Extracting Information from Spot Interest Rates and Credit Ratings using Double Higher-Order Hidden Markov Models

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January 13, 2006

Abstract

Estimating and forecasting the unobservable states of an economy are important and practically relevant topics in economics. Central bankers and regulators can use information about the market expectations on the hidden states of the economy as a reference for decision and policy makings, for instance, deciding monetary policies. Spot interest rates and credit ratings of bonds contain important information about the hidden sequence of the states of the economy. In this paper, we develop double higher-order hidden Markov chain models (DHHMMs) for extracting information about the hidden sequence of the states of an economy from the spot interest rates and credit ratings of bonds. We consider a discrete-state model described by DHHMMs and focus on the qualitative aspect

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of the unobservable states of the economy. The observable spot interest rates and credit ratings of bonds depend on the hidden states of the economy which are modelled by DHHMMs. The DHHMMs can incorporate the persistent phenomena of the time series of spot interest rates and the credit ratings. We employ the maximum likelihood method and the EM algorithm, namely Viterbi's algorithm, to uncover the optimal hidden sequence of the states of the economy which can be interpreted the "best" estimate of the sequence of the underlying economic states generating the spot interest rates and credit ratings of the bonds. Then, we develop an efficient maximum likelihood estimation method to estimate the unknown parameters in our model. Numerical experiment will be conducted to illustrate the implementation of the model.

Key words: Spot Interest Rates, Credit Ratings, Optimal Hidden Economic States, Double Higher-Order Hidden Markov Model, Long Range Dependence

1 Introduction

Estimating and forecasting the hidden states of the economy are important and practically relevant research topics in economics, in particular macroeconomics analysis. Economic agents can infer the hidden states of the economic conditions from some observable economic information, such as stock indices, interest rates and price indices. The information about the states of the economy may be useful for economic policy making. Gerlach and Yiu (2004) mentioned that many central bankers, regulators and economic researchers are routinely and regularly producing estimates and indicators that are important measures to summarize the hidden states of the macroeconomic activity in a certain region. These estimates and indicators provide them with an important piece of information to forecast future economic growth. They also pointed out that the states of the economy can be used to form reliable estimates and forecasts of business cycles since the evolution of the states of the economy are in cycles.

Broadly speaking, business cycles can be defined as the recurring growths and recessions in overall economy or economic activity as reflected in some macroeconomic series, such as production, employment, prices and wages, etc. According to Kennedy (2001), a "stylised" presentation of the business cycle consists of six phases, namely "Peak", "Recession", "Contraction", "Trough", "Recovery" and "Expansion". "Peak-to-Peak" or "Trough-to-Trough" forms ome complete cycle. The pioneering work of Burns and Mitchell (1946) developed various indices of business cycles using a large numbers of economic variables and applied these indices to summarize the states of an economy in the United States. Stock and Watson (1989, 1991, 1998) introduced a single-index model or a dynamic factor model to provide a formal definition of the hidden states of the economy, which is then used to investigate business cycles and fluctuations. They supposed that the co-movements among observable economic variables are driven by a common element, which is described by a single dynamics of hidden states of the economy. Many economic researchers have adopted the dynamic factor model by Stock and Watson to compute various economic indicators of the hidden states of the economy in various regions or countries, for instance, Camba-Mendaz (2001) and Garcia-Ferrer and Poncela (2002) for European countries, Bandholz and Funke (2003) for Germany, Chen and Lin (2000) for Taiwan, Fukuda and Onodera (2001) for Japan and Gerlach and Yiu (2004) for Hong Kong SAR. Gerlach and Yiu (2004) adopted the dynamic factor model to construct quarter estimates of the states of the economy in Hong Kong using some observable economic variables, such as the Hang Seng Index, a residential property price index, retail sales and total exports. They supposed that the dynamics of the hidden states of the economy is the common driver of the four economic variables and adopted the Principal Component Analysis (PCA) to get some insights in the unobservable common component. Then, they formulated the dynamic factor model in a state-space form and used the Kalman filtering technique and maximum likelihood method to extract the unobservable common component, which represents the unobservable state of the economy in Hong Kong.

In this paper, we develop discrete-time double higher-order hidden Markov chain models (DHHMMs) for extracting information about the hidden states of an economy from spot interest rates and credit ratings of bonds in the economy. We may interpret the hidden states of the economy as different phases of business cycles. In general, the hidden states of the economy may be interpreted as the hidden states of economic activity, growth, technology and inflationary level, etc. We employ DHHMMs for specifying the dependence between the hidden states of the economy and observable economic variables. The basic idea of DHHMMs is to describe both the observable and hidden sequences as higher-order Markov chains models (HHMMs). Our model has similar spirit with that of using of Kalman filtering for extracting the hidden states of the economy in Gerlach and Yiu (2004). The dynamic factor model in the literature focuses on the quantitative aspect of the analysis. There is relatively little amount of work that concerns qualitative analysis of the hidden states of the economy. It might be beneficial to consider both qualitative and quantitative aspects of the hidden states of the economy for analysing the issue. Here, we focus on qualitative aspect of the analysis and describe the hidden states of the economy as discrete variables using DHHMMs. In particular, the hidden states are described by a higher-order hidden Markov chain model (HHMM) with discrete state space. One example of such discrete hidden economic states can be the six phases of a "stylised" presentation of business cycles described in above. Spot interest rates and credit ratings of bonds contain important information about the hidden states of the economy.

Using spot interest rates for extracting information about the hidden states of the economy is supported by some literature. Thomas, Allen and Morkel-Kingsbury (2002) explored the dependency between spot interest rates and the hidden states of the economy by using a hidden Markov chain model. One can infer the hidden states of the economy from observable spot interest rates. Gerlach (2003) investigated and interpreted the informational content of term structures of interest rates and its impact on the states of the economy. Ang and Bekaert (2002) adopted the regime switching interest rate models to explain the U.S. business cycles and explore their potential implications on macro-economic activity. Wilkie (1986) discussed the importance of introducing the level of inflation as a driving factor for interest rates and term structures. He supposed that the yield on Consols, which are long-term government securities, responds to the changes in the inflation rate evaluated from observable Retail Prices Index series. His viewpoint is in line with the methods adopted by economists, such as Sargent (1973) and Friedman and Schwartz (1982). He also postulated that inflation is the driving force for the other investment variables, such as interest rates, but not vice versa. Using credit ratings for extracting information about the hidden states of an economy is also supported by the literature (see, for instance, Bangia, Diebold and Schuermann (2000) Monfort and Mulder

(2000) and Thomas, Allen and Morkel-Kingsbury (2002)). In general, one may consider other observable economic and investment variables to infer or learn the hidden states of an economy. Brandt, Zeng and Zhang (2004) considered an economy in which agents need to learn the hidden states of the economy in the context of Bayesian and alterative learning rules. They postulated that the dividend growth rate follows a Markov mean-switching process driven by a hidden Markov chain process, which presents the hidden states of an economy, and investigated the properties of equilibrium stock returns. The model in Brandt, Zeng and Zhang (2004) belongs to the class of incomplete information models, which include the models in Detemple (1986), Wang (1993), Moore and Schaller (1996) and Brennan and Xia (1998), etc.

We describe the dependency of the spot interest rates and credit ratings of bonds on the hidden states of the economy by the DHHMMs. In this way, we can incorporate the long-range dependence of spot interest rates, credit ratings and the states of the economy since both the observable and hidden sequences in DHHMMs are described by high-order Markov chain models. We adopt the maximum likelihood method and an EM algorithm, called Viterbi's algorithm developed in Viterbi (1967), to uncover the optimal hidden sequence of the states of the economy which can be interpreted as the "best" estimate of the sequence of the underlying economic states that generate the observable spot interest rates and credit ratings of the bonds. We develop a method based on maximum likelihood metbod for estimating the model parameters in the transition probability matrices, which provide important information about market expectations on future states of the economy based on the states in the past. Our model can provide some insights in interpreting the informational content of spot interest rates and credit ratings of bonds for estimating and forecasting the states of an economy. Numerical experiment for the implementation of the model will be presented. We organize our paper in the following way.

Section 2 presents the main idea of DHHMMs for spot interest rates, credit ratings and the hidden states of the economy. We discuss the maximum likelihood method and Viterbi's algorithm for extracting information on the hidden states of the economy from spot interest rates and credit ratings in Section 3. An optimal hidden sequence of the states of the economy will be found. Section 4 provides the maximum likelihood estimates for the unknown parameters of the transition probabilities based on the optimal hidden economic states. We show that these maximum likelihood estimates are unbiased. We also determine the "optimal" order of the long-range dependence of the spot interest rates, the credit ratings of the bonds and the hidden economic states by maximizing the log-likelihood function. In Section 5, we present the numerical results for the implementation of our models. The final section concludes this paper and proposes some possible topics for further investigation.

2 The Model

We consider a discrete-time economy consisting of several bonds. As in Thomas, Allen and Morkel-Kingsbury (2002), we suppose that spot interest rates and credit ratings of the bonds are observable and that their transition probabilities depend on the hidden states of the economy. It has been documented that spot interest rates may exhibit long-range dependence (see Duan and Jacobs (1996) and Meade and Maier (2003)). We assume that the spot interest rates process is governed by a DHHMM with its transition probabilities matrix depending on the long-memory economic conditions described by a higher-order hidden Markov chain model (HHMM). Then, we adopt the DHHMM to describe the long-range dependence of the credit ratings. There is some evidence that ratings agencies behave cyclically when they set credit ratings for debts (see Monfort and Mulder (2000), Reisen (2000) and Allen and Saunders (2003)). As a consequence, the credit ratings respond to the changes in cyclical conditions in business cycles and may exhibit the effect of long-range dependence. Bangia, Diebold and Schuermann (2000) and Nickell, Perraudin and Varotto (2000) provided evidence on the impact of macroeconomic and industry conditions on the transitions of credit ratings. Although the basic idea of our model is similar with that of Thomas, Allen and Morkel-Kingsbury (2002), we concern different aspects of the economic issue. They focused on pricing bonds and determining the term structures of credit risk spreads and the risk premiums while we concern extracting information about the hidden states of an economy based on the observable market variables, such as spot interest rates and credit ratings of bonds. Our model is described as follows:

First, write $\mathcal{T}$ for the time index set $\{0, 1, \dots\}$ of the economy. Let $\{V_t\}_{t \in \mathcal{T}}$ denote a process representing the hidden states of an economy. We assume that $\{V_t\}_{t\in\mathcal{T}}$ is an n^{th}-order discrete-time homogeneous Markov chain process taking values in the state space $\mathcal{V} := \{v_1, v_2, \ldots, v_M\}$. We may interpret v_1 as the "best" economic condition, v_2 as the second "best" economic condition and v_M as the "worst" economic condition. This provides a qualitative description to the economic situation. The state transition probabilities matrix $A = \{a(i_{t+n})\}$ of the n^{th}-order Markov chain $\{V_t\}_{t\in\mathcal{T}}$ are given by:

$$a(i_{t+n}) = P[V_{t+n} = v_{i_{t+n}} | V_t = v_{i_t}, \cdots, V_{t+n-1} = v_{i_{t+n-1}}] \ , \quad 1 \le i_t, \ldots, i_{t+n-1} \le M \ . \tag{2.1}$$

The order n represents the degree of the long-range dependence of the hidden states of the economy. The following example illustrate the basic idea of the n^{th}-order Markov chain process for the hidden states of the economy.

