Fixed-sign solutions for a system of singular focal boundary value problems

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Abstract

We consider a system of focal boundary value problems where the nonlinearities may be singular in the independent variable and may also be singular in the dependent arguments. Using Schauder fixed point theorem, we establish criteria such that the system of boundary value problems has at least one fixed-sign solution.

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1. Introduction

In this paper we shall consider the following system of focal boundary value problems:

\[
\begin{aligned}
&(-1)^{m_i - p_i} u_i^{(m_i)}(t) = f_i(t, \tilde{u}(t)), \quad \text{a.e. } t \in [0, 1], \\
&u_i^{(j)}(0) = 0, \quad 0 \leq j \leq p_i - 1, \\
&u_i^{(j)}(1) = 0, \quad p_i \leq j \leq m_i - 1, \\
i = 1, 2, \ldots, n,
\end{aligned}
\]

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where

\[ \tilde{u} \equiv (u_1, u_1', \ldots, u^{(m_1-1)}_1, u_2, \ldots, u^{(m_2-1)}_2, \ldots, u_n, \ldots, u^{(m_n-1)}_n). \]

For each \(1 \leq i \leq n, m_i \geq 2\) and \(1 \leq p_i \leq m_i - 1\) are fixed. The nonlinearities \(f_i, 1 \leq i \leq n, \) in (F) may be singular in the independent variable and may also be singular at \(u_i^{(j)} = 0\) where \(j \in \{0, 1, \ldots, m_i - 1\}\) and \(i \in \{1, 2, \ldots, n\}\).

By using Schauder fixed point theorem, we shall develop existence criteria for a fixed-sign solution of the above system. A solution \(u = (u_1, u_2, \ldots, u_n)\) of (F) is said to be of fixed sign if for each \(1 \leq i \leq n, \theta_i u_i(t) \geq 0\) for \(t \in [0, 1]\), where \(\theta_i \in \{1, -1\}\) is fixed. We remark that positive solution is a special case of fixed-sign solution when \(\theta_i = 1\) for all \(1 \leq i \leq n\).

The importance of boundary value problems, both from a theoretical perspective as well as for their applications in the physical and engineering sciences, has been well documented in the literature; see, for example, the work [1,5,10,11] and references cited therein. In particular, focal boundary value problems have received a lot of attention in the literature, the reader is referred to [2,7,12,19,20] and references therein. However, there are only a handful of papers in the literature [3,4,6,8,9,13–18] that focus on singular boundary value problems. For instance, a particular case of (F)

\[
y''(t) + f(t, y(t), y'(t)) = 0, \quad \text{a.e. } t \in [0, 1],
\]

\[
y(0) = y'(1) = 0
\]

(1.1)

has been discussed in [3], and the existence of a positive solution is established. In [13,14], by using concavity properties, iterations and a fixed point theorem for operators which are decreasing with respect to a cone in a Banach space, the existence of a positive solution is established for the following singular boundary value problems:

\[
(-1)^{n-k} y^{(n)}(t) = f(t, y(t)), \quad t \in (0, 1),
\]

\[
y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1, \quad y^{(j)}(0) = 0, \quad 0 \leq j \leq n - k - 1,
\]

(1.2)

where \(1 \leq k \leq n - 1\) is fixed; and

\[
(-1)^{n-k} y^{(n)}(t) = f(t, y(t)), \quad t \in (0, 1),
\]

\[
y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1, \quad y^{(j)}(0) = 0, \quad q \leq j \leq n - k + q - 1,
\]

(1.3)

where \(1 \leq k \leq n - 1\) and \(0 \leq q \leq k - 1\) are fixed. In both (1.2) and (1.3), \(f(t, x)\) has a singularity at \(x = 0\) and is decreasing in \(x\). Using another technique which involves the method of a priori estimates, the degree theory arguments and the Vitali convergence theorem, the following higher-order singular boundary value problems have been shown to possess a positive solution in [4,16]:

\[
-y^{(m)}(t) = \hat{f}(t, y(t), y'(t), \ldots, y^{(m-1)}(t)), \quad t \in [0, 1],
\]

\[
y^{(j)}(0) = 0, \quad 0 \leq j \leq m - 2, \quad y^{(p)}(1) = 0, \quad 1 \leq p \leq m - 1 \text{ is fixed},
\]

(1.4)

where \(f(t, x_0, x_1, \ldots, x_{m-1})\) can be singular at \(x_i = 0, 0 \leq i \leq m - 2\), but not at \(x_{m-1} = 0\);

\[
(-1)^{m} y^{(2m)}(t) = \hat{f}(t, y(t), y'(t), \ldots, y^{(2m-2)}(t)), \quad t \in [0, 1],
\]

\[
y^{(2j)}(0) = y^{(2j)}(1) = 0, \quad 0 \leq j \leq m - 1,
\]

(1.5)

where \(\hat{f}(t, x_0, x_1, \ldots, x_{2m-2})\) can be singular at \(x_i = 0, 0 \leq i \leq 2m - 2\); and
\((-1)^my^{(2m+1)}(t) = \tilde{f}(t, y(t), y'(t), \ldots, y^{(2m)}(t)), \quad t \in [0, 1],\)

\[\begin{align*}
y^{(2m)}(0) &= 0, \\
y^{(2j)}(0) &= y^{(2j)}(1) = 0, \quad 0 \leq j \leq m - 1,
\end{align*}\]

where \(\tilde{f}(t, x_0, x_1, \ldots, x_{2m})\) can be singular at \(x_i = 0, 0 \leq i \leq 2m\).

Our present work obviously generalizes (1.1)–(1.6) to systems. In addition, we have also generalized the problem (1.1) to higher-order and more general boundary conditions. By using a different technique involving Schauder fixed point theorem, we obtain the existence of fixed-sign solutions, which include positive solutions as special case. Moreover, our criteria are easily verifiable and do not require the monotonicity condition on the nonlinear terms (which is needed in (1.2) and (1.3)). Our results thus extend, improve and complement those in the literature.

The paper is outlined as follows. In Section 2 we shall state the necessary fixed point theorem. The existence results for a fixed-sign solution of the system (F) are established, and illustrated by examples, in Sections 3 and 4, respectively.

2. Preliminaries

**Theorem 2.1** (Schauder fixed point theorem). Let \(K\) be a closed, convex subset of a normed linear space \(E\). Then every compact and continuous map \(S: K \to K\) has at least one fixed point.

We also require a compactness criterion.

**Theorem 2.2** (Arzela–Ascoli theorem). Let \(M \subseteq C[0, T]\). If \(M\) is uniformly bounded and equicontinuous, then \(M\) is relatively compact in \(C[0, T]\).

