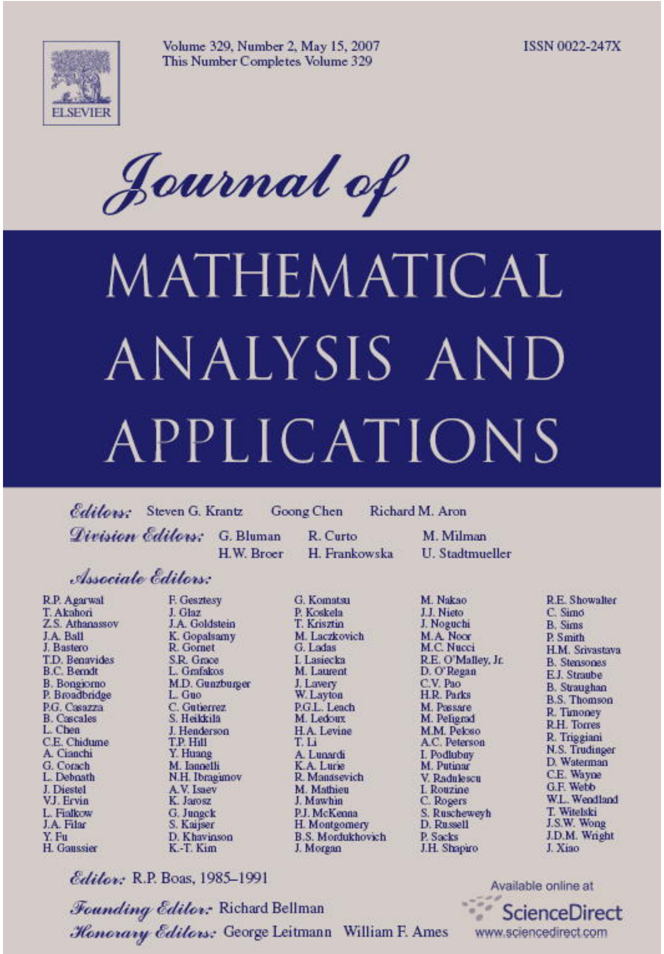
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Fixed-sign solutions for a system of singular focal boundary value problems

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Abstract

We consider a system of focal boundary value problems where the nonlinearities may be singular in the independent variable and may also be singular in the dependent arguments. Using Schauder fixed point theorem, we establish criteria such that the system of boundary value problems has at least one *fixed-sign* solution.

Keywords: System of singular boundary value problems; Fixed-sign solutions; Focal boundary conditions

1. Introduction

In this paper we shall consider the following system of focal boundary value problems:

$$\begin{cases} (-1)^{m_i-p_i}u_i^{(m_i)}(t)=f_i\big(t,\tilde{u}(t)\big), & \text{a.e. } t\in[0,1],\\ u_i^{(j)}(0)=0, & 0\leqslant j\leqslant p_i-1,\\ u_i^{(j)}(1)=0, & p_i\leqslant j\leqslant m_i-1,\\ i=1,2,\dots,n, \end{cases} \tag{F}$$

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where

$$\tilde{u} \equiv \big(u_1, u_1', \ldots, u_1^{(m_1-1)}, u_2, \ldots, u_2^{(m_2-1)}, \ldots, u_n, \ldots, u_n^{(m_n-1)}\big).$$

For each $1 \leqslant i \leqslant n$, $m_i \geqslant 2$ and $1 \leqslant p_i \leqslant m_i - 1$ are fixed. The nonlinearities f_i, $1 \leqslant i \leqslant n$, in (F) may be singular in the independent variable and may also be singular at $u_i^{(j)} = 0$ where $j \in \{0, 1, \ldots, m_i - 1\}$ and $i \in \{1, 2, \ldots, n\}$.

By using Schauder fixed point theorem, we shall develop existence criteria for a *fixed-sign solution* of the above system. A solution $u = (u_1, u_2, \ldots, u_n)$ of (F) is said to be of *fixed sign* if for each $1 \leqslant i \leqslant n$, $\theta_i u_i(t) \geqslant 0$ for $t \in [0, 1]$, where $\theta_i \in \{1, -1\}$ is fixed. We remark that *positive* solution is a special case of *fixed-sign* solution when $\theta_i = 1$ for all $1 \leqslant i \leqslant n$.

The importance of boundary value problems, both from a theoretical perspective as well as for their applications in the physical and engineering sciences, has been well documented in the literature; see, for example, the work [1,5,10,11] and references cited therein. In particular, focal boundary value problems have received a lot of attention in the literature, the reader is referred to [2,7,12,19,20] and references therein. However, there are only a handful of papers in the literature [3,4,6,8,9,13–18] that focus on singular boundary value problems. For instance, a particular case of (F)

$$\begin{aligned} &y''(t) + f\big(t, y(t), y'(t)\big) = 0, \quad \text{a.e. } t \in [0, 1], \\ &y(0) = y'(1) = 0 \end{aligned} \tag{1.1}$$

has been discussed in [3], and the existence of a *positive* solution is established. In [13,14], by using concavity properties, iterations and a fixed point theorem for operators which are decreasing with respect to a cone in a Banach space, the existence of a *positive* solution is established for the following singular boundary value problems:

$$\begin{aligned} &(-1)^{n-k} y^{(n)}(t) = f\big(t, y(t)\big), \quad t \in (0, 1), \\ &y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant k-1, \qquad y^{(j)}(0) = 0, \quad 0 \leqslant j \leqslant n-k-1, \end{aligned} \tag{1.2}$$

where $1 \leqslant k \leqslant n-1$ is fixed; and

$$\begin{aligned} &(-1)^{n-k} y^{(n)}(t) = f\big(t, y(t)\big), \quad t \in (0, 1), \\ &y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant k-1, \qquad y^{(j)}(0) = 0, \quad q \leqslant j \leqslant n-k+q-1, \end{aligned} \tag{1.3}$$

where $1 \leqslant k \leqslant n-1$ and $0 \leqslant q \leqslant k-1$ are fixed. In both (1.2) and (1.3), $f(t, x)$ has a singularity at $x = 0$ and is decreasing in x. Using another technique which involves the method of *a priori* estimates, the degree theory arguments and the Vitali convergence theorem, the following higher-order singular boundary value problems have been shown to possess a *positive* solution in [4,16]:

$$\begin{aligned} &-y^{(m)}(t) = f\big(t, y(t), y'(t), \ldots, y^{(m-1)}(t)\big), \quad t \in [0, 1], \\ &y^{(j)}(0) = 0, \quad 0 \leqslant j \leqslant m-2, \qquad y^{(p)}(1) = 0, \quad 1 \leqslant p \leqslant m-1 \text{ is fixed}, \end{aligned} \tag{1.4}$$

where $f(t, x_0, x_1, \ldots, x_{m-1})$ can be singular at $x_i = 0$, $0 \leqslant i \leqslant m-2$, but not at $x_{m-1} = 0$;

$$\begin{aligned} &(-1)^m y^{(2m)}(t) = \hat{f}\big(t, y(t), y'(t), \ldots, y^{(2m-2)}(t)\big), \quad t \in [0, 1], \\ &y^{(2j)}(0) = y^{(2j)}(1) = 0, \quad 0 \leqslant j \leqslant m-1, \end{aligned} \tag{1.5}$$

where $\hat{f}(t, x_0, x_1, \ldots, x_{2m-2})$ can be singular at $x_i = 0$, $0 \leqslant i \leqslant 2m-2$; and

$$(-1)^m y^{(2m+1)}(t) = \tilde{f}\big(t, y(t), y'(t), \ldots, y^{(2m)}(t)\big), \quad t \in [0,1],$$
$$y^{(2m)}(0) = 0, \quad y^{(2j)}(0) = y^{(2j)}(1) = 0, \quad 0 \leqslant j \leqslant m-1, \tag{1.6}$$

where $\tilde{f}(t, x_0, x_1, \ldots, x_{2m})$ can be singular at $x_i = 0$, $0 \leqslant i \leqslant 2m$.

