On the least quadratic non-residue

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Abstract. We prove that for almost all real primitive characters χ_d of modulus $|d|$, the least positive integer n_{χ_d} at which χ_d takes a value not equal to 0 and 1 satisfies $n_{\chi_d} \ll \log |d|$, and give a quite precise estimate on the size of the exceptional set. Also, we generalize Burgess' bound for $n_{\chi_{p'}}$ (with p' being a prime up to $\pm$ sign) to composite modulus $|d|$ and improve Garaev's upper bound for the least quadratic non-residue in Pajtechiĭ-Šapiro's sequence.

§ 1. Introduction

Let $q \geq 2$ be an integer and χ a non principal Dirichlet character modulo q. Here the evaluation of the least integer n_χ among all positive integers n for which $\chi(n) \neq 0, 1$ is referred as Linnik's problem. In case χ coincides with the Legendre symbol, n_χ is a least quadratic non-residue. Concerning the size of n_χ, Pólya-Vinogradov's inequality

$$\max_{x\geq 1} \Big| \sum_{n\leq x} \chi(n) \Big| \ll q^{1/2} \log q \tag{1.1}$$

implies trivially $n_\chi \ll q^{1/2} \log q$. But for prime q, Vinogradov [24] proved the better bound

$$n_\chi \ll q^{1/(2\sqrt{e})} (\log q)^2 \tag{1.2}$$

by combining a simple argument with (1.1). He also conjectured that $n_\chi \ll_\varepsilon q^\varepsilon$ for all integers $q \geq 2$ and any $\varepsilon > 0$. Under the Generalized Riemann Hypothesis (GRH), Linnik [18] settled this conjecture, and later Ankeny [1] gave a sharper estimate

$$n_\chi \ll (\log q)^2 \tag{1.3}$$

(still assuming GRH). Burgess ([3], [4], [5]) wrote a series of important papers on sharpening (1.1). His well known estimate on character sums is as follows: For any $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that

$$\Big| \sum_{n\leq x} \chi(n) \Big| \ll_\varepsilon xq^{-\delta(\varepsilon)} \tag{1.4}$$

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provided $x \ge q^{1/3+\varepsilon}$. The last condition can be improved to $x \ge q^{1/4+\varepsilon}$ if q is cubefree. When q is prime, he deduced, via (1.4) and Vinogradov's argument,

$$n_\chi \ll_\varepsilon q^{1/(4\sqrt{e})+\varepsilon}. \tag{1.5}$$

Since Burgess' estimate (1.4) on character sums holds for composite modulus, one expects a bound analogous to (1.5) for n_χ in general cases, but this seems not available in literature. Our first result is to propose such a generalisation, by modifying Vinogradov's argument.

Theorem 1. *Let ε be an arbitrarily small positive number. For all integers $q \ge 2$ and χ non principal characters* $(\operatorname{mod} q)$, *we have*

$$n_\chi \ll_\varepsilon \begin{cases} q^{1/(4\sqrt{e})+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/(3\sqrt{e})+\varepsilon} & \text{otherwise.} \end{cases}$$

The proof of Theorem 1 will be given in the Section 2.

Let us now focus on real primitive characters. Denote $\mathcal{D}$ (resp. $\mathcal{D}(Q)$) to be the set of fundamental discriminants d (resp. with $|d| \le Q$), that is, the set of non-zero integers d which are products of coprime factors of the form -4, 8, -8, p' where $p' := (-1)^{(p-1)/2}p$ (p odd prime). Also, we write $\mathcal{K}$ (resp. $\mathcal{K}(Q)$) for the set of real primitive characters (resp. with modulus $q \le Q$). Then there is a bijection between $\mathcal{D}$ and $\mathcal{K}$ given by

$$d \mapsto \chi_d(\cdot) = \left(\frac{d}{\cdot}\right)_K$$

where $\left(\frac{d}{\cdot}\right)_K$ is the Kronecker symbol. Note that the modulus of χ_d equals $|d|$ and

$$|\mathcal{D}(Q)| = |\mathcal{K}(Q)| = \frac{6}{\pi^2}Q + O(Q^{1/2}). \tag{1.6}$$

In the opposite direction of (1.2), Fridlender [12], Salié [23] and Chowla & Turán (see [10]) independently showed that there are infinitely many primes p for which

$$n_{\chi_{p'}} \gg \log p, \tag{1.7}$$

or in other words, $n_{\chi_{p'}} = \Omega(\log p)$. Under GRH, Montgomery [20] gave a stronger result $n_{\chi_{p'}} = \Omega(\log p \log_2 p)$, where $\log_k$ denotes the k-fold iterated logarithm. Without any assumption Graham & Ringrose [14] obtained $n_{\chi_{p'}} = \Omega(\log p \log_3 p)$. In view of these results, it is natural to wonder what is the size of the majority of $n_{\chi_{p'}}$, or more generally n_{χ_d}. Indeed the density of p' for which $n_{\chi_{p'}}$ satisfies (1.7) is low. This can be seen from Erdős' result [11],

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} n_{\chi_{p'}} = \text{constant}, \tag{1.8}$$

where $\pi(x)$ denotes the number of primes up to x. This result is extended and refined by Elliott in [7] and [8]. Using (1.8) or its refinement in [7], it follows, for any fixed constant $\delta > 0$, that

$$\sum_{p\le x,\, n_{\chi_{p'}} \ge \delta \log p} 1 \ll_\delta \frac{x}{(\log x)^2}. \tag{1.9}$$

In [6], Duke & Kowalski indicated: Let $\alpha > 1$ be given. Denote by $N(Q,\alpha)$ the number of primitive characters χ (not necessarily real) of modulus $q \le Q$ such that $\chi(n) = 1$ for all $n \le (\log Q)^\alpha$ and $(n,q) = 1$. Then one has

$$N(Q,\alpha) \ll_\varepsilon Q^{2/\alpha+\varepsilon}$$

for all $\varepsilon > 0$. Therefore

$$|\{|d| \le Q : \ n_{\chi_d} \ge (\log Q)^\alpha\}| \ll_\varepsilon Q^{2/\alpha+\varepsilon}.$$

However, in view of (1.6) this result is non-trivial only when $\alpha > 2$ and it tells that $n_{\chi_d} \ge (\log|d|)^{2+\varepsilon}$ for almost all fundamental discriminants d. Very recently Baier [2] improved $2+\varepsilon$ to $1+\varepsilon$ by using the large sieve inequality of Heath-Brown [15] for real primitive characters. However, the argument is unable to cover the case $\alpha = 1$ or to provide information on the sparsity of the primes p with $n_{\chi_{p'}} \gg \log p$ as in (1.9).