3. Main results

In this section we shall develop existence criteria for the system (F) where the nonlinearities \(f_i(t, \tilde{u}), 1 \leq i \leq n\), may be singular at \(u^{(j)}(i) = 0, j \in \{0, 1, \ldots, m_i - 1\}, i \in \{1, 2, \ldots, n\}\), and may also be singular in \(t\) at some subset \(\Omega\) of \([0, 1]\) with measure zero. We shall seek a fixed-sign solution of (F) in the space \(C^{m_1-1}[0, 1] \times C^{m_2-1}[0, 1] \times \cdots \times C^{m_n-1}[0, 1]\).

Let \(g_i(t, s)\) be the Green’s function of the boundary value problem

\[\begin{align*}
y^{(m_i)}(t) &= 0, \quad t \in [0, 1], \\
y^{(j)}(0) &= 0, \quad 0 \leq j \leq p_i - 1, \quad y^{(j)}(1) = 0, \quad p_i \leq j \leq m_i - 1.
\end{align*}\]

It is known that [5, p. 211]

\[
g_i(t,s) = \frac{1}{(m_i - 1)!} \left\{ \sum_{k=0}^{p_i-1} \binom{m_i-1}{k} t^k (-s)^{m_i-k-1}, \quad 0 \leq s \leq t \leq 1, \right. \\
\left. - \sum_{k=p_i}^{m_i-1} \binom{m_i-1}{k} t^k (-s)^{m_i-k-1}, \quad 0 \leq t \leq s \leq 1, \right. 
\]

and for \((t, s) \in [0, 1] \times [0, 1],\)

\[
\begin{align*} 
&\left\{ \begin{array}{ll}
(-1)^{m_i-p_i} \frac{\partial^j}{\partial t^j} g_i(t, s) \geq 0, & 0 \leq j \leq p_i - 1, \\
(-1)^{m_i-j} \frac{\partial^j}{\partial t^j} g_i(t, s) \geq 0, & p_i \leq j \leq m_i - 1.
\end{array} \right.
\end{align*}
\]
Lemma 3.1. Let $N_i = \{0, 1, \ldots, m_i - 1\}$ and $g_i^{(j)}(t, s) \equiv \frac{\partial^j}{\partial t^j} g_i(t, s)$. The following hold for $(t, s) \in [0, 1] \times [0, 1]$:

(a) $0 \leq (-1)^{m_i-p_i} g_i^{(j)}(t, s) \leq (-1)^{m_i-p_i} g_i^{(j)}(1, s)$, $0 \leq j \leq p_i - 1$, (3.3)

and

$$\left| (-1)^{m_i-p_i} g_i^{(j)}(t, s) \right| = (-1)^{j-p_i} \cdot (-1)^{m_i-p_i} g_i^{(j)}(t, s) \leq \left| (-1)^{m_i-p_i} g_i^{(j)}(0, s) \right|, \quad p_i \leq j \leq m_i - 1.$$ (3.4)

Combining (3.3) and (3.4) gives

$$\left| (-1)^{m_i-p_i} g_i^{(j)}(t, s) \right| \leq v_{ij}(s)$$

where

$$v_{ij}(s) = \begin{cases} (-1)^{m_i-p_i} g_i^{(j)}(1, s), & 0 \leq j \leq p_i - 1, \\ \left| (-1)^{m_i-p_i} g_i^{(j)}(0, s) \right|, & p_i \leq j \leq m_i - 1, \\ \frac{- (m_i-p_i)}{(m_i-j-1)!} \sum_{k=0}^{p_i-j-1} (m_i-j)!(s)^{m_i-j-k-1}, & 0 \leq j \leq p_i - 1, \\ \frac{- (m_i-j)}{m_i-j-1}! (s)^{m_i-j-k-1}, & p_i \leq j \leq m_i - 1. \end{cases}$$ (3.5)

(b) $(-1)^{m_i-p_i} g_i^{(j)}(t, s) \geq 0$ for $j \in N_i^+$ and $(-1)^{m_i-p_i} g_i^{(k)}(t, s) \leq 0$ for $k \in N_i^-$, where

$N_i^+ = \{0, 1, \ldots, p_i\} \cup \{p_i+2, p_i+3, \ldots\} \cap N_i$

and

$N_i^- = \{p_i+1, p_i+3, \ldots\} \cap N_i$.

Clearly, $N_i = N_i^+ \cup N_i^-$.

Proof. From (3.1) we find for $0 \leq j \leq m_i - 1$,

$$g_i^{(j)}(t, s) = \frac{1}{(m_i-j)!} \left\{ \begin{array}{l} \sum_{k=j}^{p_i-1} \binom{m_i-j}{k} t^k (-s)^{m_i-k-1}, \\
0 \leq s \leq t \leq 1, \\
- \sum_{k=\max\{j, p_i\}}^{m_i-j-1} \binom{m_i-j}{k} t^{-j} (-s)^{m_i-k-1}, \\
0 \leq t \leq s \leq 1, \end{array} \right.$$ (3.6)

where $k^{(j)} = k(k-1) \cdots (k-j+1)$ (with $k^{(0)} = 1$). Rewriting (3.6) gives

$$g_i^{(j)}(t, s) = \frac{1}{(m_i-j-1)!} \left\{ \begin{array}{l} \sum_{k=0}^{p_i-j-1} \binom{m_i-j}{k} t^k (-s)^{m_i-j-k-1}, \\
0 \leq s \leq t \leq 1, \\
- \sum_{k=\max\{0, p_i-j\}}^{m_i-j-1} \binom{m_i-j}{k} t^{-j} (-s)^{m_i-j-k-1}, \\
0 \leq t \leq s \leq 1. \end{array} \right.$$ (3.7)

Since from (3.2) we have

$$(-1)^{m_i-p_i} g_i^{(j)}(t, s) \geq 0, \quad 0 \leq j \leq p_i,$$

it is clear that (3.3) follows for $(t, s) \in [0, 1] \times [0, 1]$.
Next, from (3.2) again we get
\[ (-1)^{m_i - p_i} g_i^{(p_i)}(t, s) \geq 0 \quad \text{and} \quad (-1)^{m_i - p_i} g_i^{(p_i+1)}(t, s) \leq 0, \quad (t, s) \in [0, 1] \times [0, 1]. \]
Thus,
\[ 0 \leq (-1)^{m_i - p_i} g_i^{(p_i)}(t, s) \leq (-1)^{m_i - p_i} g_i^{(p_i)}(0, s), \quad (t, s) \in [0, 1] \times [0, 1]. \]
Continuing further from (3.2), we have
\[ (-1)^{m_i - p_i} g_i^{(p_i+1)}(t, s) \leq 0 \quad \text{and} \quad (-1)^{m_i - p_i} g_i^{(p_i+2)}(t, s) \geq 0, \quad (t, s) \in [0, 1] \times [0, 1]. \]
Hence,
\[ \left| (-1)^{m_i - p_i} g_i^{(p_i+1)}(t, s) \right| = (-1)^{m_i - p_i} g_i^{(p_i+1)}(t, s) \leq \left| (-1)^{m_i - p_i} g_i^{(p_i+1)}(0, s) \right|. \]
Proceeding in a similar manner, we see that for every \( p_i \leq j \leq m_i - 1 \) the inequality (3.4) is true.