Our present work obviously generalizes (1.1)–(1.6) to *systems*. In addition, we have also generalized the problem (1.1) to *higher-order* and *more general* boundary conditions. By using a *different* technique involving Schauder fixed point theorem, we obtain the existence of *fixed-sign* solutions, which include *positive* solutions as special case. Moreover, our criteria are easily verifiable and do not require the monotonicity condition on the nonlinear terms (which is needed in (1.2) and (1.3)). Our results thus extend, improve and complement those in the literature.

The paper is outlined as follows. In Section 2 we shall state the necessary fixed point theorem. The existence results for a fixed-sign solution of the system (F) are established, and illustrated by examples, in Sections 3 and 4, respectively.

2. Preliminaries

Theorem 2.1 *(Schauder fixed point theorem). Let K be a closed, convex subset of a normed linear space E. Then every compact and continuous map $S : K \to K$ has at least one fixed point.*

We also require a compactness criterion.

Theorem 2.2 *(Arźela–Ascoli theorem). Let $M \subseteq C[0,T]$. If M is uniformly bounded and equicontinuous, then M is relatively compact in $C[0,T]$.*

3. Main results

In this section we shall develop existence criteria for the system (F) where the nonlinearities $f_i(t, \tilde{u})$, $1 \leqslant i \leqslant n$, may be singular at $u_i^{(j)} = 0$, $j \in \{0, 1, \ldots, m_i - 1\}$, $i \in \{1, 2, \ldots, n\}$, and may also be singular in t at some subset Ω of $[0,1]$ with measure zero. We shall seek a fixed-sign solution of (F) in the space $C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$.

Let $g_i(t,s)$ be the Green's function of the boundary value problem

$$y^{(m_i)}(t) = 0, \quad t \in [0,1],$$
$$y^{(j)}(0) = 0, \quad 0 \leqslant j \leqslant p_i - 1, \qquad y^{(j)}(1) = 0, \quad p_i \leqslant j \leqslant m_i - 1.$$

It is known that [5, p. 211]

$$g_i(t,s) = \frac{1}{(m_i-1)!} \begin{cases} \sum_{k=0}^{p_i-1} \binom{m_i-1}{k} t^k (-s)^{m_i-k-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ -\sum_{k=p_i}^{m_i-1} \binom{m_i-1}{k} t^k (-s)^{m_i-k-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{3.1}$$

and for $(t,s) \in [0,1] \times [0,1]$,

$$\begin{cases} (-1)^{m_i-p_i} \dfrac{\partial^j}{\partial t^j} g_i(t,s) \geqslant 0, & 0 \leqslant j \leqslant p_i - 1, \\ (-1)^{m_i-j} \dfrac{\partial^j}{\partial t^j} g_i(t,s) \geqslant 0, & p_i \leqslant j \leqslant m_i - 1. \end{cases} \tag{3.2}$$

Lemma 3.1. *Let $N_i = \{0, 1, \ldots, m_i - 1\}$ and $g_i^{(j)}(t,s) \equiv \frac{\partial^j}{\partial t^j}\, g_i(t,s)$. The following hold for $(t,s) \in [0,1] \times [0,1]$:*

(a)
$$0 \leqslant (-1)^{m_i - p_i} g_i^{(j)}(t,s) \leqslant (-1)^{m_i - p_i} g_i^{(j)}(1,s), \quad 0 \leqslant j \leqslant p_i - 1, \tag{3.3}$$

and

$$\begin{aligned} \left|(-1)^{m_i - p_i} g_i^{(j)}(t,s)\right| &= (-1)^{j - p_i} \cdot (-1)^{m_i - p_i} g_i^{(j)}(t,s) \\ &\leqslant \left|(-1)^{m_i - p_i} g_i^{(j)}(0,s)\right|, \quad p_i \leqslant j \leqslant m_i - 1. \end{aligned} \tag{3.4}$$

Combining (3.3) *and* (3.4) *gives*

$$\begin{aligned} &\left|(-1)^{m_i - p_i} g_i^{(j)}(t,s)\right| \\ &\quad \leqslant \nu_{ij}(s) \\ &\quad \equiv \begin{cases} (-1)^{m_i - p_i} g_i^{(j)}(1,s), & 0 \leqslant j \leqslant p_i - 1, \\ |(-1)^{m_i - p_i} g_i^{(j)}(0,s)|, & p_i \leqslant j \leqslant m_i - 1, \end{cases} \\ &\quad = \begin{cases} \frac{(-1)^{m_i - p_i}}{(m_i - j - 1)!} \sum_{k=0}^{p_i - j - 1} \binom{m_i - j - 1}{k} (-s)^{m_i - j - k - 1}, & 0 \leqslant j \leqslant p_i - 1, \\ \frac{1}{(m_i - j - 1)!}\, s^{m_i - j - 1}, & p_i \leqslant j \leqslant m_i - 1. \end{cases} \end{aligned} \tag{3.5}$$

(b) $(-1)^{m_i - p_i} g_i^{(j)}(t,s) \geqslant 0$ *for* $j \in N_i^+$ *and* $(-1)^{m_i - p_i} g_i^{(k)}(t,s) \leqslant 0$ *for* $k \in N_i^-$*, where*

$$N_i^+ = \big(\{0, 1, \ldots, p_i\} \cup \{p_i + 2, p_i + 4, \ldots\}\big) \cap N_i$$

and

$$N_i^- = \{p_i + 1, p_i + 3, \ldots\} \cap N_i.$$

Clearly, $N_i = N_i^+ \cup N_i^-$.

Proof. From (3.1) we find for $0 \leqslant j \leqslant m_i - 1$,

$$g_i^{(j)}(t,s) = \frac{1}{(m_i - 1)!} \begin{cases} \sum_{k=j}^{p_i - 1} \binom{m_i - 1}{k} k^{(j)} t^{k-j} (-s)^{m_i - k - 1}, \\ \quad 0 \leqslant s \leqslant t \leqslant 1, \\ -\sum_{k=\max\{j, p_i\}}^{m_i - 1} \binom{m_i - 1}{k} k^{(j)} t^{k-j} (-s)^{m_i - k - 1}, \\ \quad 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{3.6}$$

where $k^{(j)} = k(k-1)\cdots(k-j+1)$ (with $k^{(0)} = 1$). Rewriting (3.6) gives

$$g_i^{(j)}(t,s) = \frac{1}{(m_i - j - 1)!} \begin{cases} \sum_{k=0}^{p_i - j - 1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i - j - k - 1}, \\ \quad 0 \leqslant s \leqslant t \leqslant 1, \\ -\sum_{k=\max\{0, p_i - j\}}^{m_i - j - 1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i - j - k - 1}, \\ \quad 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{3.7}$$

Since from (3.2) we have

$$(-1)^{m_i - p_i} g_i^{(j)}(t,s) \geqslant 0, \quad 0 \leqslant j \leqslant p_i,$$

it is clear that (3.3) follows for $(t,s) \in [0,1] \times [0,1]$.

Next, from (3.2) again we get

$$(-1)^{m_i-p_i} g_i^{(p_i)}(t,s) \geqslant 0 \quad \text{and} \quad (-1)^{m_i-p_i} g_i^{(p_i+1)}(t,s) \leqslant 0, \quad (t,s) \in [0,1]\times[0,1].$$

Thus,

$$0 \leqslant (-1)^{m_i-p_i} g_i^{(p_i)}(t,s) \leqslant (-1)^{m_i-p_i} g_i^{(p_i)}(0,s), \quad (t,s) \in [0,1]\times[0,1].$$

Continuing further from (3.2), we have

$$(-1)^{m_i-p_i} g_i^{(p_i+1)}(t,s) \leqslant 0 \quad \text{and} \quad (-1)^{m_i-p_i} g_i^{(p_i+2)}(t,s) \geqslant 0, \quad (t,s) \in [0,1]\times[0,1].$$

Hence,

$$\left|(-1)^{m_i-p_i} g_i^{(p_i+1)}(t,s)\right| = -(-1)^{m_i-p_i} g_i^{(p_i+1)}(t,s) \leqslant \left|(-1)^{m_i-p_i} g_i^{(p_i+1)}(0,s)\right|.$$

Proceeding in a similar manner, we see that for every $p_i \leqslant j \leqslant m_i - 1$ the inequality (3.4) is true.

