^{-\frac32+\varepsilon}\ll P^{-\frac{6}{25}}r_0^{-\frac65+\varepsilon}.$$

Writing

$$\mathfrak{F}=\sum_{r_5\le P}\sum_{\chi_5 \bmod r_5}{}^{*}\sum_{|\gamma_5|\le T_5}\frac{U_5^{\beta_5-1}}{(1+|\gamma_5|)^{\xi_5}}\cdots\sum_{r_2\le P}\sum_{\chi_2 \bmod r_2}{}^{*}\sum_{|\gamma_2|\le T_2}\frac{U_2^{\beta_2-1}}{(1+|\gamma_2|)^{\xi_2}}r_0^{-\frac65+\varepsilon},$$

we have

$$\begin{aligned}
&\sum_{N/2<n\le N}J_{w2}^2(n)\\
&\ll P^{-\frac{6}{25}}\mathfrak{F}\sum_{r_5\le P}\sum_{\chi_5 \bmod r_5}{}^{*}\sum_{|\gamma_5|\le T_5}\frac{U_5^{\beta_5-1}}{(1+|\gamma_5|)^{\xi_5}}\cdots\sum_{r_2\le P}\sum_{\chi_2 \bmod r_2}{}^{*}\sum_{|\gamma_2|\le T_2}\frac{U_2^{\beta_2-1}}{(1+|\gamma_2|)^{\xi_2}}\\
&\quad\sum_{\substack{q\le P\\ P^{\frac45}<r_0|q}}\varphi^{-\frac{13}{2}}(q)\sum_{N/2<n\le N}|B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|^2. \qquad (4.21)
\end{aligned}$$

By definition (4.1)

$$\begin{aligned}
&\sum_{N/2<n\le N}|B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|^2\\
&=\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,q)=1}}^{q}\prod_{j=2}^{5}C(\chi_j\chi^0,a)\overline{C(\chi_j\chi^0,b)}\sum_{N/2<n\le N}e\left(-\frac{(a-b)n}{q}\right).
\end{aligned}$$

For $1\le a,\ b\le q$, the last sum over n is $O(q)$ unless $a=b$, in which case it is $O(N)$. So by (3.2), one has

$$\sum_{N/2<n\le N}|B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|^2\ll Nq^5d^{14}(q)+q^7d^{14}(q)\ll Nq^{5+\varepsilon},$$

since $q \le P < N^{\frac{1}{2}}$. Therefore for $r_0 > P^{\frac{4}{5}}$,

$$\sum_{\substack{q\le P\\ r_0|q}} \varphi^{-\frac{13}{2}}(q) \sum_{N/2<n\le N} |B(n,q,\chi_2\chi^0,...,\chi_5\chi^0)|^2 \ll N \sum_{\substack{q\le P\\ r_0|q}} q^{-\frac{3}{2}+\varepsilon} \ll NP^{-\frac{6}{25}} r_0^{-\frac{6}{5}+\varepsilon}.$$

Putting this in (4.21), we get

$$\sum_{N/2<n\le N} J_{w2}^2(n) \quad \ll \quad N\{P^{-\frac{6}{25}}\mathfrak{F}\}^2 = NP^{-\frac{12}{25}}\mathfrak{F}^2.$$

We now estimate $\mathfrak{F}$ in the same way as for $J_{w1}(n)$ in (4.17)-(4.20). Making use of Lemma 4.3 to the triple sums over $(r_k, \chi_k, |\gamma|_k)$ for $k = 2, 3, 4$ in $\mathfrak{F}$ successively, and then Lemma 4.4 for $k = 5$, we finally arrive at

$$\mathfrak{F} \ll L^9 H(6/5-\varepsilon, \xi_5, 1, P, T_5, U_5) \ll L^9,$$

and hence

$$\sum_{N/2<n\le N} J_{w2}^2(n) \ll NP^{-\frac{12}{25}} L^9.$$

By a standard argument, this proves that for all $n \in [N/2, N]$ with at most $O(NP^{-\frac{12}{25}+\varepsilon})$ exceptions, one has

$$J_{w2}(n) \ll L^{-A}.$$

This finishes the estimate for $J_{w2}(n)$, and hence for $J_w(n)$.