The explicit expression in (3.5) follows from (3.7). Part (b) is also now immediate. \( \square \)

Throughout we shall denote
\[
\begin{align*}
    u &= (u_1, u_2, \ldots, u_n), \\
    \tilde{u} &= (u_1, u'_1, \ldots, u_{1^{(m_1-1)}}, u_2, \ldots, u_{2^{(m_2-1)}}, \ldots, u_n, \ldots, u_{n^{(m_n-1)}}), \\
    \tilde{x} &= (x_{1,1}, x_{1,1}, \ldots, x_{1,m_1-1}, x_{2,0}, \ldots, x_{2,m_2-1}, \ldots, x_{n,0}, \ldots, x_{n,m_n-1}),
\end{align*}
\]
where \( x_{i,j} \)'s are real numbers, and for each \( 0 \leq j \leq m_i - 1 \) and \( 1 \leq i \leq n \),
\[
\begin{align*}
    \tilde{u}_i &= (u_1, u'_1, \ldots, u_{i^{(m_i-1)}}), \\
    \tilde{x}_i &= (x_{i,0}, x_{i,1}, \ldots, x_{i,m_i-1}), \\
    (0, \infty)_{ij} &= \begin{cases} (0, \infty), & \text{if } \theta_i = 1, \ j \in N^+_i \ or \ \theta_i = -1, \ j \in N^-_i, \\ (-\infty, 0), & \text{if } \theta_i = -1, \ j \in N^+_i \ or \ \theta_i = 1, \ j \in N^-_i. \end{cases}
\end{align*}
\]
The definition of \( (0, \infty)_{ij} \) is similar.

**Theorem 3.1.** For each \( 1 \leq i \leq n \), let \( \theta_i \in \{1, -1\} \) be fixed and the following conditions be satisfied:

1. \( f_i : [0, 1] \times \prod_{j=1}^n \prod_{k=0}^{m_j-1} (0, \infty)_{jk} \rightarrow \mathbb{R} \), where \( t \mapsto f_i(t, \tilde{x}) \) is measurable for all \( \tilde{x} \in \prod_{j=1}^n \prod_{k=0}^{m_j-1} (0, \infty)_{jk} \), and \( \tilde{x} \mapsto f_i(t, \tilde{x}) \) is continuous for a.e. \( t \in (0, 1) \);
2. for any \( r > 0 \), there exists \( h_{r,i} : [0, 1] \rightarrow \mathbb{R} \), \( h_{r,i}(t) > 0 \) for a.e. \( t \in [0, 1] \), \( h_{r,i} \in L^1[0, 1] \) such that for all \( |x_{j,k}| \in (0, r) \), \( 0 \leq k \leq m_j - 1, 1 \leq j \leq n \),
\[
    \theta_i f_i(t, \tilde{x}) \geq h_{r,i}(t) \quad \text{for a.e. } t \in [0, 1];
\]
3. for any \( r > 0 \), with \( \int_0^1 |g_j^{(k)}(t, s)| h_{r,j,s}(s) \, ds \leq r \) for \( t \in [0, 1] \), \( 0 \leq k \leq m_j - 1 \) and \( 1 \leq j \leq n \), there exists \( H_{r,i} : [0, 1] \rightarrow \mathbb{R} \), \( H_{r,i}(t) \geq 0 \) for a.e. \( t \in [0, 1] \), \( H_{r,i} \in L^1[0, 1] \) such that
\[
    \theta_i f_i(t, \tilde{x}) \leq H_{r,i}(t) \quad \text{for a.e. } t \in [0, 1] \text{ and } |x_{j,k}| \in \left[ \int_0^1 |g_j^{(k)}(t, s)| h_{r,j,s}(s) \, ds, r \right], \ 0 \leq k \leq m_j - 1, 1 \leq j \leq n;\]
(C4) There exists $M_i > 0$ such that for $t \in [0, 1]$ and $0 \leq j \leq m_i - 1$,

$$M_i \geq \int_0^1 |g^{(j)}_i(t,s)| H_{M_{i,i}}(s) \, ds \geq \int_0^1 |g^{(j)}_i(t,s)| h_{M_{i,i}}(s) \, ds.$$ 

Then, (F) has a fixed-sign solution $u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ such that for a.e. $t \in [0,1]$ and any $1 \leq i \leq n$,

$$\theta_i u^{(j)}_i(t) > 0 \quad \text{for all } j \in N_i^+ \quad \text{and} \quad \theta_i u^{(k)}_i(t) < 0 \quad \text{for all } k \in N_i^-.$$

(3.8)

**Proof.** To begin, we define a closed convex subset of $B = C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ as

$$D = \left\{ u \in B \left| \int_0^1 (-1)^{m_i-p_i} g^{(j)}_i(t,s) H_{M_{i,i}}(s) \, ds \geq \theta_i u^{(j)}_i(t) \geq \int_0^1 (-1)^{m_i-p_i} g^{(j)}_i(t,s) h_{M_{i,i}}(s) \, ds \geq 0 \right. \right\}$$

and

$$\begin{align*}
&\left. \int_0^1 (-1)^{m_i-p_i} g^{(k)}_i(t,s) H_{M_{i,i}}(s) \, ds \right) \\
&\leq \theta_i u^{(k)}_i(t) \leq \int_0^1 (-1)^{m_i-p_i} g^{(k)}_i(t,s) h_{M_{i,i}}(s) \, ds \leq 0
\end{align*}$$

for $t \in [0,1]$, $j \in N_i^+$, $k \in N_i^-$, $1 \leq i \leq n$.

Note that the second equality follows from Lemma 3.1(b).

Let the operator $S : D \to B$ be defined by

$$S u(t) = (S_1 u(t), S_2 u(t), \ldots, S_n u(t)), \quad t \in [0,1],$$

(3.9)

where

$$S_i u(t) = \int_0^1 (-1)^{m_i-p_i} g_i(t,s) f_i(s,\tilde{u}(s)) \, ds, \quad t \in [0,1], \quad 1 \leq i \leq n.$$  

(3.10)

Clearly, a fixed point of the operator $S$ is a solution of the system (F). Indeed, a fixed point of $S$ obtained in $D$ will be a fixed-sign solution of the system (F).
First we shall show that $S$ maps $D$ into $D$. Let $u \in D$. By (C4) it is clear that

$$M_i \geq \int_{0}^{1} |g_i^{(j)}(t,s)| h_{M_i,i}(s) \, ds \geq |\theta_i u^{(j)}(t)| \geq \int_{0}^{1} |g_i^{(j)}(t,s)| h_{M_i,i}(s) \, ds > 0,$$

a.e. $t \in [0,1]$, $0 \leq j \leq m_i - 1$, $1 \leq i \leq n$. \hfill (3.11)