Our second result is to supplement the case $\alpha = 1$, using the large sieve inequality of Elliott-Montgomery-Vaughan (see [9] and [21]). We obtain an almost all result, which is strong enough to yield a tighter estimate on the low density of exceptional non-residues than in (1.9).

Theorem 2. *For $2 \le P \le Q$, define*

$$\mathcal{E}(Q,P) := \{d \in \mathcal{D}(Q) : \chi_d(p) = 1 \ \text{ for } \ P < p \le 2P \ \text{ and } \ p \nmid |d|\}.$$

Then there are two absolute positive constants C and c such that

$$|\mathcal{E}(Q,P)| \ll Qe^{-c(\log Q)/\log_2 Q} \tag{1.10}$$

holds uniformly for $Q \ge 10$ and $C\log Q \le P \le (\log Q)^2$. In particular we have

$$n_{\chi_d} \ll \log|d| \tag{1.11}$$

for all but except $O\big(Qe^{-c(\log Q)/\log_2 Q}\big)$ characters $\chi_d \in \mathcal{K}(Q)$.

Sections 3 and e are devoted to the proof of Theorem 2.

Theorem 3 (essentially due to Graham & Ringrose [14]) shows that the upper bound for exceptional real primitive characters set is optimal. Graham & Ringrose considered a problem of the quasi-random graphs (Paley graphs) which leads to study the lower bound for the sum of the right-hand side of (6.5) below. This will also be the essential part of our proof of Theorem 3. We shall provide the salient points along the line of arguments in [14] to prove Theorem 3, see Sections 5 and 6.

Theorem 3. *For any fixed constant $\delta > 0$, there are a sequence of positive real numbers $\{Q_n\}_{n=1}^\infty$ with $Q_n \to \infty$ and a positive constant c such that*

$$\sum_{\substack{Q_n^{1/2} < p \le Q_n \\ n_{\chi_p} \ge \delta \log p}} 1 \gg_\delta Q_n e^{-c(\log Q_n)/\log_2 Q_n}. \tag{1.12}$$

Further if we assume that both $\mathbf{L}_1(s,P_y)$ and $\mathbf{L}_4(s,P_y)$ defined in (5.3) below have no exceptional zeros in the region (5.4), then (1.12) holds for all $Q \ge 10$.

Finally we consider the least quadratic non-residue problem in Pajtechiĭ-Šapiro's sequence $\{[n^c]\}_{n=1}^\infty$, where $c > 1$ is a constant and $[t]$ denotes the integral part of $t \in \mathbb{R}$. Denote by $n_{\chi_{p'},c}$

the least positive integer n such that $[n^c]$ is a quadratic non-residue $(\bmod p)$. Garaev [13] proved that for $1 < c < \frac{12}{11}$ and any $\varepsilon > 0$, one has

$$n_{\chi_{p'},c} \ll_{c,\varepsilon} p^{3/(8(3-2c)\sqrt{e})+\varepsilon} \tag{1.13}$$

for all primes p. He pointed out also that by the method of exponential pairs the range of c and the exponent of p can be improved to $1 < c < \frac{12}{11} + 0.00257\cdots$ and $1/(8(1-\theta_2 c)\sqrt{e})$, respectively, where $\theta_2 = 0.66451\cdots$. Here we propose a further improvement by applying a recent result of Robert & Sargos [22], and give an almost result based on Theorem 2.

Theorem 4. *Let $1 < c < \frac{32}{29}$. Then for all primes p and any $\varepsilon > 0$, we have*

$$n_{\chi_{p'},c} \ll_{c,\varepsilon} p^{9/((64-40c)\sqrt{e})+\varepsilon}.$$

For all but except $O\big(Qe^{-c(\log Q)/\log_2 Q}\big)$ primes p with $p \le Q$, we have

$$n_{\chi_{p'},c} \ll_{c,\varepsilon} (\log p)^{9/(16-10c)+\varepsilon}.$$

We prove Theorem 4 in Section 7.

Our range of c is larger than $\frac{12}{11} + 0.00257\cdots$ ($\frac{32}{29} = \frac{12}{11} + 0.01253\cdots$.) and our exponent is definitely better than (1.13) but is smaller than $1/(8(1-\theta_2 c)\sqrt{e})$ only when $c > 1/(9\theta_2 - 5) = 1.019794\cdots$. It is possible to give a slightly better result with Huxley's estimates for exponential sums [16, § 18.5]. We can also generalize Theorem 4 to composite modulus $|d|$ as in Theorem 1, but with smaller range of c and larger exponent of $|d|$.