To conclude the proof of Lemma 4.1, we need to sketch how to bound the other terms in J. As an example, we consider the case $\prod_{k=2}^5 Z_k(\lambda) = W_2(\lambda)W_3(\lambda)W_4(\lambda)V_5(\lambda)$. Denote by $J_v(n)$ the corresponding term in (4.11). Similar to (4.12) and (4.13), we have

$$J_v(n) \quad = \quad \sum_{q\le P} \sum_{\chi_2 \bmod q} \cdots \sum_{\chi_4 \bmod q} \frac{B(n,q,\chi_2,...,\chi_4,\chi^0)}{\varphi^4(q)} \sum_{|\gamma_2|\le T_2} \cdots \sum_{|\gamma_4|\le T_4} \mathfrak{I}(n;\rho_2,...,\rho_4,1),$$

with

$$\mathfrak{I}(n;\rho_2,...,\rho_4,1) = \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left\{ \prod_{k=2}^4 \Psi_k(\lambda,\rho_k) \right\} \Phi_5(\lambda) e(-n\lambda) d\lambda.$$

By (3.16) and (4.14), the above right-hand side is

$$\ll \frac{U_2^{\beta_2}}{(1+|\gamma_2|)^{\xi_2}} \cdots \frac{U_4^{\beta_4}}{(1+|\gamma_4|)^{\xi_4}} U_5 \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \frac{d\lambda}{(1+|\lambda|N)^{19/10+\eta}} \ll N^{\frac{17}{60}} \prod_{k=2}^4 \frac{U_k^{\beta_k-1}}{(1+|\gamma_k|)^{\xi_k}}.$$

Therefore

$$J_v(n) \quad \ll \quad N^{\frac{17}{60}} \sum_{r_4\le P} \sum_{\chi_4 \bmod r_4}{}^{*} \sum_{|\gamma_4|\le T_4} \frac{U_4^{\beta_4-1}}{(1+|\gamma_4|)^{\xi_4}} \cdots \sum_{r_2\le P} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{|\gamma_2|\le T_2} \frac{U_2^{\beta_2-1}}{(1+|\gamma_2|)^{\xi_2}}$$

$$\times \sum_{\substack{q\le P\\ r_0|q}} \frac{|B(n,q,\chi_2\chi^0,...,\chi_4\chi^0,\chi^0)|}{\varphi^4(q)} =: N^{\frac{17}{60}}(J_{v1}(n) + J_{v2}(n)),$$

where $J_{v1}(n)$ and $J_{v2}(n)$ denote the contributions from those with $r_0 \le P^{\frac{4}{5}}$ and $P^{\frac{4}{5}} < r_0 \le P$, respectively. Here $r_0 = [r_2, r_3, r_4]$. By similar method as used previously in estimating $J_{w1}(n)$ and $J_{w2}(n)$, we get $J_{v1}(n),\ J_{v2}(n) \ll L^{-A}$ for $\theta = 2/33 + \eta$ and for all $n \in [N/2, N]$ but at most $O(NP^{-\frac{12}{25}+\varepsilon})$ exceptions. This completes the proof of Lemma 4.1. □

5. The singular series

Let $B(n,q) = B(n,q,\chi^0,...,\chi^0)$ be defined by (4.1). Define

$$A(n,q) = \frac{B(n,q)}{\varphi^4(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n,q). \tag{5.1}$$

This $\mathfrak{S}(n)$ is the singular series appearing in Theorem 2.

Lemma 5.1. *$A(n,q)$ is multiplicative in q.*

Proof. Suppose $q = q_1q_2$ with $(q_1,q_2) = 1$. Then

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2} \left(\prod_{k=2}^{5} C_k(q_1q_2,a)\right) e\left(-\frac{an}{q_1q_2}\right) \\ &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} \sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} \left(\prod_{k=2}^{5} C_k(q_1q_2,a_1q_2+a_2q_1)\right) e\left(-\frac{a_1n}{q_1}\right) e\left(-\frac{a_2n}{q_2}\right). \end{aligned} \tag{5.2}$$

For $(q_1,q_2) = 1$, it is easy to check that

$$C_k(q_1q_2, a_1q_2 + a_2q_1) = C_k(q_1,a_1)C_k(q_2,a_2).$$

Thus it follows from (5.2) that

$$B(n,q_1q_2) = B(n,q_1)B(n,q_2),$$

and consequently

$$A(n,q_1q_2) = \frac{B(n,q_1q_2)}{\varphi^4(q_1q_2)} = A(n,q_1)A(n,q_2). \qquad \square$$

The following lemma is Lemma 8.3 in Hua [4].

Lemma 5.2. *Let p be a prime and $p^\alpha \| k$. Suppose $p \nmid a$. Then $C_k(p^t,a) = 0$ whenever*

$$t \ge \begin{cases} \alpha+3, & if\ p=2 \text{ and } k \text{ is even};\\ \alpha+2, & otherwise. \end{cases}$$

Lemma 5.3. *Let $A(n,q)$ be as defined in (5.1). Then*

(i) *We have*

$$\sum_{q \ge X} |A(n,q)| = O\left(X^{-\frac{1}{2}+\varepsilon} d(n)\right).$$

Hence $\sum_{q=1}^{\infty} A(n,q)$ is absolutely convergent and $\mathfrak{S}(n) \ll d(n)$.