Hence, it follows from (C2) that

$$\theta_i f_i(t, \tilde{u}) \geq h_{M_i,i}(t), \quad \text{a.e. } t \in [0,1], \ 1 \leq i \leq n.$$ \hfill (3.12)

In view of Lemma 3.1(b), we find

$$|\theta_i (S_i u)^{(j)}(t)| = \int_{0}^{1} |g_i^{(j)}(t,s)| \theta_i f_i(s, \tilde{u}(s)) \, ds \geq \int_{0}^{1} |g_i^{(j)}(t,s)| h_{M_i,i}(s) \, ds,$$

$t \in [0,1]$, $0 \leq j \leq m_i - 1$, $1 \leq i \leq n$. \hfill (3.13)

Also, from (C3) and (3.11) we have

$$\theta_i f_i(t, \tilde{u}) \leq h_{M_i,i}(t), \quad \text{a.e. } t \in [0,1], \ 1 \leq i \leq n,$$

and so

$$|\theta_i (S_i u)^{(j)}(t)| = \int_{0}^{1} |g_i^{(j)}(t,s)| \theta_i f_i(s, \tilde{u}(s)) \, ds \leq \int_{0}^{1} |g_i^{(j)}(t,s)| h_{M_i,i}(s) \, ds,$$

$t \in [0,1]$, $0 \leq j \leq m_i - 1$, $1 \leq i \leq n$. \hfill (3.15)

Having obtained (3.13) and (3.15), we have shown that $S : D \rightarrow D$.

Next, we shall prove that $S : D \rightarrow D$ is continuous. Let $\{u^k\}$ be a sequence in $D$ and $u^k \rightarrow u$ in $B$. Then, applying (3.5) we find for $t \in [0,1]$, $0 \leq j \leq m_i - 1$ and $1 \leq i \leq n$,

$$|(S_i u^k)^{(j)}(t) - (S_i u)^{(j)}(t)| \leq \int_{0}^{1} |g_i^{(j)}(t,s)| \cdot |f_i(s, \tilde{u}^k(s)) - f_i(s, \tilde{u}(s))| \, ds$$

$$\leq \int_{0}^{1} v_{ij}(s) |f_i(s, \tilde{u}^k(s)) - f_i(s, \tilde{u}(s))| \, ds$$

$$\leq 2 \left( \sup_{s \in [0,1]} v_{ij}(s) \right) \int_{0}^{1} h_{M_i,i}(s) \, ds < \infty.$$  

Thus, together with (C1), the Lebesgue dominated convergence theorem gives for each $0 \leq j \leq m_i - 1$ and $1 \leq i \leq n$,

$$\sup_{t \in [0,1]} \left| (S_i u^k)^{(j)}(t) - (S_i u)^{(j)}(t) \right| \leq \int_{0}^{1} v_{ij}(s) \left| f_i(s, \tilde{u}^k(s)) - f_i(s, \tilde{u}(s)) \right| \, ds \rightarrow 0$$

as $k \rightarrow \infty$. Hence, $S$ is continuous.
Finally, we shall check that $S : D \to D$ is compact. Let $u \in D$. Then, by (3.14) and (C4) we have

$$\sup_{t \in [0, 1]} |(S_t u)^{(j)}(t)| \leq \sup_{t \in [0, 1]} \int_0^1 \left| g_i^{(j)}(t, s) \right| H_{M_i, i}(s) \, ds \leq M_i, \quad 0 \leq j \leq m_i - 1, \ 1 \leq i \leq n.$$ 

Now, let $t, t' \in [0, 1]$ with $t' < t$. Using the expression (3.7), we obtain for $0 \leq j \leq m_i - 1$ and $1 \leq i \leq n$,

$$|(S_t u)^{(j)}(t) - (S_t u)^{(j)}(t')| \cdot (m_i - j - 1)!$$

$$= \left| \int_0^t \sum_{k=0}^{p_i - j - 1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i - j - k - 1} f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$- \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (-s)^{m_i - j - k - 1} f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$- \left[ \int_0^t \sum_{k=0}^{p_i - j - 1} \binom{m_i - j - 1}{k} (t')^k (-s)^{m_i - j - k - 1} f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k (-s)^{m_i - j - k - 1} f_i(s, \bar{u}(s)) \, ds \right]$$

$$\leq \left| \int_0^t \sum_{k=0}^{p_i - j - 1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i - j - k - 1} \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} t^k \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_0^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k \theta_i f_i(s, \bar{u}(s)) \, ds \right]$$

$$\leq \left[ \int_0^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i - j - k - 1} \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} t^k \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_0^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k \theta_i f_i(s, \bar{u}(s)) \, ds \right]$$

$$\leq \left[ \int_0^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i - j - k - 1} H_{M_i, i}(s) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} t^k \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_0^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k \theta_i f_i(s, \bar{u}(s)) \, ds \right.$$ 

$$+ \left[ \int_{t'}^t \sum_{k=0}^{m_i - j - 1} \binom{m_i - j - 1}{k} (t')^k \theta_i f_i(s, \bar{u}(s)) \, ds \right]$$
Then, as \( t' \to t \). A similar argument also holds if \( t' > t \). Now Theorem 2.2 guarantees that \( S \) is compact. Hence, we conclude from Theorem 2.1 that \( S \) has a fixed point in \( D \). The proof is complete. \( \square \)

**Remark 3.1.** In Theorem 3.1, the condition \((C3)\) can be replaced by the following:

\((C3)'\) for any \( r > 0 \) with \( \int_0^1 |g_j^{(k)}(t,s)|h_{r,j}(s)\, ds \leq r \) for \( t \in [0,1] \), \( 0 \leq k \leq m_j - 1 \) and \( 1 \leq j \leq n \), let

\[
H_{r,i}(t) = \sup \left\{ \theta_i f_i(t, \tilde{x}) \mid |x_{j,k}| \in \left[ \int_0^1 |g_j^{(k)}(t,s)|h_{r,j}(s)\, ds, r \right], 0 \leq k \leq m_j - 1, 1 \leq j \leq n \right\}
\]

and assume \( H_{r,i} \in L^1[0,1] \).

If \( f_i, 1 \leq i \leq n \), are nonsingular, i.e., \( f_i : [0,1] \times \mathbb{R}^{m_1+m_2+\cdots+m_n} \to \mathbb{R} \), then by using a similar argument as in Theorem 3.1, we obtain the following result.