The explicit expression in (3.5) follows from (3.7). Part (b) is also now immediate. □

Throughout we shall denote

$$\begin{aligned} &u = (u_1, u_2, \ldots, u_n),\\ &\tilde{u} = \left(u_1, u_1', \ldots, u_1^{(m_1-1)}, u_2, \ldots, u_2^{(m_2-1)}, \ldots, u_n, \ldots, u_n^{(m_n-1)}\right),\\ &\tilde{x} = (x_{1,0}, x_{1,1}, \ldots, x_{1,m_1-1}, x_{2,0}, \ldots, x_{2,m_2-1}, \ldots, x_{n,0}, \ldots, x_{n,m_n-1}), \end{aligned}$$

where $x_{i,j}$'s are real numbers, and for each $0 \leqslant j \leqslant m_i - 1$ and $1 \leqslant i \leqslant n$,

$$\begin{aligned} &\tilde{u}_i = \left(u_i, u_i', \ldots, u_i^{(m_i-1)}\right),\\ &\tilde{x}_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,m_i-1}),\\ &(0,\infty)_{ij} = \begin{cases} (0,\infty), & \text{if } \theta_i = 1,\ j \in N_i^+ \text{ or } \theta_i = -1,\ j \in N_i^-,\\ (-\infty,0), & \text{if } \theta_i = -1,\ j \in N_i^+ \text{ or } \theta_i = 1,\ j \in N_i^-. \end{cases} \end{aligned}$$

The definition of $[0,\infty)_{ij}$ is similar.

Theorem 3.1. *For each $1 \leqslant i \leqslant n$, let $\theta_i \in \{1,-1\}$ be fixed and the following conditions be satisfied*:

(C1) $f_i : [0,1] \times \prod_{j=1}^n \prod_{k=0}^{m_j-1} (0,\infty)_{jk} \to \mathbb{R}$, *where $t \mapsto f_i(t,\tilde{x})$ is measurable for all $\tilde{x} \in \prod_{j=1}^n \prod_{k=0}^{m_j-1} (0,\infty)_{jk}$, and $\tilde{x} \mapsto f_i(t,\tilde{x})$ is continuous for a.e. $t \in (0,1)$;*

(C2) *for any $r > 0$, there exists $h_{r,i} : [0,1] \to \mathbb{R}$, $h_{r,i}(t) > 0$ for a.e. $t \in [0,1]$, $h_{r,i} \in L^1[0,1]$ such that for all $|x_{j,k}| \in (0,r]$, $0 \leqslant k \leqslant m_j - 1$, $1 \leqslant j \leqslant n$,*

$$\theta_i f_i(t,\tilde{x}) \geqslant h_{r,i}(t) \quad \textit{for a.e. } t \in [0,1];$$

(C3) *for any $r > 0$, with $\int_0^1 |g_j^{(k)}(t,s)| h_{r,j}(s)\,ds \leqslant r$ for $t \in [0,1]$, $0 \leqslant k \leqslant m_j - 1$ and $1 \leqslant j \leqslant n$, there exists $H_{r,i} : [0,1] \to \mathbb{R}$, $H_{r,i}(t) \geqslant 0$ for a.e. $t \in [0,1]$, $H_{r,i} \in L^1[0,1]$ such that*

$$\theta_i f_i(t,\tilde{x}) \leqslant H_{r,i}(t)$$

for a.e. $t \in [0,1]$ and $|x_{j,k}| \in [\int_0^1 |g_j^{(k)}(t,s)| h_{r,j}(s)\,ds, r]$, $0 \leqslant k \leqslant m_j - 1$, $1 \leqslant j \leqslant n$;

(C4) *there exists* $M_i > 0$ *such that for* $t \in [0,1]$ *and* $0 \leqslant j \leqslant m_i - 1$,

$$M_i \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| H_{M_i,i}(s)\,ds \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| h_{M_i,i}(s)\,ds.$$

Then, (F) *has a fixed-sign solution* $u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ *such that for a.e.* $t \in [0,1]$ *and any* $1 \leqslant i \leqslant n$,

$$\theta_i u_i^{(j)}(t) > 0 \quad \text{for all } j \in N_i^+ \quad \text{and} \quad \theta_i u_i^{(k)}(t) < 0 \quad \text{for all } k \in N_i^-. \tag{3.8}$$

Proof. To begin, we define a closed convex subset of $B = C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ as

$$\begin{aligned} D = \Bigg\{ u \in B \;\Bigg|\; & \int_0^1 (-1)^{m_i-p_i} g_i^{(j)}(t,s) H_{M_i,i}(s)\,ds \\ & \geqslant \theta_i u_i^{(j)}(t) \geqslant \int_0^1 (-1)^{m_i-p_i} g_i^{(j)}(t,s) h_{M_i,i}(s)\,ds \geqslant 0 \\ & \text{and } \int_0^1 (-1)^{m_i-p_i} g_i^{(k)}(t,s) H_{M_i,i}(s)\,ds \\ & \leqslant \theta_i u_i^{(k)}(t) \leqslant \int_0^1 (-1)^{m_i-p_i} g_i^{(k)}(t,s) h_{M_i,i}(s)\,ds \leqslant 0 \\ & \text{for } t \in [0,1],\ j \in N_i^+,\ k \in N_i^-,\ 1 \leqslant i \leqslant n \Bigg\} \\ = \Bigg\{ u \in B \;\Bigg|\; & \int_0^1 \left|g_i^{(j)}(t,s)\right| H_{M_i,i}(s)\,ds \geqslant \left|\theta_i u_i^{(j)}(t)\right| \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| h_{M_i,i}(s)\,ds \\ & \text{for } t \in [0,1],\ 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant n \Bigg\}. \end{aligned}$$

Note that the second equality follows from Lemma 3.1(b).

Let the operator $S : D \to B$ be defined by

$$Su(t) = \big(S_1 u(t), S_2 u(t), \ldots, S_n u(t)\big), \quad t \in [0,1], \tag{3.9}$$

where

$$S_i u(t) = \int_0^1 (-1)^{m_i-p_i} g_i(t,s) f_i\big(s, \tilde{u}(s)\big)\,ds, \quad t \in [0,1],\ 1 \leqslant i \leqslant n. \tag{3.10}$$

Clearly, a fixed point of the operator S is a solution of the system (F). Indeed, a fixed point of S obtained in D will be a *fixed-sign* solution of the system (F).

First we shall show that S maps D into D. Let $u \in D$. By (C4) it is clear that

$$M_i \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| H_{M_i,i}(s)\,ds \geqslant \left|\theta_i u_i^{(j)}(t)\right| \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| h_{M_i,i}(s)\,ds > 0,$$
$$\text{a.e. } t \in [0,1],\ 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant n. \tag{3.11}$$

Hence, it follows from (C2) that

$$\theta_i f_i(t, \tilde{u}) \geqslant h_{M_i,i}(t), \quad \text{a.e. } t \in [0,1],\ 1 \leqslant i \leqslant n. \tag{3.12}$$

In view of Lemma 3.1(b), we find

$$\left|\theta_i (S_i u)^{(j)}(t)\right| = \int_0^1 \left|g_i^{(j)}(t,s)\right| \theta_i f_i\big(s, \tilde{u}(s)\big)\,ds \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| h_{M_i,i}(s)\,ds,$$
$$t \in [0,1],\ 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant n. \tag{3.13}$$

Also, from (C3) and (3.11) we have

$$\theta_i f_i(t, \tilde{u}) \leqslant H_{M_i,i}(t), \quad \text{a.e. } t \in [0,1],\ 1 \leqslant i \leqslant n, \tag{3.14}$$

and so

$$\left|\theta_i (S_i u)^{(j)}(t)\right| = \int_0^1 \left|g_i^{(j)}(t,s)\right| \theta_i f_i\big(s, \tilde{u}(s)\big)\,ds \leqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| H_{M_i,i}(s)\,ds,$$
$$t \in [0,1],\ 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant n. \tag{3.15}$$

Having obtained (3.13) and (3.15), we have shown that $S : D \to D$.