§ 2. Vinogradov's argument and proof of Theorem 1

Without loss of generality we assume $n_\chi \ge q^{1/(4\sqrt{e})}$ (otherwise there is nothing to prove). Let x be a number specified later but satisfy

$$q > x \ge \begin{cases} q^{1/4+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/3+\varepsilon} & \text{otherwise.} \end{cases}$$

By Burgess' well known estimate (1.4) on character sums, for any $\varepsilon > 0$ there are two positive constants C_ε and $\delta(\varepsilon) > 0$ such that

$$\begin{aligned} C_\varepsilon x q^{-\delta(\varepsilon)} &\ge \Big|\sum_{n\le x} \chi(n)\Big| \\ &\ge \sum_{\substack{n\le x\\(n,q)=1}} 1 - 2 \sum_{\substack{n\le x\\(n,q)=1,\, \chi(n)\ne 1}} 1 \\ &\ge \sum_{\substack{n\le x\\(n,q)=1}} 1 - 2 \sum_{n_\chi < p \le x} \sum_{\substack{m\le x/p\\(m,q)=1}} 1. \end{aligned} \tag{2.1}$$

As usual we denote by $\varphi(n)$ the Euler function, $\mu(n)$ the Möbius function and $\omega(n)$ the number of distinct prime factors of n. With the Möbius inversion formula, we have, for some $|\theta| \le 1$,

$$\sum_{\substack{n\le x\\(n,q)=1}} 1 = \sum_{d|q} \mu(d) \sum_{m\le x/d} 1 = \frac{\varphi(q)}{q} x + \theta 2^{\omega(q)}. \tag{2.2}$$

To estimate the last double sum on the right-hand side of (2.1), we divide the sum over p into two parts according as $n_\chi < p \le x/2^{\omega(q)}$ or $x/2^{\omega(q)} < p \le x$. By (2.2), the first part contributes at most

$$\begin{aligned}(2.3)\qquad & \sum_{n_\chi < p \le x/2^{\omega(q)}} \left(\frac{\varphi(q)}{q}\frac{x}{p} + 2^{\omega(q)}\right) \\ & \le \frac{\varphi(q)}{q}x\left\{\log\left(\frac{\log x}{\log n_\chi}\right) + O\left(e^{-\sqrt{\log n_\chi}}\right)\right\} + \frac{(1+\varepsilon)x}{\log(x2^{-\omega(q)})} \\ & \le \frac{\varphi(q)}{q}x\log\left(\frac{\log x}{\log n_\chi}\right) + (1+2\varepsilon)\frac{x}{\log x}.\end{aligned}$$

Note that $2^{\omega(q)} \ll x^\varepsilon$ and $n_\chi \ge q^{1/(4\sqrt{e})}$. For the second part, we interchange the summations and apply the Rankin trick,

$$\begin{aligned}\sum_{x/2^{\omega(q)} < p \le x}\ \sum_{\substack{m \le x/p \\ (m,q)=1}} 1 &\le \sum_{\substack{1 \le m \le 2^{\omega(q)} \\ (m,q)=1}}\ \sum_{p \le x/m} 1 \\ &\ll \frac{x}{\log x}\sum_{\substack{1 \le m \le 2^{\omega(q)} \\ (m,q)=1}} \frac{1}{m} \\ &\le \frac{x}{\log x}\prod_{\substack{p \le 2^{\omega(q)} \\ (p,q)=1}}\left(1-\frac{1}{p}\right)^{-1} \\ &= \frac{\varphi(q)}{q}\frac{x}{\log x}\prod_{\substack{p > 2^{\omega(q)} \\ p \mid q}}\left(1-\frac{1}{p}\right)^{-1} \times \prod_{p \le 2^{\omega(q)}}\left(1-\frac{1}{p}\right)^{-1}.\end{aligned}$$

In virtue of the simple estimates

$$\begin{aligned}&\prod_{\substack{p > 2^{\omega(q)} \\ p \mid q}}\left(1-\frac{1}{p}\right)^{-1} \ll \exp\left\{\sum_{\substack{p > 2^{\omega(q)} \\ p \mid q}}\frac{1}{p}\right\} \ll \exp\left\{\frac{\omega(q)}{2^{\omega(q)}}\right\} \ll 1, \\ &\prod_{p \le 2^{\omega(q)}}\left(1-\frac{1}{p}\right)^{-1} \ll \exp\left\{\sum_{p \le 2^{\omega(q)}}\frac{1}{p}\right\} \ll \omega(q),\end{aligned}$$

it follows immediately that

$$(2.4)\qquad \sum_{x/2^{\omega(q)} < p \le x}\ \sum_{\substack{m \le x/p \\ (m,q)=1}} 1 \ll \frac{\varphi(q)}{q}x\frac{\omega(q)}{\log x}.$$

Inserting (2.2), (2.3) and (2.4) into (2.1), we conclude

$$C_\varepsilon x q^{-\delta(\varepsilon)} \ge \frac{\varphi(q)}{q}x\left\{1 - 2\log\left(\frac{\log x}{\log n_\chi}\right)\right\} - 2^{\omega(q)} - (1+2\varepsilon)\frac{x}{\log x} - C_\varepsilon\frac{\varphi(q)}{q}x\frac{\omega(q)}{\log x}.$$

¿From this we deduce that

$$\begin{aligned}\log\left(\frac{\log x}{\log n_\chi}\right) &\ge \frac{1}{2} - \frac{C_\varepsilon}{2}\frac{q^{1-\delta(\varepsilon)}}{\varphi(q)} - \frac{(1/2+\varepsilon)q}{\varphi(q)\log x} - \frac{C_\varepsilon}{2}\frac{\omega(q)}{\log x} \\ &\ge \frac{1}{2} - C_\varepsilon\left(\frac{q}{\varphi(q)\log x} + \frac{\omega(q)}{\log x}\right)\end{aligned}$$

provided $q \geq q_0(\varepsilon)$. Since $q/\varphi(q)\log x + \omega(q)/\log x \ll (\log_2 q)^{-1}$, the preceeding inequality implies

$$n_\chi \ll x^{1/\sqrt{e}} \exp\left\{O\left(\frac{q}{\varphi(q)} + \omega(q)\right)\right\},$$

which gives the required result, by taking

$$x = \begin{cases} q^{1/4+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/3+\varepsilon} & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem 1. □

§ 3. A large sieve inequality of Montgomery-Vaughan

Our key tool for proving Theorem 2 is a large sieve inequality of Montgomery & Vaughan in [21, page 1050] following from [21, Lemma 2]. Here we state a slightly refined version. Their original statement absorbs the factors $(6/\log P)^j$ and $\{6/(\log P)^2\}^j$ in the implied constant. We reproduce here their proof with a minuscule modification.