(ii) *There exists an absolute positive constant c^* such that, for $n \equiv 0(\mathrm{mod}2)$,*

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}. \tag{5.3}$$

Proof. By Lemmas 5.1 and 5.2, one has

$$\sum_{q=1}^{\infty} A(n,q) = \sum_{\substack{q=1 \\ q\ square-free}}^{\infty} A(n,q). \tag{5.4}$$

Write

$$S_k(p,a) = \sum_{m=1}^{p} e\left(\frac{am^k}{p}\right), \quad \text{and} \quad \mathfrak{R}(p,a) = \prod_{k=2}^{5} C_k(p,a) - \prod_{k=2}^{5} S_k(p,a).$$

Then

$$A(n,p) = \frac{1}{(p-1)^4}\sum_{a=1}^{p-1}\left\{\prod_{k=2}^{5} S_k(p,a)\right\} e\left(-\frac{an}{p}\right) + \frac{1}{(p-1)^4}\sum_{a=1}^{p-1}\mathfrak{R}(p,a)e\left(-\frac{an}{p}\right). \tag{5.5}$$

Applying (3.2) and noticing that $S_k(p,a) = C_k(p,a)+1$, we find that the second term in (5.5) is $\le c_2 p^{-\frac{3}{2}}$, since $S_k(p,a) \ll p^{1/2}$. On the other hand, by Lemma 20 in Roth [14], for $p \nmid n$, one has

$$\left|\sum_{a=1}^{p-1}\left\{\prod_{k=2}^{5}(S_k(p,a)/p)\right\} e\left(-\frac{an}{p}\right)\right| \le c_3 p^{-\frac{3}{2}}.$$

So the first term in (5.5) is also $\le c_4 p^{-\frac{3}{2}}$. Let $c_5 = 2\max(c_2, c_4)$, then we have proved that for $p \nmid n$,

$$|A(n,p)| \le c_5 p^{-\frac{3}{2}}. \tag{5.6}$$

Moreover, an application of (3.2) reveals that $|B(n,p)| \le 2048p^2(p-1)$, and therefore

$$|A(n,p)| \le c_6 p^{-1}. \tag{5.7}$$

Let $c_7 = \max(c_5, c_6)$. Then for square-free q,

$$\begin{aligned} |A(n,q)| &= \left(\prod_{\substack{p|q \\ p\nmid n}} |A(n,p)|\right)\left(\prod_{\substack{p|q \\ p|n}} |A(n,p)|\right) \le \left(\prod_{\substack{p|q \\ p\nmid n}} \left(c_7 p^{-\frac{3}{2}}\right)\right)\left(\prod_{\substack{p|q \\ p|n}} \left(c_7 p^{-1}\right)\right) \\ &= c_7^{\omega(q)}\left(\prod_{p|q} p^{-\frac{3}{2}}\right)\left(\prod_{p|(n,q)} p^{\frac{1}{2}}\right) \ll q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}}. \end{aligned}$$

Hence by (5.4),

$$\begin{aligned} \sum_{q\ge X}^{\infty} |A(n,q)| &\ll \sum_{q\ge X} q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}} = \sum_{d|n} d^{-1+\varepsilon} \sum_{qd\ge X} q^{-\frac{3}{2}+\varepsilon} \\ &\ll X^{-\frac{1}{2}+\varepsilon}\sum_{d|n} d^{-\frac{1}{2}+\varepsilon} \le X^{-\frac{1}{2}+\varepsilon} d(n). \end{aligned}$$

This proves Lemma 5.3 (i).

To prove (ii) of Lemma 5.3, we first note that

$$\begin{aligned}\mathfrak{S}(n) &= \prod_p (1+A(n,p)) = \left(\prod_{p\le 2c_7} (1+A(n,p))\right)\\ &\times \left(\prod_{\substack{p>2c_7\\ p\nmid n}} (1+A(n,p))\right)\left(\prod_{\substack{p>2c_7\\ p\mid n}} (1+A(n,p))\right).\end{aligned}$$

By (5.6) and (5.7), one has

$$\prod_{\substack{p>2c_7\\ p\nmid n}} (1+A(n,p)) \ge \prod_{p>2c_7} \left(1-c_7p^{-\frac{3}{2}}\right) \ge c_8 > 0, \tag{5.8}$$

and

$$\prod_{\substack{p>2c_7\\ p\mid n}} (1+A(n,p)) \ge \prod_{\substack{p>2c_7\\ p\mid n}} \left(1-c_7p^{-1}\right) \ge c_9(\log\log n)^{-c_7}. \tag{5.9}$$