**Theorem 3.2.** For each \( 1 \leq i \leq n \), let \( \theta_i \in \{1,-1\} \) be fixed and the following conditions be satisfied:

\((C5)\) \( f_i : [0,1] \times \prod_{j=1}^n \prod_{k=0}^{m_j-1} [0, \infty) \to \mathbb{R} \) where \( t \mapsto f_i(t, \tilde{x}) \) is measurable for all \( \tilde{x} \in \prod_{j=1}^n \prod_{k=0}^{m_j-1} [0, \infty) \), and \( \tilde{x} \mapsto f_i(t, \tilde{x}) \) is continuous for a.e. \( t \in (0,1) \);

\((C6)\) for any \( r > 0 \), there exists \( H_{r,i} : [0,1] \to \mathbb{R} \), \( H_{r,i}(t) \geq 0 \) for a.e. \( t \in [0,1] \), \( H_{r,i} \in L^1[0,1] \) such that for all \( |x_{j,k}| \in [0,r] \), \( 0 \leq k \leq m_j - 1, 1 \leq j \leq n \),

\[
0 \leq \theta_i f_i(t, \tilde{x}) \leq H_{r,i}(t) \quad \text{for a.e. } t \in [0,1];
\]

\((C7)\) there exists \( M_i > 0 \) such that for \( t \in [0,1] \) and \( 0 \leq j \leq m_i - 1 \),

\[
M_i \geq \int_0^1 |g_i^{(j)}(t,s)|H_{M,i}(s)\, ds \geq 0.
\]

Then, \((F)\) has a fixed-sign solution \( u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1] \) such that for any \( t \in [0,1] \) and any \( 1 \leq i \leq n \),

\[
\theta_i u_i^{(j)}(t) \geq 0 \quad \text{for all } j \in N_i^+ \quad \text{and} \quad \theta_i u_i^{(k)}(t) \leq 0 \quad \text{for all } k \in N_i^-.
\]
Remark 3.2. Lemma 3.1(a) can be used to ‘simplify’ conditions (C4) and (C7). The easier-to-check but stronger conditions are

(C4)' there exists $M_i > 0$ such that for $0 \leq j \leq m_i - 1$,

$$M_i \geq \int_0^1 v_{ij}(s)H_{M_i,i}(s)\,ds$$

and $H_{M_i,i}(s) \geq h_{M_i,i}(s)$ for a.e. $s \in [0, 1]$,

and

(C7)' there exists $M_i > 0$ such that for $0 \leq j \leq m_i - 1$,

$$M_i \geq \int_0^1 v_{ij}(s)H_{M_i,i}(s)\,ds \geq 0.$$

In fact, using (3.5) from (C4)' we get

$$M_i \geq \int_0^1 v_{ij}(s)H_{M_i,i}(s)\,ds \geq \int_0^1 |g^{(j)}_i(t,s)|H_{M_i,i}(s)\,ds \geq \int_0^1 |g^{(j)}_i(t,s)|h_{M_i,i}(s)\,ds,$$

$$t \in [0, 1], 0 \leq j \leq m_i - 1.$$

Hence, (C4)' implies (C4). Similarly, it can be easily seen that (C7)' implies (C7).

Theorem 3.3. Let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$, be fixed. For each $1 \leq i \leq n$, suppose (C1) and (C2) hold and the following conditions are satisfied:

(C8) for $(t, \tilde{x}) \in [0, 1] \times \prod_{j=1}^n \prod_{k=0}^{m_j - 1} (0, \infty)_{jk}$,

$$\theta_i f_i(t, \tilde{x}) \leq \mu_i(t) \prod_{j=1}^n [\alpha_j(\tilde{x}_j) + \beta_j(\tilde{x}_j)],$$

where $\mu_i : [0, 1] \to \mathbb{R}$, $\mu_i(t) > 0$ for a.e. $t \in [0, 1]$, for each $1 \leq j \leq n$, $\alpha_j > 0$, $\beta_j \geq 0$ are continuous on $\prod_{k=0}^{m_j - 1} (0, \infty)_{jk}$, and if $a \leq |x_{j,k}| \leq b$ for some $k \in \{0, 1, \ldots, m_j - 1\}$, then

$$\alpha_j(x_{j,0}, \ldots, y_{j,k}a, \ldots, x_{j,m_j - 1}) \geq \alpha_j(x_{j,0}, \ldots, x_{j,k}, \ldots, x_{j,m_j - 1}) \geq \alpha_j(x_{j,0}, \ldots, y_{j,k}b, \ldots, x_{j,m_j - 1})$$

and

$$\beta_j(x_{j,0}, \ldots, y_{j,k}a, \ldots, x_{j,m_j - 1}) \leq \beta_j(x_{j,0}, \ldots, x_{j,k}, \ldots, x_{j,m_j - 1}) \leq \beta_j(x_{j,0}, \ldots, y_{j,k}b, \ldots, x_{j,m_j - 1}),$$
where
\[
\gamma_{j,k} = \begin{cases} 
\theta_j, & \text{if } k \in N_j^+; \\
-\theta_j, & \text{if } k \in N_j^-; 
\end{cases}
\]
(C9) for any constants \(c_{j,k} > 0, d_{j,\ell} > 0, 0 \leq k \leq p_j - 1, 0 \leq \ell \leq m_j - 1, 1 \leq j \leq n,\)
\[
\frac{1}{\mu_i(t)} \left\{ \prod_{j=1}^{n} \alpha_j \left( \gamma_{j,0}[c_{j,0}t^{m_j-1} + d_{j,0}], \gamma_{j,1}[c_{j,1}t^{m_j-2} + d_{j,1}] \right), \ldots, \gamma_{j,p_j-1}[c_{j,p_j-1}t^{m_j-p_j} + d_{j,p_j-1}], \gamma_{j,p_j+1}[1 - t]^{m_j-p_j-2}, \ldots, \gamma_{j,m_j-1}d_{j,m_j-1} \right\} dt < \infty;
\]
(C10) there exists \(M_i > 0\) such that for \(t \in [0, 1]\) and \(0 \leq k \leq m_i - 1,\)
\[
M_i \geq \int_0^1 g^{(k)}_i(t, s) \frac{\mu_i(s)}{|x_j,k|} \left\{ \prod_{j=1}^{n} \beta_j \left( \gamma_{j,0}M_i, \gamma_{j,1}M_i, \ldots, \gamma_{j,m_j-1}M_i \right) \right\}
+ \alpha_j \left( \gamma_{j,0} \int_0^1 |g_j(s, x)| h_{M_i,j}(x) dx, \gamma_{j,1} \int_0^1 |g'_j(s, x)| h_{M_i,j}(x) dx, \ldots, \gamma_{j,m_j-1} \int_0^1 |g^{(m_j-1)}_j(s, x)| h_{M_i,j}(x) dx \right) \right\} ds
\geq \int_0^1 g^{(k)}_i(t, s) h_{M_i,i}(s) ds.
\]
Then, (F) has a fixed-sign solution \(u \in C^m_1[0, 1] \times C^m_2[0, 1] \times \cdots \times C^m_n[0, 1]\) satisfying (3.8) for a.e. \(t \in [0, 1]\) and any \(1 \leq i \leq n.\)