Lemma 1. *We have*

$$\sum_{d\in\mathcal{D}(Q)} \bigg|\sum_{P<p\leq 2P} \frac{\chi_d(p)}{p}\bigg|^{2j} \ll Q\left(\frac{6j}{P\log P}\right)^j + \left(\frac{6P}{(\log P)^2}\right)^j \tag{3.1}$$

uniformly for $2 \leq P \leq Q$ *and* $j \geq 1$. *The implied constant is absolute.*

Proof. Since $\chi_d(n)$ is completely multiplicative on n, we can write

$$\bigg(\sum_{P<p\leq 2P} \frac{\chi_d(p)}{p}\bigg)^j = \sum_{P^j<m\leq(2P)^j} \frac{a_j(m)}{m}\chi_d(m),$$

where

$$a_j(m) := |\{(p_1,\dots,p_j) : p_1\cdots p_j = m,\ P < p_i \leq 2P\}|.$$

By Lemma 2 of [21] with the choice of parameters

$$X = P^j, \qquad Y = (2P)^j \qquad \text{and} \qquad a_m = a_j(m)/m,$$

it follows that as $a_j(m_1)a_j(m_2) \leq a_{2j}(n^2)$ for $n^2 = m_1m_2$,

$$\sum_{d\in\mathcal{D}(Q)} \bigg|\sum_{P<p\leq 2P} \frac{\chi_d(p)}{p}\bigg|^{2j} \ll Q \sum_{P^j<n\leq(2P)^j} \frac{a_{2j}(n^2)}{n^2} + \bigg(\sum_{P<p\leq 2P} \frac{1}{p^{1/2}}\bigg)^{2j}. \tag{3.2}$$

Writing $n = p_1^{\nu_1}\cdots p_i^{\nu_i}$ with $\nu_1 + \cdots + \nu_i = j$, we have

$$\begin{aligned} a_{2j}(n^2) &= \frac{(2j)!}{(2\nu_1)!\cdots(2\nu_i)!} \\ &= \frac{(2j)!}{j!}\frac{\nu_1!}{(2\nu_1)!}\cdots\frac{\nu_i!}{(2\nu_i)!}a_j(n). \end{aligned}$$

¿From this, it is easy to see $a_{2j}(n^2) \le j^j a_j(n)$, and thus

$$\begin{aligned}\sum_{P^j<n\le(2P)^j} \frac{a_{2j}(n^2)}{n^2} &\le j^j \sum_{P^j<n\le(2P)^j} \frac{a_j(n)}{n^2}\\ &= \bigg(j\sum_{P<p\le 2P}\frac{1}{p^2}\bigg)^j\\ &\le \bigg(\frac{6j}{P\log P}\bigg)^j.\end{aligned}$$

Inserting this into (3.2) and using the estimate

$$\sum_{P<p\le 2P}\frac{1}{p^{1/2}} \le \frac{6P^{1/2}}{\log P},$$

we obtain the required result (3.1). □

§ 4. Proof of Theorem 2

Define

$$\mathcal{E}^*(Q,P) := \big\{d\in\mathcal{D}(Q) : Q^{1/2}\le |d|\le Q \ \text{ and } \ \chi_d(p)=1 \ \ (P<p\le 2P,\ p\nmid |d|)\big\}.$$

Let $C\log Q\le P\le(\log Q)^2$. For $d\in\mathcal{E}^*(Q,P)$, we invoke the prime number theorem to deduce

$$\begin{aligned}\sum_{P<p\le 2P}\frac{\chi_d(p)}{p} &= \sum_{P<p\le 2P}\frac{1}{p} - \sum_{P<p\le 2P,\ p||d|}\frac{1}{p}\\ &\ge \frac{\log 2+o(1)}{\log P} - \frac{\{1+o(1)\}\log Q}{P\log_2 Q}\\ &\ge \frac{\log 2 - 2/C + o(1)}{\log P}\\ &> \frac{1}{2\log P},\end{aligned}$$

provided C is sufficiently large. It is apparent from (3.1) that

$$\begin{aligned}\frac{|\mathcal{E}^*(Q,P)|}{(2\log P)^{2j}} &\le \sum_{d\in\mathcal{D}(Q)}\bigg|\sum_{P<p\le 2P}\frac{\chi_d(p)}{p}\bigg|^{2j}\\ &\ll Q\bigg(\frac{6j}{P\log P}\bigg)^j + \bigg(\frac{6P}{(\log P)^2}\bigg)^j.\end{aligned}$$

Hence we obtain

$$|\mathcal{E}^*(Q,P)| \ll Q(12j\log P/P)^j + (12P)^j$$

uniformly for $C\log Q\le P\le(\log Q)^2$ and $j\ge 1$. Taking

$$j = \bigg[\frac{\log Q}{48\log P}\bigg]+1,$$

a simple calculation shows that

$$|\mathcal{E}^*(Q,P)| \ll Qe^{-c(\log Q)/\log_2 Q}$$

with $c=(\log 2)/48$. This implies (1.10).

Finally let

$$\mathcal{E}^*(Q) := \big\{d\in\mathcal{D}(Q) : d\le Q^{1/2}\big\}\cup\mathcal{E}^*(Q, C\log Q).$$

Then by (1.10), we have

$$|\mathcal{E}^*(Q)| \ll Qe^{-c(\log Q)/\log_2 Q};$$

and for any $d\in\mathcal{D}(Q)\smallsetminus\mathcal{E}^*(Q)$ there is a prime number $p\asymp\log Q\asymp\log|d|$ such that $\chi_d(p)\ne 1$, which implies (1.11). The proof is complete.