Moreover, let $M(n,p)$ denote the number of solutions of the equation

$$x^2+y^3+z^4+w^5 \equiv n(\text{mod} p), \text{ with } 1 \le x,y,z,w \le p-1.$$

Then $1+A(n,p) = p\varphi^{-4}(p)M(n,p)$. By Prachar [11], one has $M(n,p)>0$ for $p>2$; while a direct calculation shows that $M(n,2)=1$ for $n\equiv 0(\text{mod}2)$. Therefore

$$\left(\prod_{p\le 2c_7} (1+A(n,p))\right) \ge c_{10} > 0. \tag{5.10}$$

Collecting estimates (5.8)-(5.10) and writing $c^* = c_7$, we come to the desired conclusion. □

6. Estimation of I and the proof of Theorem 1

Lemma 6.1. *Let I be as defined in* (3.23). *Then for even integers $n \in [N/2, N]$, we have*

$$I = \mathfrak{S}(n)\mathfrak{J}(n) + O\left(N^{\frac{17}{60}}L^{-A}\right),$$

where $\mathfrak{J}(n)$ is a mutiple integral which satisfies

$$N^{\frac{17}{60}} \ll \mathfrak{J}(n) \ll N^{\frac{17}{60}}. \tag{6.1}$$

Proof. By definition

$$I = \sum_{q\le P} \frac{B(n,q)}{\varphi^4(q)} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \Phi_2(\lambda)\cdots\Phi_5(\lambda)e(-n\lambda)d\lambda.$$

Define

$$\mathfrak{J}(n) = \int_{-\infty}^{\infty} \Phi_2(\lambda)\cdots\Phi_5(\lambda)e(-n\lambda)d\lambda. \tag{6.2}$$

Then by making use of (3.16) and on recalling (2.1), we have

$$\int_{\frac{1}{qQ}}^{\infty} |\Phi_2(\lambda)\cdots\Phi_5(\lambda)|\, d\lambda \ll \int_{\frac{1}{qQ}}^{\infty} \frac{U_2\cdots U_5}{(1+|\lambda|N)^4} d\lambda \ll N^{\frac{17}{60}} \frac{q^3}{P^{3(1+\eta)}},$$

and in view of $|B(n,q)| \ll q^{3+\varepsilon}$,

$$I = \mathfrak{I}(n) \sum_{q\le P} A(n,q) + O\left(N^{\frac{17}{60}} P^{-\eta}\right). \tag{6.3}$$

Moreover by Lemma 5.3,

$$\sum_{q\le P} A(n,q) = \mathfrak{S}(n) + O\left(P^{-\frac{1}{2}+\varepsilon} d(n)\right),$$

with $\mathfrak{S}(n)$ satisfies (5.3). So (6.3) becomes

$$I = \mathfrak{S}(n)\mathfrak{I}(n) + O\left(N^{\frac{17}{60}} L^{-A}\right),$$

provided that (6.1) is true, which will be established in the following.

To prove (6.1), one notes that an application of Fourier transformation formula reveals

$$\mathfrak{I}(n) = \frac{1}{120} \int_{\mathfrak{D}} u_2^{-\frac{1}{2}} u_3^{-\frac{2}{3}} u_4^{-\frac{3}{4}} (n-u_2-u_3-u_4)^{-\frac{4}{5}} du_2 du_3 du_4,$$

where $\mathfrak{D}$ is the set of all vectors (u_2, u_3, u_4) subject to

$$U_k^k \le u_k \le (2U_k)^k, \quad k=2,\ 3,\ 4, \quad \text{and} \quad U_5^5 \le n-u_2-u_3-u_4 \le (2U_5)^5.$$

From this the second inequality in (6.1) follows immediately. To bound $\mathfrak{I}(n)$ from below, we define

$$\mathfrak{D}^* = \{(u_2,u_3,u_4) : U_k^k \le u_k \le (3U_k/2)^k, k=2,\ 3,\ 4\}.$$

Then for $(u_2,u_3,u_4)\in\mathfrak{D}^*$, one easily deduces from $N/2 < n \le N$ and $U_k^k = N/32$ that

$$U_5^5 < n-u_2-u_3-u_4 \le (2U_5)^5.$$

Thus $\mathfrak{D}^*$ is a subset of $\mathfrak{D}$, and consequently

$$\mathfrak{I}(n) \ge \frac{1}{120} \int_{\mathfrak{D}^*} u_2^{-\frac{1}{2}} u_3^{-\frac{2}{3}} u_4^{-\frac{3}{4}} u_5^{-\frac{4}{5}} du_2 du_3 du_4 \gg N^{\frac{17}{60}}.$$

This proves (6.1). □

Proof of Theorem 2. The absolute convergence and positivity of $\mathfrak{S}(n)$ have been proved in Lemma 5.3. Other assertions of Theorem 2 follow from (3.24), Lemmas 4.1 and 6.1. □

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