Example 2.1: Suppose that the state of the economy in the next period depends on the current state and the state in the previous period (i.e. $n = 2$). There are three possible states of the economy $\{v_1, v_2, v_3\}$, where v_1, v_2 and v_3 represent the "Good" state, the "Neutral" state and the "Bad" state, respectively. Suppose further that the current state of the economy is "Neutral" and the state in the previous period is "Bad". Then, $a(1)$ represents the transition probability that the state of the economy in the next period is "Good" given that the current state of the economy is "Neutral" and the state in the previous period is "Bad".

□

In general, we assume that the transition probabilities are unknown and we will discuss the estimation of the transition probabilities in Section 4. The n^{th}-order Markov property of $\{V_t\}_{t\in\mathcal{T}}$ can describe the long-range dependence of the states of the economy induced by different periods of business cycles. We also need to specify the initial state conditional probabilities $\Pi := \{\pi(i_j)\}$ for the n^{th}-order Markov chain as follows:

$$\pi(i_j) = P[V_j = v_{i_j} | V_1 = v_{i_1}, V_2 = v_{i_2}, \ldots, V_{j-1} = v_{i_{j-1}}] \ , \quad 1 \le j \le n \ . \tag{2.2}$$

In particular, when $n = 2$, we need to specify the following initial state conditional probabilities:

$$\pi(i_1) = P[V_1 = v_{i_1} | V_0 = v_{i_0}] \ , \tag{2.3}$$

and

$$\pi(i_2) = P[V_2 = v_{i_2} | V_0 = v_{i_0}, V_1 = v_{i_1}] \ , \tag{2.4}$$

for any $v_{i_0}, v_{i_1}, v_{i_2} \in \mathcal{V}$.

Then, we modify the Markov chain lattice model for spot interest rates described by Pliska (2003) (Chapter 6 therein), which has been employed in Thomas, Allen and Morkel-Kingsbury (2002). In particular, we assume that the one-step-ahead spot interest rate has three possible states given the current level of the spot interest rate, one going up, one staying the same and one going down. We further suppose that the time series of spot interest rates follows a DHHMM. In this way, the dynamics of spot interest rates follows a higher-order Markov chain model. This can incorporate the long-range dependence of spot interest rates.

Following Pliska (2003) and Thomas, Allen and Morkel-Kingsbury (2002), we define a state process $\{I_t\}_{t\in\mathcal{T}}$, which governs the movements of spot interest rates $\{r_t\}_{t\in\mathcal{T}}$. In particular, given the value taken by I_t and the current hidden state V_t, the current spot interest rate r_t can be determined. For each $t \in \mathcal{T}$, we suppose that I_t can take values in the state space $\mathcal{I} = \{0, 1, \ldots, H\}$. In order to take into account the long-range dependence of both the dynamics of spot interest rates and the hidden states of economy, we model the state process $\{I_t\}_{t\in\mathcal{T}}$ for spot interest rates as a (l, n)-order double hidden Markov chain process. The orders l and n describe the degrees of the long-range dependence of spot interest rates and the hidden states of the economy, respectively. To highlight the main idea of the model, we simplify our discussion by considering the case that there is only one sequence of spot interest rates with one driving state process at this moment. We will then consider the case that there are several bonds and each of them has a driving state process. The main idea of the (l, n)-order DHHMM is presented in the sequel.

Let $\vec{I_t} := (I_t, I_{t-1}, \ldots, I_{t-l+1})$ and $\vec{i_t} := (i_t, i_{t-1}, \ldots, i_{t-l+1})$. Then, we assume that the transition probabilities matrix $B_1 := \{b_{\vec{i}_t,v}(j)\}$ of the state process $\{I_t\}_{t\in\mathcal{T}}$ for spot interest rates when $\vec{I}_t = \vec{i}_t$ and the hidden state $V_{t+1} = v$ satisfy:

$$\begin{aligned}
b_{\vec{i}_t,v}(i_t+1) &:= P[I_{t+1} = i_t + 1 | \vec{I}_t = \vec{i}_t, V_{t+1} = v] := p_u(\vec{i}_t, v) \\
b_{\vec{i}_t,v}(i_t) &:= P[I_{t+1} = i_t | \vec{I}_t = \vec{i}_t, V_{t+1} = v] := p_m(\vec{i}_t, v) \\
b_{\vec{i}_t,v}(i_t-1) &:= P[I_{t+1} = i_t - 1 | \vec{I}_t = \vec{i}_t, V_{t+1} = v] = 1 - p_u(\vec{i}_t, v) - p_m(\vec{i}_t, v) := p_d(\vec{i}_t, v) \ ,
\end{aligned} \tag{2.5}$$

where $v \in \mathcal{V}$ and $i \in \mathcal{I}$.

Note that $p_u(\vec{i}_t, v)$, $p_m(\vec{i}_t, v)$ and $1 - p_u(\vec{i}_t, v) - p_m(\vec{i}_t, v)$ are the conditional probabilities that the spot interest rate at time $t+1$ goes up, that it stays the same as the spot interest rate at time t and that it goes down, when the state process for the current and the past spot interest rates $\vec{I}_t$ is $\vec{i}_t$ and the hidden state V_{t+1} of the economy is v.

In addition, we need to impose the following conditions:

$$\begin{aligned}
p_u(\vec{i}_t, v) &= 0 \quad \text{if } i_t = H \\
p_d(\vec{i}_t, v) &= 0 \quad \text{if } i_t = 0 \ .
\end{aligned} \tag{2.6}$$

The first condition avoids the situation that the spot interest rate goes up at time $t+1$ when the spot interest rate at time t is at the greatest level $i_t = H$ while the second conditon avoids the situation that the spot interest rate goes down at time $t+1$ when the spot interest rate at time t is at the lowest level $i_t = 0$.

Example 2.2: Consider the dynamics of the hidden states of the economy with $n = 2$ and three possible economic states, say “Good”, “Neutral” and “Bad” in Example 2.1. Assume that the state process $\{I_t\}_{t \in \mathcal{T}}$ for spot interest rates can take values in the state space $\mathcal{I} := \{1, 2, 3\}$ and $l = 2$. In this case, the state process follows a $(2,2)$-order DHHMM. Let t denote the current time. Suppose further that the hidden state of the economy at time $t+1$ is “Neutral” (i.e. $V_{t+1} = v_2$) and that the values of the state process for spot interest rates at time $t-1$ and time t are given by:

$$I_{t-1} = 1 \ , \quad I_t = 2 \ .$$

Given this information, the conditional probabilities that the spot interest rate at time $t+1$ goes up, that it stays the same as the spot interest rate at time t and that it goes down are given by $b_{(2,1),v_2}(3)$, $b_{(2,1),v_2}(2)$ and $b_{(2,1),v_2}(1)$, respectively.

□

The initial distribution Π_1 for $\{I_t\}_{t\in\mathcal{T}}$ is specified as follows:

$$\begin{aligned}
b_{\vec{i}_{j-1},v}(i_{j-1}+1) &:= P[I_j = i_{j-1}+1|\vec{I}_{j-1} = \vec{i}_{j-1}, V_j = v] := p_u(\vec{i}_{j-1}, v) \\
b_{\vec{i}_{j-1},v}(i_{j-1}) &:= P[I_j = i_{j-1}|\vec{I}_{j-1} = \vec{i}_{j-1}, V_j = v] := p_m(\vec{i}_{j-1}, v) \\
b_{\vec{i}_{j-1},v}(i_{j-1}-1) &:= P[I_j = i_{j-1}-1|\vec{I}_{j-1} = \vec{i}_{j-1}, V_j = v] := 1 - p_u(\vec{i}_{j-1}, v) - p_m(\vec{i}_{j-1}, v) \\
&\quad := p_d(\vec{i}_{j-1}, v), \quad 1 \leq j \leq l.
\end{aligned} \tag{2.7}$$

In the DHHMMs, the dynamics of spot interest rates depends on the hidden states of the economy, but not vice versa. This is in line with the models in Thomas, Allen and Morkel-Kingsbury (2002) and Wilkie (1986). However, one can infer the hidden states of the economy from observable dynamics spot interest rates in the context of the DHHMMs since the latter contains information about the former. In practice, it is more realistic to consider the case that there are several bonds instead of one in an economy. We shall adopt the above DHHMM for the state process for spot interest rate to model the dynamics of spot interest rates for several bonds in the economy.