§ 5. Graham-Ringrose's method

In this section, we shall state and extend the main results of ([14], Theorems 2, 3 and 4) for our purposes. For characters of certain moduli, Graham & Ringrose [14] obtained a wide zero-free region and good zero density estimates for the corresponding Dirichlet L-functions. The main ingredient of their method is an q-analogue of van der Corput's result, which can be stated as follows: *Suppose that $q=2^\nu r$, where $0\le\nu\le 3$ and r is an odd squarefree integer, and that χ is a non-prinicipal charater* mod q. *Let p be the largest prime factor of q. Suppose that k is a non-negative integer, and $K=2^k$. Finally, assume that $N\le M$. Then*

$$\sum_{M<n\le M+N}\chi(n) \ll M^{1-\frac{k+3}{8K-2}}p^{\frac{k^2+3k+4}{32K-8}}q^{\frac{1}{8K-2}}d(q)^{\frac{32k^2+11k+8}{16K-4}}(\log q)^{\frac{k+3}{8K-2}}\sigma_{-1}(q), \tag{5.1}$$

where $\sigma_a(q) := \sum_{d\mid q} d^a$ and $d(q):=\sigma_0(q)$. The implied constant is absolute.

Recall that for any odd prime p,

$$\chi_8(p) = \Big(\frac{2}{p}\Big),\qquad \chi_{q'}(p) = \Big(\frac{q}{p}\Big)_K = \Big(\frac{q}{p}\Big)\quad (q \text{ odd prime, } q' := (-1)^{(q-1)/2}q)$$

by definition. For squarefree $m\ge 2$, the character $\chi_m := \prod_{p\mid m}\chi_{p'}$ for odd m or $\chi_m := \chi_8\chi_{m'}$ for $m=2m'$ is a real primitive of modulus m or $4m$, respectively. By convention, we set $\chi_1\equiv 1$. Moreover, if χ_4 is the real primitive character mod 4, i.e. $\chi_4(n)=(-1)^{(n-1)/2}$ for odd n, then $\chi_{4m} := \chi_4\chi_m$ is also a real primitive character of modulus $4m$.

Let

$$P_y := \prod_{p\le y} p = e^{\{1+o(1)\}y}\qquad (y\to\infty), \tag{5.2}$$

and define for $\ell = 1$ or 4,

$$\mathbf{L}_\ell(s,P_y) := \prod_{m\mid P_y} L(s,\chi_{\ell m}), \tag{5.3}$$

where $L(s,\chi_{\ell m})$ is the Dirichlet L-function associated to $\chi_{\ell m}$. Denote by $N_\ell(\alpha)$ the number of zeros of $\mathbf{L}_\ell(s,P_y)$ in the rectangle

$$\alpha\le\sigma\le 1\qquad\text{and}\qquad |\tau|\le\log P_y.$$

Here and in the sequel we implicitly define the real numbers σ and τ by the relation $s=\sigma+i\tau$.

The next lemmas 2, 3 and 4 are trivial extensions of Theorems 2, 3 and 4 of [14], respectively.

Lemma 2. *Let $y \geq 100$. Then there is an absolute positive constant C_1 such that the L-function $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_y)$ has at most one zero in the region*

$$\sigma \geq 1 - \frac{C_1(\log_2 P_y)^{1/2}}{\log P_y} \qquad \textit{and} \qquad |\tau| \leq \log P_y. \tag{5.4}$$

The exceptional zero, if exists, is real.

Proof. As the crucial estimate (5.1) holds for all non-principal primitive characters of modulus $q = 2^\nu r \geq 2$ with $0 \leq \nu \leq 3$ and r being odd squarefree. Consider the case $\nu = 0$ or 3, and $\nu = 2$ or 3, respectively. We see that (5.1) applies to χ_m and χ_{4m} for any $m|P_y$. It follows that [14, Lemma 6.1] is valid for these characters. Proceeding with the same argument, we have [14, Lemma 6.2] for our L-function $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_y)$ in place of $\mathbf{L}(s, P_y)$ there. Then the same proof of [14, Theorem 2] will give the desired result. (Note that the value of ϕ suffers a negligible change when P_y is replaced by $4P_y$ or $8P_y$.) The exceptional zero must be real, for otherwise, its conjugate is another zero in the specified region. □

Lemma 3. *Let C_1 be as in Lemma 2. There is a sequence of positive real numbers $\{y_n\}_{n=1}^\infty$ with $y_n \to \infty$ such that both $\mathbf{L}_1(s, P_{y_n})$ and $\mathbf{L}_4(s, P_{y_n})$ have no zeros in the region*

$$\sigma \geq 1 - \eta(y_n) \qquad \textit{and} \qquad |\tau| \leq \log P_{y_n}, \tag{5.5}$$

where

$$\eta(y) := \frac{C_1(\log_2 P_y)^{1/2}}{2 \log P_y}.$$

Proof. Similar to [14, Theorem 3], our proof is also based on an interesting argument attributed to Maier [19]. Suppose that for some y, the product $\mathbf{L}_1(s, P_y)\mathbf{L}_4(s, P_y)$ has an exceptional zero in the region (5.4). That is, it has a real zero $\beta > 1 - 2\eta(y)$. In view of (5.2), we can take $y_n \geq y$ such that

$$\eta(y_n) < 1 - \beta < 2\eta(y_n). \tag{5.6}$$

By Lemma 2, β is the only exceptional zero of $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_{y_n})$ in the region

$$\sigma > 1 - 2\eta(y_n) \qquad \text{and} \qquad |\tau| \leq \log P_{y_n}.$$

Together with the first inequality in (5.6), this forces $\prod_{\ell=1,4} \mathbf{L}_\ell(s, P_{y_n})$ to have no zero in the region (5.5). It follows that we can find a sequence of positive real numbers $\{y_n\}_{n=1}^\infty$ with $y_n \to \infty$ such that both $\mathbf{L}_1(s, P_{y_n})$ and $\mathbf{L}_4(s, P_{y_n})$ have no zero in this region. □

Lemma 4. *Let $\ell = 1$ or 4 and $y \geq 100$. Then there is an absolute constant C_2 such that*

$$N_\ell(\alpha) \ll \begin{cases} \exp\left\{\dfrac{C_2(1-\alpha)\log P_y}{\sqrt{\log_2 P_y}} + \dfrac{\log_3 P_y}{2}\right\} & \textit{if } \alpha \geq 1 - \eta_1(y), \\ \exp\left\{\dfrac{C_2(1-\alpha)\log P_y}{\log(1/(1-\alpha))}\right\} & \textit{if } \alpha < 1 - \eta_1(y), \end{cases} \tag{5.7}$$

where

$$k_0(y) := \left[(\log_2 P_y)^{1/2}\right] \qquad \text{and} \qquad \eta_1(y) := \frac{k_0(y)}{2(2^{k_0(y)} - 2)}.$$

