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Bohr's inequalities for Hilbert space operators ☆

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Abstract

The classical Bohr's inequality states that

$$|z+w|^2 \leqslant p|z|^2 + q|w|^2$$

for all $z, w \in \mathbb{C}$ and all $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. In this paper, Bohr's inequality is generalized to the context of Hilbert space operators for all positive conjugate exponents $p, q \in \mathbb{R}$. In particular, the parallelogram law is recovered and some other interesting operator inequalities and established.

Keywords: Bohr's inequality

1. Introduction

The classical Bohr's inequality [4,10] states that for any $z, w \in \mathbb{C}$ and any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$|z+w|^2 \leqslant p|z|^2 + q|w|^2,$$

with equality if and only if $w = (p-1)z$. Over the years, various generalizations of Bohr's inequality have been obtained. These include, among others, the works of Archbold [2], Mąkowski [7], Bergström [3], Mitrinović [8,9], Mitrinović et al. [10], Vasić and Kečkić [16],

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Rassias [14,15], Delbosco [5], and Pečarić and Janić [12]. In 1989, Pečarić and Dragomir generalized Bohr's inequality to the context of normed vector spaces. In [11] they showed that if $(X, \|\ \|)$ is a normed vector space and $p, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|v + w\|^2 \leqslant p\|v\|^2 + q\|w\|^2$$

for any $v, w \in X$. Recently, Hirzallah further generalized the inequality to the context of operator algebras. In [6], it was shown that if $\mathbb{H}$ is a complex separable Hilbert space and $B(\mathbb{H})$ is the algebra of all bounded linear operators on $\mathbb{H}$, then for any $A, B \in B(\mathbb{H})$ and $q \geqslant p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$|A - B|^2 + \big|(1-p)A - B\big|^2 \leqslant p|A|^2 + q|B|^2,$$

where $|X| := (X^*X)^{1/2}$.

It is worthwhile noting that in [6], only the situation where $q \geqslant p > 1$ is considered. Equivalently, the conjugate exponents p, q are only restricted to $q \geqslant 2$ and $1 < p \leqslant 2$, while other combinations are left alone.

In this paper, we continue working in the setting as that in [6], but without restriction to the conjugate exponents p, q. Meanwhile, we also investigate the situation of equality in detail and make connection with the parallelogram law for the Banach algebra $B(\mathbb{H})$.

2. Bohr's inequality and the parallelogram law in $B(\mathbb{H})$

Let $\mathbb{H}$ be a complex separable Hilbert space and $B(\mathbb{H})$ the algebra of all bounded linear operators on $\mathbb{H}$. For any $X \in B(\mathbb{H})$, write $|X| = (X^*X)^{1/2}$.

Theorem 1. *For any $A, B \in B(\mathbb{H})$ and any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $1 < p \leqslant 2$, then*

(i) $|A - B|^2 + |(1-p)A - B|^2 \leqslant p|A|^2 + q|B|^2$, *and*
(ii) $|A - B|^2 + |A - (1-q)B|^2 \geqslant p|A|^2 + q|B|^2$.

Furthermore, in both (i) *and* (ii)*, the equality holds if and only if $p = q = 2$ or $(1-p)A = B$.*

Proof. (i) We have as in [6]

$$|A - B|^2 = |A|^2 + |B|^2 - (A^*B + B^*A)$$

and

$$\big|(1-p)A - B\big|^2 = (1-p)^2|A|^2 + |B|^2 - (1-p)(A^*B + B^*A).$$

Thus

$$\begin{aligned}
&|A - B|^2 + \big|(1-p)A - B\big|^2 - p|A|^2 - q|B|^2 \\
&\quad = \big(p^2 - 3p + 2\big)|A|^2 + (2-q)|B|^2 + (p-2)(A^*B + B^*A) \\
&\quad = (p-2)(p-1)|A|^2 + \left(\frac{p-2}{p-1}\right)|B|^2 + (p-2)(A^*B + B^*A) \\
&\quad = (p-2)\left[(p-1)|A|^2 + \frac{1}{p-1}|B|^2 + (A^*B + B^*A)\right]
\end{aligned}$$