The DHHMM can be specified given appropriate values for the order n of the HHMM for the hidden states of the economy, the number of hidden economic states M, the order l of the higher-order Markov model for the state process of spot interest rates, the number of possible values H taken by the state process, the initial conditional probabilities Π and the transition probabilities A for the hidden economic states, the initial conditional probabilities Π_1 and transition probabilities B_1 for the state process. Suppose $I = I_1 I_2 \ldots I_T$ are the observable sequence generated from the DHHMM, where T is the number of observations in the sequence of spot interest rates. Now, for each $q = 1, 2, \ldots, Q$, we suppose that the q^{th} sequence $I^q := \{I^q_t\}_{t\in\mathcal{T}}$ follows the above DHHMM and that $I^1, I^2, \ldots, I^Q$ are independent sequences, which represent the state processess for the Q bonds in the economy. We further suppose that the spot interest rate r^q_t at time t of the q^{th} bond is a function $r^q(t, I^q_t, V_t)$ of time t, the level of I^q_t

and the level of V_t at time t. If we know that $I_t^q = i_t^q$ and $V_t = v$, then the spot interest rate for the q^{th} bond r_t^q can be determined completely as $r^q(t, i_t^q, v)$. If we are given that the historical values of the state process $\vec{I}_t^q = \vec{i}_t^q$ and that $V_t = v$, the three possible values of the spot interest rate for the q^{th} bond in the next period are $r^q(t+1, i_t^q+1, v)$, $r^q(t+1, i_t^q, v)$ and $r^q(t+1, i_t^q-1, v)$ with probabilities $p_u(\vec{i}_t, v)$, $p_m(\vec{i}_t, v)$ and $p_d(\vec{i}_t, v)$. To simplify our notations, we write Λ_1 for (A, B_1, Π_1), which represents the set of unknown parameters of the DHHMM for spot interest rates and hidden economic states. In practice, the transition probabilities $p_u(\vec{i}_t, v)$, $p_m(\vec{i}_t, v)$ and $p_d(\vec{i}_t, v)$ are unknown to market participants. We shall discuss the estimation issue of the transition probabilities in Section 4.

Similar method can be employed to model the dynamics of credit ratings of the bonds in the economy. We shall employ the DHHMM to model the dependence of the dynamics of credit ratings on the hidden states of the economy. For each $q = 1, 2, \ldots, Q$, let $C^q := \{C_t^q\}_{t\in\mathcal{T}}$ denote the process of credit ratings for the q^{th} bond. We suppose that $C^1, C^2, \ldots, C^Q$ are independent and identically distributed random sequences and that the evolution of the credit ratings $\{C_t^q\}_{t\in\mathcal{T}}$ for the q^{th} bond is described by a (m, n)-order DHHMM. The orders m and n represent the degrees of the long-range dependence of the dynamics of credit ratings and the hidden states of the economy. For each $q = 1, 2, \ldots, Q$ and $t \in \mathcal{T}$, C_t^q can take values in $\mathcal{X} := \{0, 1, 2, \ldots, N\}$, where the rating 0 is given to the bonds with no credit risk. The most secure rating is 1 and the least secure rating is $N - 1$. The rating N represents that the bond issuer goes bankrupt. Note that both 0 and M are absorbing states since the bond issuer remains credit risk-free (or bankrupt) once it becomes credit risk-free (or bankrupt)

Let $\vec{c}_t^q$ denote the vector $(c_{t-m+1}^q, c_{t-m+2}^q, \ldots, c_{t-1}^q, c_t^q)$. We assume that the common conditional probability $b_{\vec{c}_t^q, v}(c_{t+1}^q)$ that the credit rating C_{t+1}^q of the q^{th} bond is in state c_{t+1}^q at time $t + 1$ given that the hidden state of the economy V_{t+1} is $v \in \mathcal{V}$ and that the current and the past credit ratings $C_t^q = c_t^q, C_{t-1}^q = c_{t-1}^q, \ldots, C_{t-m+1}^q = c_{t-m+1}^q$ is given by:

$$b_{\vec{c}_t^q, v}(c_{t+1}^q) = P[C_{t+1}^q = c_{t+1}^q \mid C_t^q = c_t^q, C_{t-1}^q = c_{t-1}^q, \ldots, C_{t-m+1}^q = c_{t-m+1}^q, V_{t+1} = v] \,, \qquad (2.8)$$

where $m \leq t \leq T - 1$.

Let B_2 denote the transition probabilities $\{b_{\vec{c}_t^q, v}(c_{t+1}^q)\}$. Note that the transition probabilities

B_2 do not depend on the index q for distinguishing different bonds and are common transition probabilities for the credit ratings processes $C^1, C^2, \dots, C^Q$.

In order to determine the DHHMM for the credit ratings $\{C_t^q\}_{t\in\mathcal{T}}$ of the q^{th} bond, we assume that the conditional probability distributions for the initial credit ratings are given by:

$$b_{\vec{c}^q_{j-1},v}(c_j^q) = P[C_j^q = c_j^q \mid C_{j-1}^q = c_{j-1}^q, C_{j-2}^q = c_{j-2}^q \dots, C_1^q = c_1^q, V_j = v] \ , \quad 1 \le j \le m \ , \quad (2.9)$$

where $\sum_{c_j^q=0}^{N} b_{\vec{c}^q_{j-1},v}(c_j^q) = 1$, for each $v \in \mathcal{V}$.

To simplify the notations, we write Π_2 for the initial conditional probabilities $\{b_{\vec{c}^q_{j-1},v}(c_j^q)\}$ for the credit ratings process $\{C_t^q\}_{t\in\mathcal{T}}$. Note also that the initial conditional probabilities Π_2 are the same for all credit ratings processes $C^1, C^2, \dots, C^Q$. Again, we may use $\Lambda_2 := (A, B_2, \Pi_2)$ to indicate the set of unknown parameters of the DHHMM for credit ratings and hidden economic states.

For each $q = 1, 2, \dots, Q$, the multivariate processes (C_t^q, t, V_t) and (C_t^q, I_t^q, t, V_t) are finite-state time-homogeneous HHMMs. The credit rating model considered by Jarrow et al. (1997) assumed that the credit rating C_t^q and the state of the spot interest rate I_t^q are independent. See Jarrow et al. (1997) and Thomas, Allen and Morkel-Kingsbury (2002) for discussion. As in Thomas, Allen and Morkel-Kingsbury (2002), the dependence of the credit rating C_t^q and the state of the spot interest rate I_t^q for the q^{th} bond are described by their mutual dependency on the underlying hidden state of the economy V_t, for each $q = 1, 2, \dots, Q$. Thomas, Allen and Morkel-Kingsbury (2002) also pointed out that the spot interest rate process driven by the underlying hidden states of the economy can be reduced to the Markov lattice interest rate model by Pliska (2003) when there is only one economic state for the hidden economic conditions. For further simplifying our notations, we adopt $\Lambda := (\Lambda_1, \Lambda_2)$ to indicate the complete set of unknown parameters of our model. For each $q = 1, 2, \dots, Q$, let $\mathcal{O}^q$ denote the bivariate observation (I^q, C^q) about the dynamics of spot interest rates and credit ratings of the q^{th} bond. That is, $\mathcal{O}^q = O_1^q O_2^q \dots O_T^q = (I_1^q, C_1^q)(I_2^q, C_2^q) \dots (I_T^q, C_T^q)$, where T is the number of observations. Based on the observations $\{\mathcal{O}^q | q = 1, 2, \dots, Q\}$, we determine the optimal hidden economic states and the estimates of unknown model parameters by the method of maximum likelihood estimation and Viterbi's algorithm. We also determine the orders of the long-range dependence of the spot interest rates and the credit ratings of the bonds from the observations by maximizing the log-likelihood function.

3 The Optimal Hidden Economic Conditions

In this section, we attempt to uncover an optimal hidden sequence of the states of economy which can be interpreted the "best" estimate of the dynamics of the underlying economic states that generates the observable dynamics of spot interest rates and credit ratings of the bonds. The model provide a general and flexible way to facilitate the filtering of the time-varying dynamics of the hidden states of the economy using the observable dynamics of spot interest rates and credit ratings of the bonds. The optimal hidden dynamics of the states of the economy can provide regulators and market participants with an important piece of information about the temporal behavior of the evolution of the states of the economy. It also serves as an indicator to describe the economic condition of a certain region which is of practical interest to regulators and central bankers. Gerlach and Yiu (2004) mentioned that it is important to adopt the information about the "true" states of economy for economic policy making. The optimal hidden dynamics of the states of the economy is also very useful for forecasting the future states of the economy, future spot interest rates and credit ratings of the bonds (see Nagayasu (2004)). Central bankers or monetary authorities can also use the optimal hidden dynamics of the states of the economy as a reference to decide future monetary policies. The optimal hidden dynamics of the states of the economy may also be used to detect business cycles and the turning points of business cycles.

We employ the method of maximum likelihood estimation and Viterbi's algorithm to uncover the optimal hidden dynamics of the states of the economy. The optimality criteria is given by the likelihood function which is the probability of obtaining a hidden sequence of the states of economic conditions given observations about the spot interest rates and credit ratings of the bonds. The idea behind this estimation method for the hidden dynamics of the economic states resembles the maximum likelihood estimation in the statistical literature. We will use the optimal hidden dynamics of the states of the economy obtained in this section to estimate the unknown parameters in our model, including the unknown parameters in the transition probabilities in both the hidden and observable sequences. The estimated model can then be used for forecasting future spot interest rates and credit ratings of the bonds.

First, we use the likelihood function $P[V|\Lambda, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q]$ as the optimality criteria; that is, we find the optimal hidden sequence of the states of the economy that $P[V|\Lambda, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q]$ can be maximized. Note that $P[V|\Lambda, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q]$ is the conditional probability for the dynamics of the hidden economic states V given the information about the prior estimates of the unknown parameters Λ and observable credit ratings and spot interest rates for the Q bonds, namely $\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q$. By Bayes' rule, we have:

$$P[V|\Lambda, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q] = \frac{P[V, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q|\Lambda]}{P[\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q|\Lambda]} \ . \tag{3.1}$$

Then, maximizing $P[V|\Lambda, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q]$ is equivalent to maximizing $P[V, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q|\Lambda]$.

Suppose we are given the observations $\mathcal{O}^q := O^q_1, O^q_2, \ldots, O^q_T$ of the spot interest rates and the credit ratings of the q^{th} bond, where $O^q_t = (j^q_t, c^q_t)$, for $t \in \mathcal{T}$, $q = 1, 2 \ldots, Q$ and T is the number of the observations. A formal technique in the optimization literature, namely the Viterbi algorithm proposed in Viterbi (1967), can be used to find an optimal "best" hidden sequence of the states of the economy $V := V_1, V_2, \ldots, V_T$ that maximizes $P[V, \mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q|\Lambda]$. The Viterbi algorithm can be described as follows:

First, we define the quantity $\delta_t(i_{t-n+1}, \cdots, i_t)$, namely the best score with the highest probability, of the first $t-n$ hidden economic states along the "best" hidden states of the economy at time t, which accounts for the first t observations and ends in state v_{i_t}, as follows:

$$\delta_t(i_{t-n+1}, \cdots, i_t) = \max_{v_1, v_2, \cdots, v_{t-n}} P[V_1 = v_{i_1}, \cdots, V_t = v_{i_t}, O^1_1, \cdots, O^1_t, \cdots, O^Q_1, \cdots, O^Q_t|\Lambda] \ , \tag{3.2}$$

where $n + 1 \leq t \leq T$.