Proof. The case of $\ell = 1$ has been done in [14, Sections 7 and 8] and $N_4(\alpha)$ can be treated in the same way by applying (5.1) to our χ_{4m}. $\square$

§ 6. Proof of Theorem 3

In this section, we denote by p and q prime numbers. Define

$$\mathbb{P}_y := \{p : p \equiv 1 \,(\mathrm{mod}\, 4) \text{ and } \chi_p(q) = 1 \text{ for all } q \le y\}.$$

Clearly we have $n_{\chi_p} > y$ for any $p \in \mathbb{P}_y$. We shall first show that the set $\mathbb{P}_y$ is not too small for suitable y.

Proposition. *Let $\delta > 0$ be a fixed small constant and $y(x)$ be an strictly increasing function defined on $[120, \infty)$ satisfying*

$$(\log x)e^{-\delta(\log_2 x)^{1/2}} \le y(x) \le \delta(\log x)\log_3 x. \tag{6.1}$$

Then there are a positive constant $c = c(\delta)$ and a sequence of positive real numbers $\{x_n\}_{n=1}^{\infty}$ with $x_n \to \infty$ such that

$$\sum_{\substack{x_n^{1/2} < p \le x_n \log x_n \\ p \in \mathbb{P}_{y(x_n)}}} 1 \gg x_n e^{-cy(x_n)/\log y(x_n)}. \tag{6.2}$$

Further if we assume that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ have no zeros in the region (5.4) for all $y \ge 100$, then there is a positive constant c such that for all $x \ge 100$ we have

$$\sum_{\substack{x^{1/2} < p \le x \log x \\ p \in \mathbb{P}_y}} 1 \gg x\, e^{-cy(x)/\log y(x)}. \tag{6.3}$$

Proof. First let $10 \le y \le x^{1/2}$. As usual, $\pi(y)$ denotes the number of prime numbers $\le y$. Clearly we have

$$2^{-\pi(y)-1}\big(1 + \chi_4(p)\big)\prod_{q \le y}\big(1 + \chi_p(q)\big) = \begin{cases} 1 & \text{if } p \in \mathbb{P}_y, \\ 0 & \text{if } p \notin \mathbb{P}_y. \end{cases} \tag{6.4}$$

When p and q are odd primes with $p \equiv 1 \,(\mathrm{mod}\, 4)$, i.e. $\chi_4(p) = 1$, we infer by quadratic reciprocity law that

$$\chi_p(q) = \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = \chi_{q'}(p) \quad (q' := (-1)^{(q-1)/2}q).$$

Note also for odd prime p,

$$\chi_p(2) = \left(\frac{p}{2}\right)_K = \left(\frac{2}{p}\right) = \chi_8(p).$$

Thus we can replace $\chi_p(q)$ by $\left(\frac{q}{p}\right)$ in (6.4) to write

$$\sum_{\substack{x^{1/2} < p \le x \log x \\ p \in \mathbb{P}_y}} 1 = \frac{1}{2^{\pi(y)+1}} \sum_{x^{1/2} < p \le x \log x} \big(1 + \chi_4(p)\big)\prod_{q \le y}\left(1 + \left(\frac{q}{p}\right)\right).$$

It is convenient to introduce the weight factor $(\log p)\left(e^{-p/(2x)} - e^{-p/x}\right)$ to the summands,

$$\sum_{\substack{x^{1/2}<p\le x\log x\\ p\in\mathbb{P}_y}} 1 \ge \frac{1}{2^{\pi(y)+2}\log x}\sum_{x^{1/2}<p\le x\log x}(\log p)\left(e^{-p/(2x)} - e^{-p/x}\right)\times$$
$$\times\left(1+\chi_4(p)\right)\prod_{q\le y}\left(1+\left(\frac{q}{p}\right)\right).$$

We want to relax the range of the sum over p. To this end, we observe that by the prime number theorem and integration by parts,

$$\begin{aligned}\frac{1}{2^{\pi(y)}\log x}&\sum_{x\log x<p\le x^2}(\log p)\big(e^{-p/(2x)} - e^{-p/x}\big)\big(1+\chi_4(p)\big)\prod_{q\le y}\left(1+\left(\frac{q}{p}\right)\right)\\ &\ll \sum_{x\log x<p\le x^2}\big(e^{-p/(2x)} - e^{-p/x}\big)\\ &\ll x^{1/2}/\log x.\end{aligned}$$

Combining this with the preceeding inequality, we obtain

$$\sum_{\substack{x^{1/2}<p\le x\log x\\ p\in\mathbb{P}_y}} 1 \ge \frac{1}{(\log x)2^{\pi(y)+2}}\sum_{m\mid P_y}\big(S_x(m)+S_x(4m)\big)+O\bigg(\frac{x^{1/2}}{\log x}\bigg), \tag{6.5}$$

where $\ell = 1$ or 4, and

$$S_x(\ell m) := \sum_{x^{1/2}<p\le x^2}(\log p)\big(e^{-p/(2x)} - e^{-p/x}\big)\chi_{\ell m}(p).$$

By the Perron formula, we can write

$$S_x(\ell m) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s,\chi_{\ell m})(2^s-1)\Gamma(s)x^s\,\mathrm{d}s + O\big(x^{1/2}\log x\big). \tag{6.6}$$

We shift the line of integration to $\sigma = -\frac{3}{4}$. The function $(2^s-1)\Gamma(s)x^s$ has no pole in the strip $-\frac{3}{4}\le\sigma\le 2$ since the pole of $\Gamma(s)$ at $s=0$ is canceled by the zero of (2^s-1). Thus the only poles of the integrand in (6.6) occur at $s=1$ if $\ell m = 1$ (note that $L(s,\chi_1)$ is the Riemann ζ-function), or at the zeros $\rho(\ell m) = \beta(\ell m) + i\gamma(\ell m)$ of $L(s,\chi_{\ell m})$. It follows that

$$S_x(\ell m) = \delta_{\ell m,1}x - \sum_{\rho(\ell m)}(2^{\rho(\ell m)}-1)\Gamma(\rho(\ell m))x^{\rho(\ell m)} + O\big(x^{1/2}\log x\big),$$

where $\delta_{j,1} = 1$ if $j=1$ and 0 otherwise, and the sum is over all zeros with $0\le\beta(\ell m)<1$.