$$= (p-2)\left|\sqrt{p-1}A + \frac{1}{\sqrt{p-1}}B\right|^2$$
$$\leqslant 0, \tag{2.1}$$

hence

$$|A-B|^2 + \left|(1-p)A - B\right|^2 \leqslant p|A|^2 + q|B|^2,$$

with equality if and only if $p = 2$ (hence $q = 2$) or $\sqrt{p-1}A + \frac{1}{\sqrt{p-1}}B = 0$, that is, $p = q = 2$ or $(1-p)A = B$.

(ii) Similar to (i), we have

$$|A-B|^2 = |A|^2 + |B|^2 - (A^*B + B^*A)$$

and

$$\left|A - (1-q)B\right|^2 = (1-q)^2|B|^2 + |A|^2 - (1-q)(A^*B + B^*A).$$

Thus

$$|A-B|^2 + \left|A-(1-q)B\right|^2 - p|A|^2 - q|B|^2$$
$$= \left(q^2 - 3q + 2\right)|B|^2 + (2-p)|A|^2 + (q-2)(A^*B + B^*A)$$
$$= (q-2)(q-1)|B|^2 + \left(\frac{q-2}{q-1}\right)|A|^2 + (q-2)(A^*B + B^*A)$$
$$= (q-2)\left[(q-1)|B|^2 + \frac{1}{q-1}|A|^2 + (A^*B + B^*A)\right]. \tag{2.2}$$

Since $1 < p \leqslant 2$, we have $q \geqslant 2$ and so

$$|A-B|^2 + \left|A-(1-q)B\right|^2 - p|A|^2 - q|B|^2 = (q-2)\left|\sqrt{q-1}B + \frac{1}{\sqrt{q-1}}A\right|^2 \geqslant 0,$$

hence

$$|A-B|^2 + \left|A-(1-q)B\right|^2 \geqslant p|A|^2 + q|B|^2,$$

with equality if and only if $q = 2$ (hence $p = 2$) or $\sqrt{q-1}B + \frac{1}{\sqrt{q-1}}A = 0$, that is, $p = q = 2$ or $(1-q)B = A$; or equivalently, $p = q = 2$ or $(1-p)A = B$. □

Remark 1. Part (i) of Theorem 1 is equivalent to Theorem 1 in [6]. In fact, this follows immediately from the fact that $1 < p \leqslant 2 \Leftrightarrow 1 < p \leqslant q$.

Remark 2. By combining (i) and (ii) in Theorem 1, we have, for any $1 < p \leqslant 2$,

$$|A-B|^2 + \left|(1-p)A - B\right|^2 \leqslant p|A|^2 + q|B|^2 \leqslant |A-B|^2 + \left|A-(1-q)B\right|^2.$$

In particular, if $p = 2$, then $q = 2$ and we have

$$|A-B|^2 + |A+B|^2 \leqslant 2|A|^2 + 2|B|^2 \leqslant |A-B|^2 + |A+B|^2$$

and so we arrive at the parallelogram law

$$|A-B|^2 + |A+B|^2 = 2|A|^2 + 2|B|^2. \tag{2.3}$$

Equivalently, this is also obtained by directly writing out the equality in (i) or (ii) for the case $p = 2$.

Remark 3. In case $\mathbb{H} = \mathbb{C}$, we have $B(\mathbb{H}) = \mathbb{C}$ and in this case (2.3) reduces to

$$|a-b|^2 + |a+b|^2 = 2|a|^2 + 2|b|^2$$

for all $a, b \in \mathbb{C}$, which is the usual parallelogram law in $\mathbb{C}$.