Let $\vec{j}^q_t = (j^q_1, j^q_2, \ldots, j^q_t)$ denote the observed spot interest rates of the q^{th} bond up to and including time t, $q = 1, 2, \ldots, Q$. By the method of induction, we obtain:

$$\delta_{t+1}(i_{t-n+2}, \cdots, i_{t+1}) = \max_{v_{t-n+1}} \delta_t(i_{t-n+1}, \ldots, i_t) a(i_{t+1}) \prod_{q=1}^{Q} \{ b_{\vec{j}^q_t, v_{i_{t+1}}}(j^q_{t+1}) b_{\vec{c}^q_t, v_{i_{t+1}}}(c^q_{t+1}) \} \ , \tag{3.3}$$

where $j^q_{t+1} = j^q_t - 1, j^q_t, j^q_t + 1$ and $q = 1, 2, \ldots, Q$.

In order to find the optimal sequence of the hidden states of the economy, we have to keep track of the argument that maximizes the function $\delta_t(i_{t-n+1},\cdots,i_t)$, for each t and i_{t-n+1}, $\cdots$, i_t. To simplify the matter, we do the optimization by considering the following array:

$$\psi_t(i_{t-n+1},\cdots,i_t). \tag{3.4}$$

We adopt the three procedures of Viterbi's algorithm for finding the "best" or "optimal" state sequence of the economy, namely initialization, recursion and termination. We describe the main idea of the procedures as follows:

- (U1) Initialization:

$$\begin{aligned}
&\delta_n(i_1,\ldots,i_n) = P[V_1 = v_{i_1},\ldots,V_n = v_{i_n}, O_1^1,\ldots,O_n^1,\ldots,O_1^Q,\ldots,O_n^Q \mid \Lambda] \\
= &\prod_{j=1}^{n}\prod_{q=1}^{Q} b_{\vec{c}_{j-1}^q, v_{i_j}}(c_j^q) b_{\vec{k}_{j-1}^q, v_{i_j}}(k_j^q)\pi(i_j) \ ,
\end{aligned}$$

  where $k_j^q = k_{j-1}^q - 1, k_{j-1}^q, k_{j-1}^q + 1$ and $1 \le i_1, i_2, \cdots, i_n \le N$.

- (U2) Recursion:

$$\begin{aligned}
&\delta_{t+1}(i_{t-n+2},\ldots,i_{t+1}) \\
= &\max_{v_1,\ldots,v_{t-n+1}} P[V_1 = v_{i_1},\ldots,V_{t+1} = v_{i_{t+1}}, O_1^1,\ldots,O_{t+1}^1,\ldots,O_1^Q,\ldots,O_{t+1}^Q \mid \Lambda] \\
= &\max_{v_{t-n+1}} \delta_t(i_{t-n+1},\ldots,i_t)a(i_{t+1})\prod_{q=1}^{Q}\{b_{\vec{j}_t^q, v_{i_{t+1}}}(j_{t+1}^q) b_{\vec{c}_t^q, v_{i_{t+1}}}(c_{t+1}^q)\} \ ,
\end{aligned}$$

  where $j_{t+1}^q = j_t^q - 1, j_t^q, j_t^q + 1$ and $q = 1, 2, \ldots, Q$.

  Then, we set

$$\psi_{t+1}(i_{t-n+2},\cdots,i_{t+1}) = \text{argmax}_{V_{t-n+1}}\{\delta_t(i_{t-n+1},\cdots,i_t) \times a(i_{t+1})\}.$$

- (U3) Termination

$$P^* = \max_{1 \le i_{T-n+1},\ldots,i_T \le N}\{\delta_T(i_{T-n+1},\ldots,i_T)\}$$

  and

$$(i_{T-n+1}^*,\cdots,i_T^*) = \text{argmax}_{1 \le i_{T-n+1},\cdots,i_T \le N}\{\delta_T(i_{T-n+1},\ldots,i_T)\}$$

4 Estimation of Model Parameters

We consider the estimation of the unknown parameters in our model given the "optimal" hidden sequence of the states of the economy in the last section by adopting the maximum likelihood method given the values of the initial distributions of the hidden sequence and observable sequences. First, we provide initial values to all relevant parameters and estimate the "optimal" hidden sequence of states of the economy by the procedures (U1), (U2) and (U3) in the last section. All of the model parameters are then re-estimated again. For double higher-order Markov chain Models, we develop the maximum likelihood estimates for the unknown model parameters and show that these estimators are unbiased. We present the main idea of our approach in the sequel.

First, there is no doubt that the "true" hidden sequence of the states of economy is not observable, and hence, it is not possible to measure how accurate the resulting "optimal" hidden sequence of the states of economy as a proxy for the "true" one is. By using the "optimal" hidden sequence of the states of economy, we can provide a possible way to construct the model parameters. For each state of the economy $v_i \in \mathcal{V}$ $(i = 1, 2, \ldots, M)$, we develop DHHMMs for the observable spot interest rates and credit ratings of the Q bonds. Hence, there are $2QM$ double higher-order observable Markov chain models for the spot interest rates and the credit ratings of the Q bonds. We assume that all observable double higher-order Markov chain models are independent with each other. From the observable sequence of the spot interest rates and credit ratings $\mathcal{O}^q := O_1^q O_2^q \ldots O_T^q$ of the q^{th} bond, we have $(T - m)$ sub-sequences in the form of $O_t^q O_{t+1}^q \ldots O_{t+m}^q$, for $t = 1, 2, \ldots, T - m$ and $q = 1, 2, \ldots, Q$. If two sub-sequences are generated by the same current hidden state of the economy V_{t+m}, they will be put into the same Markov chain models for the spot interest rates and the credit ratings since they are generated from the same model. It can be shown that under our estimation procedure, the conditional probability of the observation and hidden sequence given by new model parameters $\bar{\Lambda}$, $P[\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q, V|\bar{\Lambda}]$, is always greater than or equal to that of old parameters $P[\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q, V|\Lambda']$. This is a desirable result since the estimation of the model parameters Λ can always be improved by more iterations in terms of maximizing the likelihood function. One can then use the new parameters $\bar{\Lambda}$ to re-estimate the hidden sequence of states of the economy, and this process can be done iteratively. The procedures of the iterative estimation method stops when the conditional probability $P[\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q, V|\Lambda]$ converge and satisfy the following stopping criterion:

$$|P[\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q, V|\bar{\Lambda}] - P[\mathcal{O}^1, \mathcal{O}^2, \ldots, \mathcal{O}^Q, V|\Lambda']| < \epsilon \ , \tag{4.1}$$

for some ϵ which is the upper bound for the degree of accuracy.

The stopping criterion stops the iterative estimation procedure when the increase or improvement in the likelihood function by generating one more iteration for the estimates is less than a given small positive number ϵ.

Now, we are going to present the convergence criterion for the model parameters Λ. Recall that $o_t^q := (j_t^q, c_t^q)$ $(q = 1, 2, \ldots, Q)$ is the realization of the observation O_t^q at time t. For a given hidden sequence of states of the economy $V_1V_2 \ldots V_T$, we can define the following quantities:

1. $A(\vec{v}, v_j)$ denote the transition frequency from hidden state sequence $\vec{v} = (v_{i_1}, v_{i_2}, \cdots, v_{i_n})$ to v_j is occurred.

2. For each $q = 1, 2, \ldots, Q$, let $E_l^q(\vec{i}, h, v_j)$ be the transition frequency of the q-th spot interest rate from $\vec{i} = (i_{j_1}, i_{j_2}, \cdots, i_{j_l})$ to $i_{j_{l+1}} = i_h$ given the current hidden state $v_{j_{l+1}} = v_j$. Let $E_l(\vec{i}, h, v_j) = \sum_{q=1}^{Q} E_l^q(\vec{i}, h, v_j)$, which represents the sum of transition frequencies of the observable sequences of spot interest rates for all of the Q bonds from $\vec{i} = (i_{j_1}, i_{j_2}, \cdots, i_{j_l})$ to $i_{j_{l+1}} = i_h$ given the current hidden state $v_{j_{l+1}} = v_j$.

3. For each $q = 1, 2, \ldots, Q$, let $E_m^q(\vec{c}, u, v_j)$ be the transition frequency of the q-th credit rating from $\vec{c} = (c_{j_1}, c_{j_2}, \cdots, c_{j_m})$ to $c_{j_{m+1}} = c_u$ given the current hidden state $v_{j_{l+1}} = v_j$. Let $E_m(\vec{c}, u, v_j) = \sum_{q=1}^{Q} E_m^q(\vec{i}, u, v_j)$, which represents the sum of transition frequencies of the observable sequences of credit ratings for all of the Q bonds from $\vec{c} = (c_{j_1}, c_{j_2}, \cdots, c_{j_m})$ to $c_{j_{m+1}} = c_u$ given the current hidden state $v_{j_{l+1}} = v_j$

Then, we present the convergence criteria in the following proposition:

Proposition 1 *For a given optimal hidden states of the economy $V = V_1 \dots V_T$, if the estimated parameters satisfy the following criteria,*

1. $$\overline{A(\vec{v}, v_j)} = \frac{A(\vec{v}, v_j)}{\sum_{j=1}^{M} A(\vec{v}, v_j)} \quad , \quad \vec{v} = (v_{i_1}, v_{i_2}, \cdots, v_{i_n})$$

2. $$\overline{b_{\vec{i}, v_j}(h)} = \frac{E_l(\vec{i}, h, v_j)}{\sum_{h=max\{0, i_{j_n}-1\}}^{min\{i_{j_n}+1\}} E_l(\vec{i}, h, v_j)} \quad , \quad \vec{i} = (i_{j_1}, i_{j_2}, \cdots, i_{j_l})$$

3. $$\overline{b_{\vec{c}, v_j}(u)} = \frac{E_m(\vec{c}, u, v_j)}{\sum_{u=0}^{N} E_m(\vec{c}, u, v_j)} \quad , \quad \vec{c} = (c_{j_1}, c_{j_2}, \cdots, c_{j_m})$$

Then, given the values of the initial parameters $\pi(i_j)$, $b_{\vec{c}_{j-1}, v_{i_j}}(c_j)$ and $b_{\vec{k}_{j-1}, v_{i_j}}(k_j)$, the estimators maximize the following likelihood function:

$$L(l, m, n) = \left(\prod_{k=1}^{n} \pi(i_k) \right) \left(\prod_{h=n+1}^{T} a(i_h) \right) \left(\prod_{q=1}^{Q} \prod_{t=1}^{T} b_{\vec{c}^q_{t-1}, v_{i_t}}(c^q_t) b_{\vec{j}^q_{t-1}, v_{i_t}}(j^q_t) \right) . \tag{4.2}$$

Hence, the estimation procedure converges to a local maximum, i.e.,

$$P[\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^Q, V|\bar{\Lambda}] \geq P[\mathcal{O}^1, \mathcal{O}^2, \dots, \mathcal{O}^Q, V|\Lambda'].$$

Proposition 1 says that if the estimators given by (1), (2) and (3), they maximize the likelihood function and the estimation procedure converges; that is, more iterations can improve the estimation results until the stopping criterion is satisfied.