Next, we shall prove that $S : D \to D$ is continuous. Let $\{u^k\}$ be a sequence in D and $u^k \to u$ in B. Then, applying (3.5) we find for $t \in [0,1]$, $0 \leqslant j \leqslant m_i - 1$ and $1 \leqslant i \leqslant n$,

$$\begin{aligned}
\left|\big(S_i u^k\big)^{(j)}(t) - (S_i u)^{(j)}(t)\right| &\leqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| \cdot \left|f_i\big(s, \tilde{u}^k(s)\big) - f_i\big(s, \tilde{u}(s)\big)\right| ds \\
&\leqslant \int_0^1 \nu_{ij}(s) \left|f_i\big(s, \tilde{u}^k(s)\big) - f_i\big(s, \tilde{u}(s)\big)\right| ds \\
&\leqslant 2\Big(\sup_{s \in [0,1]} \nu_{ij}(s)\Big) \int_0^1 H_{M_i,i}(s)\,ds < \infty.
\end{aligned}$$

Thus, together with (C1), the Lebesgue dominated convergence theorem gives for each $0 \leqslant j \leqslant m_i - 1$ and $1 \leqslant i \leqslant n$,

$$\sup_{t \in [0,1]} \left|\big(S_i u^k\big)^{(j)}(t) - (S_i u)^{(j)}(t)\right| \leqslant \int_0^1 \nu_{ij}(s) \left|f_i\big(s, \tilde{u}^k(s)\big) - f_i\big(s, \tilde{u}(s)\big)\right| ds \to 0$$

as $k \to \infty$. Hence, S is continuous.

Finally, we shall check that $S : D \to D$ is compact. Let $u \in D$. Then, by (3.14) and (C4) we have

$$\sup_{t\in[0,1]} \left|(S_i u)^{(j)}(t)\right| \leqslant \sup_{t\in[0,1]} \int_0^1 \left|g_i^{(j)}(t,s)\right| H_{M_i,i}(s)\,ds \leqslant M_i, \quad 0 \leqslant j \leqslant m_i - 1,\ 1 \leqslant i \leqslant n.$$

Now, let $t, t' \in [0,1]$ with $t' < t$. Using the expression (3.7), we obtain for $0 \leqslant j \leqslant m_i - 1$ and $1 \leqslant i \leqslant n$,

$$\begin{aligned}
&\left|(S_i u)^{(j)}(t) - (S_i u)^{(j)}(t')\right| \cdot (m_i - j - 1)! \\
&= \Bigg| \int_0^t \sum_{k=0}^{p_i-j-1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i-j-k-1} f_i\big(s, \tilde u(s)\big)\,ds \\
&\quad - \int_t^1 \sum_{k=\max\{0,p_i-j\}}^{m_i-j-1} \binom{m_i - j - 1}{k} t^k (-s)^{m_i-j-k-1} f_i\big(s, \tilde u(s)\big)\,ds \\
&\quad - \int_0^{t'} \sum_{k=0}^{p_i-j-1} \binom{m_i - j - 1}{k} (t')^k (-s)^{m_i-j-k-1} f_i\big(s, \tilde u(s)\big)\,ds \\
&\quad + \int_{t'}^1 \sum_{k=\max\{0,p_i-j\}}^{m_i-j-1} \binom{m_i - j - 1}{k} (t')^k (-s)^{m_i-j-k-1} f_i\big(s, \tilde u(s)\big)\,ds \Bigg| \\
&\leqslant \int_0^{t'} \sum_{k=0}^{p_i-j-1} \binom{m_i - j - 1}{k} s^{m_i-j-k-1} \big[t^k - (t')^k\big] \theta_i f_i\big(s, \tilde u(s)\big)\,ds \\
&\quad + \int_{t'}^t \sum_{k=0}^{p_i-j-1} \binom{m_i - j - 1}{k} s^{m_i-j-k-1} t^k \theta_i f_i\big(s, \tilde u(s)\big)\,ds \\
&\quad + \int_{t'}^t \sum_{k=\max\{0,p_i-j\}}^{m_i-j-1} \binom{m_i - j - 1}{k} s^{m_i-j-k-1} (t')^k \theta_i f_i\big(s, \tilde u(s)\big)\,ds \\
&\quad + \int_t^1 \sum_{k=\max\{0,p_i-j\}}^{m_i-j-1} \binom{m_i - j - 1}{k} s^{m_i-j-k-1} \big[t^k - (t')^k\big] \theta_i f_i\big(s, \tilde u(s)\big)\,ds \\
&\leqslant \int_0^{t'} \sum_{k=0}^{p_i-j-1} \binom{m_i - j - 1}{k} s^{m_i-j-k-1} \big[t^k - (t')^k\big] H_{M_i,i}(s)\,ds \\
&\quad + \int_{t'}^t \sum_{k=0}^{p_i-j-1} \binom{m_i - j - 1}{k} s^{m_i-j-k-1} H_{M_i,i}(s)\,ds
\end{aligned}$$

$$+\int_{t'}^{t} \sum_{k=\max\{0,p_i-j\}}^{m_i-j-1} \binom{m_i-j-1}{k} s^{m_i-j-k-1} H_{M_i,i}(s)\,ds$$

$$+\int_{t}^{1} \sum_{k=\max\{0,p_i-j\}}^{m_i-j-1} \binom{m_i-j-1}{k} s^{m_i-j-k-1} \big[t^k-(t')^k\big] H_{M_i,i}(s)\,ds \to 0$$

as $t' \to t$. A similar argument also holds if $t' > t$. Now Theorem 2.2 guarantees that S is compact.

Hence, we conclude from Theorem 2.1 that S has a fixed point in D. The proof is complete. □

Remark 3.1. In Theorem 3.1, the condition (C3) can be replaced by the following:

(C3)′ for any $r > 0$ with $\int_0^1 |g_j^{(k)}(t,s)|h_{r,j}(s)\,ds \leqslant r$ for $t \in [0,1]$, $0 \leqslant k \leqslant m_j - 1$ and $1 \leqslant j \leqslant n$, let

$$H_{r,i}(t) = \sup\left\{\theta_i f_i(t,\tilde{x}) \;\middle|\; |x_{j,k}| \in \left[\int_0^1 \big|g_j^{(k)}(t,s)\big|h_{r,j}(s)\,ds, r\right],\right.$$
$$\left. 0 \leqslant k \leqslant m_j - 1,\ 1 \leqslant j \leqslant n\right\}$$

and assume $H_{r,i} \in L^1[0,1]$.

If f_i, $1 \leqslant i \leqslant n$, are nonsingular, i.e., $f_i : [0,1] \times \mathbb{R}^{m_1+m_2+\cdots+m_n} \to \mathbb{R}$, then by using a similar argument as in Theorem 3.1, we obtain the following result.

Theorem 3.2. *For each $1 \leqslant i \leqslant n$, let $\theta_i \in \{1,-1\}$ be fixed and the following conditions be satisfied*:

(C5) $f_i : [0,1] \times \prod_{j=1}^{n}\prod_{k=0}^{m_j-1}[0,\infty)_{jk} \to \mathbb{R}$ *where $t \mapsto f_i(t,\tilde{x})$ is measurable for all $\tilde{x} \in \prod_{j=1}^{n}\prod_{k=0}^{m_j-1}[0,\infty)_{jk}$, and $\tilde{x} \mapsto f_i(t,\tilde{x})$ is continuous for a.e. $t \in (0,1)$;*

(C6) *for any $r > 0$, there exists $H_{r,i} : [0,1] \to \mathbb{R}$, $H_{r,i}(t) \geqslant 0$ for a.e. $t \in [0,1]$, $H_{r,i} \in L^1[0,1]$ such that for all $|x_{j,k}| \in [0,r]$, $0 \leqslant k \leqslant m_j - 1$, $1 \leqslant j \leqslant n$,*

$$0 \leqslant \theta_i f_i(t,\tilde{x}) \leqslant H_{r,i}(t) \quad \text{for a.e. } t \in [0,1];$$

(C7) *there exists $M_i > 0$ such that for $t \in [0,1]$ and $0 \leqslant j \leqslant m_i - 1$,*

$$M_i \geqslant \int_0^1 \big|g_i^{(j)}(t,s)\big| H_{M_i,i}(s)\,ds \geqslant 0.$$

Then, (F) *has a fixed-sign solution $u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ such that for any $t \in [0,1]$ and any $1 \leqslant i \leqslant n$,*

$$\theta_i u_i^{(j)}(t) \geqslant 0 \quad \text{for all } j \in N_i^{+} \quad \text{and} \quad \theta_i u_i^{(k)}(t) \leqslant 0 \quad \text{for all } k \in N_i^{-}.$$