Corollary 1. *For any $A, B \in B(\mathbb{H})$ and any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $p > 2$, then*

(i) $|A-B|^2 + |(1-p)A - B|^2 \geqslant p|A|^2 + q|B|^2$, *and*
(ii) $|A-B|^2 + |A-(1-q)B|^2 \leqslant p|A|^2 + q|B|^2$.

Furthermore, in both (i) *and* (ii), *the equality holds if and only if $(1-p)A = B$.*

Proof. This follows from Theorem 1 by interchanging p and q. □

Combining Theorem 1 and Corollary 1, we have the following result which takes the form of the original Bohr's inequality.

Corollary 2. *For any $A, B \in B(\mathbb{H})$ and any $p, q \in \mathbb{R}$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,*

$$|A+B|^2 \leqslant p|A|^2 + q|B|^2,$$

with equality if and only if $(p-1)A = B$.

Proof. This is immediate from (i) of Theorem 1 for the case $1 < p \leqslant 2$, and (ii) of Corollary 1 for the case $p > 2$. □

Corollary 3. *For any $A, B \in B(\mathbb{H})$ and any $p > 1$,*

$$\pm(A^*B + B^*A) \leqslant (p-1)|A|^2 + \frac{1}{p-1}|B|^2.$$

Furthermore,

$$A^*B + B^*A = (p-1)|A|^2 + \frac{1}{p-1}|B|^2$$

if and only if $(1-p)A = -B$, and

$$-(A^*B + B^*A) = (p-1)|A|^2 + \frac{1}{p-1}|B|^2$$

if and only if $(1-p)A = B$.

Proof. Applying Corollary 2 to the pair A, B, we have

$$|A|^2 + |B|^2 - (A^*B + B^*A) \leqslant p|A|^2 + \frac{p}{p-1}|B|^2,$$

that is,

$$-(A^*B + B^*A) \leqslant (p-1)|A|^2 + \frac{1}{p-1}|B|^2,$$

with equality if and only if $(1-p)A = B$. Similarly, applying Corollary 2 to the pair $A, -B$, we have

$$|A|^2 + |B|^2 + (A^*B + B^*A) \leqslant p|A|^2 + \frac{p}{p-1}|B|^2,$$

that is,

$$(A^*B + B^*A) \leqslant (p-1)|A|^2 + \frac{1}{p-1}|B|^2,$$

with equality if and only if $(1-p)A = -B$. □

Theorem 2. *For any $A, B \in B(\mathbb{H})$ and any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, if $p < 1$, then*

(i) $|A-B|^2 + |(1-p)A - B|^2 \geqslant p|A|^2 + q|B|^2$, *and*
(ii) $|A-B|^2 + |A-(1-q)B|^2 \geqslant p|A|^2 + q|B|^2$.

Furthermore, in both (i) *and* (ii)*, the equality holds if and only if* $(1-p)A = B$.

Proof. (i) Since $p < 1$, by (2.1)

$$\begin{aligned} &|A-B|^2 + \left|(1-p)A - B\right|^2 - p|A|^2 - q|B|^2 \\ &\quad = (2-p)\left[(1-p)|A|^2 + \frac{1}{1-p}|B|^2 - (A^*B + B^*A)\right] \\ &\quad = (2-p)\left|\sqrt{1-p}A - \frac{1}{\sqrt{1-p}}B\right|^2 \\ &\quad \geqslant 0, \end{aligned}$$

so

$$|A-B|^2 + \left|(1-p)A - B\right|^2 \geqslant p|A|^2 + q|B|^2,$$

with equality if and only if $(1-p)A = B$.

(ii) Since $p < 1$, we have $q < 1$ and so by (2.2),

$$\begin{aligned} &|A-B|^2 + \left|A - (1-q)B\right|^2 - p|A|^2 - q|B|^2 \\ &\quad = (2-q)\left[(1-q)|B|^2 + \frac{1}{1-q}|A|^2 - (A^*B + B^*A)\right] \\ &\quad = (2-q)\left|\sqrt{1-q}B - \frac{1}{\sqrt{1-q}}A\right|^2 \\ &\quad \geqslant 0, \end{aligned}$$

with equality if and only if $(1-q)B - A = 0$, or equivalently, $(1-p)A = B$. □

Remark 4. When $\mathbb{H} = \mathbb{C}$, again $B(\mathbb{H}) = \mathbb{C}$ and Theorem 1, Corollary 1, and Theorem 2 reduce to the following.