The next proposition shows that these estimators are unbiased.

Proposition 2 *The estimators satisfy the following unbiased conditions:*

(i) $\mathbf{E}(A(\vec{v}, v_j)) = a(\vec{v}, v_j) \mathbf{E} \left(\sum_{j=1}^{M} A(\vec{v}, v_j) \right).$

(ii) $\mathbf{E}(E_l(\vec{i}, h, v_j)) = b_{\vec{i}, v_j}(h) \mathbf{E} \left(\sum_{j=1}^{M} E_l(\vec{i}, h, v_j) \right).$

(iii) $\mathbf{E}(E_m(\vec{c},q,v_j)) = b_{\vec{c},q,v_j}(q)\mathbf{E}\left(\sum_{j=1}^{M} E_m(\vec{c},q,v_j)\right).$

where $\mathbf{E}(X)$ is the expectation of a random variable X.

The unbiasedness of an estimator is a desirable statistical property of an estimator. It means that the long-run average of the estimator is equal to the target parameter to be estimated.

Based on these results, we show how to determine the orders of long-range dependence for credit ratings, spot interest rates and the hidden economic states. ¿From the results of the two propositions equation (4.1), we notice that the stopping criterion can be re-defined as:

$$log(L(l,m,n|\bar{\Lambda})) - log(L(l,m,n|\bar{\Lambda}')) < \sigma \ , \tag{4.3}$$

where $\sigma := log\epsilon$.

Recall that the likelihood function $L(l,m,n)$ in our context is determined as follows:

$$L(l,m,n) = \left(\prod_{k=1}^{n} \pi(i_k)\right)\left(\prod_{h=n+1}^{T} a(i_h)\right)\left(\prod_{q=1}^{Q}\prod_{t=1}^{T} b_{\vec{c}^q_{t-1},v_{i_t}}(c^q_t) b_{\vec{j}^q_{t-1},v_{i_t}}(j^q_t)\right) \ , \tag{4.4}$$

where $\vec{c}^q_{t-1} := (c^q_{\max(1,t-m)},...,c^q_{t-1})$ and $\vec{j}^q_{t-1} := (j^q_{max(1,t-l)},...,j^q_{t-1})$, $t = 2,3,\ldots,T$. The log-likelihood function $\tilde{l}(l,m,n)$ is given by:

$$\tilde{l}(l,m,n) = \sum_{k=1}^{n} \log \pi(i_k) + \sum_{h=n+1}^{T} \log a(i_h) + \sum_{q=1}^{Q}\sum_{t=1}^{T} \log b_{\vec{c}^q_{t-1},v_{i_t}}(c^q_t) + \log b_{\vec{j}^q_{t-1},v_{i_t}}(j^q_t) \ . \tag{4.5}$$

We consider some possible models with orders $l,m,n \in \{1,2,3\}$ and select the "best" one with the maximum the log-likelihood. That is, the "optimal" orders l^*,m^*,n^* are determined by:

$$(l^*,m^*,n^*) = \arg \max_{l,m,n\in\{1,2,3\}} \tilde{l}(l,m,n) \ . \tag{4.6}$$

These optimal orders are useful in determining the degrees of the long-range dependence of spot interest rates, credit ratings and the hidden states of economy.

The numerical results will be presented and discussed in the next section.

5 An Example

We adopt an example to illustrate the estimation method in our model. The following scenario is considered.

1. There is only one bond in the economy. (i.e. Q=1)
2. The hidden states of the economy follow a second-order Markov chain model with two different states, namely "Good" and "bad". (I.e., n=2 and M=2).
3. The time series of credit rating f the bond follows a second-order DHHMM with two states, namely "default" or "no default". (i.e., m=2, N=1).
4. The time series of spot interest rates follows a first-order DHHMM with two states, namely "up" or "down". (i.e., l=1, H=1).

Since we consider Q=1, we omit the superscript q in both credit rating and spot interest rates. Once we have above information, we can randomly generate the initial distributions and the transition probabilities matrices for both hidden sequence and observable sequences. Based on the values of those parameters, we can further generate the hidden sequence and observable sequences. Note that we do not have any prior information on the hidden sequence and all transition probabilities matrices and that we can only observe the initial distributions and observable sequences. We suppose that the following initial distributions are given:

$$\pi(i_1) = \begin{bmatrix} 0.5752 \\ 0.4248 \end{bmatrix}^T, \quad \pi(i_2) = \begin{bmatrix} 0.938 & 0.062 \\ 0.1315 & 0.8685 \end{bmatrix} \quad b_{\vec{i}_0,1} = \begin{bmatrix} 0.8551 \\ 0.1449 \end{bmatrix}^T, \quad b_{\vec{i}_0,2} = \begin{bmatrix} 0.9218 \\ 0.0782 \end{bmatrix}^T$$

$$b_{\vec{c}_0,1}(c_1) = \begin{bmatrix} 0.6668 \\ 0.3332 \end{bmatrix}^T \quad b_{\vec{c}_0,2}(c_1) = \begin{bmatrix} 0.6017 \\ 0.3983 \end{bmatrix}^T$$

$$b_{\vec{c}_1,1}(c_2) = \begin{bmatrix} 0.4168 & 0.5832 \\ 0.7243 & 0.2757 \end{bmatrix} \quad b_{\vec{c}_1,2}(c_2) = \begin{bmatrix} 0.4592 & 0.5408 \\ 0.6728 & 0.3272 \end{bmatrix}$$

and observable sequences of credit ratings and spot interest rate of the bonds are given by:

$$C = \{2, 1, 2, 2, 1, 1, 2, 1, 2, 2\}, \quad I = \{1, 2, 2, 1, 2, 1, 1, 1, 2, 1\}.$$

Initially, there is no prior knowledge on all state transition probabilities, $a(i_{t+n})$, $b_{\vec{i}_{j-1},v}(i_j)$ and $b_{\vec{c}_t,v}(c_{t+1})$. Therefore, we initialize them by random generations and the results are as follows:

$$\tilde{a}(i_{t+n}) = \begin{bmatrix} 0.8623 & 0.1377 \\ 0.1183 & 0.8817 \\ 0.9496 & 0.0504 \\ 0.1091 & 0.8909 \end{bmatrix},$$

$$\tilde{b}_{\vec{c}_t,1}(c_{t+1}) = \begin{bmatrix} 0.4073 & 0.5927 \\ 0.5263 & 0.4737 \\ 0.0096 & 0.9904 \\ 0.9952 & 0.0048 \end{bmatrix}, \quad \tilde{b}_{\vec{c}_t,2}(c_{t+1}) = \begin{bmatrix} 0.7846 & 0.2154 \\ 0.5394 & 0.4606 \\ 0.7540 & 0.2460 \\ 0.4740 & 0.5260 \end{bmatrix},$$

$$\tilde{b}_{\vec{i}_{j-1},1}(i_j) = \begin{bmatrix} 0.6906 & 0.3094 \\ 0.4608 & 0.5392 \end{bmatrix}, \quad \tilde{b}_{\vec{i}_{j-1},2}(i_j) = \begin{bmatrix} 0.2472 & 0.7528 \\ 0.6386 & 0.3614 \end{bmatrix}.$$

¿From above information, we can employ procedures (U1), (U2) and (U3) to estimate the optimal hidden sequence of the states of the economy:

$$\tilde{V} = 2, 2, 2, 2, 2, 2, 1, 1, 1, 1$$

¿From the optimal hidden sequence, we can see that the dynamics of the hidden states of the economy are persistent.

The log-likelihood can be obtained by:

$$\begin{aligned}
\tilde{l}(1,2,2) &= \sum_{k=1}^{2} \log \pi(i_k) + \sum_{h=3}^{10} \log a(i_h) + \sum_{t=1}^{10} \log b_{\vec{c}_{t-1},v_{i_t}}(c_t) + \sum_{t=1}^{10} \log b_{\vec{j}_{t-1},v_{i_t}}(j_t) \\
&= (\log 0.4248 + \log 0.8685) + (4\log(0.8909) + \log 0.1091 + \log 0.9496 + 2\log 0.8623) \\
&\quad +(\log 0.3983 + \log 0.6728 + \log 0.246 + \log 0.4606 + \log 0.474 + \log 0.754 + \log 0.5927 \\
&\quad + \log 0.5263 + \log 0.9904 + \log 0.4737) + (\log 0.9218 + \log 0.7528 + \log 0.3614 \\
&\quad + \log 0.6386 + \log 0.7528 + \log 0.6386 + \log 0.6906 + \log 0.6906 + \log 0.3094 + \log 0.4608) \\
&= -15.6238
\end{aligned}$$

¿From the optimal hidden sequence of the states of the economy, we can re-estimate all the state transition probabilities matrices by Proposition 1. Then, the states of the economy are given by:

$$\tilde{a}(i_{t+n}) = \begin{bmatrix} 1.0 & 0.0 \\ 0.5 & 0.5 \\ 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix}, \quad \tilde{b}_{\vec{c}_t,1}(c_{t+1}) = \begin{bmatrix} 0.0 & 1.0 \\ 0.5 & 0.5 \\ 0.0 & 1.0 \\ 0.5 & 0.5 \end{bmatrix}, \quad \tilde{b}_{\vec{c}_t,2}(c_{t+1}) = \begin{bmatrix} 0.5 & 0.5 \\ 0.0 & 1.0 \\ 0.5 & 0.5 \\ 1.0 & 0.0 \end{bmatrix},$$

and

$$\tilde{b}_{\vec{i}_{j-1},1}(i_j) = \begin{bmatrix} 0.6667 & 0.3333 \\ 1.0000 & 0.0000 \end{bmatrix}, \quad \tilde{b}_{\vec{i}_{j-1},2}(i_j) = \begin{bmatrix} 0.0000 & 1.0000 \\ 0.6667 & 0.3333 \end{bmatrix}.$$

Again, we can re-estimate the optimal hidden sequence of the states of the economy based on above transition probabilities matrices and re-estimate the log-likelihood function until convergence occurs. We found that the log-likelihood function converges when it is equal to -11.4891.