**Proof.** We shall show that (C3) and (C4) are satisfied, then the conclusion will be immediate from Theorem 3.1. Let \(1 \leq i \leq n.\) In order to choose \(H_{r,i}\) which is described in (C3), we use (C8) to obtain for a.e. \(t \in [0, 1], |x_j,k| \in \left[ \int_0^1 |g_j^{(k)}(t, s)| h_{r,j}(s) ds, r \right], 0 \leq k \leq m_j - 1, 1 \leq j \leq n,\)
\[
\theta_i f_i(t, \tilde{x}) \leq \mu_i(t) \prod_{j=1}^{n} \beta_j \left( \gamma_{j,0}r, \gamma_{j,1}r, \ldots, \gamma_{j,m_j-1}r \right)
+ \alpha_j \left( \gamma_{j,0} \int_0^1 |g_j(t, s)| h_{r,j}(s) ds, \gamma_{j,1} \int_0^1 |g'_j(t, s)| h_{r,j}(s) ds, \ldots, \gamma_{j,m_j-1} \int_0^1 |g^{(m_j-1)}_j(t, s)| h_{r,j}(s) ds \right)
\equiv H_{r,i}(t).
\]
Observed that we have picked $H_{r,i}(t)$ to be the right-hand side of (3.16). Hence, (C10) implies (C4).

Now, (C3) is fulfilled if we can show that $H_{r,i} \in L^1[0,1]$. To proceed, we shall first use (3.7) and Lemma 3.1(b) to find for $t \in [0,1]$, $0 \leq k \leq p_j - 1$ and $1 \leq j \leq n$,

$$
\int_{0}^{1} \left| g_j^{(k)}(t,s) \right| h_{r,j}(s) \, ds \cdot (m_j - k - 1)!
$$

$$
= \int_{0}^{1} \left| \sum_{\ell=0}^{p_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) t^\ell (-s)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds
$$

$$
+ \int_{t}^{1} \left| \sum_{\ell=0}^{m_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) t^\ell (-s)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds
$$

$$
= t^{m_j-k-1} \int_{0}^{t} \left| \sum_{\ell=0}^{p_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) \left( \frac{-s}{t} \right)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds
$$

$$
+ \int_{t}^{1} \left| \sum_{\ell=0}^{m_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) t^\ell (-s)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds. \tag{3.17}
$$

For the first integral in (3.17), we see that

$$
\int_{0}^{t} \left| \sum_{\ell=0}^{p_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) \left( \frac{-s}{t} \right)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds
$$

$$
\leq \int_{0}^{t} \sum_{\ell=0}^{p_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) h_{r,j}(s) \, ds \to 0
$$

as $t \to 0^+$. Hence, this integral extends to a continuous function on $[0,1]$, and there exists some constant $a_{j,k} > 0$ such that

$$
\int_{0}^{t} \left| \sum_{\ell=0}^{p_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) \left( \frac{-s}{t} \right)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds \geq a_{j,k} > 0, \quad t \in [0,1]. \tag{3.18}
$$

Further, for the second integral in (3.17), it is clear that

$$
\int_{t}^{1} \left| \sum_{\ell=0}^{m_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) t^\ell (-s)^{m_j-k-\ell-1} \right| h_{r,j}(s) \, ds
$$

$$
\leq \int_{t}^{1} \sum_{\ell=0}^{m_j-k-1} \left( \frac{m_j - k - 1}{\ell} \right) h_{r,j}(s) \, ds \to 0
$$

as $t \to 1^-$. Therefore, this integral also extends to a continuous function on $[0,1]$, and there exists some constant $b_{j,k} > 0$ such that
\[ \int_0^1 \left| \sum_{\ell=p_j-k}^{m_j-k-1} \binom{m_j-k-1}{\ell} t^{\ell} (-s)^{m_j-k-\ell-1} h_{r,j}(s) \right| ds \geq b_{j,k} > 0, \quad t \in [0, 1]. \] (3.19)

Using (3.18) and (3.19) in (3.17), we find

\[ \left| \int_0^1 \left| g^{(k)}_j (t,s) h_{r,j}(s) \right| ds \right| = \int_0^1 \left| g^{(k)}_j (t,s) h_{r,j}(s) \right| ds \geq c_{j,k} t^{m_j-k-1} + d_{j,k}, \]

\[ t \in [0, 1], \quad 0 \leq k \leq p_j - 1, \quad 1 \leq j \leq n, \] (3.20)

where \( c_{j,k} = \frac{a_{j,k}}{(m_j-k-1)!} \) and \( d_{j,k} = \frac{b_{j,k}}{(m_j-k-1)!} \).

Next, for \( t \in [0, 1], \quad p_j \leq k \leq m_j - 1 \) and \( 1 \leq j \leq n \), using (3.7) and Lemma 3.1(b) again we obtain

\[
\int_0^1 \left| g^{(k)}_j (t,s) h_{r,j}(s) \right| ds = \int_0^1 \left| g^{(k)}_j (t,s) h_{r,j}(s) \right| ds \geq \frac{d_{j,k}}{(m_j-k-1)!}.
\] (3.23)

Since

\[ \int_t^1 \left( \frac{s-t}{1-t} \right)^{m_j-k-1} h_{r,j}(s) ds \leq \int_t^1 h_{r,j}(s) ds \to 0 \]

as \( t \to 1^- \), this integral extends to a continuous function on [0, 1], and there exists some constant \( l_{j,k} > 0 \) such that

\[ \int_t^1 \left( \frac{s-t}{1-t} \right)^{m_j-k-1} h_{r,j}(s) ds \geq l_{j,k} > 0, \quad t \in [0, 1]. \] (3.22)

Using (3.22) in (3.21), we get

\[ \int_0^1 \left| g^{(k)}_j (t,s) h_{r,j}(s) \right| ds = \int_0^1 \left| g^{(k)}_j (t,s) h_{r,j}(s) \right| ds \geq d_{j,k} (1-t)^{m_j-k-1}, \]

\[ t \in [0, 1], \quad p_j \leq k \leq m_j - 1, \quad 1 \leq j \leq n, \] (3.23)

where \( d_{j,k} = \frac{l_{j,k}}{(m_j-k-1)!} \).
Having obtained (3.20) and (3.23), it follows from the definition of \( H_{r,i}(t) \) in (3.16) that

\[
H_{r,i}(t) \leq \mu_i(t) \prod_{j=1}^{n} \left[ \beta_j(y_{j,0}, y_{j,1}r, \ldots, y_{j,m_j-1}r) \right] \\
+ \alpha_j(y_{j,0}[c_{j,0}t^{m_j-1} + d_{j,0}], y_{j,1}[c_{j,1}t^{m_j-2} + d_{j,1}], \ldots, y_{j,p_j-1}[c_{j,p_j-1}t^{m_j-p_j} + d_{j,p_j-1}], y_{j,p_j-1}(1-t)^{m_j-p_j-1}, \\
y_{j,p_j}d_{j,p_j}(1-t)^{m_j-p_j-2}, \ldots, y_{j,m_j-1}d_{j,m_j-1} \right) \\
\in L^1[0,1] \quad \text{(by (C9)).}
\]