For any $a, b \in \mathbb{C}$ and any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$,

(i) if $1 < p \leqslant 2$,

$$|a-b|^2 + \left|(1-p)a-b\right|^2 \leqslant p|a|^2 + q|b|^2,$$
$$|a-b|^2 + \left|a-(1-q)b\right|^2 \geqslant p|a|^2 + q|b|^2,$$

with equality if and only if $p = q = 2$ or $(1-p)a = b$;

(ii) if $p > 2$,

$$|a-b|^2 + \left|(1-p)a-b\right|^2 \geqslant p|a|^2 + q|b|^2,$$
$$|a-b|^2 + \left|a-(1-q)b\right|^2 \leqslant p|a|^2 + q|b|^2,$$

with equality if and only if $(1-p)a = b$;

(iii) if $p < 1$,

$$|a-b|^2 + \left|(1-p)a-b\right|^2 \geqslant p|a|^2 + q|b|^2,$$
$$|a-b|^2 + \left|a-(1-q)b\right|^2 \geqslant p|a|^2 + q|b|^2,$$

with equality if and only if $(1-p)a = b$.

Furthermore, Corollary 2 gives

(iv) if $p > 1$,

$$|a-b|^2 \leqslant p|a|^2 + q|b|^2,$$

with equality if and only if $(1-p)a = b$, which is exactly the classical Bohr's inequality.

Theorem 3. *Let $A, B \in B(\mathbb{H})$ and $\alpha, \beta \in \mathbb{R}$ be nonzero constants.*

(a) *If $\alpha\beta > 0$ with, say, $|\alpha| \geqslant |\beta| > 0$, then*

$$|A-B|^2 + \frac{1}{\alpha^2}|\beta A + \alpha B|^2 \leqslant \frac{\alpha+\beta}{\alpha}|A|^2 + \frac{\alpha+\beta}{\beta}|B|^2,$$

with equality if and only if $\alpha = \beta$ or $\beta A + \alpha B = 0$.

(b) *If $\alpha\beta < 0$ with, say, $\alpha > 0 > \beta$,*

(i) *if $\alpha > 0 > \beta \geqslant -\alpha$, then*

$$|A-B|^2 + \frac{1}{\alpha^2}|\beta A - \alpha B|^2 \leqslant \frac{\alpha-\beta}{\alpha}|A|^2 - \frac{\alpha-\beta}{\beta}|B|^2,$$

with equality if and only if $\alpha + \beta = 0$ or $\beta A - \alpha B = 0$;

(ii) *if $\alpha > 0 > -\alpha \geqslant \beta$, then*

$$|A-B|^2 + \frac{1}{\beta^2}|\alpha A - \beta B|^2 \leqslant -\frac{\alpha-\beta}{\beta}|A|^2 + \frac{\alpha-\beta}{\alpha}|B|^2,$$

with equality if and only if $\alpha + \beta = 0$ or $\alpha A - \beta B = 0$.

Proof. (a) If $\alpha \geqslant \beta > 0$, write

$$p = \frac{\alpha+\beta}{\alpha}, \qquad q = \frac{\alpha+\beta}{\beta}.$$

Then

$$\frac{1}{p}+\frac{1}{q}=1 \quad \text{and} \quad 1<p\leqslant 2\leqslant q.$$

Hence Theorem 1 applies and we have

$$|A-B|^2+\left|-\frac{\beta}{\alpha}A-B\right|^2\leqslant\frac{\alpha+\beta}{\alpha}|A|^2+\frac{\alpha+\beta}{\beta}|B|^2$$