Remark 3.2. Lemma 3.1(a) can be used to 'simplify' conditions (C4) and (C7). The easier-to-check but stronger conditions are

(C4)′ there exists $M_i > 0$ such that for $0 \leqslant j \leqslant m_i - 1$,

$$M_i \geqslant \int_0^1 \nu_{ij}(s) H_{M_i,i}(s)\, ds$$

and $H_{M_i,i}(s) \geqslant h_{M_i,i}(s)$ for a.e. $s \in [0, 1]$,

and

(C7)′ there exists $M_i > 0$ such that for $0 \leqslant j \leqslant m_i - 1$,

$$M_i \geqslant \int_0^1 \nu_{ij}(s) H_{M_i,i}(s)\, ds \geqslant 0.$$

In fact, using (3.5) from (C4)′ we get

$$M_i \geqslant \int_0^1 \nu_{ij}(s) H_{M_i,i}(s)\, ds \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| H_{M_i,i}(s)\, ds \geqslant \int_0^1 \left|g_i^{(j)}(t,s)\right| h_{M_i,i}(s)\, ds,$$
$$t \in [0,1],\ 0 \leqslant j \leqslant m_i - 1.$$

Hence, (C4)′ implies (C4). Similarly, it can be easily seen that (C7)′ implies (C7).

Theorem 3.3. *Let $\theta_i \in \{1, -1\}$, $1 \leqslant i \leqslant n$, be fixed. For each $1 \leqslant i \leqslant n$, suppose* (C1) *and* (C2) *hold and the following conditions are satisfied*:

(C8) *for $(t, \tilde{x}) \in [0,1] \times \prod_{j=1}^n \prod_{k=0}^{m_j-1} (0, \infty)_{jk}$,*

$$\theta_i f_i(t, \tilde{x}) \leqslant \mu_i(t) \prod_{j=1}^n \left[\alpha_j(\tilde{x}_j) + \beta_j(\tilde{x}_j)\right],$$

where $\mu_i : [0,1] \to \mathbb{R}$, $\mu_i(t) > 0$ for a.e. $t \in [0,1]$, for each $1 \leqslant j \leqslant n$, $\alpha_j > 0$, $\beta_j \geqslant 0$ are continuous on $\prod_{k=0}^{m_j-1} (0, \infty)_{jk}$, and if $a \leqslant |x_{j,k}| \leqslant b$ for some $k \in \{0, 1, \ldots, m_j - 1\}$, then

$$\begin{aligned}\alpha_j(x_{j,0}, \ldots, \gamma_{j,k} a, \ldots, x_{j,m_j-1}) &\geqslant \alpha_j(x_{j,0}, \ldots, x_{j,k}, \ldots, x_{j,m_j-1})\\ &\geqslant \alpha_j(x_{j,0}, \ldots, \gamma_{j,k} b, \ldots, x_{j,m_j-1})\end{aligned}$$

and

$$\begin{aligned}\beta_j(x_{j,0}, \ldots, \gamma_{j,k} a, \ldots, x_{j,m_j-1}) &\leqslant \beta_j(x_{j,0}, \ldots, x_{j,k}, \ldots, x_{j,m_j-1})\\ &\leqslant \beta_j(x_{j,0}, \ldots, \gamma_{j,k} b, \ldots, x_{j,m_j-1}),\end{aligned}$$

where

$$\gamma_{j,k} = \begin{cases} \theta_j, & \text{if } k \in N_j^+, \\ -\theta_j, & \text{if } k \in N_j^-; \end{cases}$$

(C9) *for any constants* $c_{j,k} > 0$, $d_{j,\ell} > 0$, $0 \leqslant k \leqslant p_j - 1$, $0 \leqslant \ell \leqslant m_j - 1$, $1 \leqslant j \leqslant n$,

$$\int_0^1 \mu_i(t) \Biggl\{ \prod_{j=1}^n \alpha_j \bigl(\gamma_{j,0}\bigl[c_{j,0} t^{m_j-1} + d_{j,0}\bigr], \gamma_{j,1}\bigl[c_{j,1} t^{m_j-2} + d_{j,1}\bigr], \ldots, \\ \gamma_{j,p_j-1}\bigl[c_{j,p_j-1} t^{m_j-p_j} + d_{j,p_j-1}\bigr], \gamma_{j,p_j} d_{j,p_j} (1-t)^{m_j-p_j-1}, \\ \gamma_{j,p_j+1} d_{j,p_j+1} (1-t)^{m_j-p_j-2}, \ldots, \gamma_{j,m_j-1} d_{j,m_j-1} \bigr) \Biggr\} dt < \infty;$$

(C10) *there exists* $M_i > 0$ *such that for* $t \in [0,1]$ *and* $0 \leqslant k \leqslant m_i - 1$,

$$M_i \geqslant \int_0^1 \bigl| g_i^{(k)}(t,s) \bigr| \mu_i(s) \Biggl\{ \prod_{j=1}^n \Biggl[\beta_j(\gamma_{j,0} M_i, \gamma_{j,1} M_i, \ldots, \gamma_{j,m_j-1} M_i) \\ + \alpha_j \Biggl(\gamma_{j,0} \int_0^1 \bigl| g_j(s,x) \bigr| h_{M_i,j}(x)\, dx, \gamma_{j,1} \int_0^1 \bigl| g_j'(s,x) \bigr| h_{M_i,j}(x)\, dx, \ldots, \\ \gamma_{j,m_j-1} \int_0^1 \bigl| g_j^{(m_j-1)}(s,x) \bigr| h_{M_i,j}(x)\, dx \Biggr) \Biggr] \Biggr\} ds \\ \geqslant \int_0^1 \bigl| g_i^{(k)}(t,s) \bigr| h_{M_i,i}(s)\, ds.$$

Then, (F) *has a fixed-sign solution* $u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ *satisfying* (3.8) *for a.e.* $t \in [0,1]$ *and any* $1 \leqslant i \leqslant n$.

Proof. We shall show that (C3) and (C4) are satisfied, then the conclusion will be immediate from Theorem 3.1. Let $1 \leqslant i \leqslant n$. In order to choose $H_{r,i}$ which is described in (C3), we use (C8) to obtain for a.e. $t \in [0,1]$, $|x_{j,k}| \in [\int_0^1 |g_j^{(k)}(t,s)| h_{r,j}(s)\, ds, r]$, $0 \leqslant k \leqslant m_j - 1$, $1 \leqslant j \leqslant n$,

$$\theta_i f_i(t,\tilde{x}) \leqslant \mu_i(t) \prod_{j=1}^n \Biggl[\beta_j(\gamma_{j,0} r, \gamma_{j,1} r, \ldots, \gamma_{j,m_j-1} r) \\ + \alpha_j \Biggl(\gamma_{j,0} \int_0^1 \bigl| g_j(t,s) \bigr| h_{r,j}(s)\, ds, \gamma_{j,1} \int_0^1 \bigl| g_j'(t,s) \bigr| h_{r,j}(s)\, ds, \ldots, \\ \gamma_{j,m_j-1} \int_0^1 \bigl| g_j^{(m_j-1)}(t,s) \bigr| h_{r,j}(s)\, ds \Biggr) \Biggr] \\ \equiv H_{r,i}(t). \tag{3.16}$$

Observed that we have picked $H_{r,i}(t)$ to be the right-hand side of (3.16). Hence, (C10) implies (C4).