This completes the proof. \( \square \)

**Remark 3.3.** Once again in view of Lemma 3.1(a), the condition (C10) can be replaced by the following, which is easier to check but stronger:

(C10)' there exists \( M_i > 0 \) such that for \( 0 \leq k \leq m_i - 1 \),

\[
M_i \geq \int_0^1 v_{ik}(s)\mu_i(s) \left\{ \prod_{j=1}^{n} \left[ \beta_j(y_{j,0}M_i, y_{j,1}M_i, \ldots, y_{j,m_j-1}M_i) \right] \\
+ \alpha_j(y_{j,0} \int_0^1 |g_j(s,x)|h_{M_i,j}(x) \, dx, y_{j,1} \int_0^1 |g'_j(s,x)|h_{M_i,j}(x) \, dx, \ldots, \\
y_{j,m_j-1} \int_0^1 |g_j^{(m_j-1)}(s,x)|h_{M_i,j}(x) \, dx \right) \right\} \, ds
\]

and

\[
\mu_i(s) \left\{ \prod_{j=1}^{n} \left[ \beta_j(y_{j,0}M_i, y_{j,1}M_i, \ldots, y_{j,m_j-1}M_i) \right] \\
+ \alpha_j(y_{j,0} \int_0^1 |g_j(s,x)|h_{M_i,j}(x) \, dx, y_{j,1} \int_0^1 |g'_j(s,x)|h_{M_i,j}(x) \, dx, \ldots, \\
y_{j,m_j-1} \int_0^1 |g_j^{(m_j-1)}(s,x)|h_{M_i,j}(x) \, dx \right) \right\} \geq h_{M_i,i}(s)
\]

for a.e. \( s \in [0,1] \).
As an application of Theorem 3.3, we consider a special case of the system (F), viz,

\[
\begin{aligned}
(-1)^{m_i-p_i} & u_i^{(m_i)}(t) = \theta_i u_i(t) \prod_{j=1}^{n} \left[ \alpha_j \left( \tilde{u}_j(t) \right) + \beta_j \left( \tilde{u}_j(t) \right) \right], \quad \text{a.e. } t \in [0, 1], \\
\left\{ \begin{array}{l}
u_i^{(j)}(0) = 0, \quad 0 \leq j \leq p_i - 1, \\
u_i^{(j)}(1) = 0, \quad p_i \leq j \leq m_i - 1,
\end{array} \right. \\
& i = 1, 2, \ldots, n,
\end{aligned}
\]

where for each \(1 \leq i \leq n, m_i \geq 2, 1 \leq p_i \leq m_i - 1\) and \(\theta_i \in \{1, -1\}\) are fixed.

**Theorem 3.4.** Let \(\theta_i \in \{1, -1\}, 1 \leq i \leq n,\) be fixed. For each \(1 \leq i \leq n,\) suppose (C9) holds and the following conditions are satisfied:

(C11) \(\mu_i : [0, 1] \to \mathbb{R}, \mu_i(t) > 0\) for a.e. \(t \in [0, 1],\) for each \(1 \leq j \leq n, \alpha_j > 0, \beta_j \geq 0\) are continuous on \(\prod_{k=0}^{m_j-1} (0, \infty)_{jk},\) and if \(a \leq |x_{j,k}| \leq b\) for some \(k \in \{0, 1, \ldots, m_j - 1\},\) then

\[
\alpha_j(x_{j,0}, \ldots, y_{j,k}a, \ldots, x_{j,m_j-1}) \geq \alpha_j(x_{j,0}, \ldots, x_{j,k}, \ldots, x_{j,m_j-1}) \\
\geq \alpha_j(x_{j,0}, \ldots, y_{j,k}b, \ldots, x_{j,m_j-1})
\]

and

\[
\beta_j(x_{j,0}, \ldots, y_{j,k}a, \ldots, x_{j,m_j-1}) \leq \beta_j(x_{j,0}, \ldots, x_{j,k}, \ldots, x_{j,m_j-1}) \\
\leq \beta_j(x_{j,0}, \ldots, y_{j,k}b, \ldots, x_{j,m_j-1})
\]

where \(y_{j,k}\) is defined in (C8);

(C12) there exists \(M_i > 0\) such that for \(t \in [0, 1]\) and \(0 \leq k \leq m_i - 1,\)

\[
M_i \geq \frac{1}{\int_0^1 \left| g_j^{(k)}(t, s) \right| \mu_i(s) \left\{ \prod_{j=1}^{n} \beta_j(y_{j,0}M_i, y_{j,1}M_i, \ldots, y_{j,m_j-1}M_i) \right\} ds}
\]

\[
+ \alpha_j \left[ \prod_{l=1}^{n} \alpha_l(y_{l,0}M_i, y_{l,1}M_i, \ldots, y_{l,m_l-1}M_i) \right] \int_0^1 \left| g_{j}(s, x) \right| \mu_j(x) dx,
\]

\[
\times \int_0^1 \left| g_j'(s, x) \right| \mu_j(x) dx, \ldots,
\]

\[
\gamma_{j,m_j-1} \left[ \prod_{l=1}^{n} \alpha_l(y_{l,0}M_i, y_{l,1}M_i, \ldots, y_{l,m_l-1}M_i) \right]
\]

\[
\times \int_0^1 \left| g_j^{(m_j-1)}(s, x) \right| \mu_j(x) dx \right] ds.
\]
\[
\geq \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \int_0^1 |g_i^{(k)}(t,s)| \mu_i(s) ds.
\]

Then, \((F)'\) has a fixed-sign solution \(u \in C^{m-1}_1[0,1] \times C^{m-2}_1[0,1] \times \cdots \times C^{m_{n-1}}_1[0,1]\) satisfying (3.8) for a.e. \(t \in [0,1]\) and any \(1 \leq i \leq n\).