or

$$|A-B|^2+\frac{1}{\alpha^2}|\beta A+\alpha B|^2\leqslant\frac{\alpha+\beta}{\alpha}|A|^2+\frac{\alpha+\beta}{\beta}|B|^2,$$

with equality if and only if

$$\frac{\alpha+\beta}{\alpha}=2 \quad \text{or} \quad -\frac{\beta}{\alpha}A=B,$$

that is,

$$\alpha=\beta \quad \text{or} \quad \beta A+\alpha B=0.$$

If $0>\beta\geqslant\alpha$, then $-\alpha\geqslant-\beta\geqslant 0$ and so from above,

$$|A-B|^2+\frac{1}{(-\alpha)^2}|-\beta A-\alpha B|^2\leqslant\frac{-\alpha-\beta}{-\alpha}|A|^2+\frac{-\alpha-\beta}{-\beta}|B|^2,$$

or

$$|A-B|^2+\frac{1}{\alpha^2}|\beta A+\alpha B|^2\leqslant\frac{\alpha+\beta}{\alpha}|A|^2+\frac{\alpha+\beta}{\beta}|B|^2,$$

with equality if and only if

$$\alpha=\beta \quad \text{or} \quad \beta A+\alpha B=0.$$

(b) (i) If $\alpha>0>\beta\geqslant-\alpha$, the assertion follows immediately by applying (a) to $\alpha\geqslant-\beta>0$.
(ii) If $\alpha>0>-\alpha\geqslant\beta$, the assertion follows immediately by applying (a) to $-\beta\geqslant\alpha>0$. □

3. Applications

Interesting inequalities on operators in $B(\mathbb{H})$ can easily be derived from the Bohr-type inequalities obtained in Section 2 above. For this we first observe the following generalization of Adamović's result [1] to $B(\mathbb{H})$.

Lemma 1. *For any $A_i\in B(\mathbb{H})$, $i=1,\dots,n$,*

$$\left|\sum_{i=1}^n A_i\right|^2-\left(\sum_{i=1}^n|A_i|\right)^2=\sum_{1\leqslant i<j\leqslant n}\left[|A_i+A_j|^2-\left(|A_i|+|A_j|\right)^2\right].$$

Proof. We use induction on n. Clearly the result holds for $n=2$. Suppose now that the result holds for $n=k$, then

$$\begin{aligned}
&\left|\sum_{i=1}^{k+1} A_i\right|^2 - \left(\sum_{i=1}^{k+1} |A_i|\right)^2 \\
&= \left|\sum_{i=1}^{k} A_i + A_{k+1}\right|^2 - \left(\sum_{i=1}^{k} |A_i| + |A_{k+1}|\right)^2 \\
&= \left|\sum_{i=1}^{k} A_i\right|^2 - \left(\sum_{i=1}^{k} |A_i|\right)^2 + \sum_{i=1}^{k}(A_i^* A_{k+1} + A_{k+1}^* A_i) - 2|A_{k+1}|\sum_{i=1}^{k}|A_i| \\
&= \sum_{1\leqslant i<j\leqslant n} \left[|A_i + A_j|^2 - \left(|A_i| + |A_j|\right)^2\right] \\
&\quad + \sum_{i=1}^{k}(A_i^* A_{k+1} + A_{k+1}^* A_i) - 2|A_{k+1}|\sum_{i=1}^{k}|A_i| \\
&= \sum_{1\leqslant i<j\leqslant n} \left[A_i^* A_j + A_j^* A_i - 2|A_i||A_j|\right] \\
&\quad + \sum_{i=1}^{k}(A_i^* A_{k+1} + A_{k+1}^* A_i) - 2|A_{k+1}|\sum_{i=1}^{k}|A_i| \\
&= \sum_{1\leqslant i<j\leqslant n+1} \left[A_i^* A_j + A_j^* A_i - 2|A_i||A_j|\right] \\
&= \sum_{1\leqslant i<j\leqslant n+1} \left[|A_i + A_j|^2 - \left(|A_i| + |A_j|\right)^2\right]. \qquad \square
\end{aligned}$$