Now, (C3) is fulfilled if we can show that $H_{r,i} \in L^1[0,1]$. To proceed, we shall first use (3.7) and Lemma 3.1(b) to find for $t \in [0,1]$, $0 \leqslant k \leqslant p_j - 1$ and $1 \leqslant j \leqslant n$,

$$\begin{aligned}
&\int_0^1 \left|g_j^{(k)}(t,s)\right| h_{r,j}(s)\,ds \cdot (m_j-k-1)! \\
&\quad = \int_0^t \left|\sum_{\ell=0}^{p_j-k-1} \binom{m_j-k-1}{\ell} t^\ell (-s)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds \\
&\qquad + \int_t^1 \left|\sum_{\ell=p_j-k}^{m_j-k-1} \binom{m_j-k-1}{\ell} t^\ell (-s)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds \\
&\quad = t^{m_j-k-1} \int_0^t \left|\sum_{\ell=0}^{p_j-k-1} \binom{m_j-k-1}{\ell} \left(\frac{-s}{t}\right)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds \\
&\qquad + \int_t^1 \left|\sum_{\ell=p_j-k}^{m_j-k-1} \binom{m_j-k-1}{\ell} t^\ell (-s)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds. \qquad (3.17)
\end{aligned}$$

For the first integral in (3.17), we see that

$$\begin{aligned}
&\int_0^t \left|\sum_{\ell=0}^{p_j-k-1} \binom{m_j-k-1}{\ell} \left(\frac{-s}{t}\right)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds \\
&\quad \leqslant \int_0^t \sum_{\ell=0}^{p_j-k-1} \binom{m_j-k-1}{\ell} h_{r,j}(s)\,ds \to 0
\end{aligned}$$

as $t \to 0^+$. Hence, this integral extends to a continuous function on $[0,1]$, and there exists some constant $a_{j,k} > 0$ such that

$$\int_0^t \left|\sum_{\ell=0}^{p_j-k-1} \binom{m_j-k-1}{\ell} \left(\frac{-s}{t}\right)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds \geqslant a_{j,k} > 0, \quad t \in [0,1]. \qquad (3.18)$$

Further, for the second integral in (3.17), it is clear that

$$\begin{aligned}
&\int_t^1 \left|\sum_{\ell=p_j-k}^{m_j-k-1} \binom{m_j-k-1}{\ell} t^\ell (-s)^{m_j-k-\ell-1}\right| h_{r,j}(s)\,ds \\
&\quad \leqslant \int_t^1 \sum_{\ell=p_j-k}^{m_j-k-1} \binom{m_j-k-1}{\ell} h_{r,j}(s)\,ds \to 0
\end{aligned}$$

as $t \to 1^-$. Therefore, this integral also extends to a continuous function on $[0,1]$, and there exists some constant $b_{j,k} > 0$ such that

$$\int_t^1 \left| \sum_{\ell=p_j-k}^{m_j-k-1} \binom{m_j-k-1}{\ell} t^\ell (-s)^{m_j-k-\ell-1} \right| h_{r,j}(s)\,ds \geqslant b_{j,k} > 0, \quad t \in [0,1]. \tag{3.19}$$

Using (3.18) and (3.19) in (3.17), we find

$$\left| \int_0^1 \left| g_j^{(k)}(t,s) \right| h_{r,j}(s)\,ds \right| = \int_0^1 \left| g_j^{(k)}(t,s) \right| h_{r,j}(s)\,ds \geqslant c_{j,k} t^{m_j-k-1} + d_{j,k},$$
$$t \in [0,1],\ 0 \leqslant k \leqslant p_j - 1,\ 1 \leqslant j \leqslant n, \tag{3.20}$$

where $c_{j,k} = \frac{a_{j,k}}{(m_j-k-1)!}$ and $d_{j,k} = \frac{b_{j,k}}{(m_j-k-1)!}$.

Next, for $t \in [0,1]$, $p_j \leqslant k \leqslant m_j - 1$ and $1 \leqslant j \leqslant n$, using (3.7) and Lemma 3.1(b) again we obtain

$$\begin{aligned}
&\int_0^1 \left| g_j^{(k)}(t,s) \right| h_{r,j}(s)\,ds \cdot (m_j-k-1)! \\
&\quad = \int_t^1 \left| \sum_{\ell=0}^{m_j-k-1} \binom{m_j-k-1}{\ell} t^\ell (-s)^{m_j-k-\ell-1} \right| h_{r,j}(s)\,ds \\
&\quad = \int_t^1 \left| (t-s)^{m_j-k-1} \right| h_{r,j}(s)\,ds \\
&\quad = (1-t)^{m_j-k-1} \int_t^1 \left(\frac{s-t}{1-t} \right)^{m_j-k-1} h_{r,j}(s)\,ds.
\end{aligned} \tag{3.21}$$

Since

$$\int_t^1 \left(\frac{s-t}{1-t} \right)^{m_j-k-1} h_{r,j}(s)\,ds \leqslant \int_t^1 h_{r,j}(s)\,ds \to 0$$

as $t \to 1^-$, this integral extends to a continuous function on $[0,1]$, and there exists some constant $l_{j,k} > 0$ such that

$$\int_t^1 \left(\frac{s-t}{1-t} \right)^{m_j-k-1} h_{r,j}(s)\,ds \geqslant l_{j,k} > 0, \quad t \in [0,1]. \tag{3.22}$$

Using (3.22) in (3.21), we get

$$\left| \int_0^1 \left| g_j^{(k)}(t,s) \right| h_{r,j}(s)\,ds \right| = \int_0^1 \left| g_j^{(k)}(t,s) \right| h_{r,j}(s)\,ds \geqslant d_{j,k} (1-t)^{m_j-k-1},$$
$$t \in [0,1],\ p_j \leqslant k \leqslant m_j - 1,\ 1 \leqslant j \leqslant n, \tag{3.23}$$

where $d_{j,k} = \frac{l_{j,k}}{(m_j-k-1)!}$.

Having obtained (3.20) and (3.23), it follows from the definition of $H_{r,i}(t)$ in (3.16) that

$$\begin{aligned}
H_{r,i}(t) \leqslant \mu_i(t) \prod_{j=1}^{n} \bigl[& \beta_j(\gamma_{j,0}r, \gamma_{j,1}r, \ldots, \gamma_{j,m_j-1}r) \\
& + \alpha_j\bigl(\gamma_{j,0}\bigl[c_{j,0}t^{m_j-1} + d_{j,0}\bigr], \gamma_{j,1}\bigl[c_{j,1}t^{m_j-2} + d_{j,1}\bigr], \ldots, \\
& \quad \gamma_{j,p_j-1}\bigl[c_{j,p_j-1}t^{m_j-p_j} + d_{j,p_j-1}\bigr], \gamma_{j,p_j}d_{j,p_j}(1-t)^{m_j-p_j-1}, \\
& \quad \gamma_{j,p_j+1}d_{j,p_j+1}(1-t)^{m_j-p_j-2}, \ldots, \gamma_{j,m_j-1}d_{j,m_j-1}\bigr)\bigr] \\
\in L^1[0,1] \quad & \text{(by (C9))}.
\end{aligned}$$

This completes the proof. $\square$

Remark 3.3. Once again in view of Lemma 3.1(a), the condition (C10) can be replaced by the following, which is easier to check but stronger:

(C10)$'$ there exists $M_i > 0$ such that for $0 \leqslant k \leqslant m_i - 1$,

$$\begin{aligned}
M_i \geqslant \int_0^1 \nu_{ik}(s)\mu_i(s)\Biggl\{\prod_{j=1}^{n}\Biggl[& \beta_j(\gamma_{j,0}M_i, \gamma_{j,1}M_i, \ldots, \gamma_{j,m_j-1}M_i) \\
& + \alpha_j\Biggl(\gamma_{j,0}\int_0^1 \bigl|g_j(s,x)\bigr|h_{M_i,j}(x)\,dx, \gamma_{j,1}\int_0^1 \bigl|g_j'(s,x)\bigr|h_{M_i,j}(x)\,dx, \ldots, \\
& \quad \gamma_{j,m_j-1}\int_0^1 \bigl|g_j^{(m_j-1)}(s,x)\bigr|h_{M_i,j}(x)\,dx\Biggr)\Biggr]\Biggr\}\,ds
\end{aligned}$$

and

$$\begin{aligned}
\mu_i(s)\Biggl\{\prod_{j=1}^{n}\Biggl[& \beta_j(\gamma_{j,0}M_i, \gamma_{j,1}M_i, \ldots, \gamma_{j,m_j-1}M_i) \\
& + \alpha_j\Biggl(\gamma_{j,0}\int_0^1 \bigl|g_j(s,x)\bigr|h_{M_i,j}(x)\,dx, \gamma_{j,1}\int_0^1 \bigl|g_j'(s,x)\bigr|h_{M_i,j}(x)\,dx, \ldots, \\
& \quad \gamma_{j,m_j-1}\int_0^1 \bigl|g_j^{(m_j-1)}(s,x)\bigr|h_{M_i,j}(x)\,dx\Biggr)\Biggr]\Biggr\} \geqslant h_{M_i,i}(s)
\end{aligned}$$

for a.e. $s \in [0,1]$.