**Proof.** Taking \(h_{r,i}(t) = \mu_i(t) \prod_{l=1}^{n} \alpha_l(\gamma_l,0r, \gamma_l,1r, \ldots, \gamma_l,m_l-1r)\), the conclusion follows immediately from Theorem 3.3. \(\square\)

**Remark 3.4.** Applying Lemma 3.1(a) again, the condition (C12) can be replaced by a stronger but easier-to-verify version:

\((C12)'\) there exists \(M_i > 0\) such that for \(0 \leq k \leq m_i - 1\),

\[
M_i \geq \int_0^1 v_{ik}(s) \mu_i(s) \left\{ \prod_{j=1}^{n} \left[ \beta_j(\gamma_{j,0}M_i, \gamma_{j,1}M_i, \ldots, \gamma_{j,m_j-1}M_i) \right] \right. \\
+ \alpha_j \left( \gamma_{j,0} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \int_0^1 |g_j(s,x)| \mu_j(x) dx, \\
\gamma_{j,1} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \\
\times \int_0^1 |g_j'(s,x)| \mu_j(x) dx, \ldots, \\
\gamma_{j,m_j-1} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \\
\times \int_0^1 |g_j^{(m_j-1)}(s,x)| \mu_j(x) dx \right) \right\} ds
\]

and

\[
\prod_{j=1}^{n} \left[ \beta_j(\gamma_{j,0}M_i, \gamma_{j,1}M_i, \ldots, \gamma_{j,m_j-1}M_i) \right] \\
+ \alpha_j \left( \gamma_{j,0} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \int_0^1 |g_j(s,x)| \mu_j(x) dx, \\
\gamma_{j,1} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \\
\times \int_0^1 |g_j'(s,x)| \mu_j(x) dx, \ldots, \\
\gamma_{j,m_j-1} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_l,0M_i, \gamma_l,1M_i, \ldots, \gamma_l,m_l-1M_i) \right] \\
\times \int_0^1 |g_j^{(m_j-1)}(s,x)| \mu_j(x) dx \right) \right\} ds
\]
\[
\gamma_{j,m_j-1} \left[ \prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_l, \gamma_{l,1} M_l, \ldots, \gamma_{l,m_l-1} M_l) \right] \\
\times \int_{0}^{1} \left| g^{(m_j-1)}_j(s,x) \right| \mu_j(x) dx \right) \right] \\
\geq \prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_l, \gamma_{l,1} M_l, \ldots, \gamma_{l,m_l-1} M_l)
\]

for a.e. \( s \in [0,1] \).

4. Example

We shall now illustrate our results through an example. Consider the system \( (F)' \) where the following are satisfied for each \( 1 \leq i \leq n \):

\[
\begin{cases}
\mu_i : [0,1] \to \mathbb{R}, \quad \mu_i(t) > 0 \quad \text{for a.e. } t \in [0,1], \\
\theta_i = 1, \quad \alpha_i(\tilde{x}_i) = |x_i,p_i|^{-a}, \quad \beta_i(\tilde{x}_i) = A_i |x_i,0|^b + B_i, \\
a > 0, \quad b \geq 0, \quad A_i, B_i \geq 0,
\end{cases}
\]  

(4.1)

\[
\int_{0}^{1} (1-t)^{-a[(m_1+m_2+\cdots+m_n)-(p_1+p_2+\cdots+p_n)-n]} \mu_i(t) dt < \infty
\]

(4.2)

and

\[ jna^2 + kb < 1 \quad \text{for all nonnegative integers } j, k \text{ with } j + k = n. \]  

(4.3)

Note that \( \gamma_{j,0} = \gamma_{j,p_j} = 1, \quad 1 \leq j \leq n. \) Clearly, (C9) is equivalent to (4.2) which is assumed. Moreover, (C11) is satisfied. Finally, the condition (C12)' reduces to

\[
M_i \geq \int_{0}^{1} v_{ik}(s) \mu_i(s) \prod_{j=1}^{n} \left[ A_j M_i^b + B_j + M_i^{na^2} \left( \int_{0}^{1} \left| g^{(p_j)}_j(s,x) \right| \mu_j(x) dx \right)^{-a} \right] ds
\]

(4.4)

and

\[
\prod_{j=1}^{n} \left[ A_j M_i^b + B_j + M_i^{na^2} \left( \int_{0}^{1} \left| g^{(p_j)}_j(s,x) \right| \mu_j(x) dx \right)^{-a} \right] \geq M_i^{-na}.
\]

(4.5)

The condition (4.3) ensures that (4.4) is satisfied for sufficiently large \( M_i \). Further, (4.5) is also fulfilled for large \( M_i \). Thus, applying Theorem 3.4 the system \( (F)' \) with (4.1)–(4.3) has a positive solution \( u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1] \) such that for a.e. \( t \in [0,1] \) and any \( 1 \leq i \leq n \),

\[ u^{(j)}_i(t) > 0 \quad \text{for all } j \in N_i^+ \quad \text{and } u^{(k)}_i(t) < 0 \quad \text{for all } k \in N_i^-.
\]

(4.6)
As a specific example, consider the system

\[
\begin{align*}
\mu_1(t) &= |u_1'(t)|^{-a} + A_1|u_1(t)|^b + B_1 \quad \text{a.e. } t \in [0, 1], \\
\mu_2(t) &= |u_2''(t)|^{-a} + A_2|u_2(t)|^b + B_2 \quad \text{a.e. } t \in [0, 1],
\end{align*}
\]

\[ u_1(0) = u_1''(0) = u_1'''(0) = u_1''''(1) = 0, \]

\[ u_2(0) = u_2''(0) = u_2'''(1) = u_2''''(1) = 0, \]

where for \( i = 1, 2, \)

\[ \mu_i : [0, 1] \to \mathbb{R}, \quad \mu_i(t) > 0 \quad \text{for a.e. } t \in [0, 1], \quad 0 < a < \frac{1}{2}, \quad 0 < b < \frac{1}{2}, \quad A_i, B_i > 0, \]

and

\[ \int_0^1 (1 - t)^{-2a} \mu_i(t) \, dt < \infty. \]  

(4.9)

In (4.7), we have

\[ n = 2, \quad m_1 = 4, \quad m_2 = 5, \quad p_1 = 2, \quad p_2 = 3, \quad \theta_1 = \theta_2 = 1, \]

\[ N_1^+ = \{0, 1, 2\}, \quad N_1^- = \{3\}, \quad N_2^+ = \{0, 1, 2, 3\}, \quad N_2^- = \{4\}. \]

Noting (4.8) and (4.9), the system (4.7) is a special case of (F) satisfying (4.1) and (4.2). Further, the condition (4.3) in this case is equivalent to

\[ 4a^2 < 1, \quad 2b < 1, \quad 2a^2 + b < 1, \]

which are clearly satisfied by any \( 0 < a < \frac{1}{2} \) and \( 0 < b < \frac{1}{2} \). Hence, from the above discussion we conclude that the system (4.7) with (4.8) and (4.9) has a positive solution \( u \in C^3[0, 1] \times C^4[0, 1] \) such that (see (4.6))

\[
\begin{align*}
&u_1(t) > 0, \quad u_1'(t) > 0, \quad u_1''(t) > 0, \quad u_1'''(t) < 0 \quad \text{and} \\
&u_2(t) > 0, \quad u_2'(t) > 0, \quad u_2''(t) > 0, \quad u_2'''(t) > 0, \quad u_2''''(t) < 0 \quad \text{for a.e. } t \in [0, 1].
\end{align*}
\]

(4.10)

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References