6 Numerical Experiment

We present a numerical experiment to illustrate the implementation of our model to estimate the "optimal" hidden sequence of the states of economy using simulated data on the spot interest rates and credit ratings of the bonds. In this experiment, we suppose that there are two bonds in the economy. We assume that the credit rating and the spot interest rate for each bond at each fixed time period can take five possible states and six possible states, respectively. We further suppose that there are three possible hidden states of the economy, namely "good", "neutral" and "bad". First, we specify the model parameters in the DHHMMs and assume that the specified parameters are "true" model parameters. The hypothetical DHHMMs with specified parameters are then considered as the "true" underlying model that are used for simulating the realizations of the spot interest rates and the credit ratings of the two bonds. Hence, the simulated data for the spot interest rates and the credit ratings of the bonds are assumed to be the observations in our model. They are then used to investigate the model and the estimators of the model parameters. Since there are no prior information about the order of hidden sequence, credit ratings as well as spot interest rates, we simulate 27 scenarios with different l=1,2,3, m=1,2,3 and n=1,2,3. We simulate each scenario 1000 times and the average results are presented in Table 1.

l	1	1	1	1	1	1	1	1	1
n	1	1	1	2	2	2	3	3	3
m	1	2	3	1	2	3	1	2	3
Iteration	2.29	2.30	2.11	2.01	2.02	2.01	2.01	2.01	2.02
Time (sec.)	0.05	0.05	0.07	0.11	0.12	0.14	0.50	0.52	0.54
Log-likelihood	-34.57	-26.68	-26.32	-34.76	-26.78	-26.44	-34.99	-26.73	-26.34
l	2	2	2	2	**2**	2	2	2	2
n	1	1	1	2	**2**	2	3	3	3
m	1	2	3	1	**2**	3	1	2	3
Iteration	2.17	2.20	2.11	2.01	2.20	2.09	2.00	2.00	2.00
Time (sec.)	0.05	0.06	0.08	0.12	0.14	0.15	0.52	0.53	0.56
Log-likelihood	-31.65	-24.04	-24.00	-31.88	**-23.74**	-23.71	-31.76	-24.05	-23.59
l	3	3	3	3	3	3	3	3	3
n	1	1	1	2	2	2	3	3	3
m	1	2	3	1	2	3	1	2	3
Iteration	2.22	2.19	2.08	2.01	2.21	2.12	2.00	2.00	2.09
Time (sec.)	0.10	0.11	0.12	0.17	0.19	0.20	0.57	0.58	0.62
Log-likelihood	-31.49	-23.95	-23.87	-31.79	-23.50	-23.65	-31.72	-23.94	-23.49

Table 1. Various log-likelihood results with Q=2

For providing a graphical analysis of the results, we visualize the results by creating the two-dimensional plots for the above results for the values of the log-likelihood functions against the orders m and displaying the case of n=1, n=2 and n=3 in the plots of Figure 1, Figure 2 and Figure 3 respectively. In each Figure, we plot 3 different curves for representing the log-likelihood result of each l. From these Figures, we observe that the value of a log-likelihood function increases as either m or l does. This indicates that the higher the degree of the long-range dependence of either the credit ratings and the spot interest rate, the higher the value of the log-likelihood function. In particular, there is a sharp increase in the value of the log-likelihood function when either m or l increases from 1 to 2 in all cases while there is only a very small increase in the value of the log-likelihood function when either m or l increases from 2 to 3. This reveals that it is not unreasonable to determine the degree of the long-range dependence for the dynamics of both credit rating and spot interest rate as 2.

In our setting, we assume that the "true" orders of hidden sequences of the states of the economy, the credit ratings and the spot interest rates are all equal to 2. From the above numerical results, we notice that the value of the log-likelihood function corresponding to the the "true" orders of hidden and observable sequences is very close to the maximum value of the log-likelihood functions for all 27 scenarios. This illustrates the ability of our model in recovering the "true" orders of the hidden and observable sequences. To illustrate this graphically, we visualize the results by plotting the number of iterations, computational time and the values of the log-likelihood function against each simulation in Figure 4, Figure 5 and Figure 6, respectively. To provide more information in the stability of our results, we report the average result for the Markov transition probability matrix for credit ratings based on 1000 simulation runs when the hidden state of the economy is 2. We also report the result for the estimated stationary probability vector for credit ratings. We compare the estimated Markov transition probability matrix $\tilde{b}_{\vec{c}_t,2}(c_{t+1})$ with the "true" one $b_{\vec{c}_t,2}(c_{t+1})$ and the estimated stationary probability vector $\hat{C}$ with the "true" one C.

$$\tilde{b}_{\vec{c}_t,2}(c_{t+1}) = \begin{bmatrix} 0.139 & 0.139 & 0.444 & 0.139 & 0.139 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1382 & 0.1382 & 0.1382 & 0.1382 & 0.4472 \\ 0.1552 & 0.1552 & 0.1552 & 0.3792 & 0.1552 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1676 & 0.1676 & 0.1676 & 0.3296 & 0.1676 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2967 & 0.1282 & 0.3187 & 0.1282 & 0.1282 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2646 & 0.1198 & 0.2646 & 0.1198 & 0.2311 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1474 & 0.1474 & 0.1474 & 0.1474 & 0.4104 \\ 0.1434 & 0.4264 & 0.1434 & 0.1434 & 0.1434 \\ 0.4312 & 0.1422 & 0.1422 & 0.1422 & 0.1422 \\ 0.132 & 0.132 & 0.132 & 0.132 & 0.472 \\ 0.1382 & 0.1382 & 0.1382 & 0.1382 & 0.4472 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1552 & 0.3792 & 0.1552 & 0.1552 & 0.1552 \end{bmatrix}, \quad b_{\vec{c}_t,2}(c_{t+1}) = \begin{bmatrix} 0.0263 & 0.2703 & 0.2367 & 0.2741 & 0.1926 \\ 0.2835 & 0.1278 & 0.0977 & 0.2612 & 0.2297 \\ 0.2366 & 0.2245 & 0.2001 & 0.252 & 0.0868 \\ 0.0957 & 0.2178 & 0.3366 & 0.3162 & 0.0337 \\ 0.2527 & 0.1496 & 0.202 & 0.2514 & 0.1443 \\ 0.0956 & 0.1233 & 0.2311 & 0.2013 & 0.3486 \\ 0.1231 & 0.0494 & 0.3436 & 0.3048 & 0.1791 \\ 0.1948 & 0.1569 & 0.0146 & 0.3558 & 0.2779 \\ 0.094 & 0.2314 & 0.231 & 0.0847 & 0.3589 \\ 0.1762 & 0.3753 & 0.1791 & 0.0346 & 0.2348 \\ 0.1863 & 0.1583 & 0.2187 & 0.3631 & 0.0737 \\ 0.0269 & 0.2519 & 0.4684 & 0.2233 & 0.0295 \\ 0.0707 & 0.3396 & 0.168 & 0.0781 & 0.3435 \\ 0.3784 & 0.176 & 0.2463 & 0.0755 & 0.1237 \\ 0.2774 & 0.2081 & 0.141 & 0.3312 & 0.0423 \\ 0.2259 & 0.1959 & 0.1207 & 0.2842 & 0.1733 \\ 0.0938 & 0.3027 & 0.1454 & 0.4036 & 0.0545 \\ 0.1188 & 0.21 & 0.3964 & 0.0203 & 0.2545 \\ 0.2053 & 0.2248 & 0.3093 & 0.2272 & 0.0334 \\ 0.2376 & 0.127 & 0.2095 & 0.2148 & 0.211 \\ 0.3012 & 0.1807 & 0.2536 & 0.0495 & 0.2149 \\ 0.3237 & 0.0332 & 0.0384 & 0.3262 & 0.2785 \\ 0.0492 & 0.2381 & 0.0237 & 0.3494 & 0.3396 \\ 0.2411 & 0.1361 & 0.2554 & 0.1321 & 0.2353 \\ 0.0052 & 0.3049 & 0.4705 & 0.0912 & 0.1282 \end{bmatrix}$$

and

$$\hat{C} = \begin{bmatrix} 0.3000 & 0.3001 \\ 0.2999 & 0.3002 \\ 0.3000 & 0.3000 \\ 0 & 0.3000 \\ 0.3001 & 0.3001 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3000 & 0.1000 \\ 0.1000 & 0.2000 \\ 0.2000 & 0.2000 \\ 0 & 0.2000 \\ 0.4000 & 0.3000 \end{bmatrix}$$

where $\hat{C}_{.j}$ is the average of 1000 simulated stationary vectors of the j^{th} credit rating while $C_{.j}$ is the "true" stationary vector of the j^{th} credit rating. We observe that both of the estimated transition probability matrix and the estimated stationary probability vector are not very close to their corresponding "true" values. This may be attributed to the limited length of the sample data sequence, say $T = 10$, in our numerical results, which leads to the situation that some of the unknown model parameters have not been fully recovered.

The numerical experiment in this section was done by a PC with CPU=1.2Ghz and RAM=512Mb. The computational time of generating the numerical results is less than 1 second for each scenario. This illustrates the effectiveness of the proposed model. Also, from the numerical results, we can see that the dynamics of credit ratings, spot interest rates and the hidden economic states exhibit some degrees of long-range dependence with order 2 or 3.