Theorem 4. *For any $A_i \in B(\mathbb{H})$, $i = 1, \ldots, n$, and any $p_{ij}, q_{ij} \in \mathbb{R}$ with $\frac{1}{p_{ij}} + \frac{1}{q_{ij}} = 1$, $1 \leqslant i < j \leqslant n$,*

(i) *if $p_{ij} > 1$ for all $1 \leqslant i < j \leqslant n$,*

$$\left|\sum_{i=1}^{n} A_i\right|^2 \leqslant \sum_{k=1}^{n}\left[1 + \sum_{j=k+1}^{n}(p_{kj} - 1) + \sum_{j=1}^{k-1}(q_{jk} - 1)\right]|A_k|^2;$$

(ii) *if $p_{ij} < 1$ for all $1 \leqslant i < j \leqslant n$,*

$$\left|\sum_{i=1}^{n} A_i\right|^2 \geqslant \sum_{k=1}^{n}\left[1 + \sum_{j=k+1}^{n}(p_{kj} - 1) + \sum_{j=1}^{k-1}(q_{jk} - 1)\right]|A_k|^2.$$

Furthermore, the equalities in (i) *and* (ii) *hold if and only if $(p_{ij} - 1)A_i = A_j$ for all $1 \leqslant i < j \leqslant n$.*

Proof. (i) By Lemma 1,

$$\left|\sum_{i=1}^{n} A_i\right|^2 - \left(\sum_{i=1}^{n} |A_i|\right)^2 = \sum_{1\leqslant i<j\leqslant n} \left[|A_i + A_j|^2 - \left(|A_i| + |A_j|\right)^2\right].$$

Equivalently we have

$$\left|\sum_{i=1}^{n} A_i\right|^2 - \sum_{i=1}^{n} |A_i|^2 = \sum_{1\leqslant i<j\leqslant n} \big[|A_i + A_j|^2 - \big(|A_i|^2 + |A_j|^2\big)\big].$$

Applying Corollary 2 to $|A_i + A_j|$, this gives

$$\left|\sum_{i=1}^{n} A_i\right|^2 - \sum_{i=1}^{n} |A_i|^2 \leqslant \sum_{1\leqslant i<j\leqslant n} \big[(p_{ij}-1)|A_i|^2 + (q_{ij}-1)|A_j|^2\big],$$

with equality if and only if $(p_{ij}-1)A_i = A_j$ for all $1 \leqslant i < j \leqslant n$, that is,

$$\begin{aligned}\left|\sum_{i=1}^{n} A_i\right|^2 &\leqslant \sum_{i=1}^{n} |A_i|^2 + \sum_{1\leqslant i<j\leqslant n} \big[(p_{ij}-1)|A_i|^2 + (q_{ij}-1)|A_j|^2\big]\\ &= \sum_{k=1}^{n}\left[1 + \sum_{j=k+1}^{n} (p_{kj}-1) + \sum_{j=1}^{k-1}(q_{jk}-1)\right]|A_k|^2,\end{aligned}$$

with equality if and only if $(p_{ij}-1)A_i = A_j$ for all $1 \leqslant i < j \leqslant n$.

(ii) It follows from Corollary 4 by arguments similar to the proof of (i). □

Remark 5. In [10, Theorem 5, p. 504] and [13, Theorem 5], it was proved that if x_i, $i = 1, \ldots, n$, are elements in an unitary vector space X and $a_{ij} > 0$, $1 \leqslant i < j \leqslant n$, then

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 \leqslant \sum_{k=1}^{n}\left(1 + \sum_{j=k+1}^{n} a_{kj} + \sum_{j=1}^{k-1} \frac{1}{a_{jk}}\right)\|x_k\|^2.$$

Note that this is equivalent to (i) of Theorem 4.

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