As an application of Theorem 3.3, we consider a special case of the system (F), viz,

$$\begin{cases} (-1)^{m_i-p_i}u_i^{(m_i)}(t)=\theta_i\mu_i(t)\prod_{j=1}^{n}\left[\alpha_j\left(\tilde u_j(t)\right)+\beta_j\left(\tilde u_j(t)\right)\right], & \text{a.e. } t\in[0,1],\\ u_i^{(j)}(0)=0, \quad 0\leqslant j\leqslant p_i-1,\\ u_i^{(j)}(1)=0, \quad p_i\leqslant j\leqslant m_i-1,\\ i=1,2,\dots,n, \end{cases} \tag{F$'$}$$

where for each $1\leqslant i\leqslant n$, $m_i\geqslant 2$, $1\leqslant p_i\leqslant m_i-1$ and $\theta_i\in\{1,-1\}$ are fixed.

Theorem 3.4. *Let $\theta_i\in\{1,-1\}$, $1\leqslant i\leqslant n$, be fixed. For each $1\leqslant i\leqslant n$, suppose* (C9) *holds and the following conditions are satisfied*:

(C11) *$\mu_i:[0,1]\to\mathbb{R}$, $\mu_i(t)>0$ for a.e. $t\in[0,1]$, for each $1\leqslant j\leqslant n$, $\alpha_j>0$, $\beta_j\geqslant 0$ are continuous on $\prod_{k=0}^{m_j-1}(0,\infty)_{jk}$, and if $a\leqslant|x_{j,k}|\leqslant b$ for some $k\in\{0,1,\dots,m_j-1\}$, then*

$$\begin{aligned}\alpha_j(x_{j,0},\dots,\gamma_{j,k}a,\dots,x_{j,m_j-1})&\geqslant\alpha_j(x_{j,0},\dots,x_{j,k},\dots,x_{j,m_j-1})\\&\geqslant\alpha_j(x_{j,0},\dots,\gamma_{j,k}b,\dots,x_{j,m_j-1})\end{aligned}$$

and

$$\begin{aligned}\beta_j(x_{j,0},\dots,\gamma_{j,k}a,\dots,x_{j,m_j-1})&\leqslant\beta_j(x_{j,0},\dots,x_{j,k},\dots,x_{j,m_j-1})\\&\leqslant\beta_j(x_{j,0},\dots,\gamma_{j,k}b,\dots,x_{j,m_j-1})\end{aligned}$$

where $\gamma_{j,k}$ is defined in (C8);

(C12) *there exists $M_i>0$ such that for $t\in[0,1]$ and $0\leqslant k\leqslant m_i-1$,*

$$\begin{aligned}M_i\geqslant\int_0^1\left|g_i^{(k)}(t,s)\right|\mu_i(s)\Bigg\{\prod_{j=1}^{n}\Bigg[&\beta_j(\gamma_{j,0}M_i,\gamma_{j,1}M_i,\dots,\gamma_{j,m_j-1}M_i)\\&+\alpha_j\Bigg(\gamma_{j,0}\Bigg[\prod_{l=1}^{n}\alpha_l(\gamma_{l,0}M_i,\gamma_{l,1}M_i,\dots,\gamma_{l,m_l-1}M_i)\Bigg]\int_0^1\left|g_j(s,x)\right|\mu_j(x)\,dx,\\&\quad\gamma_{j,1}\Bigg[\prod_{l=1}^{n}\alpha_l(\gamma_{l,0}M_i,\gamma_{l,1}M_i,\dots,\gamma_{l,m_l-1}M_i)\Bigg]\\&\quad\times\int_0^1\left|g_j'(s,x)\right|\mu_j(x)\,dx,\dots,\\&\quad\gamma_{j,m_j-1}\Bigg[\prod_{l=1}^{n}\alpha_l(\gamma_{l,0}M_i,\gamma_{l,1}M_i,\dots,\gamma_{l,m_l-1}M_i)\Bigg]\\&\quad\times\int_0^1\left|g_j^{(m_j-1)}(s,x)\right|\mu_j(x)\,dx\Bigg)\Bigg]\Bigg\}\,ds\end{aligned}$$

$$\geqslant \left[\prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_i, \gamma_{l,1} M_i, \ldots, \gamma_{l,m_l-1} M_i)\right] \int_0^1 \left|g_i^{(k)}(t,s)\right| \mu_i(s)\, ds.$$

Then, (F)$'$ *has a fixed-sign solution* $u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ *satisfying* (3.8) *for a.e.* $t \in [0,1]$ *and any* $1 \leqslant i \leqslant n$.

Proof. Taking $h_{r,i}(t) = \mu_i(t) \prod_{l=1}^{n} \alpha_l(\gamma_{l,0} r, \gamma_{l,1} r, \ldots, \gamma_{l,m_l-1} r)$, the conclusion follows immediately from Theorem 3.3. □

Remark 3.4. Applying Lemma 3.1(a) again, the condition (C12) can be replaced by a stronger but easier-to-verify version:

(C12)$'$ there exists $M_i > 0$ such that for $0 \leqslant k \leqslant m_i - 1$,

$$\begin{aligned} M_i \geqslant \int_0^1 \nu_{ik}(s)\mu_i(s) \Bigg\{ \prod_{j=1}^{n} \Bigg[& \beta_j(\gamma_{j,0} M_i, \gamma_{j,1} M_i, \ldots, \gamma_{j,m_j-1} M_i) \\ & + \alpha_j \Bigg(\gamma_{j,0} \left[\prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_i, \gamma_{l,1} M_i, \ldots, \gamma_{l,m_l-1} M_i)\right] \int_0^1 \left|g_j(s,x)\right| \mu_j(x)\, dx, \\ & \quad \gamma_{j,1} \left[\prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_i, \gamma_{l,1} M_i, \ldots, \gamma_{l,m_l-1} M_i)\right] \\ & \quad \times \int_0^1 \left|g_j'(s,x)\right| \mu_j(x)\, dx, \ldots, \\ & \quad \gamma_{j,m_j-1} \left[\prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_i, \gamma_{l,1} M_i, \ldots, \gamma_{l,m_l-1} M_i)\right] \\ & \quad \times \int_0^1 \left|g_j^{(m_j-1)}(s,x)\right| \mu_j(x)\, dx \Bigg) \Bigg] \Bigg\} ds \end{aligned}$$

and

$$\begin{aligned} \prod_{j=1}^{n} \Bigg\{ & \beta_j(\gamma_{j,0} M_i, \gamma_{j,1} M_i, \ldots, \gamma_{j,m_j-1} M_i) \\ & + \alpha_j \Bigg(\gamma_{j,0} \left[\prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_i, \gamma_{l,1} M_i, \ldots, \gamma_{l,m_l-1} M_i)\right] \int_0^1 \left|g_j(s,x)\right| \mu_j(x)\, dx, \\ & \quad \gamma_{j,1} \left[\prod_{l=1}^{n} \alpha_l(\gamma_{l,0} M_i, \gamma_{l,1} M_i, \ldots, \gamma_{l,m_l-1} M_i)\right] \end{aligned}$$

$$\times \int_0^1 \left|g_j'(s,x)\right| \mu_j(x)\,dx, \ldots,$$

$$\gamma_{j,m_j-1}\left[\prod_{l=1}^n \alpha_l(\gamma_{l,0}M_i, \gamma_{l,1}M_i, \ldots, \gamma_{l,m_l-1}M_i)\right]$$

$$\times \int_0^1 \left|g_j^{(m_j-1)}(s,x)\right| \mu_j(x)\,dx\Bigg)\Bigg\}$$

$$\geqslant \prod_{l=1}^n \alpha_l(\gamma_{l,0}M_i, \gamma_{l,1}M_i, \ldots, \gamma_{l,m_l-1}M_i)$$

for a.e. $s \in [0,1]$.