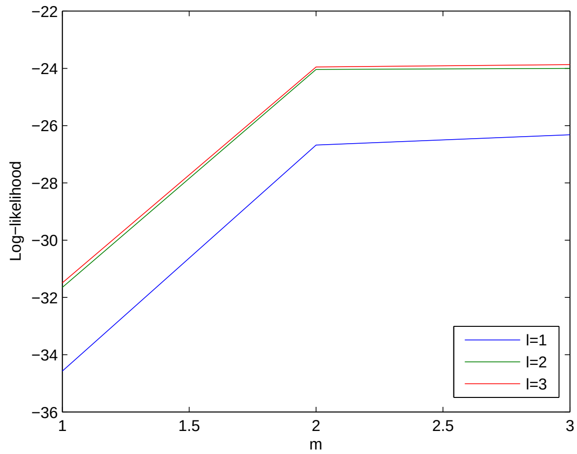

Figure 1: Various log-likelihood results with Q=2 and n=1

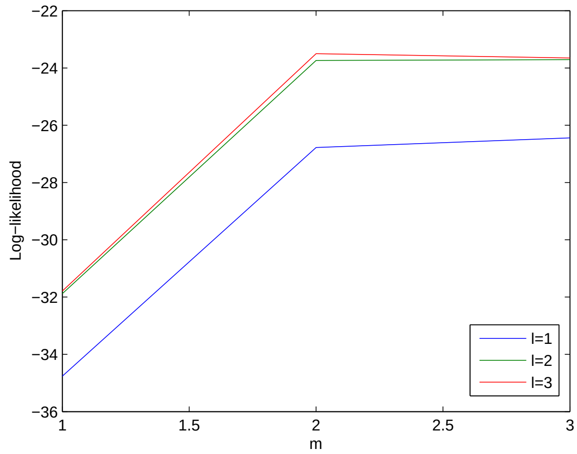

Figure 2: Various log-likelihood results with Q=2 and n=2

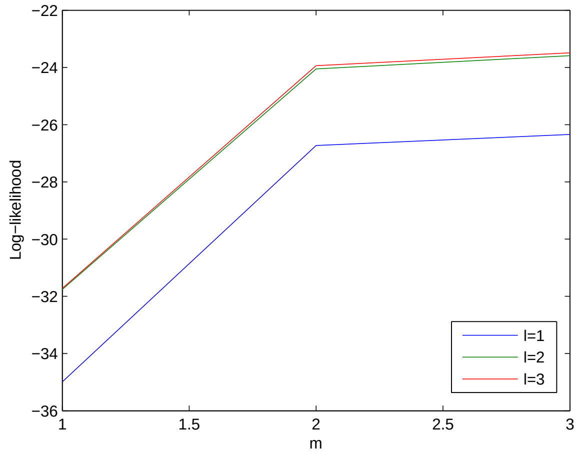

Figure 3: Various log-likelihood results with Q=2 and n=3

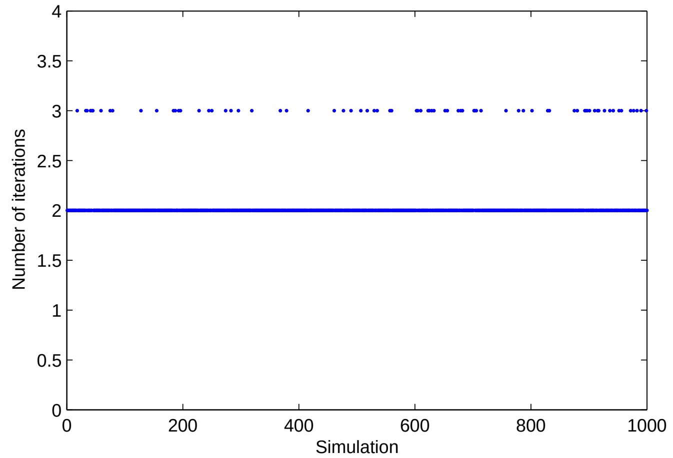

Figure 4: The simulation result for l=2, n=2 and m=2

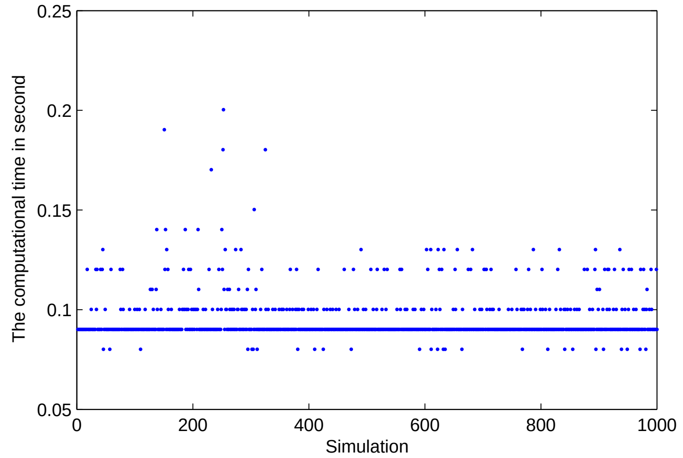

Figure 5: The simulation result for l=2, n=2 and m=2

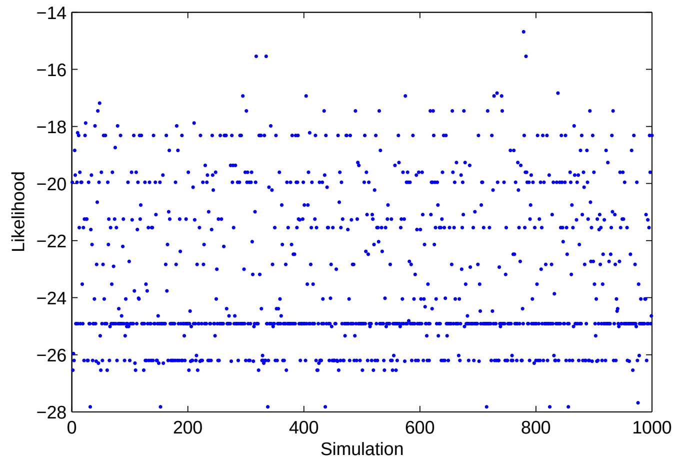

Figure 6: The simulation result for l=2, n=2 and m=2

7 Conclusion and Further Research

We have developed DHHMMs for extracting information about the unobservable states of the economy from the spot interest rates and credit ratings of bonds. The dependency of the spot interest rates and the economic states and the relationship between the credit ratings and economic states were modelled by the DHHMMs. We have adopted the maximum likelihood estimation and Viterbi's algorithm to uncover the optimal hidden sequence of the states of economy. An efficient maximum likelihood estimation method has been developed to estimate the model parameters in the transition probability matrices. We have also employed the maximum of log-likelihood function to determine the orders of the DHHMMs. We have provided numerical examples that illustrate the implementation of our model.

For further investigation, it may be interesting to adopt our model to investigate the relationship between other observable economic or indicator variables and the unobservable states of the economy. We may develop some statistical tests for the hypotheses about the significance of the impact of various observable economic variables or indicator variables on the unobservable states of the economy. The model we have developed in this paper assumed that the spot interest rates and the credit ratings are conditionally independent given the current level of hidden economic state. It is very interesting to develop a simple model to incorporate the conditional dependency of the spot interest rates and the credit ratings. In this case, we can provide a more general and flexible model to describe the impact of the levels of the hidden economic states on the dependency or the (non-linear) association of the spot interest rates and the credit ratings. One possible way to model the impact of the unobservable states of the economy on the conditional dependency of the observable economic variables is to consider the modification of the multivariate Markov chain model by Ching et el. (2002). Another way is to modify the DHHMM we adopted in this paper and provide parameterizations for describing the conditional dependency. From the perspective of corporate finance, it is very interesting to adopt our idea for extracting the unobservable dynamics of the market values of a corporation from the observable spot interest rates and credit ratings. Finally, it is also interesting to further explore the applications of our model for investigating business cycles and fluctuations and its implications of macroeconomic analysis.

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Proof of Preposition 1: First, we notice that

$$
\begin{aligned}
L(l,m,n) &= \left(\prod_{k=1}^{n}\pi(i_k)\right)\left(\prod_{h=n+1}^{T}a(i_h)\right)\left(\prod_{q=1}^{Q}\prod_{t=1}^{T}b_{\vec{c}^q_{t-1},v_{i_t}}(c^q_t)b_{\vec{j}^q_{t-1},v_{i_t}}(j^q_t)\right)\\
&= \left(\prod_{k=1}^{n}\pi(i_k)\right)\left(\prod_{h=n+1}^{T}a(i_h)\right)\left[\prod_{q=1}^{Q}\left(\prod_{k=1}^{m}b_{\vec{c}^q_{k-1},v_{i_k}}(c^q_k)\prod_{h=m+1}^{T}b_{\vec{c}^q_{h-1},v_{i_h}}(c^q_h)\right)\right.\\
&\quad \left.\left(\prod_{k=1}^{l}b_{\vec{j}^q_{k-1},v_{i_k}}(j^q_k)\prod_{h=l+1}^{T}b_{\vec{j}^q_{h-1},v_{i_h}}(j^q_h)\right)\right]\\
&= \left(\prod_{k=1}^{n}\pi(i_k)\right)\left[\prod_{q=1}^{Q}\left(\prod_{k=1}^{m}b_{\vec{c}^q_{k-1},v_{i_k}}(c^q_k)\right)\left(\prod_{k=1}^{l}b_{\vec{j}^q_{k-1},v_{i_k}}(j^q_k)\right)\right]\\
&\quad \left(\prod_{h=n+1}^{T}a(i_h)\right)\left[\prod_{q=1}^{Q}\left(\prod_{h=m+1}^{T}b_{\vec{c}^q_{h-1},v_{i_h}}(c^q_h)\right)\left(\prod_{h=l+1}^{T}b_{\vec{j}^q_{h-1},v_{i_h}}(j^q_h)\right)\right],
\end{aligned}
$$

where $\vec{c}^q_{t-1} := (c^q_{\max(1,t-m)}, ..., c^q_{t-1})$ and $\vec{j}^q_{t-1} := (j^q_{max(1,t-l)}, ..., j^q_{t-1})$, $t = 2, 3, \ldots, T$.