We write $N(T,\chi_{\ell m})$ for the number of zeros of $L(s,\chi_{\ell m})$ in the rectangle $0<\beta(\ell m)<1$ and $|\gamma|\le T$. Then we have the classical bound

$$N(T,\chi_{\ell m}) \ll T\log(Tm), \tag{6.7}$$

which implies, for any $\alpha\in(0,1)$,

$$N_\ell(\alpha) \le \sum_{m\mid P_y} N(\log P_y,\chi_{\ell m}) \ll 2^{\pi(y)}y^2. \tag{6.8}$$

On the other hand, by means of $(2^s-1)\Gamma(s)x^s \ll x^\sigma|\tau|e^{-(\pi/2)|\tau|}$, the contribution of the zeros with $|\gamma(\ell m)| \geq \log P_y$ to $S_x(\ell m)$ is $\ll 1$. Let ε be an arbitrarily small positive number. The zeros with $\beta(\ell m) \leq 1-\varepsilon$ and $|\gamma(\ell m)| \leq \log P_y$ contribute

$$\ll x^{1-\varepsilon} N(\log P_y, \chi_{\ell m}) \ll x^{1-\varepsilon}(\log P_y)^2 \ll x^{1-\varepsilon} y^2.$$

Combining these with (6.5), we conclude

$$\sum_{\substack{x^{1/2}<p\leq x\log x\\ p\in\mathbb{P}_y}} 1 \geq \frac{x}{(\log x)2^{\pi(y)+2}} + O\biggl(x^{1-\varepsilon}2^{\pi(y)}y^2 + \frac{T_1(x,y)+T_4(x,y)}{(\log x)2^{\pi(y)}}\biggr) \tag{6.9}$$

uniformly for $x \geq 10$ and $1 \leq y \leq x^{1/2}$, where

$$\begin{aligned} T_\ell(x,y) &:= \sum_{m|P_y} \sum_{\substack{\rho(\ell m)\\ \beta(\ell m)\geq 1-\varepsilon,\ |\gamma(\ell m)|\leq \log P_y}} x^{\beta(\ell m)} \\ &= -\int_{1-\varepsilon}^1 x^\alpha \,\mathrm{d}N_\ell(\alpha). \end{aligned}$$

It remains to estimate $T_\ell(x,y)$. ¿From now on we take $y=y(x)$. By integration by parts and by using (6.8), we can deduce

$$T_\ell(x,y) \ll x^{1-\varepsilon}2^{\pi(y)}y^2 + x(\log x)I_\ell, \tag{6.10}$$

where

$$I_\ell := \int_0^\varepsilon x^{-\beta} N_\ell(1-\beta)\,\mathrm{d}\beta.$$

Let $\eta=\eta(y)$ and $\eta_1=\eta_1(y)$ be defined as in Lemmas 3 and 4, respectively. Set $\eta_2 := 2y(x)/(\log x)\log y$. It is easy to verify that $0<\eta<\eta_1<\eta_2<\varepsilon$. (The inequality $\eta_1<\eta_2$ governs the lower bound of $y(x)$ in (6.1).) Thus we can divide the interval $[0,\varepsilon]$ into four subintervals $[0,\eta]$, $[\eta,\eta_1]$, $[\eta_1,\eta_2]$ and $[\eta_2,\varepsilon]$, and denote by $I_{\ell,0}$, $I_{\ell,1}$, $I_{\ell,2}$ and $I_{\ell,3}$ the corresponding contribution to I_ℓ. Plainly we have

$$\frac12\log_3 P_y \leq \frac{\eta}{4}\log x, \quad \frac{C_2\log P_y}{\sqrt{\log_2 P_y}} \leq \frac14\log x, \quad \frac{C_2\log P_y}{\log(1/\eta_2)} \leq \frac12\log x, \quad \frac{y}{\log y} = \frac{\eta_2}{2}\log x.$$

(The third inequality governs the upper bound of $y(x)$ in (6.1).) ¿From Lemma 4 and (6.8), we deduce that

$$\begin{aligned} I_{\ell,1} &\ll \int_\eta^{\eta_1} \exp\biggl\{-\beta\log x + \frac{C_2\beta\log P_y}{\sqrt{\log_2 P_y}} + \frac12\log_3 P_y\biggr\}\,\mathrm{d}\beta \ll \frac{x^{-\eta/2}}{\log x}, \\ I_{\ell,2} &\ll \int_{\eta_1}^{\eta_2} \exp\biggl\{-\beta\log x + \frac{C_2\beta\log P_y}{\log(1/\beta)}\biggr\}\,\mathrm{d}\beta \ll \frac{x^{-\eta_1/2}}{\log x}, \\ I_{\ell,3} &\ll \int_{\eta_2}^{\varepsilon} \exp\biggl\{-\beta\log x + \frac{y}{\log y}\biggr\}\,\mathrm{d}\beta \ll \frac{x^{-\eta_2/2}}{\log x}. \end{aligned}$$

Hence, all of them satisfy

$$I_{\ell,i} = o\bigl((\log x)^{-1}\bigr) \qquad (i=1,2,3).$$

If we assume that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ have no zeros in the region (5.4) for all $y \geq 100$, then $I_{\ell,0} = 0$. Otherwise we use Lemma 3 to ensure the existence of $\{y_n\}_{n=1}^{\infty}$ such that $I_{\ell,0} = 0$.