4. Example

We shall now illustrate our results through an example. Consider the system (F)$'$ where the following are satisfied for each $1 \leqslant i \leqslant n$:

$$\begin{cases} \mu_i : [0,1] \to \mathbb{R}, \quad \mu_i(t) > 0 \quad \text{for a.e. } t \in [0,1], \\ \theta_i = 1, \quad \alpha_i(\tilde{x}_i) = |x_{i,p_i}|^{-a}, \quad \beta_i(\tilde{x}_i) = A_i|x_{i,0}|^b + B_i, \\ \quad a > 0, \quad b \geqslant 0, \quad A_i, B_i \geqslant 0, \end{cases} \tag{4.1}$$

$$\int_0^1 (1-t)^{-a[(m_1+m_2+\cdots+m_n)-(p_1+p_2+\cdots+p_n)-n]} \mu_i(t)\,dt < \infty \tag{4.2}$$

and

$$jna^2 + kb < 1 \quad \text{for all nonnegative integers } j, k \text{ with } j+k = n. \tag{4.3}$$

Note that $\gamma_{j,0} = \gamma_{j,p_j} = 1$, $1 \leqslant j \leqslant n$. Clearly, (C9) is equivalent to (4.2) which is assumed. Moreover, (C11) is satisfied. Finally, the condition (C12)$'$ reduces to

$$M_i \geqslant \int_0^1 \nu_{ik}(s)\mu_i(s) \prod_{j=1}^n \left[A_j M_i^b + B_j + M_i^{na^2}\left(\int_0^1 \left|g_j^{(p_j)}(s,x)\right|\mu_j(x)\,dx\right)^{-a}\right] ds \tag{4.4}$$

and

$$\prod_{j=1}^n \left[A_j M_i^b + B_j + M_i^{na^2}\left(\int_0^1 \left|g_j^{(p_j)}(s,x)\right|\mu_j(x)\,dx\right)^{-a}\right] \geqslant M_i^{-na}. \tag{4.5}$$

The condition (4.3) ensures that (4.4) is satisfied for sufficiently large M_i. Further, (4.5) is also fulfilled for large M_i. Thus, applying Theorem 3.4 the system (F)$'$ with (4.1)–(4.3) has a positive solution $u \in C^{m_1-1}[0,1] \times C^{m_2-1}[0,1] \times \cdots \times C^{m_n-1}[0,1]$ such that for a.e. $t \in [0,1]$ and any $1 \leqslant i \leqslant n$,

$$u_i^{(j)}(t) > 0 \quad \text{for all } j \in N_i^+ \quad \text{and} \quad u_i^{(k)}(t) < 0 \quad \text{for all } k \in N_i^-. \tag{4.6}$$

As a specific example, consider the system

$$\begin{cases} u_1^{(4)}(t) = \mu_1(t)\{|u_1''(t)|^{-a} + A_1|u_1(t)|^b + B_1\}\{|u_2'''(t)|^{-a} + A_2|u_2(t)|^b + B_2\}, \\ \quad \text{a.e. } t \in [0,1], \\ u_2^{(5)}(t) = \mu_2(t)\{|u_1''(t)|^{-a} + A_1|u_1(t)|^b + B_1\}\{|u_2'''(t)|^{-a} + A_2|u_2(t)|^b + B_2\}, \\ \quad \text{a.e. } t \in [0,1], \\ u_1(0) = u_1'(0) = u_1''(1) = u_1'''(1) = 0, \\ u_2(0) = u_2'(0) = u_2''(0) = u_2'''(1) = u_2^{(4)}(1) = 0, \end{cases} \tag{4.7}$$

where for $i = 1, 2$,

$$\mu_i : [0,1] \to \mathbb{R}, \quad \mu_i(t) > 0 \quad \text{for a.e. } t \in [0,1], \quad 0 < a < \frac{1}{2}, \quad 0 \leqslant b < \frac{1}{2}, \quad A_i, B_i \geqslant 0, \tag{4.8}$$

and

$$\int_0^1 (1-t)^{-2a}\mu_i(t)\,dt < \infty. \tag{4.9}$$

In (4.7), we have

$$n = 2, \quad m_1 = 4, \quad p_1 = 2, \quad m_2 = 5, \quad p_2 = 3, \quad \theta_1 = \theta_2 = 1,$$
$$N_1^+ = \{0,1,2\}, \quad N_1^- = \{3\}, \quad N_2^+ = \{0,1,2,3\}, \quad N_2^- = \{4\}.$$

Noting (4.8) and (4.9), the system (4.7) is a special case of (F)$'$ satisfying (4.1) and (4.2). Further, the condition (4.3) in this case is equivalent to

$$4a^2 < 1, \quad 2b < 1, \quad 2a^2 + b < 1,$$

which are clearly satisfied by any $0 < a < \frac{1}{2}$ and $0 \leqslant b < \frac{1}{2}$. Hence, from the above discussion we conclude that the system (4.7) with (4.8) and (4.9) has a positive solution $u \in C^3[0,1] \times C^4[0,1]$ such that (see (4.6))

$$\begin{cases} u_1(t) > 0, \quad u_1'(t) > 0, \quad u_1''(t) > 0, \quad u_1'''(t) < 0 \quad \text{and} \\ u_2(t) > 0, \quad u_2'(t) > 0, \quad u_2''(t) > 0, \quad u_2'''(t) > 0, \quad u_2^{(4)}(t) < 0 \quad \text{for a.e. } t \in [0,1]. \end{cases} \tag{4.10}$$

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References

[1] R.P. Agarwal, Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore, 1986.
[2] R.P. Agarwal, Focal Boundary Value Problems for Differential and Difference Equations, Kluwer, Dordrecht, 1998.
[3] R.P. Agarwal, D. O'Regan, Singular boundary value problems with integrable singularities, Miskolc Math. Notes 2 (2004) 119–127.
[4] R.P. Agarwal, D. O'Regan, I. Rachůnková, S. Staněk, Two-point higher-order BVPs with singularities in phase variables, Comput. Math. Appl. 46 (2003) 1799–1826.

[5] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer, Dordrecht, 1999.
[6] L.E. Bobisud, J.E. Calvert, W.D. Royalty, Some existence results for singular boundary value problems, Differential Integral Equations 6 (1993) 553–571.
[7] K.L. Boey, P.J.Y. Wong, Two-point right focal eigenvalue problems on time scale, Appl. Math. Comput. 167 (2005) 1281–1303.
[8] G. Bonanno, An existence theorem of positive solutions to a singular nonlinear boundary value problem, Comment. Math. Univ. Carolin. 36 (1995) 609–614.
[9] G. Bonanno, Positive solutions to nonlinear singular second-order boundary value problems, Ann. Polon. Math. 64 (1996) 237–251.
[10] W.S. Cheung, J. Ren, Twin positive solutions for quasi-linear multi-point boundary value problems, Nonlinear Anal. 62 (2005) 167–177.
[11] W.S. Cheung, J. Ren, Positive solution for m-point boundary value problems, J. Math. Anal. Appl. 303 (2005) 565–575.
[12] C.J. Chyan, P.J.Y. Wong, Triple solutions of focal boundary value problems on time scale, Comput. Math. Appl. 49 (2005) 963–979.
[13] P.W. Eloe, J. Henderson, Singular nonlinear $(k, n-k)$ conjugate boundary value problems, J. Differential Equations 133 (1997) 136–151.
[14] J. Henderson, W. Yin, Singular $(k, n-k)$ boundary value problems between conjugate and right focal, J. Comput. Appl. Math. 88 (1998) 57–69.
[15] E.K. Lee, Y.H. Lee, Multiple positive solutions of singular two point boundary value problems for second-order impulsive differential equations, Appl. Math. Comput. 158 (2004) 745–759.
[16] I. Rachůnková, S. Staněk, A singular boundary value problem for odd-order differential equations, J. Math. Anal. Appl. 291 (2004) 741–756.
[17] J. Wang, W. Gao, Z. Zhang, Singular nonlinear boundary value problems arising in boundary layer theory, J. Math. Anal. Appl. 233 (1999) 246–256.
[18] Z. Wei, C. Pang, Positive solutions of some singular m-point boundary value problems at non-resonance, Appl. Math. Comput. 171 (2005) 433–449.
[19] P.J.Y. Wong, Positive solutions of difference equations with two-point right focal boundary conditions, J. Math. Anal. Appl. 224 (1998) 34–58.
[20] P.J.Y. Wong, Triple fixed-sign solutions for a system of third-order generalized right focal boundary value problems, J. Math. Anal. Appl., in press.