For estimating the model parameters, we suppose that the initial distributions $\pi(i_j)$, $b_{\vec{c}^q_{k-1},v_{i_k}}(c^q_k)$ and $b_{\vec{j}^q_{k-1},v_{i_k}}(j^q_k)$ for the hidden economic states, spot interest rates and credit ratings are known in advance. Hence, the estimators obtained by maximizing $L(l,m,n)$ are the same as those obtained by maximizing the following likelihood:

$$
\hat{L}(l,m,n) = \left(\prod_{h=n+1}^{T}a(i_h)\right)\left[\prod_{q=1}^{Q}\left(\prod_{h=m+1}^{T}b_{\vec{c}^q_{h-1},v_{i_h}}(c^q_h)\right)\left(\prod_{h=l+1}^{T}b_{\vec{j}^q_{h-1},v_{i_h}}(j^q_h)\right)\right].
$$

We can rewrite the likelihood $\hat{L}(l,m,n)$ and formulate the problem of maximum likelihood estimation as follows:

$$
\begin{aligned}
\hat{L}(l,m,n) &= \prod_{v_1=1}^{M}\prod_{v_2=1}^{M}...\prod_{v_n=1}^{M}\prod_{j=1}^{M} a(\vec{v},v_j)^{A(\vec{v},v_j)}\left[\prod_{q=1}^{Q}\left(\prod_{j=1}^{M}\prod_{c_1=0}^{N}\prod_{c_2=0}^{N}...\prod_{c_m=0}^{N}\prod_{u=0}^{N} b_{\vec{c},v_j}(u)^{E_m^q(\vec{c},u,v_j)}\right)\right] \\
&\left[\prod_{q=1}^{Q}\left(\prod_{j=1}^{M}\prod_{i_1=0}^{H}\prod_{i_2=max\{0,i_1-1\}}^{min\{i_1+1,H\}}...\prod_{i_l=max\{0,i_{l-1}-1\}}^{min\{i_{l-1}+1,H\}}\prod_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} b_{\vec{i},v_j}(h)^{E_l^q(\vec{i},h,v_j)}\right)\right] \\
&= \prod_{v_1=1}^{M}\prod_{v_2=1}^{M}...\prod_{v_n=1}^{M}\prod_{j=1}^{M} a(\vec{v},v_j)^{A(\vec{v},v_j)}\left(\prod_{j=1}^{M}\prod_{c_1=0}^{N}\prod_{c_2=0}^{N}...\prod_{c_m=0}^{N}\prod_{u=0}^{N} b_{\vec{c},v_j}(u)^{\sum_{q=1}^{Q}E_m^q(\vec{c},u,v_j)}\right) \\
&\left(\prod_{j=1}^{M}\prod_{i_1=0}^{H}\prod_{i_2=max\{0,i_1-1\}}^{min\{i_1+1,H\}}...\prod_{i_l=max\{0,i_{l-1}-1\}}^{min\{i_{l-1}+1,H\}}\prod_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} b_{\vec{i},v_j}(h)^{\sum_{q=1}^{Q}E_l^q(\vec{i},h,v_j)}\right) \\
&= \prod_{v_1=1}^{M}\prod_{v_2=1}^{M}...\prod_{v_n=1}^{M}\prod_{j=1}^{M} a(\vec{v},v_j)^{A(\vec{v},v_j)}\left(\prod_{j=1}^{M}\prod_{c_1=0}^{N}\prod_{c_2=0}^{N}...\prod_{c_m=0}^{N}\prod_{u=0}^{N} b_{\vec{c},v_j}(u)^{E_m(\vec{c},u,v_j)}\right) \\
&\left(\prod_{j=1}^{M}\prod_{i_1=0}^{H}\prod_{i_2=max\{0,i_1-1\}}^{min\{i_1+1,H\}}...\prod_{i_l=max\{0,i_{l-1}-1\}}^{min\{i_{l-1}+1,H\}}\prod_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} b_{\vec{i},v_j}(h)^{E_l(\vec{i},h,v_j)}\right) ,
\end{aligned}
$$

subject to the following constraints:

$$
\sum_{j=1}^{M} a(\vec{v},v_j)=1, \quad \forall\vec{v}, \qquad \sum_{h=min\{0,i_l-1\}}^{max\{i_l+1,H\}} b_{\vec{i},v_j}(h)=1, \quad \forall\vec{i},v_j \qquad \sum_{u=0}^{N} b_{\vec{c},v_j}(u)=1, \quad \forall\vec{c},v_j \ ,
$$

where $\vec{v}=[v_1,v_2,...,v_n]$, $\vec{i}=[i_1,i_2,...,i_l]$ and $\vec{c}=[c_1,c_2,...,c_m]$. Here, we assume $0^0=1$.

Furthermore, maximizing the likelihood $\hat{L}(l,m,n)$ is equivalent to maximizing the log-likelihood $\tilde{l}$=log $\hat{L}(l,m,n)$. The corresponding log-likelihood $\tilde{l}$ is given by:

$$
\begin{aligned}
\tilde{l} &= \sum_{v_1=1}^{M}\sum_{v_2=1}^{M}...\sum_{v_n=1}^{M}\sum_{j=1}^{M} A(\vec{v},v_j)\log a(\vec{v},v_j) + \sum_{j=1}^{M}\sum_{c_1=0}^{N}\sum_{c_2=0}^{N}...\sum_{c_m=0}^{N}\sum_{u=0}^{N} E_m(\vec{c},u,v_j)\log b_{\vec{c},v_j}(u) \\
&+\sum_{j=1}^{M}\sum_{i_1=0}^{H}\sum_{i_2=max\{0,i_1-1\}}^{min\{i_1+1,H\}}...\sum_{i_l=max\{0,i_{l-1}-1\}}^{min\{i_{l-1}+1,H\}}\sum_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} E_l(\vec{i},h,v_j)\log b_{\vec{i},v_j}(h) \ .
\end{aligned}
$$

By considering both the log-likelihood function and the constraints, we maximize the following function:

$$
\begin{aligned}
\tilde{l}_c \;=\; & \sum_{v_1,\ldots,v_n=1}^{M} \sum_{j=1}^{M} A(\vec{v},v_j)\log a(\vec{v},v_j) + \sum_{j=1}^{M}\sum_{i_1,\ldots,i_l}\sum_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} E_l(\vec{i},h,v_j)\log b_{\vec{i},v_j}(h) \\
& + \sum_{j=1}^{M}\sum_{c_1,\ldots,c_m=0}^{N}\sum_{u=0}^{N} E_m(\vec{c},u,v_j)\log b_{\vec{c},v_j}(u) + \sum_{v_1,\ldots,v_n=1}^{M}\mu(\vec{v})\left(1-\sum_{j=1}^{M}a(\vec{v},v_j)\right) \\
& + \sum_{j=1}^{M}\sum_{i_1,\ldots,i_l}\rho(\vec{i},v_j)\left(1-\sum_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} b_{\vec{i},v_j}(h)\right) + \sum_{j=1}^{M}\sum_{c_1,\ldots,c_m=0}^{N}\sigma(\vec{c},v_j)\left(1-\sum_{u=0}^{N}b_{\vec{c},v_j}(u)\right) \;,
\end{aligned}
$$

where $\mu(\vec{v})$, $\rho(\vec{i},v_j)$ and $\sigma(\vec{c},v_j)$ are the Lagrange multipliers.

Since

$$
\frac{\partial^2\tilde{l}_c}{\partial(a(\vec{v},v_j))} = -\frac{A(\vec{v},v_j)}{(a(\vec{v},v_j))^2} < 0, \quad \frac{\partial^2\tilde{l}_c}{\partial(b_{\vec{i},v_j}(h))^2} = -\frac{E_l(\vec{i},h,v_j)}{(b_{\vec{i},v_j}(h))^2} < 0, \quad \frac{\partial^2\tilde{l}_c}{\partial(b_{\vec{c},v_j}(u))^2} = -\frac{E_m(\vec{c},u,v_j)}{(b_{\vec{c},v_j}(u))^2} < 0\,,
$$

the function $\hat{L}$ is concave in $a(\vec{v},v_j)$, $b_{\vec{i},v_j}(h)$ and $b_{\vec{c},v_j}(u)$. By solving the following equations

$$
\frac{\partial\tilde{l}_c}{\partial a(\vec{v},v_j)} = 0\;, \quad \frac{\partial\tilde{l}_c}{\partial b_{\vec{i},v_j}(h)} = 0 \quad \text{and} \quad \frac{\partial\tilde{l}_c}{\partial b_{\vec{c},v_j}(u)} = 0\;,
$$

we obtain:

$$
\frac{A(\vec{v},v_j)}{a(\vec{v},v_j)} - \mu(\vec{v}) = 0 \quad \text{or} \quad a(\vec{v},v_j)\mu(\vec{v}) = A(\vec{v},v_j).
$$

$$
\frac{E_l(\vec{i},h,v_j)}{b_{\vec{i},v_j}(h)} - \rho(\vec{i},v_j) = 0 \quad \text{or} \quad b_{\vec{i},v_j}(h)\rho(\vec{i},v_j) = E_l(\vec{i},h,v_j).
$$

$$
\frac{E_m(\vec{c},u,v_j)}{b_{\vec{c},v_j}(u)} - \sigma(\vec{c},v_j) = 0 \quad \text{or} \quad b_{\vec{c},u,v_j}(u)\sigma(\vec{c},v_j) = E_m(\vec{c},u,v_j).
$$

Therefore,

$$
\sum_{j=1}^{M} a(\vec{v},v_j)\mu(\vec{v}) = \mu(\vec{v}) = \sum_{j=1}^{M} A(\vec{v},v_j)\;.
$$

$$
\sum_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} b_{\vec{i},v_j}(h)\rho(\vec{i},v_j) = \rho(\vec{i},v_j) = \sum_{h=max\{0,i_l-1\}}^{min\{i_l+1,H\}} E_l(\vec{i},h,v_j)\;.
$$

$$
\sum_{u=0}^{N} b_{\vec{c},v_j}(u)\sigma(\vec{c},v_j) = \sigma(\vec{c},v_j) = \sum_{u=0}^{N} E_m(\vec{c},u,v_j)\;.
$$

Hence, the result follows. □

Proof of Preposition 2: (i) Let T denote the length of the sequence and $a(\vec{v}, v_j)$ the transition probability matrix. Suppose X_j denotes the steady state probability that the process is in state j. Then,

$$\mathbf{E}(A(\vec{v}, v_j)) = T \cdot X_j \cdot a(\vec{v}, v_j) \ ,$$

and

$$\mathbf{E}\left(\sum_{j=1}^{M} A(\vec{v}, v_j) \right) = T \cdot X_j \cdot \left(\sum_{j=1}^{m} a(\vec{v}, v_j) \right) = T \cdot X_j \ .$$

Therefore,

$$\mathbf{E}(A(\vec{v}, v_j)) = a(\vec{v}, v_j) \cdot \mathbf{E}\left(\sum_{j=1}^{m} A(\vec{v}, v_j) \right) \ .$$

Condition (ii) and (iii) can be done in a similar way. Hence, we omit the proof here. □