With (6.10), our conclusion is

$$T_\ell(x_n, y_n) = o\bigg(\frac{x_n}{(\log x_n)2^{\pi(y_n)}}\bigg) \qquad (n \to \infty),$$

or

$$T_\ell(x, y) = o\bigg(\frac{x}{(\log x)2^{\pi(y)}}\bigg) \qquad (x \to \infty)$$

under the assumption that both $\mathbf{L}_1(s, P_y)$ and $\mathbf{L}_4(s, P_y)$ have no exceptional zeros. Clearly this and (6.9) imply the required result. This completes the proof of Proposition. $\square$

Now we are ready to prove Theorem 3.

Taking $Q_n = x_n \log x_n$ and $y(x) = 100\delta \log x$ in Proposition and noticing that $p \in \mathbb{P}_y \Rightarrow n_{\chi_p} \geq y$, we have

$$\sum_{\substack{(Q_n/\log Q_n)^{1/2} < p \leq Q_n \\ n_{\chi_p} \geq 100\delta \log Q_n}} 1 \gg Q_n e^{-c_1 (\log Q_n)/\log_2 Q_n}.$$

It implies the first assertion of Theorem 3, and the second one can be treated similarly. This concludes Theorem 3. $\square$

§ 7. Proof of Theorem 4

Let $1 < c < \frac{32}{29}$ and ε be an arbitrary but sufficiently small positive constant. The upshot is to show

$$n_{\chi_{p'},c} \ll n_{\chi_{p'}}^{9/(16-10c)+\varepsilon} \tag{7.1}$$

whenever $n_{\chi_{p'}} \geq N_0(c, \epsilon)$ for some suitably large constant $N_0(c, \varepsilon)$ depending only on c and ε. Once (7.1) is established, the required results follow from Burgess' upper bound (1.5) or (1.11).

To prove (7.1), we make use of the observation that the integer $mn_{\chi_{p'}}$ is quadratic non-residue for any integer $m < n_{\chi_{p'}}$. Now, we want to find a positive M $(< \frac{1}{2}n_{\chi_{p'}})$ as small as possible such that

$$[n^c] = mn_{\chi_{p'}} \tag{7.2}$$

for some integers $m \in (M, 2M]$ and $n > 1$. This implies

$$n_{\chi_{p'},c} \ll (Mn_{\chi_{p'}})^{1/c} \tag{7.3}$$

which leads to (7.1) with a suitable estimate on M.

Apparently, (7.2) is equivalent to

$$(mn_{\chi_{p'}})^{1/c} \leq n < (mn_{\chi_{p'}} + 1)^{1/c}. \tag{7.4}$$

Denote by $\{x\}$ the fractional part of x. Then (7.4) holds if

$$0 < \{(mn_{\chi_{p'}} + 1)^{1/c}\} \leq (2^{1/c-2}/c)(Mn_{\chi_{p'}})^{1/c-1} =: \Delta < 1 \quad (c > 1), \tag{7.5}$$

since

$$(mn_{\chi_{p'}}+1)^{1/c}-(mn_{\chi_{p'}})^{1/c}\ge (1/c)(2Mn_{\chi_{p'}})^{1/c-1}.$$

Let $\delta_\Delta(t)$ be the periodic function of period 1 such that $\delta_\Delta(t)=1$ if $t\in(0,\Delta]$ and $=0$ if $t\in(\Delta,1]$. Then (7.5) will follow from

$$\sum_{M<m\le 2M}\delta_\Delta\big((mn_{\chi_{p'}}+1)^{1/c}\big)>0. \tag{7.6}$$

Introducing the function $\psi(t):=\frac12-\{t\}$, we can express

$$\delta_\Delta(t)=\Delta+\psi(\Delta-t)-\psi(-t).$$

Thus we have

$$\sum_{M<m\le 2M}\delta_\Delta\big((mn_{\chi_{p'}}+1)^{1/c}\big)=\Delta M+R,$$

where

$$R:=\sum_{M<m\le 2M}\Big(\psi\big(\Delta-(mn_{\chi_{p'}}+1)^{1/c}\big)-\psi\big(-(mn_{\chi_{p'}}+1)^{1/c}\big)\Big).$$

Consider respectively

$$f(t)=\Delta-((M+t)n_{\chi_{p'}}+1)^{1/c},\qquad f(t)=-((M+t)n_{\chi_{p'}}+1)^{1/c}.$$

Then the treatment of R is reduced to the sum $\sum_{M<m\le 2M}\psi(f(m))$, which can be handled using a recent result in [22] via third derivative of $f(t)$. Applying Theorem 2 of [22], we obtain

$$R\ll_{c,\varepsilon}\Big\{M\big(M^{1/c-3}n_{\chi_{p'}}^{1/c}\big)^{3/19}+M^{3/4}+\big(M^{1/c-3}n_{\chi_{p'}}^{1/c}\big)^{-1/3}\Big\}M^{\varepsilon^2}.$$

Thus (7.6) will hold provided

$$M^{1-\varepsilon}\ge n_{\chi_{p'}}^{(19c-16)/(16-10c)}.$$

Taking $M=n_{\chi_{p'}}^{(19c-16)/(16-10c)+\varepsilon}$, it follows that

$$R\le C_0(c,\varepsilon)n_{\chi_{p'}}^{\varepsilon(10c-16)/19c}M^{\varepsilon^2}\Delta M$$

for $n_{\chi_{p'}}\ge N_1(c,\varepsilon)$ where $C_0(c,\varepsilon)$ and $N_1(c,\varepsilon)$ are absolute constants depending only on c and ε. The hypothesis $1<c<\frac{32}{29}$ yields that $M<\frac12 n_{\chi_{p'}}$ for all sufficently large $n_{\chi_{p'}}$. Furthermore, this hypothesis ensures that the exponent of $n_{\chi_{p'}}$ is negative and hence R is suppressed by ΔM for all large $n_{\chi_{p'}}$. Consequently, we derive (7.6) for $n_{\chi_{p'}}\ge N_2(c,\epsilon)$, and therefore (7.1) by inserting the value of M into (7.3). The proof of Theorem 4 is thus complete. □

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